

Exam 1

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Math 222

1. Find the integral $\int \frac{3x^2+3x+1}{2x^3+2x^2+x} dx$.
2. Find the integral $\int \frac{x^2}{(1+x^2)^{\frac{3}{2}}} dx$.
3. Find the integral $\int x^5 \cos(x^3) dx$.
4. Find the integral $\int \frac{dx}{1+\sin(x)}$.
5. Find the integral $\int \sqrt{x} \ln(x) dx$.
6. Find the integral $\int \sin(2x) \cos(x) dx$.
7. Find the general solution of $\frac{dy}{dx} = \frac{\sec^3(y)}{4-x^2}$. (You do not need to solve for y explicitly.)
8. Find the arclength of the curve $y = 2e^{\frac{x}{2}}$ for $x = 0$ to $x = 1$.
9. Determine whether the improper integral $\int_1^\infty \frac{x}{\sqrt{x^3+1}} dx$ is convergent or divergent and explain why.
10. If the curve $y = \frac{x^3}{3}$ for x between 0 and 1 is rotated about the x -axis, what is the resulting surface area?

Bonus Extra Credit 10 point Problem. Work only if you have done the problems 1-10 completely. Show how to derive Simpson's (local) rule:

$$\int_x^{x+2h} f(t) dt \approx \frac{h}{3} (f(x) + 4f(x+h) + f(x+2h))$$

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as expressions of t for the parameterized curve

$$x = t^2 + t$$

$$y = t^2 + 1$$

2. Find the length of the curve

$$x = e^t - t$$

$$y = 4e^{t/2}$$

$$0 \leq t \leq 1$$

3. Find the area enclosed by one loop of the (polar coordinate) curve

$$r = \sin(5\theta)$$

4. Find the value of

$$\sum_{n=0}^{\infty} 4\left(\frac{2}{3}\right)^n$$

or show that it diverges and explain why.

5. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4 + 1}$$

converges or diverges. Explain why.

6. Determine whether the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{n}$$

converges absolutely, converges conditionally, or diverges. Explain why.

7. Determine whether the series

$$\sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{n}{n+1}\right)^2$$

converges absolutely, converges conditionally, or diverges. Explain why.

8. Find the radius of convergence and interval of convergence of the series:

$$\sum_{n=0}^{\infty} \frac{3^n (x+1)^n}{n+1}$$

9. Find the power series ($\sum_{n=0}^{\infty} a_n x^n$) representation for the function

$$g(x) = \frac{1}{1+9x^2}$$

and determine the interval of convergence.

10. Find the power series ($\sum_{n=0}^{\infty} a_n x^n$) representation for the function

$$f(x) = xe^{-x}$$

What is $f^{(100)}(0)$? (This is the 100th derivative at zero.)

1. Find $\int \sec^3(x)dx$. Show all details.
2. Find $\int \frac{x^3}{x^2-1}dx$.
3. Find the values of p for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

is convergent.

4. Find an equation $ax + by + cz = d$ for the plane consisting of all points which are equidistant from the points $(1, -1, 1)$ and $(0, 1, 3)$.
5. Find the volume of the parallelopiped determined by the vectors $\mathbf{i} + \mathbf{j}$, $\mathbf{i} - \mathbf{k}$ and $\mathbf{j} + 2\mathbf{k}$.
6. Find the velocity, acceleration, and speed of a particle with the position function:

$$\mathbf{r}(t) = e^t \mathbf{i} + e^{-t} \mathbf{j} + t \mathbf{k}$$

7. Which of the following are always true, and which are at least sometimes false? ($\{a_n\}$ denotes an arbitrary sequence of real numbers. $\sum a_n$ denotes an arbitrary series.)

Circle either T or F.

