Answers and Hints

(34) $\frac{11}{6}$

(36) $\frac{2}{3}$

(47) $-\ln(1+\cos^2(x))+C$

(51) $\frac{1}{4}(\ln(2x^2))^2+C$

(61) (a) $\frac{1}{2}(x\sqrt{1+x^2}+\ln|x+\sqrt{1+x^2}|\,)+C$

(b) $\frac{1}{2}\,\frac{x}{1+x^2}+\frac{1}{2}\,\arctan(x)+C$

(c) $\ln|x+\sqrt{1+x^2}|+C$

(64) $\arcsin(x-1)+C$

(72) $\frac{1}{3}\arctan(x+1)+C$

(75) $\displaystyle\int x^n \ln x \,\mathrm{d}x = \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2}+C.$

(76) $\displaystyle\int e^{ax}\sin bx \,\mathrm{d}x = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+C.$

(77) $\displaystyle\int e^{ax}\cos bx \,\mathrm{d}x = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+C.$

(81) $\int_0^\pi \sin^{14} x dx = \frac{13\cdot11\cdot9\cdot7\cdot5\cdot3\cdot1}{14\cdot12\cdot10\cdot8\cdot6\cdot4\cdot2}\pi$

(82) $\int \cos^n x dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int\cos^{n-2}x dx$; $\int_0^{\pi/4}\cos^4 x dx = \frac{7}{16}+\frac{3}{32}\pi$

(83) Hint: first integrate x^m.

(84) $x\ln x - x + C$

(85) $x(\ln x)^2 - 2x\ln x + 2x + C$

(87) Substitute $u=\ln x$.

(88) $\int_0^{\pi/4}\tan^5 x dx = \frac{1}{4}(1)^4 - \frac{1}{2}(1)^2 + \int_0^{\pi/4}\tan x dx = -\frac{1}{4}+\ln\frac{1}{2}\sqrt{2}$

(95) $1+\frac{4}{x^3-4}$

(96) $1+\frac{2x+4}{x^3-4}$

(97) $1-\frac{x^2+x+1}{x^3-4}$

(98) $\frac{x^3-1}{x^2-1} = x+\frac{x-1}{x^2-1}$. You can simplify this further: $\frac{x^3-1}{x^2-1} = x+\frac{x-1}{x^2-1} = x+\frac{1}{x+1}$.

(99) $x^2+6x+8=(x+3)^2-1=(x+4)(x+2)$ so $\frac{1}{x^2+6x+8}=\frac{1/2}{x+2}+\frac{-1/2}{x+4}$ and $\int\frac{dx}{x^2+6x+8} = \frac{1}{2}\ln(x+2)-\frac{1}{2}\ln(x+4)+C$.

(100) $\int\frac{dx}{x^2+6x+10} = \arctan(x+3)+C$.

(101) $\frac{1}{5}\int\frac{dx}{x^2+4x+5} = \frac{1}{5}\arctan(x+2)+C$

(102) We add

$$\begin{aligned}\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1} &= \frac{A(x+1)(x-1)+Bx(x-1)+Cx(x+1)}{x(x+1)(x-1)}\\ &= \frac{(A+B+C)x^2+(C-B)x-A}{x(x+1)(x-1)}.\end{aligned}$$

The numerators must be equal, i.e.

$$x^2+3=(A+B+C)x^2+(C-B)x-A$$

for all x, so equating coefficients gives a system of three linear equations in three unknowns A, B, C:

$$\begin{cases} A+B+C=1\\ C-B=0\\ -A=3\end{cases}$$

so $A=-3$ and $B=C=2$, i.e.

$$\frac{x^2+3}{x(x+1)(x-1)}=-\frac{3}{x}+\frac{2}{x+1}+\frac{2}{x-1}$$

and hence

$$\int\frac{x^2+3}{x(x+1)(x-1)}\,\mathrm{d}x=-3\ln|x|+2\ln|x+1|+2\ln|x-1|+\text{constant}.$$

(103) To solve

$$\frac{x^2+3}{x(x+1)(x-1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1},$$

multiply by x:

$$\frac{x^2+3}{(x+1)(x-1)}=A+\frac{Bx}{x+1}+\frac{Cx}{x-1}$$

and plug in $x=0$ to get $A=-3$; then multiply by $x+1$:

$$\frac{x^2+3}{x(x-1)}=\frac{A(x+1)}{x}+B+\frac{C(x+1)}{x-1}$$

and plug in $x=-1$ to get $B=2$; finally multiply by $x-1$:

$$\frac{x^2+3}{x(x+1)}=\frac{A(x-1)}{x}+\frac{B(x-1)}{x+1}+C,$$

and plug in $x=1$ to get $C=2$.

(104) Apply the method of equating coefficients to the form

$$\frac{x^2+3}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}.$$

In this problem, the Heaviside trick can still be used to find C and B; we get $B=-3$ and $C=4$. Then

$$\frac{A}{x}-\frac{3}{x^2}+\frac{4}{x-1}=\frac{Ax(x-1)+3(x-1)+4x^2}{x^2(x-1)}$$

so $A=-3$. Hence

$$\int\frac{x^2+3}{x^2(x-1)}\,\mathrm{d}x=-3\ln|x|+\frac{3}{x}+4\ln|x-1|+\text{constant}.$$

(109) $\frac{1}{2}(x^2+\ln|x^2-1|)+C$

(110) $\frac{1}{4}\ln|e^x-1|-\frac{1}{4}\ln|e^x+1|+\frac{1}{2}\arctan(e^x)+C$

(112) $\arctan(e^x+1)+C$

(113) $x-\ln(1+e^x)+C$

(116) $-\ln|x| + \frac{1}{x} + \ln|x-1| + C$

(124) $\int_0^a x \sin x dx = \sin a - a\cos a$

(125) $\int_0^a x^2 \cos x dx = (a^2+2)\sin a + 2a\cos a$

(126) $\int_3^4 \frac{x\,dx}{\sqrt{x^2-1}} = \left[\sqrt{x^2-1}\right]_3^4 = \sqrt{15} - \sqrt{8}$

(127) $\int_{1/4}^{1/3} \frac{x\,dx}{\sqrt{1-x^2}} = \left[-\sqrt{1-x^2}\right]_{1/4}^{1/3} = \frac{1}{4}\sqrt{15} - \frac{1}{3}\sqrt{8}$

(128) same as previous problem after substituting $x = 1/t$

(129) $\frac{1}{2}\ln|x^2+2x+17| + \frac{1}{4}\arctan(\frac{x+1}{4}) + C$

(132) $\ln|x| + \frac{1}{x} + \ln|x-1| - \ln|x+1| + C$

(142) $x^2\ln(x+1) - \frac{x^2}{2} + x - \ln(x+1) + C$

(145) $x^2\arctan(\sqrt{x}) - \sqrt{x} + \arctan(\sqrt{x}) + C$

(148) $\tan(x) - \sec(x) + C$

(150) $\frac{1}{4}\ln(\frac{(x+1)^2}{x^2+1}) + \frac{1}{2}\arctan(x) + C$

(160) $\int \sqrt{1+x^2}\,\mathrm{d}x = \int \frac{(t^2+1)^2}{4t^3}\,\mathrm{d}t$

(164) $t = \sqrt{y^2-1} + y$

(165)

$$\int \frac{\mathrm{d}y}{(y^2-1)^{1/2}} = \int \frac{1}{t}\,dt$$

(169)

$$z = \frac{1}{y}$$

$$t = \sqrt{y^2-1} + y = \sqrt{(\frac{1}{z})^2 - 1} + \frac{1}{z}$$

(170)

$$\int \sqrt{1-z^2}\ \mathrm{d}z = \int \frac{2(t^2-1)^2}{(t^2+1)^3}\,dt$$

(171)

$$\int \frac{\mathrm{d}z}{\sqrt{1-z^2}} = \int \frac{-2\,\mathrm{d}t}{1+t^2}$$

(175) $\cos(\theta) = \sqrt{1-z^2}$ and $d\theta = \frac{dz}{\sqrt{1-z^2}}$, therefore

$$\int r(sin(\theta), cos(\theta))\,d\theta = \int \frac{r(z, \sqrt{1-z^2})}{\sqrt{1-z^2}}\,dz$$

(176)

$$\int \frac{r(\frac{1}{y}, \frac{x}{y})}{\frac{x}{y}}\,(\frac{-1}{y^2})(\frac{dy}{dt})\,dt$$

where

$$x = \frac{1}{2}(t - \frac{1}{t}) \qquad y = \frac{1}{2}(t + \frac{1}{t}) \qquad \frac{dy}{dt} = \frac{1}{2}(1 - \frac{1}{t^2})$$

(177)

$$t = \sqrt{(\frac{1}{z})^2 - 1} + \frac{1}{z} \qquad \text{where } z = \sin(\theta)$$

(179) Use Taylor's formula : $Q(x) = 43 + 19(x-7) + \frac{11}{2}(x-7)^2$.

A different, correct, but more laborious (clumsy) solution is to say that $Q(x) = Ax^2 + Bx + C$,, compute $Q'(x) = 2Ax + B$ and $Q''(x) = 2A$. Then

$$Q(7) = 49A + 7B + C = 43, \qquad Q'(7) = 14A + B = 19, \qquad Q''(7) = 2A = 11.$$

This implies $A = 11/2$, $B = 19 - 14A = 19 - 77 = -58$, and $C = 43 - 7B - 49A = 179\frac{1}{2}$.

