Kepler's Law's

Kepler's first law: Planets move in a plane in an ellipse with the sun at one focus.

Kepler's second law: The position vector from the sun to a planet sweeps out area at a constant rate.

Kepler's third law: The square of the period of a planet is proportional to the cube of its mean distance from the sun. The mean distance is the average of the closest distance and the furthest distance. The period is the time required to go once around the sun.

Let $\vec{p} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position of a planet in space where x, y and z are all function of time t. Assume the sun is at the origin. Newton's law of gravity implies that

$$\frac{d^2\vec{p}}{dt^2} = \alpha \frac{\vec{p}}{||\vec{p}||^3} \tag{1}$$

where α is $-GM$, G is a universal gravitational constant and M is the mass of the sun. It does not depend on the mass of the planet.

First let us show that planets move in a plane. By the product rule

$$\frac{d}{dt}(\vec{p} \times \frac{d\vec{p}}{dt}) = (\frac{d\vec{p}}{dt} \times \frac{d\vec{p}}{dt}) + (\vec{p} \times \frac{d^2\vec{p}}{dt^2}) \tag{2}$$

By (1) and the fact that the cross product of parallel vectors is $\vec{0}$ the right hand side of (2) is $\vec{0}$. It follows that there is a constant vector $\vec{c}$ such that at all times

$$\vec{p} \times \frac{d\vec{p}}{dt} = \vec{c} \tag{3}$$

Thus we can conclude that both the position and velocity vector lie in the plane with normal vector $\vec{c}$. Without loss of generality we assume that $\vec{c} = \beta\vec{k}$ for some scaler β and $\vec{p} = x\vec{i} + y\vec{j}$. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$ where we consider r and θ as functions of t. If we calculate the derivative of $\vec{p}$ we get

$$\frac{d\vec{p}}{dt} = \left[\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}\right]\vec{i} + \left[\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}\right]\vec{j} \tag{4}$$

Since $\vec{p} \times \frac{d\vec{p}}{dt} = \beta\vec{k}$ we have

$$r\cos(\theta)(\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}) - r\sin(\theta)(\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}) = \beta \tag{5}$$

After multiplying out and simplifying this reduces to

$$r^2\frac{d\theta}{dt} = \beta \tag{6}$$

The area swept out from time t_0 to time t_1 by a curve in polar coordinates is

$$A = \frac{1}{2}\int_{t_0}^{t_1} r^2\frac{d\theta}{dt}dt \tag{7}$$

By (6) A is proportional to $t_1 - t_0$. This is Kepler's second law.

We will now prove Kepler's third law for the special case of a circle. So let T be the time it takes the planet to go around the sun one time and let r be its distance from the sun. We will show that

$$\frac{T^2}{r^3} = -\frac{(2\pi)^2}{\alpha} \tag{8}$$

The second law implies that $\theta(t)$ is a linear function of t and so in fact

$$\frac{d\theta}{dt} = \frac{2\pi}{T} \tag{9}$$

Since r is constant we have that $\frac{dr}{dt} = 0$ and so (4) simplifies to

$$\frac{d\vec{p}}{dt} = \left[-r\sin(\theta)\frac{2\pi}{T}\right]\vec{i} + \left[r\cos(\theta)\frac{2\pi}{T}\right]\vec{j} \tag{10}$$

Differentiating once more we get

$$\frac{d^2\vec{p}}{dt^2} = \left[-r\cos(\theta)\left(\frac{2\pi}{T}\right)^2\right]\vec{i} + \left[-r\sin(\theta)\left(\frac{2\pi}{T}\right)^2\right]\vec{j} = -\left(\frac{2\pi}{T}\right)^2\vec{p} \tag{11}$$

Noting that $r = ||\vec{p}||$ and using (1) we get

$$\frac{\alpha}{r^3} = -\left(\frac{2\pi}{T}\right)^2 \tag{12}$$

from which (8) immediately follows.

Complete derivations of the three laws from Newton's law of gravity can be found in T.M.Apostal, Calculus vol I , Blaisdel(1967), p.545-548. Newton deduced the law of gravity from Kepler's laws. The argument can be found in L.Bers, Calculus vol II , Holt,Rinhart,and Winston(1969), p.748-754.

The planet earth is 93 million miles from the sun. The year has 365 days. The moon is 250,000 miles from the earth and circles the earth once every 28 days. The earth's diameter is 7850 miles. In the first four problems you may assume orbits are circular. Use only the data in this paragraph.

701. The former planet Pluto takes 248 years to orbit the sun. How far is Pluto from the sun?

702. A communication satellite is to orbit the earth around the equator at such a distance so as to remain above the same spot on the earth's surface at all times. What is the distance from the center of the earth such a satellite should orbit?

703. Find the ratio of the masses of the earth and the sun.

704. The Kmart7 satellite is to be launched into polar earth orbit by firing it from a large cannon. This is possible since the satellite is very small, consisting of a single blinking blue light. Polar orbit means that the orbit passes over both the north and south poles. Let p(t) be the point on the earth's surface at which the blinking blue light is directly overhead at time t. Find the largest orbit that the Kmart7 can have so that every person on earth will be within 1000 miles of p(t) at least once a day. You may assume that the satellite orbits the earth exactly n times per day for some integer n.

705. Let A be the total area swept out by an elliptical orbit. Show that $\beta = \frac{2A}{T}$.

706. Let E be an ellipse with one of the focal points f. Let d be the minimum distance from some point of the ellipse to f and let D be the maximum distance. In terms of d and D only what is the area of the ellipse E?

Hint: The area of an ellipse is πab where a is its minimum diameter and b is its maximum diameter. If f_1 and f_2 are the focal points of E then the sum of the distances from f_1 to p and f_2 to p is constant for all points p on E.

707. Halley's comet goes once around the sun every 77 years. Its closest approach is 53 million miles. What is its furthest distance from the sun? What is the maximum speed of the comet and what is the minimum speed?