Show all work. Circle your answer.

No notes, no books, no calculator, no cell phones, no pagers, no electronic devices at all.

Solutions will be posted shortly after the exam: www.math.wisc.edu/∼miller/m240

Name________________________________

Problem	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	10	
10	8	
11	8	
12	10	
Total	100	

1. (8 pts) Prove that $3^n \geq 3n + 1$ for every integer $n \geq 2$.

2. (8 pts) Construct a truth table for $((P \vee Q) \wedge (P \rightarrow R))$

3. (8 pts) Find $d = \gcd(45, 39)$ and find integers k and l so that $d = 45k + 39l$.

4. (8 pts) A prime triple is a prime p such that p, $p+2$, $p+4$ are all prime numbers. Prove that $3, 5, 7$ is the only prime triple. (Note: 1 is not a prime number and $1, 3, 5$ is not a prime triple.)

5. (8 pts) How many words of length 8 are there such that they are made up of the 26 letters $abc \ldots z$ and they contain exactly two a's and three b's and no other letter is repeated?

6. (8 pts) What is the probability of a seven card flush? This means 7 cards are dealt randomly out of a standard deck of 52 and all 7 cards have the same suit.

7. (8 pts) Find the general solution of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

8. (8 pts) Let $R \subseteq A \times A$ be a binary relation on the set A. Define

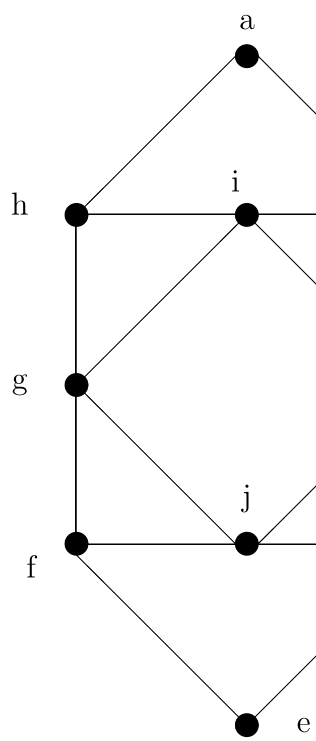
(a) R is reflexive

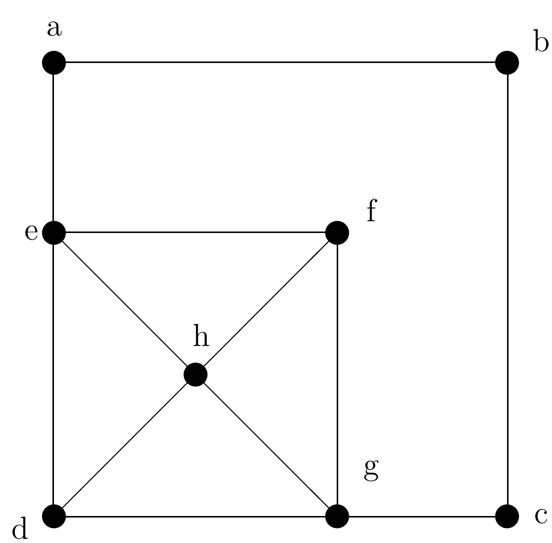
(b) R is transitive

(c) R is symmetric

(d) R is antisymmetric

9. (10 pts) In the two graphs below determine whether they have an Euler path. For each one either give such a path, or state the reason it doesn't have one.

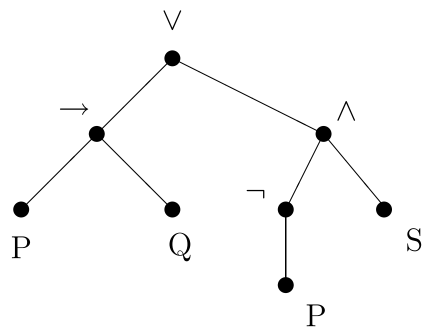




10. (8 pts) State Euler's formula relating the number of faces $|F|$, vertices $|V|$ and edges $|E|$ for a connected planar graph. Use it to prove that the complete graph K_5 on five vertices is not planar.

Hint: Use that each face must be bounded by three or more edges to write an inequality relating $|F|$ and $|E|$.

11. (8 pts) The parse tree below is determined by a formula of propositional logic.



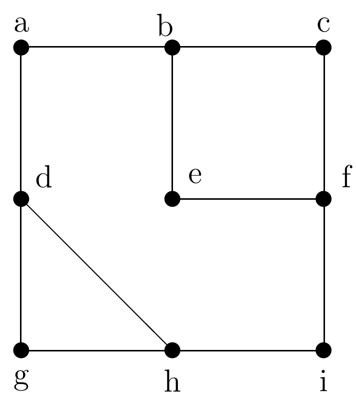
(a) Use inorder traversal to determine the infix formula.

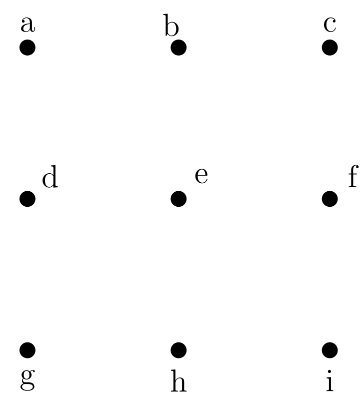
(b) Use preorder traversal to determine the prefix formula (Polish notation).