(180) $p(x) = 3 + 8(x-2) - \frac{1}{2}(x-2)^2$

(195) $T_\infty e^t = 1 + t + \frac{1}{2!}t^2 + \cdots + \frac{1}{n!}t^n + \cdots$

(196) $T_\infty e^{\alpha t} = 1 + \alpha t + \frac{\alpha^2}{2!}t^2 + \cdots + \frac{\alpha^n}{n!}t^n + \cdots$

(197) $T_\infty \sin(3t) = 3t - \frac{3^3}{3!}t^3 + \frac{3^5}{5!}t^5 + \cdots + \frac{(-1)^k 3^{2k+1}}{(2k+1)!}t^{2k+1} + \cdots$

(198) $T_\infty \sinh t = t + \frac{1}{3!}t^3 + \cdots + \frac{1}{(2k+1)!}t^{2k+1} + \cdots$

(199) $T_\infty \cosh t = 1 + \frac{1}{2!}t^2 + \cdots + \frac{1}{(2k)!}t^{2k} + \cdots$

(200) $T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - \cdots + (-1)^n 2^n t^n + \cdots$

(201) $T_\infty \frac{3}{(2-t)^2} = \frac{3}{2^2} + \frac{3\cdot 2}{2^3}t + \frac{3\cdot 3}{2^4}t^2 + \frac{3\cdot 4}{2^5}t^3 + \cdots + \frac{3\cdot(n+1)}{2^{n+2}}t^n + \cdots$ (note the cancellation of factorials)

(202) $T_\infty \ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

(203) $T_\infty \ln(2+2t) = T_\infty \ln[2\cdot(1+t)] = \ln 2 + \ln(1+t) = \ln 2 + t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

(204) $T_\infty \ln\sqrt{1+t} = T_\infty \frac{1}{2}\ln(1+t) = \frac{1}{2}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 + \cdots + \frac{(-1)^{n+1}}{2n}t^n + \cdots$

(205) $T_\infty \ln(1+2t) = 2t - \frac{2^2}{2}t^2 + \frac{2^3}{3}t^3 + \cdots + \frac{(-1)^{n+1}2^n}{n}t^n + \cdots$

(206) $T_\infty \ln\sqrt{\left(\frac{1+t}{1-t}\right)} = T_\infty\left[\frac{1}{2}\ln(1+t) - \frac{1}{2}\ln(1-t)\right] = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots + \frac{1}{2k+1}t^{2k+1} + \cdots$

(207) $T_\infty \frac{1}{1-t^2} = T_\infty\left[\frac{1/2}{1-t} + \frac{1/2}{1+t}\right] = 1 + t^2 + t^4 + \cdots + t^{2k} + \cdots$ (you could also substitute $x = -t^2$ in the geometric series $1/(1+x) = 1 - x + x^2 + \cdots$, later in this chapter we will use "little-oh" to justify this point of view.)

(208) $T_\infty \frac{t}{1-t^2} = T_\infty\left[\frac{1/2}{1-t} - \frac{1/2}{1+t}\right] = t + t^3 + t^5 + \cdots + t^{2k+1} + \cdots$ (note that this function is t times the previous function so you would think its Taylor series is just t times the taylor series of the previous function. Again, "little-oh" justifies this.)

(209) The pattern for the n^{th} derivative repeats every time you increase n by 4. So we indicate the the general terms for $n = 4m, 4m+1, 4m+2$ and $4m+3$:

$$T_\infty(\sin t + \cos t) = 1 + t - \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots + \frac{t^{4m}}{(4m)!} + \frac{t^{4m+1}}{(4m+1)!} - \frac{t^{4m+2}}{(4m+2)!} - \frac{t^{4m+3}}{(4m+3)!} + \cdots$$

(210) Use a double angle formula

$$T_\infty(2\sin t\cos t) = \sin 2t = 2t - \frac{2^3}{3!}t^3 + \cdots + \frac{2^{4m+1}}{(4m+1)!}t^{4m+1} - \frac{2^{4m+3}}{(4m+3)!}t^{4m+3} + \cdots$$

(211) $T_3 \tan t = t + \frac{1}{3}t^3$. There is no simple general formula for the n^{th} term in the Taylor series for $\tan x$.

(212) $T_\infty\left[1+t^2-\frac{2}{3}t^4\right] = 1+t^2-\frac{2}{3}t^4$

(213) $T_\infty[(1+t)^5] = 1+5t+10t^2+10t^3+5t^4+t^5$

(214) $T_\infty\sqrt[3]{1+t} = 1+\frac{1/3}{1!}t+\frac{(1/3)(1/3-1)}{2!}t^2+\cdots+\frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!}t^n+\cdots$

(215) $10!\cdot 2^6$

(216) Because of the addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\sin\beta\cos\alpha$$

you should get the same answer for f and g, since they are the same function!

The solution is

$$\begin{aligned} T_\infty \sin(x+a) &= \sin a + \cos(a)x - \frac{\sin a}{2!}x^2 - \frac{\cos a}{3!}x^3 + \cdots \\ &\cdots + \frac{\sin a}{(4n)!}x^{4n} + \frac{\cos a}{(4n+1)!}x^{4n+1} - \frac{\sin a}{(4n+2)!}x^{4n+2} - \frac{\cos a}{(4n+3)!}x^{4n+3} + \cdots \end{aligned}$$

(219)

$$f(x) = f^{(4)}(x) = \cos x, \qquad f^{(1)}(x) = f^{(5)}(x) = -\sin x,$$
$$f^{(2)}(x) = -\cos x, \qquad f^{(3)}(x) = \sin x,$$

so

$$f(0) = f^{(4)}(0) = 1, \qquad f^{(1)}(0) = f^{(3)}(0) = 0, \qquad f^{(2)}(0) = -1.$$

and hence the fourth degree Taylor polynomial is

$$T_4\{\cos x\} = \sum_{k=0}^{4}\frac{f^{(k)}(0)}{k!}x^k = 1-\frac{x^2}{2!}+\frac{x^4}{4!}.$$

The error is

$$R_4\{\cos x\} = \frac{f^{(5)}(\xi)}{5!}x^5 = \frac{(-\sin\xi)}{5!}x^5$$

for some ξ between 0 and x. As $|\sin\xi|\le 1$ we have

$$\left|\cos x - \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)\right| = |R_4(x)| \le \frac{|x^5|}{5!} < \frac{1}{5!}$$

for $|x|<1$.

Remark: Since the fourth and fifth order Taylor polynomial for the cosine are the same, it must be that $R_4(x) = R_5(x)$. It follows that $\frac{1}{6!}$ is also an upperbound.

(221)

(a) The polynomial is $p(x) = 2 + \frac{1}{12}x - \frac{1}{9\cdot 32}x^2$. Then

$$p(1) \approx 2.07986111$$

and the error satisfies:

$$|\sqrt[3]{9} - p(1)| \le \frac{10}{27} \cdot 8^{-\frac{8}{3}} \cdot \frac{1}{3!} \approx 0.00024112654321$$

The $\sqrt[3]{9}$ according to a computer is:

$$\sqrt[3]{9} \approx 2.08008382305$$

(238) The PFD of g is $g(x) = \dfrac{1}{x-2} - \dfrac{1}{x-1}$.

$g(x) = \frac{1}{2} + \big(1 - \frac{1}{2^2}\big)x + \big(1 - \frac{1}{2^3}\big)x^2 + \cdots + \big(1 - \frac{1}{2^{n+1}}\big)x^n + \cdots$.

So $g_n = 1 - 1/2^{n+1}$ and $g^{(n)}(0)$ is $n!$ times that.

(239) You could repeat the computations from problem 238, and this would get you the right answer with the same amount of work. In this case you could instead note that $h(x) = xg(x)$ so that

$$h(x) = \tfrac{1}{2}x + \big(1 - \tfrac{1}{2^2}\big)x^2 + \big(1 - \tfrac{1}{2^3}\big)x^3 + \cdots + \big(1 - \tfrac{1}{2^{n+1}}\big)x^{n+1} + \cdots$$

Therefore $h_n = 1 - 1/2^n$.

The PFD of $k(x)$ is

$$k(x) = \frac{2-x}{(x-2)(x-1)} \overset{\text{cancel!}}{=} \frac{1}{1-x},$$

the Taylor series of k is just the Geometric series.

(241) $T_\infty e^{at} = 1 + at + \dfrac{a^2}{2!}t^2 + \cdots + \dfrac{a^n}{n!}t^n + \cdots$.

(242) $e^{1+t} = e \cdot e^t$ so $T_\infty e^{1+t} = e + et + \frac{e}{2!}t^2 + \cdots + \frac{e}{n!}t^n + \cdots$

(243) Substitute $u = -t^2$ in the Taylor series for e^u.

$$T_\infty e^{-t^2} = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \cdots + \frac{(-1)^n}{n!}t^{2n} + \cdots$$

(244) PFD! The PFD of $\frac{1+t}{1-t}$ is $\frac{1+t}{1-t} = -1 + \frac{2}{1-t}$. Remembering the Geometric Series you get

$$T_\infty \frac{1+t}{1-t} = 1 + 2t + 2t^2 + 2t^3 + \cdots + 2t^n + \cdots$$

(245) Substitute $u = -2t$ in the Geometric Series $1/(1-u)$. You get

$$T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - 2^3t^3 + \cdots + \cdots + (-1)^n 2^n t^n + \cdots$$

(246) $f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n}$

(247)

$$T_\infty \frac{\ln(1+x)}{x} = \frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n-1}\frac{1}{n}x^n + \cdots}{x}$$
$$= 1 - \frac{1}{2}x + \frac{1}{3}x^2 + \cdots + (-1)^{n-1}\frac{1}{n}x^{n-1} + \cdots$$

(248)

$$T_\infty \frac{e^t}{1-t} = 1 + 2t + \big(1 + 1 + \tfrac{1}{2!}\big)t^2 + \big(1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!}\big)t^3 + \cdots + \big(1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!}\big)t^n + \cdots$$

(249) $1/\sqrt{1-t} = (1-t)^{-1/2}$ so

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \tfrac{1}{2}t + \frac{\frac{1}{2}\frac{3}{2}}{1\cdot 2}t^2 + \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}}{1\cdot 2\cdot 3}t^3 + \cdots$$

(be careful with minus signs when you compute the derivatives of $(1-t)^{-1/2}$.)

You can make this look nicer if you multiply top and bottom in the n^{th} term with 2^n:

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \frac{1}{2}t + \frac{1\cdot 3}{2\cdot 4}t^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^3 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^n + \cdots$$

(250)

$$T_\infty \frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^{2n} + \cdots$$

(251)

$$T_\infty \arcsin t = t + \frac{1}{2}\frac{t^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{t^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^7}{7} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{t^{2n+1}}{2n+1} + \cdots$$

(252) $T_4[e^{-t}\cos t] = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4$.

(253) $T_4[e^{-t}\sin 2t] = t - t^2 + \dfrac{1}{3}t^3 + o(t^4)$ (the t^4 terms cancel).

(254) PFD of $1/(2-t-t^2) = \frac{1}{(2+t)(1-t)} = \frac{-\frac{1}{3}}{2+t} + \frac{\frac{1}{3}}{1-t}$. Use the geometric series.

(255) $\sqrt[3]{1+2t+t^2} = \sqrt[3]{(1+t)^2} = (1+t)^{2/3}$. This is very similar to problem 249. The answer follows from Newton's binomial formula.

(258) 1/2

(259) Does not exist (or "$+\infty$")

(260) 1/2

(261) -1

(262) 0

(263) Does not exist (or "$-\infty$") because $e > 2$.

(264) 0.

(265) 0.

(266) 0 (write the limit as $\lim_{n\to\infty} \frac{n!+1}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} + \lim_{n\to\infty} \frac{1}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} + \lim_{n\to\infty} \frac{1}{(n+1)!}$).

