601. You are watching a buoy bobbing up and down in the water. Assume that the buoy height with respect to the surface level of the water satisfies the damped oscillator equation: $z'' + bz' + kz \equiv 0$ where b and k are positive constants.

Something has initially disturbed the buoy causing it to go up and down, but friction will gradually cause its motion to die out.

You make the following observations: At time zero the center of the buoy is at $z(0) = 0$, i.e., the position it would be in if it were at rest. It then rises up to a peak and falls down so that at time $t = 2$ it again at zero, $z(2) = 0$ descends downward and then comes back to 0 at time 4, i.e, $z(4) = 0$. Suppose $z(1) = 25$ and $z(3) = -16$.

(a) How high will z be at time $t = 5$

(b) What are b and k?

Hint: Use that $z = Ae^{\alpha t}\sin(\omega t + B)$.

Answer

(a) $z(5) = 10.24$

(b) $b = -2\ln\frac{4}{5}$ and $k = \frac{b^2+\pi^2}{4}$

End Ans

602. Contrary to what one may think the buoy does not reach its peak at time $t = 1$. For example, suppose $z = e^{-t}\sin t$. Then z is zero at both $t = 0$ and $t = \pi$. Does z have a local maximum at $t = \frac{\pi}{2}$?

Answer

No. Differentiating gives $z' = e^{-t}(\cos t - \sin t) = 0$ which is zero at $\pi/4$.

End Ans

603. In the buoy problem 601 suppose you make the following observations:

It rises up to its first peak at $t = 1$ where $z(1) = 25$ and then descends downward to a local minimum at at $t = 3$ where $z(3) = -16$.

(a) When will the buoy reach its second peak and how high will that be?

(b) What are b and k?

Note: It will not be the case that $z(0) = 0$.

Answer

Its next peak will be at $t = 5$. Its height will be same as in problem 601 and the constants b, k will be the same. The actual solution (determined by the constants A and B) will be different.

End Ans

The remain problems are optional. You do not have to do them. Your TA does not have to explain them to you.

Linear operators

Given a polynomial $p = p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $z = z(t)$ an infinitely differentiable function of t define

$$\mathcal{L}_p(z) = a_0z + a_1z^{(1)} + a_2z^{(2)} + \ldots + a_nz^{(n)}$$

where $z^{(k)} = \frac{d^kz}{dt^k}$ is the k^{th} derivative of z with respect to t.

604. Show for $\mathcal{L} = \mathcal{L}_p$ that

(a) $\mathcal{L}(z_1 + z_2) = \mathcal{L}(z_1) + \mathcal{L}(z_2)$

(b) $\mathcal{L}(Cz) = C\mathcal{L}(z)$ where C is any constant

Such an $\mathcal{L}$ is called a linear operator. Operator because it takes as input a function and then ouputs another function. Linear refers to properties (a) and (b).

605. Let r be any constant and p any polynomial. Show that $\mathcal{L}_p(e^{rt}) = p(r)e^{rt}$.

606. For p and q polynomials show that

$$\mathcal{L}_{p+q}(z) = \mathcal{L}_p(z) + \mathcal{L}_q(z)$$

607. For p and q polynomials show that

$$\mathcal{L}_{p\cdot q}(z) = \mathcal{L}_p(\mathcal{L}_q(z))$$

Here $p \cdot q$ refers to the ordinary product of the two polynomials.

608. Let α be a constant. For any u an infinitely differentiable function of t show that

(a) $\mathcal{L}_{x-\alpha}(u \cdot e^{\alpha t}) = u^{(1)}e^{\alpha t}$

(b) $\mathcal{L}_{(x-\alpha)^n}(u \cdot e^{\alpha t}) = u^{(n)}e^{\alpha t}$

Answer

Hint: $(x - \alpha)^n = (x - \alpha) \cdot (x - \alpha)^{n-1}$

609. Let α be any constant, p a polynomial, and suppose that $(x - \alpha)^n$ divides p. Show that for any $k < n$

$$\mathcal{L}_p(t^ke^{\alpha t}) \equiv 0$$

610. Suppose that $p(x) = (x-\alpha_1)^{n_1} \cdots (x-\alpha_m)^{n_m}$ where the α_i are distinct constants. Suppose that

$$z = C_1^1e^{\alpha_1 t}+C_1^2te^{\alpha_1 t}+\cdots C_1^{n_1}t^{n_1-1}e^{\alpha_1 t}+\cdots+C_m^1e^{\alpha_m t}+C_m^2te^{\alpha_m t}+\cdots C_m^{n_m}t^{n_m-1}e^{\alpha_m t}$$

Show that $\mathcal{L}_p(z) \equiv 0$.

In a more advanced course in the theory of differential equations it would be proved that every solution of $\mathcal{L}_p(z) \equiv 0$ has this form, i.e., z satisfies the above formula for some choice of the constants C_j^i.

611. Suppose $\mathcal{L}$ is a linear operator and $b = b(t)$ is a fixed function of t. Suppose that z_P is one particular solution of $\mathcal{L}(z) = b$, i.e., $\mathcal{L}(z_P) = b$. Suppose that z is any other solution of $\mathcal{L}(z) = b$. Show that $\mathcal{L}(z - z_P) \equiv 0$. Show that for any solution of the equation $\mathcal{L}(z) = b$ there is a solution z_H of the associated homogenous equation such that $z = z_P + z_H$.