(c) Use postorder traversal to determine the postfix formula (reverse Polish notation).

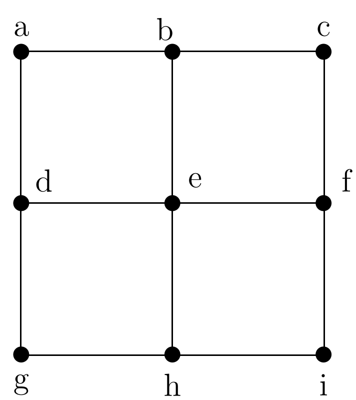
12. (10 pts) For each graph, choose a as the root and use alphabetical order on the vertices and produce a spanning tree by

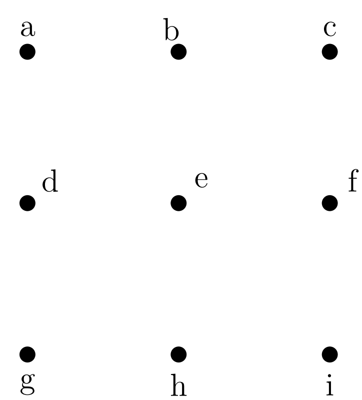
(a) using depth-first search. Draw in edges on right.





(b) using breath-first search. Draw in edges on right





Answers

1.
Base Case: $n = 2$.

$$3^2 = 9 \geq 7 = 3 \cdot 2 + 1$$

Inductive Step: Assume $n \geq 2$ and $3^n \geq 3n + 1$ then

$$3^{n+1} = 3 \cdot 3^n \geq 3(3n + 1) = 3((n + 1) + 2n) = 3(n + 1) + 6n \geq 3(n + 1) + 1$$

The last inequality is true since $6n \geq 1$ for $n \geq \frac{1}{6}$. It follows by transitivity that

$$3^{n+1} \geq 3(n + 1) + 1$$

as we needed to show.

2.

p	q	r	$(p \vee q) \wedge (p \to r)$		
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	F	T

3. $\gcd(45, 39) = 3 = (6)45 + (-7)39$

4. Given any integer p, one of p, $p+2$, or $p+4$ is divisible by 3. If p is not divisible by 3, then p must be either 1 or 2 mod 3. If it is 1 mod 3 then $p+2$ is 3 mod 3 hence divisible by 3. If it is 2 mod 3 then $p + 4$ is 6 mod 3 hence divisible by 3. If follows that the only prime triple is 3, 5, 7.

5.

$$\binom{8}{2}\binom{6}{3} \cdot 24 \cdot 23 \cdot 22$$

6.

$$\frac{4 \cdot \binom{13}{7}}{\binom{52}{7}}$$

7. $a_n = A\, 2^n \;+\; B\, 3^n$ where A and B are arbitrary constants.

8. (a) $\forall x \in A \ \ (x,x) \in R$
(b) $\forall x,y,z \in A \ \ (x,y),(y,z) \in R \to (x,z) \in R$
(c) $\forall x,y \in A \ \ (x,y) \in R \to (y,x) \in R$
(d) $\forall x,y \in A \ \ (x,y),(y,x) \in R \to x = y$

9. The first graph has no Euler path because it has four vertices of odd degree. The second graph has an Euler path starting at d and ending at f (or vice-versa) and visiting each of the 12 edges exactly once. For example: deabcgdgfehgf

10. $|F| + |V| = |E| + 2$. For K_5 we have $|V| = 5$ and $|E| = \binom{5}{2} = 10$ and so $|F| = 7$ For each face $f \in F$ let edges(f) be set of edges which bound the face. Then

$$2|E| = \sum_{f \in F} |\text{edges}(f)|$$

this is because each edge is lies on two faces. Since every face is surrounded by at least three edges we have

$$\sum_{f \in F} |\text{edges}(f)| \geq \sum_{f \in F} 3 = 3|F|$$

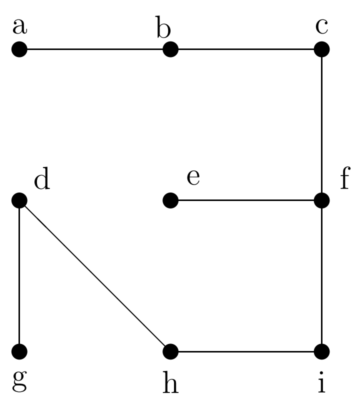
and so

$$20 = 2|E| \geq 3|F| = 21$$

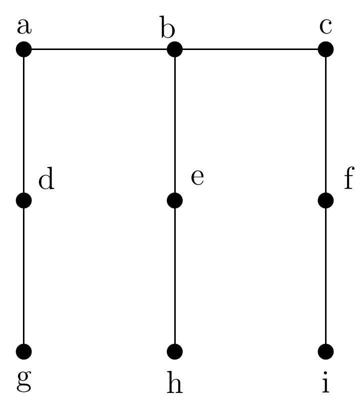
which is a contradiction.

11. (a) $(P \to Q) \vee (\neg P \wedge S)$
(b) $\vee \to PQ \wedge \neg PS$
(c) $PQ \to P \neg S \wedge \vee$

12.



(a)



(b)