(268) Use the explicit formula (13) from Example 16.13. The answer is the Golden Ratio ϕ.

(269) The remainder term $R_n(x)$ is equal to $\frac{f^{(n)}(\zeta_n)}{n!}x^n$ for some ζ_n. For either the cosine or sine and any n and ζ we have $|f^{(n)}(\zeta)| \le 1$. So $|R_n(x)| \le \frac{|x|^n}{n!}$. But we know $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ and hence $\lim_{n\to\infty} R_n(x) = 0$.

(270) The k^{th} derivative of $g(x) = \sin(2x)$ is $g^{(k)}(x) = \pm 2^k \mathrm{soc}(2x)$. Here $\mathrm{soc}(\theta)$ is either $\sin\theta$ or $\cos\theta$, depending on k. Therefore k^{th} remainder term is bounded by

$$|R_k[\sin 2x]| \le \frac{|g^{(k+1)}(c)|}{(k+1)!}|x|^{k+1} = \frac{2^{k+1}|x|^{k+1}}{(k+1)!}|\mathrm{soc}(2x)| \le \frac{|2x|^{k+1}}{(k+1)!}.$$

Since $\lim_{k\to\infty} \frac{|2x|^{k+1}}{(k+1)!} = 0$ we can use the Sandwich Theorem and conclude that $\lim_{k\to\infty} R_k[g(x)] = 0$, so the Taylor series of g converges for every x.

(274) Read the example in §18.4.

(275) $-1 < x < 1$.

(276) $-1 < x < 1$.

(277) $-1 < x < 1$.

(278) $-\frac{3}{2} < x < \frac{3}{2}$. Write $f(x)$ as $f(x) = \frac{1}{3}\frac{1}{1-(-\frac{2}{3}x)}$ and use the Geometric Series.

(279) $|x| < 2/5$

(288) The Taylor series is

$$\sin(t) = t - t^3/6 + \cdots$$

and the order one and two Taylor polynomial is the same $p(t) = t$. For any t there is a ζ between 0 and t with

$$\sin(t) - p(t) = \frac{f^{(3)}(\zeta)}{3!}t^3$$

When $f(t) = \sin(t)$, $|f^{(n)}(\zeta)| \leq 1$ for any n and ζ. Consequently,

$$|\sin(t) - p(t)| \leq \frac{t^3}{3!}$$

for nonnegative t. Hence

$$|\int_0^{\frac{1}{2}} \sin(x^2)\, dx - \int_0^{\frac{1}{2}} p(x^2)\, dx| \leq \int_0^{\frac{1}{2}} |\sin(x^2) - p(x^2)|\, dx \leq \int_0^{\frac{1}{2}} \frac{x^6}{3!}\, dx = \frac{(\frac{1}{2})^7}{3!\, 7} = \epsilon$$

Since $\int_0^{\frac{1}{2}} p(x^2)\, dx = \frac{(\frac{1}{2})^3}{3} = A$ (the approximate value) we have that

$$A - \epsilon \leq \int_0^{\frac{1}{2}} \sin(x^2)\, dx \leq A + \epsilon$$

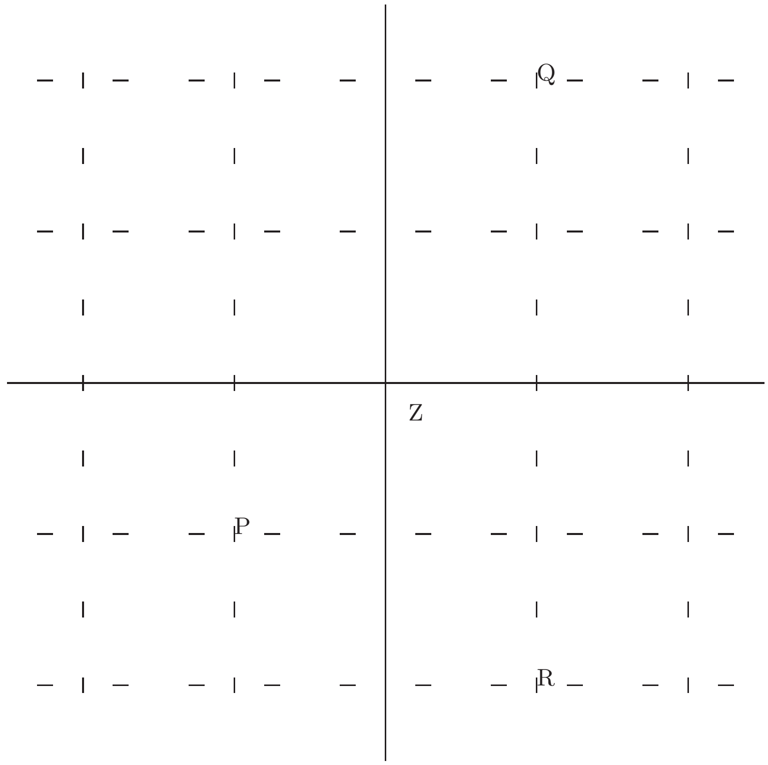
(289) (b) $\frac{43}{30}$ (c) $\frac{3}{6!\cdot 13}$

(295) (1) $-5 + 12i$ (2) $2 - 3i$ (3) $\sqrt{13}$ (4) $\frac{2}{13} - \frac{3}{13}i$

(1) 3 (2) 2 (3) $4e^{6i}$ (4) $\frac{1}{2}e^{-3i}$

(1) π (2) $\frac{3}{2}\pi$

(296) The dotted lines are one unit apart.



(297) (a) $\arg(1 + i\tan\theta) = \theta + 2k\pi$, with k any integer.

(b) $zw = 1 - \tan\theta\tan\phi + i(\tan\theta + \tan\phi)$

(c) $\arg(zw) = \arg z + \arg w = \theta + \phi$ (+ a multiple of 2π.)

(d) $\tan(\arg zw) = \tan(\theta+\phi)$ on one hand, and $\tan(\arg zw) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$ on the other hand. The conclusion is that

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

(298) $\cos 4\theta =$ real part of $(\cos\theta + i\sin\theta)^4$. Expand, using Pascal's triangle to get

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta.$

(301) To prove or disprove the statements set $z = a + bi$, $w = c + di$ and substitute in the equation. Then compare left and right hand sides.

(a) $\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(z + w)$ TRUE, *because:*
$\mathfrak{Re}(z + w) = \mathfrak{Re}(a + bi + c + di) = \mathfrak{Re}[(a + c) + (b + d)i] = a + c$ and
$\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(a + bi) + \mathfrak{Re}(c + di) = a + c.$

The other proofs go along the same lines.

(b) $\overline{z+w} = \bar{z} + \bar{w}$ TRUE. *Proof:* if $z = a + bi$ and $w = c + di$ with a, b, c, d real numbers, then

$$\begin{aligned} \mathfrak{Re}(z) = a, \quad \mathfrak{Re}(w) = c &\implies \mathfrak{Re}(z) + \mathfrak{Re}(w) = a + c \\ z + w = a + c + (b + d)i &\implies \mathfrak{Re}(z + w) = a + c. \end{aligned}$$

So you see that $\mathfrak{Re}(z) + \mathfrak{Re}(w)$ and $\mathfrak{Re}(z + w)$ are equal.

(c) $\mathfrak{Im}(z) + \mathfrak{Im}(w) = \mathfrak{Im}(z + w)$ TRUE. *Proof:* if $z = a + bi$ and $w = c + di$ with a, b, c, d real numbers, then

$$\begin{aligned} \mathfrak{Im}(z) = b, \quad \mathfrak{Im}(w) = d &\implies \mathfrak{Im}(z) + \mathfrak{Im}(w) = b + d \\ z + w = a + c + (b + d)i &\implies \mathfrak{Im}(z + w) = b + d. \end{aligned}$$

So you see that $\mathfrak{Im}(z) + \mathfrak{Im}(w)$ and $\mathfrak{Im}(z + w)$ are equal.

(d) $\overline{zw} = (\bar{z})(\bar{w})$ TRUE

(e) $\mathfrak{Re}(z)\mathfrak{Re}(w) = \mathfrak{Re}(zw)$ FALSE. *Counterexample:* Let $z = i$ and $w = i$. Then $\mathfrak{Re}(z)\mathfrak{Re}(w) = 0 \cdot 0 = 0$, but $\mathfrak{Re}(zw) = \mathfrak{Re}(i \cdot i) = \mathfrak{Re}(-1) = -1$.

(f) $\overline{z/w} = (\bar{z})/(\bar{w})$ TRUE

(g) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$ FALSE (almost true though, only off by a minus sign)

(h) $\mathfrak{Re}(iz) = i\mathfrak{Re}(z)$ FALSE. The left hand side is a real number, the right hand side is an imaginary number: they can never be equal (except when $z = 0$.)

(i) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$ same as (g), sorry.

(j) $\mathfrak{Re}(iz) = i\mathfrak{Im}(z)$ FALSE

(k) $\mathfrak{Im}(iz) = \mathfrak{Re}(z)$ TRUE

(l) $\mathfrak{Re}(\bar{z}) = \mathfrak{Re}(z)$ TRUE

(302) The number is either $\frac{1}{5}\sqrt{5} + \frac{2}{5}i\sqrt{5}$ or $-\frac{1}{5}\sqrt{5} - \frac{2}{5}i\sqrt{5}$.

(303) It is $\frac{1}{3}\sqrt{3} + i$.

(305) The absolute value is 2 and the argument is $\ln 2$.

(306) e^z can be negative, or any other complex number except zero.

If $z = x + iy$ then $e^z = e^x(\cos y + i \sin y)$, so the absolute value and argument of e^z are $|z| = e^x$ and $\arg e^z = y$. Therefore the argument can be anything, and the absolute value can be any *positive* real number, but not 0.

(307) $\dfrac{1}{e^{it}} = \dfrac{1}{\cos t + i \sin t} = \dfrac{1}{\cos t + i \sin t} \dfrac{\cos t - i \sin t}{\cos t - i \sin t} = \dfrac{\cos t - i \sin t}{\cos^2 t + \sin^2 t} = \cos t - i \sin t = e^{-it}$.

(310) $Ae^{i\beta t} + Be^{-i\beta t} = A(\cos \beta t + i \sin \beta t) + B(\cos \beta t - i \sin \beta t) = (A + B) \cos \beta t + i(A - B) \sin \beta t$.

So $Ae^{i\beta t} + Be^{-i\beta t} = 2 \cos \beta t + 3 \sin \beta t$ holds if $A + B = 2, i(A - B) = 3$. Solving these two equations for A and B we get $A = 1 - \frac{3}{2}i$, $B = 1 + \frac{3}{2}i$.

