Show all work on problems 5-9 for partial credit.

No notes, no books, no calculators, no cell phones, no pagers, no electronic devices of any kind.

Name____

Circle your Discussion Section:

DIS 301	9:55	T	B305 VAN VLECK
DIS 302	9:55	R	115 INGRAHAM
DIS 305	1:20p	T	B105 VAN VLECK
DIS 306	1:20p	R	B333 VAN VLECK

Problem	Points	Score
1	9	
2	10	
3	12	
4	10	
5	12	
6	10	
7	12	
8	15	
9	10	
Total	100	

Solutions will be posted shortly after the exam:
www.math.wisc.edu/~miller/m240

1. (9 pts) Match each of these with the statement which is closest to its negation.

Circle the letter: (a) (b) (c) of the statement closest to the **negation** of the given statement.

1. Today is Wednesday.
   - (a) Tomorrow is Wednesday.
   - (b) Yesterday was not Tuesday.
   - (c) Today is never Wednesday.
2. There is pollution in Lake Michigan.
   - (a) There is no pollution in the Great Lakes.
   - (b) Lake Erie is really polluted.
   - (c) Lake Michigan is not polluted.
3. In the summer in Wisconsin, there are many mosquitos.
   - (a) It is too cold in the winter in Wisconsin to have mosquitos.
   - (b) There are very few mosquitos in June, July, and August in Wisconsin.
   - (c) The mosquito is not really the state bird of Wisconsin.

2. (10 pts) Enter the correct letter a,b,c,d,e in each blank:

1. ________ $\equiv$ $A \setminus B = A$
2. ________ $\equiv$ $(A \setminus B) = (B \setminus A)$
3. ________ $\equiv$ $A \cap \overline{B} = A \setminus B$
4. ________ $\equiv$ $A \cap B = B$
5. ________ $\equiv$ $B \cup A = B$

(a) $A = B$

(b) $A \subseteq B$

(c) $A \cap B = \emptyset$

(d) $B \subseteq A$

(e) none of above

3. (12 pts)

The domain U for the variables and the relation C consists of the students in our class. The relation $C(x, y)$ says that x has copied the homework of y who is different from x.

For each logical formula choose a the best match in English and put the letter A B C D E F in the blank provided.

1. ________ $\equiv$ $\exists x \exists y \ \ C(x, y)$

2. ________ $\equiv$ $\forall x \exists y \ \ C(x, y)$

3. ________ $\equiv$ $\forall y \exists x \ \ C(x, y)$

4. ________ $\equiv$ $\exists x \forall y \ \ (C(x, y) \vee x = y)$

5. ________ $\equiv$ $\forall x \forall y \ \ (C(x, y) \vee x = y)$

6. ________ $\equiv$ $\exists y \forall x \ \ (x \neq y \rightarrow C(x, y))$

(A) Everybody in class has cheated off somebody.

(B) Somebody cheats off everybody else.

(C) There is a genius that everybody else cheats off of.

(D) There is at least one cheater.

(E) Everybody does all there homework together.

(F) Everybody's homework is copied by somebody.

4. (10 pts) The set $A = \{a, b\}$ has two elements.

(a) List the set $\mathcal{P}(A)$ here:

(b) How many elements does the set $\mathcal{P}(\mathcal{P}(A))$ have? Enter the number here: ______.

5. (12 pts) Determine which of the following sets are countable and which are uncountable. Circle the word countable or uncountable.

1. The negative integers.	countable	uncountable
2. The real line, $\mathbb{R}$.	countable	uncountable
3. The plane, $\mathbb{R}^2$.	countable	uncountable
4. Set of integers which are perfect squares.	countable	uncountable
5. Power set of the natural numbers, $\mathcal{P}(\mathbb{N})$.	countable	uncountable

For each of the sets that is countable exhibit a map from the natural numbers $\mathbb{N}$ onto the set.

6. (10 pts) Show that

$$(p \rightarrow q) \vee (p \rightarrow r)$$

is logically equivalent to

$$p \rightarrow (q \vee r).$$

7. (12 pts) Suppose that the domain U of the predicate $R(x, y)$ consists of the just two distinct objects $U = \{a, b\}$. Write out a propositional sentence which is equivalent to the predicate sentence:

$$\forall x \; \exists y \;\; R(x, y).$$

Hint: Your answer can use disjunctions and conjunctions, but no quantifiers.

8. (15 pts) Use a proof by contradiction to show that there is no rational number r such that $r^3 + 3r + 5 = 0$.

Hint: assume $r = a/b$ is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Show that the three cases each lead to a contradiction:

(1) a odd and b odd.
(2) a even and b odd.
(3) a odd and b even.

9. (10 pts) What is the difference between a constructive and nonconstructive existence proof? Give an example of each.

Answers

1. 1b 2c 3b The statement “Today is Wednesday” is the same as the statement “Yesterday was Tuesday”.

2. 1c 2a 3e 4d 5b

3. 1D 2A 3F 4B 5E 6C

4. (a) $\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$ (b) 16

5.

(1) the set of negative integers is countable. The mapping defined by $n \mapsto -(n+1)$ takes $\mathbb{N}$ onto the negative integers.

(4) The set of integers which are perfect squares is countable. The mapping defined by $n \mapsto n^2$ takes $\mathbb{N}$ onto it.

The other sets are uncountable.

6. They have the same truth table. All lines are true except p true, q false, r false.

7. $(R(a,a) \vee R(a,b)) \wedge (R(b,a) \vee R(b,b))$

8. First we prove a

Lemma The product of two integers is odd iff both are odd. Similarly, the sum of two integers is even iff both are even or both odd.

proof: If a and b are odd then there are integers n and m such that $a = 2n+1$ and $b = 2m+1$. But then

$$ab = (2n+1)(2m+1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$$

and hence it is odd. Conversely if either a or b is even, say a is even, then there exists an integer n such that $a = 2n$ and so $ab = 2nb$ and so ab is even.

The sums are proved similarly.

QED

It follows immediately that the product of three odd integers is odd.

Now suppose that $r^3+3r+5=0$ and for contradiction, that r is rational. Then let $r = a/b$ where a and b are integers and not both of them are even. Then

$$(\frac{a}{b})^3 + 3(\frac{a}{b}) + 5 = 0 \text{ so } a^3 + 3ab^2 + 5b^3 = 0$$

Case (1) a odd and b odd. In this case since the product of odd numbers is odd each of a^3, $3ab^2$, $5b^3$ is odd. Then since the sum of 3 odd numbers is odd, they cannot total 0 which is even.

Case (2) a even and b odd. In this case a^3 and $3ab^2$ are even and $5b^3$ is odd. Hence the total is odd and cannot be 0.

Case (3) a odd and b even. In this case a^3 is odd and both $3ab^2$ and $5b^3$ are even. Hence, again the total is odd and so cannot add up to 0.

QED

9. A proof of a statement of the form $\exists x\ P(x)$ is constructive if it actually produces an x satisfying the statement $P(x)$. A nonconstructive proof doesn't produce a particular x. An example, of a nonconstructive proof is the proof given in the book on page 91 of the existence of two irrational numbers x and y such that x^y is rational. An example of a constructive proof is showing that 223 cannot be written as the sum of 36 fifth powers of nonnegative integers, exercise 39 on page 108.