- | | |
|--------|---|
| T or F | If $a_n > 0$ for all n and $a_n \rightarrow L$, then $L > 0$. |
| T or F | If $a_n \geq 0$ for all n and $a_n \rightarrow L$, then $L \geq 0$. |
| T or F | If $\{a_n\}$ is bounded, then it converges. |
| T or F | If $\{a_n\}$ is not bounded, then it diverges. |
| T or F | If $\{a_n\}$ is decreasing, then it converges. |
| T or F | If $\{a_n\}$ is decreasing and $a_n > 0$ for all n , then it converges. |
| T or F | If $\{a_n\}$ is neither increasing nor decreasing, then it diverges. |
| T or F | If $a_n > 0$ for all n and $(-1)^n a_n \rightarrow L$, then $L = 0$. |
| T or F | If $\sum a_n$ converges, then $a_n \rightarrow 0$. |
| T or F | If $a_n \rightarrow 0$, then $\sum a_n$ converges. |
| T or F | If $\sum a_n$ and $\sum b_n$ diverge, then $\sum(a_n + b_n)$ diverges. |

- T or F If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ diverges.
- T or F If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n b_n) = (\sum a_n)(\sum b_n)$.
- T or F If $\{a_n\}$ is monotonic and bounded, then $\sum a_n$ converges.
- T or F If the partial sums of $\sum a_n$ are bounded, then $\sum a_n$ converges.
- T or F If $a_n \geq 0$ and the partial sums of $\sum a_n$ are bounded, then $\sum a_n$ converges.
- T or F $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ whenever $x \neq 1$.
- T or F If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a nonzero radius of convergence, then f is differentiable and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
- T or F If $\sum a_n x^n$ converges when $x = 1$, then it must also converge when $x = -\frac{1}{2}$.
- T or F If $a_n > 0$ and both limits $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L_1$ and $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L_2$ exist, then $L_1 = L_2$.
- T or F If $f(x)$ is a continuous positive function on the interval $[1, \infty)$ and

$$\lim_{x \rightarrow \infty} f(x) = 0,$$

then $\int_1^{\infty} f(x) dx$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges.

8. Circle all of the following vectors which are orthogonal to the vector $\langle 1, -1, 2 \rangle$

- $\langle -1, 1, -2 \rangle$
- $2\mathbf{i} + 2\mathbf{j}$
- $\mathbf{i} - \mathbf{j}$
- $\langle 0, 2, 1 \rangle$

9. Circle all the vector equations of a line below which contain the two points $(0, 1, 1)$ and $(1, -1, 1)$.

- $\mathbf{r}(t) = (\mathbf{j} + \mathbf{k}) + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$
- $\mathbf{r}(t) = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{j} + \mathbf{k})$
- $\mathbf{r}(t) = (\mathbf{j} + \mathbf{k}) + t(\mathbf{i} - 2\mathbf{j})$
- $\mathbf{r}(t) = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{i} - 2\mathbf{j})$
- $\mathbf{r}(t) = (\mathbf{i} + 2\mathbf{k}) + t(-2\mathbf{i} + 4\mathbf{j})$

10. For each of the following series fill in the blank with exactly one of the answers a, b, c, d or e which stand for:

- a) divergent b) conditionally convergent c) absolutely convergent

$\sum \frac{1}{n}$ is _____.

$\sum \frac{(-1)^n}{n}$ is _____.

$\sum \frac{1}{n^2}$ is _____.

$\sum \frac{(-1)^n}{n^2}$ is _____.

$\sum \frac{1}{\sqrt{n}}$ is _____.

$\sum \frac{(-1)^n}{\sqrt{n}}$ is _____.

In the remaining problems fill in the blank.

11. Suppose $f(x) = \sum_{n=0}^{\infty} \frac{n}{n+1} x^n$. Then $f^{(13)}(0) =$ _____.

12. The distance between the points $(1, 1, -2)$ and $(0, 2, 0)$ is _____.

13. If $r = 3$ and $\mathbf{v} = \langle 1, -2, 3 \rangle$ then $r\mathbf{v}$ is _____.

14. $|\langle 1, 2, -3 \rangle| =$ _____.