(316) (a) $z^2 + 6z + 10 = (z + 3)^2 + 1 = 0$ has solutions $z = -3 \pm i$.

(b) $z^3 + 8 = 0 \iff z^3 = -8$. Since $-8 = 8e^{\pi i + 2k\pi}$ we find that $z = 8^{1/3} e^{\frac{\pi}{3}i + \frac{2}{3}k\pi i}$ (k any integer). Setting $k = 0, 1, 2$ gives you all solutions, namely

$$\begin{aligned} k = 0 \quad &: \quad z = 2e^{\frac{\pi}{3}i} = 1 + i\sqrt{3} \\ k = 1 \quad &: \quad z = 2e^{\frac{\pi}{3}i + 2\pi i/3} = -2 \\ k = 2 \quad &: \quad z = 2e^{\frac{\pi}{3}i + 4\pi i/3} = 1 - i\sqrt{3} \end{aligned}$$

(c) $z^3-125=0$: $z_0=5$, $z_1=-\frac{5}{2}+\frac{5}{2}i\sqrt{3}$, $z_2=-\frac{5}{2}-\frac{5}{2}i\sqrt{3}$

(d) $2z^2+4z+4=0$: $z=-1\pm i$.

(e) $z^4+2z^2-3=0$: $z^2=1$ or $z^2=-3$, so the ***four*** solutions are $\pm 1, \pm i\sqrt{3}$.

(f) $3z^6=z^3+2$: $z^3=1$ or $z^3=-\frac{2}{3}$. The ***six*** solutions are therefore

$$-\tfrac{1}{2}\pm\tfrac{i}{2}\sqrt{3}, 1 \quad (\text{from } z^3=1)$$
$$-\sqrt[3]{\tfrac{2}{3}}, \sqrt[3]{\tfrac{2}{3}}\left(\tfrac{1}{2}\pm\tfrac{i}{2}\sqrt{3}\right), \quad (\text{from } z^3=-\tfrac{2}{3})$$

(g) $z^5-32=0$: The ***five*** solutions are

$$2, \quad 2\cos\tfrac{2}{5}\pi \pm 2i\sin\tfrac{2}{5}\pi, \quad 2\cos\tfrac{4}{5}\pi \pm 2i\sin\tfrac{4}{5}\pi.$$

Note that $2\cos\frac{6}{5}\pi+2i\sin\frac{6}{5}\pi = 2\cos\frac{4}{5}\pi-2i\sin\frac{4}{5}\pi$, and likewise, $2\cos\frac{8}{5}\pi+2i\sin\frac{8}{5}\pi = 2\cos\frac{2}{5}\pi-2i\sin\frac{2}{5}\pi$. (Make a drawing of these numbers to see why).

(h) $z^5-16z=0$: Clearly $z=0$ is a solution. Factor out z to find the equation $z^4-16=0$ whose solutions are ± 2, $\pm 2i$. So the ***five*** solutions are 0, ± 2, and $\pm 2i$

(i) $\sqrt{3}$, $2i$, $-\sqrt{3}$, $-2i$

(317) $f'(x)=\frac{-1}{(x+i)^2}$. In this computation you use the quotient rule, which is valid for complex valued functions.

$g'(x)=\frac{1}{x}+\frac{i}{1+x^2}$

$h'(x)=2ixe^{ix^2}$. Here we are allowed to use the Chain Rule because $h(x)$ is of the form $h_1(h_2(x))$, where $h_1(y)=e^{iy}$ is a complex valued function of a real variable, and $h_2(x)=x^2$ is a real valued function of a real variable (a "221 function").

(318) (a) Use the hint:

$$\int (\cos 2x)^4\,dx = \int \left(\frac{e^{2ix}+e^{-2ix}}{2}\right)^4 dx$$
$$= \tfrac{1}{16}\int \left(e^{2ix}+e^{-2ix}\right)^4 dx$$

The fourth line of Pascal's triangle says $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Apply this with $a=e^{2ix}$, $b=e^{-2ix}$ and you get

$$\int (\cos 2x)^4\,dx = \frac{1}{16}\int \left\{e^{8ix}+4e^{4ix}+6+4e^{-4ix}+e^{-8ix}\right\} dx$$
$$= \frac{1}{16}\left\{\frac{1}{8i}e^{8ix}+\frac{4}{4i}e^{4ix}+6x+\frac{4}{-4i}e^{-4ix}+\frac{1}{-8i}e^{-8ix}\right\}+C.$$

We could leave this as the answer since we're done with the integral. However, we are asked to simplify our answer, and since we know ahead of time that the answer is a real function we should rewrite this as a real function. There are several ways of doing this, one of which is to carefully match complex exponential terms with their complex conjugates (e.g. e^{8ix} with e^{-8ix}.) This gives us

$$\int (\cos 2x)^4\,dx = \frac{1}{16}\left\{\frac{e^{8ix}-e^{-8ix}}{8i}+\frac{e^{4ix}-e^{-4ix}}{i}+6x\right\}+C.$$

Finally, we use the formula $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$ to remove the complex exponentials. We end up with the answer

$$\int (\cos 2x)^4\,dx = \frac{1}{16}\left\{\tfrac{1}{4}\sin 8x+2\sin 4x+6x\right\}+C = \tfrac{1}{64}\sin 8x+\tfrac{1}{8}\sin 4x+\tfrac{3}{8}x+C.$$

(b) Use $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$:

$$\int e^{-2x}\bigl(\sin ax\bigr)^2\,dx = \int e^{-2x}\Bigl(\frac{e^{iax}-e^{-iax}}{2i}\Bigr)^2\,dx$$

$$= \frac{1}{(2i)^2}\int e^{-2x}\bigl(e^{2iax} - 2 + e^{-2iax}\bigr)\,dx$$

$$= -\tfrac14 \int \bigl(e^{(-2+2ia)x} - 2 + e^{(-2-2ia)x}\bigr)\,dx$$

$$(\dagger)\qquad = -\tfrac14\Bigl\{\underbrace{\frac{e^{(-2+2ia)x}}{-2+2ia}}_{A} - 2x + \underbrace{\frac{e^{(-2-2ia)x}}{-2-2ia}}_{B}\Bigr\} + C.$$

We are done with integrating. The answer must be a real function (being the integral of a real function), so we have to be able to write our answer in a real form. To get this real form we must expand the complex exponentials above, and do the division by $-2+2ia$ and $-2-2ia$. This is still a fair amount of work, but we can cut the amount of work in half by noting that the terms A and B are complex conjugates of each other, i.e. they are the same, except for the sign in front of i: you get B from A by changing all i's to $-i$'s. So once we have simplified A we immediately know B.

We compute A as follows

$$A = \frac{-2-2ia}{(-2-2ia)(-2+2ia)}\bigl(e^{-2x+2iax}\bigr)$$

$$= \frac{(-2-2ia)e^{-2x}(\cos 2ax + i\sin 2ax)}{(-2)^2 + (-2a)^2}$$

$$= \frac{e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) + i\frac{e^{-2x}}{4+4a^2}(-2a\cos 2ax - 2\sin 2ax).$$

Hence

$$B = \frac{e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) - i\frac{e^{-2x}}{4+4a^2}(-2a\cos 2ax - 2\sin 2ax).$$

and

$$A + B = \frac{2e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) = \frac{e^{-2x}}{1+a^2}(-\cos 2ax + a\sin 2ax).$$

Substitute this in (†) and you get the real form of the integral

$$\int e^{-2x}\bigl(\sin ax\bigr)^2\,dx = -\tfrac14\frac{e^{-2x}}{1+a^2}(-\cos 2ax + a\sin 2ax) + \frac{x}{2} + C.$$

(319) ***(a)*** This one can be done with the double angle formula, but if you had forgotten that, complex exponentials work just as well:

$$\int \cos^2 x\,\mathrm{d}x = \int \Bigl(\frac{e^{ix}+e^{-ix}}{2}\Bigr)^2\,dx$$

$$= \tfrac14\int \bigl\{e^{2ix} + 2 + e^{-2ix}\bigr\}\,dx$$

$$= \tfrac14\bigl\{\tfrac{1}{2i}e^{2ix} + 2x + \tfrac{1}{-2i}e^{-2ix}\bigr\} + C$$

$$= \tfrac14\Bigl\{\frac{e^{2ix}-e^{-2ix}}{2i} + 2x\Bigr\} + C$$

$$= \tfrac14\bigl\{\sin 2x + 2x\bigr\} + C$$

$$= \tfrac14\sin 2x + \frac{x}{2} + C.$$

(c), (d) using complex exponentials works, but for these integrals substituting $u = \sin x$ works better, if you use $\cos^2 x = 1 - \sin^2 x$.

(e) Use $(a-b)(a+b) = a^2 - b^2$ to compute

$$\cos^2 x \sin^2 x = \frac{(e^{ix}+e^{-ix})^2}{2^2}\frac{(e^{ix}-e^{-ix})^2}{(2i)^2} = \tfrac{1}{-16}(e^{2ix}+e^{-2ix})^2 = \tfrac{1}{-16}(e^{4ix}+2+e^{-4ix})$$

First variation: The integral is

$$\int \cos^2 x \sin^2 x \, dx = \tfrac{1}{-16}(\tfrac{1}{4i}e^{4ix} + 2x + \tfrac{1}{-4i}e^{-4ix}) + C = \tfrac{1}{-32}\sin 4x - \tfrac{1}{8}x + C.$$

Second variation: Get rid of the complex exponentials before integrating:

$$\tfrac{1}{-16}(e^{4ix}+2+e^{-4ix}) = \tfrac{1}{-16}(2\cos 4x + 2) = -\tfrac{1}{8}(\cos 4x + 1),$$

If you integrate this you get the same answer as above.

(j) and (l): Substituting complex exponentials will get you the answer, but for these two integrals you're much better off substituting $u = \cos x$ (and keep in mind that $\sin^2 x = 1 - \cos^2 x$.)

(k) See (e) above.

(335) $y(t) = 2\dfrac{Ae^t+1}{Ae^t-1}$

(336) $y = Ce^{-x^3/3}$, $C = 5e^{1/3}$

(337) $y = Ce^{-x-x^3}$, $C = e^2$

(338) Implicit form of the solution $\tan y = -\frac{x^2}{2} + C$, so $C = \tan \pi/3 = \sqrt{3}$.

Solution $y(x) = \arctan\big(\sqrt{3} - x^2/3\big)$

(339) Implicit form of the solution: $y + \frac{1}{2}y^2 + x + \frac{1}{2}x^2 = A + \frac{1}{2}A^2$. If you solve for y you get

$$y = -1 \pm \sqrt{A^2 + 2A + 1 - x^2 - 2x}$$

Whether you need the "+" or "−" depends on A.

(340) Integration gives $\dfrac{1}{2}\ln\left|\dfrac{y-1}{y+1}\right| = x + C$. Solve for y to get $\frac{y-1}{y+1} = \pm e^{2x+2C} = \big(\pm e^{2C}\big)e^{2x}$.

Let $B = \pm e^{2C}$ be the new constant and you get $\dfrac{y-1}{y+1} = Be^{2x}$ whence $y = \dfrac{1+Be^{2x}}{1-Be^{2x}}$.

The initial value $y(0) = A$ tells you that $B = \frac{A-1}{A+1}$, and therefore the solution with initial value $y(0) = A$ is $y = \dfrac{A+1+(A-1)e^{2x}}{A+1-(A-1)e^{2x}}$.

(341) $y(x) = \tan\big(\arctan(A) - x\big)$.

(342) $y = \sqrt{2(x - \frac{x^3}{3}) + 1}$

(343) $y = Ce^{-2x} - \frac{1}{3}e^x$

(344) $y = xe^{\sin x} + Ae^{\sin x}$

(345) Implicit form of the solution $\frac{1}{3}y^3 + \frac{1}{4}x^4 = C$; $C = \frac{1}{3}A^3$. Solution is $y = \sqrt[3]{A^3 - \frac{3}{4}x^4}$.

(349) General solution: $y(t) = Ae^{3t}\cos t + Be^{3t}\sin t$. Solution with given initial values has $A = 7$, $B = -10$.

(350) $y = 3e^x - e^{4x}$

(351) $y = Ae^x \sin(3x + B)$

(352) $y = Ae^t + Be^{-t} + C\cos t + D\sin t$

(353) The characteristic roots are $r = \pm\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{2}$, so the general solution is

$$y = Ae^{\frac{1}{2}\sqrt{2}t}\cos\tfrac{1}{2}\sqrt{2}t + Be^{\frac{1}{2}\sqrt{2}t}\sin\tfrac{1}{2}\sqrt{2}t + Ce^{-\frac{1}{2}\sqrt{2}t}\cos\tfrac{1}{2}\sqrt{2}t + De^{-\frac{1}{2}\sqrt{2}t}\sin\tfrac{1}{2}\sqrt{2}t.$$

(354) The characteristic equation is $r^4 - r^2 = 0$ whose roots are $r = \pm 1$ and $r = 0$ (double). Hence the general solution is $y = A + Bt + Ce^t + De^{-t}$.

(355) The characteristic equation is $r^4 + r^2 = 0$ whose roots are $r = \pm i$ and $r = 0$ (double). Hence the general solution is $y = A + Bt + C\cos t + D\sin t$.

(356) The characteristic equation is $r^3 + 1 = 0$, so we must solve

$$r^3 = -1 = e^{(\pi + 2k\pi)i}.$$

The characteristic roots are

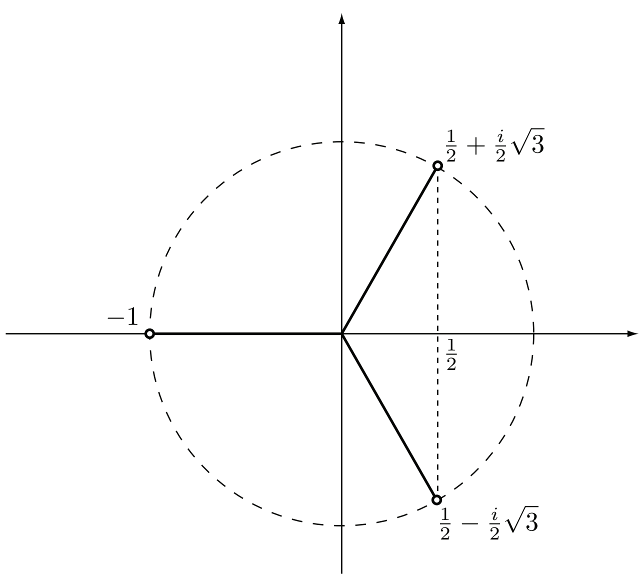
$$r = e^{\left(\frac{\pi}{3} + \frac{2}{3}k\pi\right)i}$$

where k is an integer. The roots for $k = 0, 1, 2$ are different, and all other choices of k lead to one of these roots. They are

$$k = 0: \qquad r = e^{\pi i/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{i}{2}\sqrt{3}$$

$$k = 1: \qquad r = e^{\pi i} = \cos\pi + i\sin\pi = -1$$

$$k = 2: \qquad r = e^{5\pi i/3} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{i}{2}\sqrt{3}$$



The real form of the general solution of the differential equation is therefore

$$y = Ae^{-t} + Be^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + Ce^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

(357) $y = Ae^t + Be^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + Ce^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

(358) $y(t) = c_1e^{\sqrt{3}t} + c_2e^{-\sqrt{3}t} + A\cos t + B\sin t.$

(359) Characteristic polynomial: $r^4 + 4r^2 + 3 = \big(r^2+3\big)\big(r^2+1\big)$.

Characteristic roots: $-i\sqrt{3}, -i, i, i\sqrt{3}$.

General solution: $y(t) = A_1 \cos\sqrt{3}t + B_1 \sin\sqrt{3}t + A_2 \cos t + B_2 \sin t$.

(360) Characteristic polynomial: $r^4 + 2r^2 + 2 = \big(r^2+1\big)^2 + 1$.

Characteristic roots: $r_{1,2}^2 = -1 + i$, $r_{3,4}^2 = -1 - i$.

Since $-1 + i = \sqrt{2}e^{\pi i/4 + 2k\pi}$ (k an integer) the square roots of $-1 + i$ are $\pm 2^{1/4}e^{\pi i/8} = 2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8}$. The angle $\pi/8$ is not one of the familiar angles so we don't simplify $\cos\pi/8$, $\sin\pi/8$.

Similarly, $-1 - i = \sqrt{2}e^{-\pi i/4 + 2k\pi i}$ so the square roots of $-1 - i$ are $\pm 2^{1/4}e^{-\pi i/8} = \pm 2^{1/4}\big(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\big)$.

If you abbreviate $a = 2^{1/4}\cos\frac{\pi}{8}$ and $b = 2^{1/4}\sin\frac{\pi}{8}$, then the four characteristic roots which we have found are

$$\begin{aligned} r_1 &= 2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8} = a + bi \\ r_2 &= 2^{1/4}\cos\frac{\pi}{8} - i2^{1/4}\sin\frac{\pi}{8} = a - bi \\ r_3 &= -2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8} = -a + bi \\ r_4 &= -2^{1/4}\cos\frac{\pi}{8} - i2^{1/4}\sin\frac{\pi}{8} = -a - bi \end{aligned}$$

The general solution is

$$y(t) = A_1 e^{at}\cos bt + B_1 e^{at}\sin bt + A_2 e^{-at}\cos bt + B_2 e^{-at}\sin bt$$

(363) Characteristic equation is $r^3 - 125 = 0$, i.e. $r^3 = 125 = 125e^{2k\pi i}$. The roots are $r = 5e^{2k\pi i/3}$, i.e.

$$5, \quad 5(-\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}) = -\tfrac{5}{2} + \tfrac{5}{2}i\sqrt{3}, \text{ and } 5(-\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}) = -\tfrac{5}{2} - \tfrac{5}{2}i\sqrt{3}.$$

The general solution is

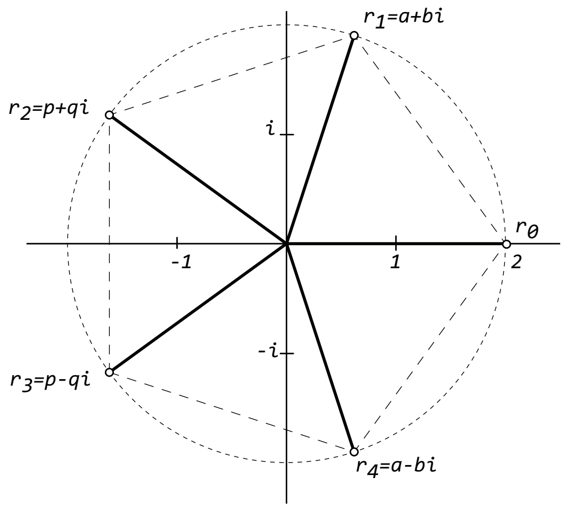
$$f(x) = c_1 e^{5x} + c_2 e^{-\frac{5}{2}x}\cos\tfrac{5}{2}\sqrt{3}x + c_3 e^{-\frac{5}{2}x}\sin\tfrac{5}{2}\sqrt{3}x.$$

(364) Try $u(x) = e^{rx}$ to get the characteristic equation $r^5 = 32$ which has solutions

$$r = 2,\ 2e^{\frac{2}{5}\pi i},\ 2e^{\frac{4}{5}\pi i},\ 2e^{\frac{6}{5}\pi i},\ 2e^{\frac{8}{5}\pi i},$$

i.e.

$$\begin{aligned} r_0 &= 2 \\ r_1 &= 2\cos\tfrac{2}{5}\pi + 2i\sin\tfrac{2}{5}\pi \\ r_2 &= 2\cos\tfrac{4}{5}\pi + 2i\sin\tfrac{4}{5}\pi \\ r_3 &= 2\cos\tfrac{6}{5}\pi + 2i\sin\tfrac{6}{5}\pi \\ r_4 &= 2\cos\tfrac{8}{5}\pi + 2i\sin\tfrac{8}{5}\pi. \end{aligned}$$



Remember that the roots come in complex conjugate pairs. By making a drawing of the roots you see that r_1 and r_4 are complex conjugates of each other, and also that r_2 and r_3 are complex conjugates of each other. So the roots are

$$2, \qquad 2\cos\tfrac{2}{5}\pi \pm 2i\sin\tfrac{2}{5}\pi, \text{ and } \qquad 2\cos\tfrac{4}{5}\pi \pm 2i\sin\tfrac{4}{5}\pi.$$

The general solution of the differential equation is

$$u(x) = c_1e^{2x} + c_2e^{ax}\cos bx + c_3e^{ax}\sin bx + c_3e^{px}\cos qx + c_3e^{px}\sin qx.$$

Here we have abbreviated

$$a = 2\cos\tfrac{2}{5}\pi, b = 2\sin\tfrac{2}{5}\pi, p = 2\cos\tfrac{4}{5}\pi, q = 2\sin\tfrac{4}{5}\pi.$$