15. The angle between the planes $x + 2y + z = 4$ and $x - 2y + 2z = 5$ is _____.

Answers

Exam 1

1. Partial fractions. $\frac{1}{4} \ln |2x^2 + 2x + 1| + \frac{1}{2} \arctan(2x + 1) + \ln |x| + C$
2. Substitute $x = \tan(\theta)$. $\ln |x + \sqrt{1 + x^2}| - \frac{x}{\sqrt{1 + x^2}} dx + C$
3. Substitute $u = x^3$ then do by parts. $\frac{1}{3} \cos(x^3) + \frac{1}{3} x^3 \sin(x^3) + C$
4. Multiply by $\frac{1 - \sin(x)}{1 - \sin(x)}$ and simplify. $\tan(x) - \sec(x) + C$
5. Substitute $u^2 = x$ then by parts. $\frac{2}{3} x^{\frac{3}{2}} \ln |x| - \frac{4}{9} x^{\frac{3}{2}} + C$
6. $-\frac{2}{3} \cos^3(x) + C$
7. $\sin(y) - \frac{1}{3} \sin^3(y) = \frac{1}{4}(-\ln |2 - x| + \ln |2 + x|) + C$.
8. Substitute $u^2 = 1 + e^x$ in the integral $\int_0^1 \sqrt{1 + e^x} dx$.

$$(2\sqrt{1+e} + \ln(\sqrt{1+e}-1) - \ln(\sqrt{1+e}+1)) - (2\sqrt{2} + \ln(\sqrt{2}-1) - \ln(\sqrt{2}+1))$$

9. It diverges because

$$\frac{x}{\sqrt{x^3+1}} \geq \frac{x}{\sqrt{x^3+x^3}} = \frac{1}{\sqrt{2}} \frac{1}{x^{\frac{1}{2}}}$$

and

$$\int_1^\infty \frac{1}{x^{\frac{1}{2}}} dx = \lim_{b \rightarrow \infty} 2b^{1/2} - 2 = \infty$$

10. $\int_0^1 2\pi \frac{x^3}{3} \sqrt{1+x^4} dx = \frac{\pi}{9} (2^{\frac{3}{2}} - 1)$.

Exam 2

1. $\frac{dy}{dx} = \frac{2t}{2t+1}, \frac{d^2y}{dx^2} = \frac{2}{(2t+1)^3}$.
2. e
3. $\frac{\pi}{20}$.
4. 12
5. Converges, explain why.
6. Converges conditionally, explain why.
7. Converges absolutely, explain why.
8. $\frac{1}{3}, [-\frac{4}{3}, -\frac{2}{3})$
9. $\frac{1}{3}, (-\frac{1}{3}, \frac{1}{3})$
10. $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n-1)!} x^n, -100$.

Final

1. $\frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) + C$
2. $\frac{1}{2}(x^2 + \ln|x^2 - 1|) + C$
3. $p > 1$
4. $-2x + 4y + 4z = 7$
5. 1
6. $\mathbf{v} = e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}$, $\mathbf{a} = e^t\mathbf{i} + e^{-t}\mathbf{j}$, $\sqrt{e^{2t} + e^{-2t} + 1}$
7. ftftf tfttf ftfff ttfft tf
8. $2\mathbf{i} + 2\mathbf{j} < 0, 2, 1 >$
9. $\mathbf{r}(t) = (\mathbf{j} + \mathbf{k}) + t(\mathbf{i} - 2\mathbf{j})$ $\mathbf{r}(t) = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{i} - 2\mathbf{j})$
10. column 1: aca, column 2: bcb
11. 5782233600
12. $\sqrt{6}$
13. $\langle 3, -6, 9 \rangle$
14. $\sqrt{14}$
15. 82.2 degrees or 97.8 degrees