(366) Characteristic polynomial is $r^3-5r^2+6r-2 = (r-1)(r^2-4r+2)$, so the characteristic roots are $r_1 = 1, r_{2,3} = 2 \pm \sqrt{2}$. General solution:

$$y(t) = c_1e^t + c_2e^{(2-\sqrt{2})t} + c_3e^{(2+\sqrt{2})t}.$$

(368) Characteristic polynomial is $r^3 - 5r^2 + 4 = (r-1)(r^2 - 4r - 4)$. Characteristic roots are $r_1 = 1, r_{2,3} = 2 \pm 2\sqrt{2}$. General solution

$$z(x) = c_1e^x + c_2e^{(2+2\sqrt{2})x} + c_3e^{(2-2\sqrt{2})x}.$$

(369) General: $y(t) = A\cos 3t + B\sin 3t$. With initial conditions: $y(t) = \sin 3t$

(370) General: $y(t) = A\cos 3t + B\sin 3t$. With initial conditions: $y(t) = -3\cos 3t$

(371) General: $y(t) = Ae^{2t} + Be^{3t}$. With initial conditions: $y(t) = e^{3t} - e^{2t}$

(372) General: $y(t) = Ae^{-2t} + Be^{-3t}$. With initial conditions: $y(t) = 3e^{-2t} - 2e^{-3t}$

(373) General: $y(t) = Ae^{-2t} + Be^{-3t}$. With initial conditions: $y(t) = e^{-2t} - e^{-3t}$

(374) General: $y(t) = Ae^t + Be^{5t}$. With initial conditions: $y(t) = \frac{5}{4}e^t - \frac{1}{4}e^{5t}$

(375) General: $y(t) = Ae^t + Be^{5t}$. With initial conditions: $y(t) = (e^{5t} - e^t)/4$

(376) General: $y(t) = Ae^{-t} + Be^{-5t}$. With initial conditions: $y(t) = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$

(377) General: $y(t) = Ae^{-t} + Be^{-5t}$. With initial conditions: $y(t) = \frac{1}{4}\bigl(e^{-t} - e^{-5t}\bigr)$

(378) General: $y(t) = e^{2t}\bigl(A\cos t + B\sin t\bigr)$. With initial conditions: $y(t) = e^{2t}\bigl(\cos t - 2\sin t\bigr)$

(379) General: $y(t) = e^{2t}\bigl(A\cos t + B\sin t\bigr)$. With initial conditions: $y(t) = e^{2t}\sin t$

(380) General: $y(t) = e^{-2t}\bigl(A\cos t + B\sin t\bigr)$. With initial conditions: $y(t) = e^{-2t}\bigl(\cos t + 2\sin t\bigr)$

(381) General: $y(t) = e^{-2t}\bigl(A\cos t + B\sin t\bigr)$. With initial conditions: $y(t) = e^{-2t}\sin t$

(382) General: $y(t) = Ae^{2t} + Be^{3t}$. With initial conditions: $y(t) = 3e^{2t} - 2e^{3t}$

(383) Characteristic polynomial: $r^3 + r^2 - r + 15 = (r+3)(r^2 - 2r + 5)$. Characteristic roots: $r_1 = -3$, $r_{2,3} = 1 \pm 2i$. General solution (real form) is

$$f(t) = c_1 e^{-3t} + Ae^t \cos 2t + Be^t \sin 2t.$$

The initial conditions require

$$f(0) = c_1 + A = 0, \quad f'(0) = -3c_1 + A + 2B = 1, \quad f''(0) = 9c_1 - 3A + 4B = 0.$$

Solve these equations to get $c_1 = -1/10$, $A = 1/10$, $B = 3/10$, and thus

$$f(t) = -\tfrac{1}{10}e^{-3t} + \tfrac{1}{10}e^t \cos 2t + \tfrac{3}{10}e^t \sin 2t.$$

(385) $y_P = \frac{1}{4}e^x + \frac{1}{2}x + \frac{1}{4}$

(386) $y = -2 + Ae^t + Be^{-t}$

(387) $y = Ae^t + Be^{-t} + te^t$

(388) $y = A\cos t + B\sin t + \frac{1}{6}t\sin t$

(389) $y = A\cos 3t + B\sin 3t + \frac{1}{8}\cos t$

(390) $y = A\cos t + B\sin t + \frac{1}{2}t\sin t$

(391) $y = A\cos t + B\sin t - \frac{1}{8}\cos 3t$

(393) **(i)** Homogeneous equation: try $z(t) = e^{rt}$, get characteristic equation $r^2 + 4r + 5 = 0$, with roots $r_{1,2} = -2 \pm i$. The general solution of the homogenous equation is therefore $z_h(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$.

To find a particular solution try $z_p(t) = Ae^{it}$. You get $(i^2 + 4i + 5)Ae^{it} = e^{it}$, i.e. $(4+4i)A = 1$, so $A = \frac{1}{4+4i} = \frac{1}{4}\frac{1}{1+i} = \frac{1}{4}\frac{1-i}{2} = \frac{1}{8} - \frac{i}{8}$. So the general solution to the inhomogeneous problem is

$$z(t) = \frac{1-i}{8}e^{it} + c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t.$$

(395) Let $X(t)$ be the rabbit population size at time t. The rate at which this population grows is dX/dt rabbits per year.

$\frac{5}{100}X$ from growth at 5% per year
$\frac{2}{100}X$ from death at 2% per year
-1000 car accidents
$+700$ immigration from Sun Prairie

Together we get

$$\frac{dX}{dt} = \frac{3}{100}X - 300.$$

This equation is both separable and first order linear, so you can choose from two methods to find the general solution, which is

$$X(t) = 10{,}000 + Ce^{0.03t}.$$

If $X(1991) = 12000$ then

$$10{,}000 + Ce^{0.03\times 1991} = 12{,}000 \implies C = 2{,}000e^{-0.03\times 1991} \text{(don't simplify yet!)}$$

Hence
$X(1994) = 10,000+2,000e^{-0.03\times1991}e^{0.03\times1994} = 10,000+2,000e^{0.03\times(1994-1991)} = 10,000+2,000e^{0.09} \approx 12,188.\ldots$

(396.ii) (i) Separate variables or find an integrating factor ($\frac{dT}{dt} - kT = -kA$). Both methods work here. You get $T(t) = A + Ce^{kt}$, where C is an arbitrary constant. Since $k < 0$ one has $\lim_{t\to\infty} e^{kt} = 0$, and hence $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} A+Ce^{kt} = A+C\cdot 0 = A$.

(ii) Given $T(0) = 180$, $A = 75$, and $T(5) = 150$. This gives the following equations:

$$A + C = 180, \quad A + Ce^{5k} = 105 \implies C = 105, \quad 5k = \ln\frac{75}{105} = \ln\frac{5}{7} = -\ln\frac{7}{5}.$$

When is $T = 90$? Solve $T(t) = 90$ for t using the values for A, C, k found above (k is a bit ugly so we substitute it at the end of the problem):

$$T(t) = A + Ce^{kt} = 75 + 105e^{kt} = 90 \implies e^{kt} = \frac{15}{105} = \frac{1}{7}.$$

Hence

$$t = \frac{\ln 1/7}{k} = -\frac{\ln 7}{k} = \frac{\ln 7}{\ln 7/5}.$$

The limit as $t \to \infty$ of the temperature is $A = 75$ degrees.

(397) (a) Let $y(t)$ be the amount of "retaw" (in gallons) in the tank at time t. Then

$$\frac{dy}{dt} = \underbrace{\frac{5}{100}y}_{\text{growth}} - \underbrace{3}_{\text{removal}}.$$

(b) $y(t) = 60 + Ce^{t/20} = 60 + (y_0 - 60)e^{t/20}$.

(c) If $y_0 = 100$ then $y(t) = 60 + 40e^{t/20}$ so that $\lim_{t\to\infty} y(t) = +\infty$.

(d) $y_0 = 60$.

(398) Finding the equation is the hard part. Let $A(t)$ be the ***volume*** of acid in the vat at time t. Then $A(0) = 25\%$ of $1000 = 250$gallons.

$A'(t) =$ the volume of acid that gets pumped in minus the volume that gets extracted per minute. Per minute 40% of 20 gallons, i.e. 8 gallons of acid get added. The vat is well mixed, and $A(t)$ out of the 1000gallons are acid, so if 20 gallons get extracted, then $\frac{A}{1000} \times 20$ of those are acid. Hence

$$\frac{dA}{dt} = 8 - \frac{A}{1000} \times 20 = 8 - \frac{A}{50}.$$

The solution is $A(t) = 400 + Ce^{-t/50} = 400 + (A(0) - 400)e^{-t/50} = 400 - 150e^{-t/50}$.

The ***concentration*** at time t is

$$\text{concentration} = \frac{A(t)}{\text{total volume}} = \frac{400 - 150e^{-t/50}}{1000} = 0.4 - 0.15r^{-t/50}.$$

If you wait for very long the concentration becomes

$$\text{concentration} = \lim_{t\to\infty} \frac{A(t)}{1000} = 0.4.$$

(399) P is the volume of polluted water in the lake at time t. At any time the fraction of the lake water which is polluted is P/V, so if 24 cubic feet are drained then $\frac{P}{V} \times 24$ of those are polluted. Here $V = 10^9$; for simplicity we'll just write V until the end of the problem. We get

$$\frac{dP}{dt} = \text{"in minus out"} = 3 - \frac{P}{V} \times 24$$

whose solution is $P(t) = \frac{1}{8}V + Ke^{-\frac{24}{V}t}$. Here K is an arbitrary constant (which we can't call C because in this problem C is the concentration).

The concentration at time t is

$$C(t) = \frac{P(t)}{V} = \frac{1}{8} + \frac{K}{V}e^{-\frac{24}{V}t} = \frac{1}{8} + \bigl(C_0 - \frac{1}{8}\bigr)e^{-\frac{24}{V}t}.$$

No matter what C_0 is you always have

$$\lim_{t\to\infty} C(t) = 0$$

because $\lim_{t\to\infty} e^{-\frac{24}{V}t} = 0$.

If $C_0 = \frac{1}{8}$ then the concentration of polluted water remains constant: $C(t) = \frac{1}{8}$.

(408) (a) $z(5) = 10.24$

(b) $b = -2\ln\frac{4}{5}$ and $k = \frac{b^2+\pi^2}{4}$

(409) No. Differentiating gives $z' = e^{-t}(\cos t - \sin t) = 0$ which is zero at $\pi/4$.

(410) Differentiating the equation $z'' + bz' + kz \equiv 0$ shows that z' is also a solution. Hence the zeros of z' are separated by the same intervals as the zeros of z. Hence its next peak will be at $t = 5$. Its height will be same as in problem 408 and the constants b, k will be the same. The actual solution (determined by the constants A and B) will be different.

(415) Hint: $(x-\alpha)^n = (x-\alpha)\cdot(x-\alpha)^{n-1}$

(420) $z = (-t + C_1)\cos t \quad + \quad (\ln|\sin t| + C_2)\sin t$

(421) Hint: you can say yourself some work if you remember that f and g were chosen to satisfy $f'z_1 + g'z_2 \equiv 0$.

(423) Hint: Multiply the first equation by d and the second by $-b$ then add them and solve for x.

(425)

$$f' = \frac{\det\begin{pmatrix} 0 & z_2 \\ b & z_2' \end{pmatrix}}{\det\begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix}} \qquad g' = \frac{\det\begin{pmatrix} z_1 & 0 \\ z_1' & b \end{pmatrix}}{\det\begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix}}$$

tells us by problem 423 that

$$f'z_1 + g'z_2 \equiv 0$$
$$f'z_1' + g'z_2' \equiv b$$

By problem 424

$$\mathfrak{L}(fz_1 + gz_2) = f'z_1' + g'z_2'$$

and so it follows that

$$\mathfrak{L}(fz_1 + gz_2) \equiv b$$

(428) (1) 3 (2) $\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$ (3) 36 (4) $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ (5) $\begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

(431) (a) Since $\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}1+x\\2+x\end{pmatrix}$ the number x would have to satisfy both $1+x=2$ and $2+x=1$. That's impossible, so there is no such x.

(b) No drawing, but $\vec{\boldsymbol{p}} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + x\begin{pmatrix}1\\1\end{pmatrix}$ is the parametric representation of a straight line through the points $(1,2)$ (when $x=0$) and $(2,3)$ (when $x=1$).

(c) x and y must satisfy $\begin{pmatrix}x+y\\2x+y\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$. Solve $x+y=2$, $2x+y=1$ to get $x=-1$, $y=3$.

(432) Every vector is a position vector. To see of which point it is the position vector translate it so its initial point is the origin.

Here $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, so $\overrightarrow{AB}$ is the position vector of the point $(-3, 3)$.

(433) One always labels the vertices of a parallelogram counterclockwise (see §45.4).

$ABCD$ is a parallelogram if $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. So $\overrightarrow{AB} + \overrightarrow{AD} \neq \overrightarrow{AC}$, and $ABCD$ is not a parallelogram.

(434) (a) As in the previous problem, we want $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. If D is the point (d_1, d_2, d_3) then $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} d_1 \\ d_2 - 2 \\ d_3 - 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$, so that $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ will hold if $d_1 = 4$, $d_2 = 0$ and $d_3 = 3$.

(b) Now we want $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$, so $d_1 = 4$, $d_2 = 2$, $d_3 = 5$.

(439) (a) $\vec{\boldsymbol{x}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - t \\ t \\ 1 + t \end{pmatrix}$.

(b) Intersection with xy plane when $z = 0$, i.e. when $t = -1$, at $(4, -1, 0)$. Intersection with xz plane when $y = 0$, when $t = 0$, at $(3, 0, 1)$ (i.e. at A). Intersection with yz plane when $x = 0$, when $t = 3$, at $(0, 3, 4)$.

(440) (a) $\vec{L}[t] = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$

(441) (a) $\vec{\boldsymbol{p}} = (\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}})/2$, $\vec{\boldsymbol{q}} = (\vec{\boldsymbol{a}} + \vec{\boldsymbol{c}})/2$, $\vec{\boldsymbol{r}} = (\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}})/2$.

(b) $\vec{\boldsymbol{m}} = \vec{\boldsymbol{a}} + \frac{2}{3}(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})$ (See Figure 18, with AX twice as long as XB). Simplify to get $\vec{\boldsymbol{m}} = \frac{1}{3}\vec{\boldsymbol{a}} + \frac{1}{3}\vec{\boldsymbol{b}} + \frac{1}{3}\vec{\boldsymbol{c}}$.

(c) Hint : find the point N on the line segment BQ which is twice as far from B as it is from Q. If you compute this carefully you will find that $M = N$.

(443) To decompose $\vec{\boldsymbol{b}}$ set $\vec{\boldsymbol{b}} = \vec{\boldsymbol{b}}_\perp + \vec{\boldsymbol{b}}_{/\!/}$, with $\vec{\boldsymbol{b}}_{/\!/} = t\vec{\boldsymbol{a}}$ for some number t. Take the dot product with $\vec{\boldsymbol{a}}$ on both sides and you get $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = t\|\vec{\boldsymbol{a}}\|^2$, whence $3 = 14t$ and $t = \frac{3}{14}$. Therefore

$$\vec{\boldsymbol{b}}_{/\!/} = \frac{3}{14}\vec{\boldsymbol{a}}, \qquad \vec{\boldsymbol{b}}_\perp = \vec{\boldsymbol{b}} - \frac{3}{14}\vec{\boldsymbol{a}}.$$

To find $\vec{\boldsymbol{b}}_{/\!/}$ and $\vec{\boldsymbol{b}} - \perp$ you now substitute the given values for $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$.

The same procedure leads to $\vec{\boldsymbol{a}}_\perp$ and $\vec{\boldsymbol{a}}_{/\!/}$: $\vec{\boldsymbol{a}}_{/\!/} = \frac{3}{2}\vec{\boldsymbol{b}}$, $\vec{\boldsymbol{a}}_\perp = \vec{\boldsymbol{a}} - \frac{3}{2}\vec{\boldsymbol{b}}$.

(444) This problem is of the same type as the previous one, namely we have to decompose one vector as the sum of a vector perpendicular and a vector parallel to the hill's surface. The only difference is that we are not given the normal to the hill so we have to find it ourselves. The equation of the hill is $12x_1 + 5x_2 = 130$ so the vector $\vec{\boldsymbol{n}} = \left(\begin{smallmatrix} 12 \\ 5 \end{smallmatrix}\right)$ is a normal.

The problem now asks us to write $\vec{\boldsymbol{f}}_{\text{grav}} = \vec{\boldsymbol{f}}_\perp + \vec{\boldsymbol{f}}_{/\!/}$, where $\vec{\boldsymbol{f}}_\perp = t\vec{\boldsymbol{n}}$ is perpendicular to the surface of the hill, and $\vec{\boldsymbol{f}}_{/\!/}$ id parallel to the surface.

Take the dot product with $\vec{\boldsymbol{n}}$, and you find $t\|\vec{\boldsymbol{n}}\|^2 = \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{f}}_{\text{grav}} \implies 169t = -5mg \implies t = -\frac{5}{169}mg$. Therefore

$$\vec{\boldsymbol{f}}_\perp = -\frac{5}{169}mg\begin{pmatrix}12\\5\end{pmatrix} = \begin{pmatrix}-\frac{60}{169}mg\\-\frac{25}{169}mg\end{pmatrix}, \qquad \vec{\boldsymbol{f}}_{/\!/} = \vec{\boldsymbol{f}}_{\text{grav}} - \vec{\boldsymbol{f}}_\perp = \begin{pmatrix}-\frac{60}{169}mg\\\frac{144}{169}mg\end{pmatrix},$$

(446) (i) $\|\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\|^2 = \|\vec{\boldsymbol{a}}\|^2 - 2\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} + \|\vec{\boldsymbol{b}}\|^2$; (ii) $\|2\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\|^2 = 4\|\vec{\boldsymbol{a}}\|^2 - 4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} + \|\vec{\boldsymbol{b}}\|^2$; (iii) $\|\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\| = \sqrt{54}$, $\|\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\| = \sqrt{62}$ and $\|2\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\| = \sqrt{130}$.

(448) Compute $\overrightarrow{AB} = -\overrightarrow{BA} = \begin{pmatrix}1\\1\end{pmatrix}$, $\overrightarrow{BC} = -\overrightarrow{CB} = \begin{pmatrix}-2\\2\end{pmatrix}$, $\overrightarrow{AC} = -\overrightarrow{CA} = \begin{pmatrix}-1\\3\end{pmatrix}$. Hence $\|\overrightarrow{AB}\| = \sqrt{2}$, $\|\overrightarrow{BC}\| = \sqrt{8} = 2\sqrt{2}$, $\|\overrightarrow{AC}\| = \sqrt{10}$.

And also $\overrightarrow{AB}\cdot\overrightarrow{AC} = 2 \implies \cos\angle A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}$.

A similar calculation gives $\cos\angle B = 0$ so we have a right triangle; and $\cos\angle C = \frac{2}{\sqrt{5}}$.

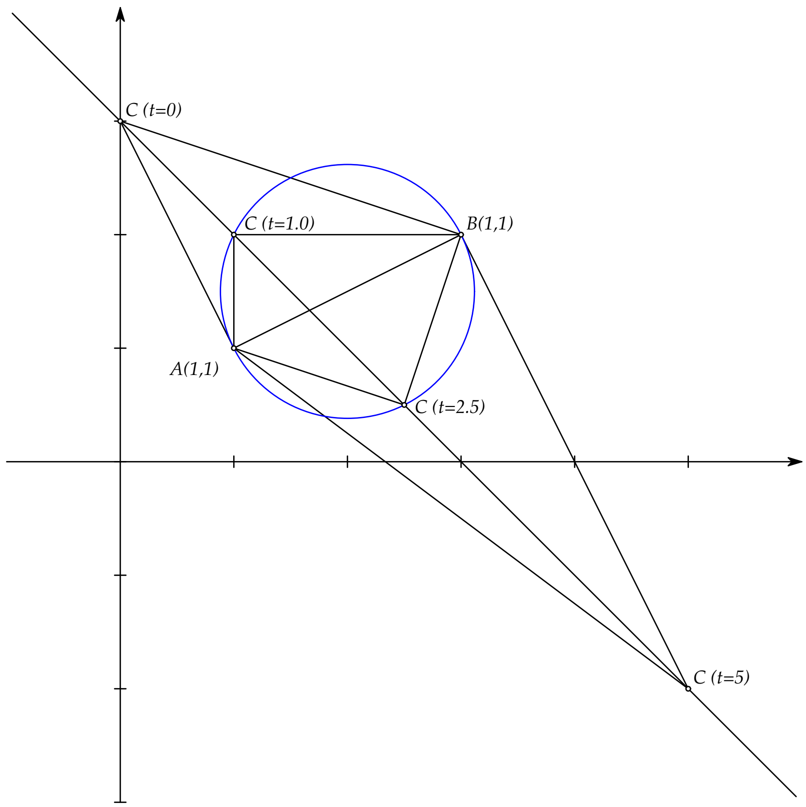
(449) $\overrightarrow{AB} = \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$, $\overrightarrow{AC} = \left(\begin{smallmatrix}t-1\\2-t\end{smallmatrix}\right)$, $\overrightarrow{BC} = \left(\begin{smallmatrix}t-3\\1-t\end{smallmatrix}\right)$.

If the right angle is at A then $\overrightarrow{AB}\cdot\overrightarrow{AC} = 0$, so that we must solve $2(t-1) + (2-t) = 0$. Solution: $t = 0$, and $C = (0, 3)$.

If the right angle is at B then $\overrightarrow{AB}\cdot\overrightarrow{BC} = 0$, so that we must solve $2(t-3) + (1-t) = 0$. Solution: $t = 5$, and $C = (5, -2)$.

If the right angle is at C then $\overrightarrow{AC}\cdot\overrightarrow{BC} = 0$, so that we must solve $(t-1)(t-3) + (2-t)(1-t) = 0$. Note that this case is different in that we get a quadratic equation, and in that there are two solutions, $t = 1$, $t = \frac{5}{2}$.

This is a complete solution of the problem, but it turns out that there is a nice picture of the solution, and that the four different points C we find are connected with the circle whose diameter is the line segment AB:



(450.i) ℓ has defining equation $-\frac{1}{2}x + y = 1$ which is of the form $\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}$ =constant if you choose $\vec{\boldsymbol{n}} = \binom{-1/2}{1}$.

(450.ii) The distance to the point D with position vector $\vec{\boldsymbol{d}}$ from the line ℓ is $\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{d}}-\vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|}$ where $\vec{\boldsymbol{a}}$ is the position vector of any point on the line. In our case $\vec{\boldsymbol{d}} = \vec{\boldsymbol{0}}$ and the point $A(0,1)$, $\vec{\boldsymbol{a}} = \overrightarrow{OA} = \binom{0}{1}$, is on the line. So the distance to the origin from the line is $\dfrac{-\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{a}}}{\|\vec{\boldsymbol{n}}\|} = \dfrac{1}{\sqrt{(1/2)^2+1^2}} = 2/\sqrt{5}$.

(450.iii) $3x + y = 2$, normal vector is $\vec{\boldsymbol{m}} = \binom{3}{1}$.

(450.iv) Angle between ℓ and m is the angle θ between their normals, whose cosine is $\cos\theta = \frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{m}}}{\|\vec{\boldsymbol{n}}\|\,\|\vec{\boldsymbol{m}}\|} = \frac{-1/2}{\sqrt{5/4}\sqrt{10}} = -\frac{1}{\sqrt{50}} = -\frac{1}{10}\sqrt{2}$.

(456.i) $\vec{\boldsymbol{0}}$ (the cross product of any vector with itself is the zero vector).

(456.iii) $(\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}) \times (\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}) = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{a}} + \vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} - \vec{\boldsymbol{b}} \times \vec{\boldsymbol{b}} = -2\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$.

(457) Not true. For instance, the vector $\vec{\boldsymbol{c}}$ could be $\vec{\boldsymbol{c}} = \vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}$, and $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$ would be the same as $\vec{\boldsymbol{c}} \times \vec{\boldsymbol{b}}$.

(458.i) A possible normal vector is $\vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$. Any (non zero) multiple of this vector is also a valid normal. The nicest would be $\frac{1}{4}\vec{\boldsymbol{n}} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

(458.ii) $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = 0$, or $\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}} = \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{a}}$. Using $\vec{\boldsymbol{n}}$ and $\vec{\boldsymbol{a}}$ from the first part we get $-4x_1 + 4x_2 - 4x_3 = -8$. Here you could replace $\vec{\boldsymbol{a}}$ by either $\vec{\boldsymbol{b}}$ or $\vec{\boldsymbol{c}}$. (Make sure you understand why; if you don't think about it, then ask someone).

(458.iii) Distance from D to $\mathcal{P}$ is $\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{d}}-\vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|} = 4/\sqrt{3} = \frac{4}{3}\sqrt{3}$. There are many valid choices of normal $\vec{\boldsymbol{n}}$ in part (i) of this problem, but they all give the same answer here.

Distance from O to $\mathcal{P}$ is $\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{0}}-\vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|} = \frac{2}{3}\sqrt{3}$.

(458.iv) Since $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{0}}-\vec{\boldsymbol{a}})$ and $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{d}}-\vec{\boldsymbol{a}})$ have the same sign the point D and the origin lie on the same side of the plane $\mathcal{P}$.

(458.v) The area of the triangle is $\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = 2\sqrt{3}$.

(458.vi) Intersection with x axis is A, the intersection with y-axis occurs at $(0,-2,0)$ and the intersection with the z-axis is B.

(459.i) Since $\vec{\boldsymbol{n}} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}-3\\1\\1\end{pmatrix}$ the plane through A,B,C has defining equation $-3x+y+z=3$. The coordinates $(2,1,3)$ of D do not satisfy this equation, so D is not on the plane ABC.

(459.ii) If E is on the plane through A,B,C then the coordinates of E satisfy the defining equation of this plane, so that $-3\cdot 1+1\cdot 1+1\cdot\alpha = 3$. This implies $\alpha = 5$.

(460.i) If $ABCD$ is a parallelogram then the vertices of the parallelogram are labeled A, B, C, D as you go around the parallelogram in a counterclockwise fashion. See the figure in §43.2. Then $\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$. Starting from this equation there are now two ways to solve this problem.

(first solution) If D is the point (d_1,d_2,d_3) then $\overrightarrow{AD} = \begin{pmatrix}d_1-1\\d_2+1\\d_3-1\end{pmatrix}$, while $\overrightarrow{AB} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$. Hence $\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$ implies $\begin{pmatrix}d_1\\d_2+2\\d_3-1\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$, and thus $d_1=0$, $d_2=1$ and $d_3=0$.

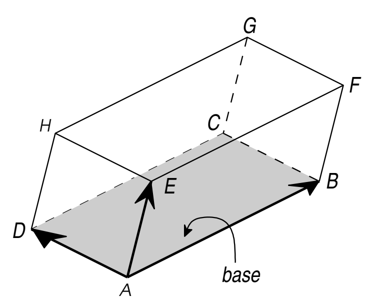
(second solution) Let $\vec{\boldsymbol{a}},\vec{\boldsymbol{b}},\vec{\boldsymbol{c}},\vec{\boldsymbol{d}}$ be the position vectors of A,B,C,D. Then $\overrightarrow{AB}=\vec{\boldsymbol{b}}-\vec{\boldsymbol{a}}$, etc. and $\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$ is equivalent to $\vec{\boldsymbol{b}}-\vec{\boldsymbol{a}}+\vec{\boldsymbol{d}}-\vec{\boldsymbol{a}}=\vec{\boldsymbol{c}}-\vec{\boldsymbol{a}}$. Since we know $\vec{\boldsymbol{a}},\vec{\boldsymbol{b}},\vec{\boldsymbol{c}}$ we can solve for $\vec{\boldsymbol{d}}$ and we get $\vec{\boldsymbol{d}}=\vec{\boldsymbol{c}}-\vec{\boldsymbol{b}}+\vec{\boldsymbol{a}} = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} + \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

(460.ii) The area of the parallelogram $ABCD$ is $\|\overrightarrow{AB}\times\overrightarrow{AD}\| = \left\|\begin{pmatrix}-1\\1\\3\end{pmatrix}\right\| = \sqrt{11}$.

(460.iii) In the previous part we computed $\overrightarrow{AB}\times\overrightarrow{AD} = \begin{pmatrix}-1\\1\\3\end{pmatrix}$, so this is a normal to the plane containing A,B,D. The defining equation for that plane is $-x+y+3z=1$. Since $ABCD$ is a parallelogram any plane containing ABD automatically contains C.

(460.iv) $(-1,0,0)$, $(0,1,0)$, $(0,0,\frac{1}{3})$.

(461.i) Here is the picture of the parallelepiped (which you can also find on page 103):



Knowing the points A, B, D we get $\overrightarrow{AB} = \left(\begin{smallmatrix}-1\\2\\0\end{smallmatrix}\right)$, $\overrightarrow{AD} = \left(\begin{smallmatrix}-2\\0\\1\end{smallmatrix}\right)$. Also, since $\begin{smallmatrix}EFGH\\ABCD\end{smallmatrix}$ is a parallelepiped, we know that all its faces are parallelogram, and thus $\overrightarrow{EF} = \overrightarrow{AB}$, etc. Hence: we find these coordinates for the points A, B, ...

$A(1, 0, 0)$, (given); $B(0, 2, 0)$, (given); $C(-2, 2, 1)$, since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \left(\begin{smallmatrix}-3\\2\\1\end{smallmatrix}\right)$;
$D(-1, 0, 1)$, (given); $E(0, 0, 2)$, (given)
$F(-1, 2, 2)$, since we know E and $\overrightarrow{EF} = \overrightarrow{AB} = \left(\begin{smallmatrix}-1\\2\\0\end{smallmatrix}\right)$
$G(-3, 2, 3)$, since we know F and $\overrightarrow{FG} = \overrightarrow{EH} = \overrightarrow{AD} = \left(\begin{smallmatrix}-2\\0\\1\end{smallmatrix}\right)$
$H(-2, 0, 3)$, since we know E and $\overrightarrow{EH} = \overrightarrow{AD} = \left(\begin{smallmatrix}-2\\0\\1\end{smallmatrix}\right)$.

(461.ii) The area of $ABCD$ is $\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{21}$.

(461.iii) The volume of $\mathfrak{P}$ is the product of its height and the area of its base, which we compute in the previous and next problems. So height$= \frac{\text{volume}}{\text{area base}} = \frac{6}{\sqrt{21}} = \frac{2}{7}\sqrt{21}$.

(461.iv) The volume of the parallelepiped is $\overrightarrow{AE} \cdot (\overrightarrow{AB} \times \overrightarrow{AD}) = 6$.

Sketching Parametrized Curves

(463) The straight line $y = x + 1$, traversed from the top right to the bottom left as t increases from $-\infty$ to $+\infty$.

(464) The diagonal $y = x$ traversed from left to right, from upwards.

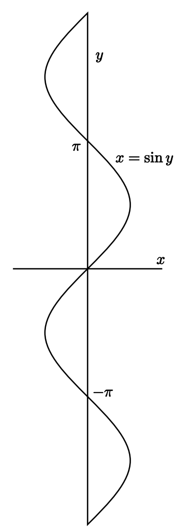
(465) The diagonal $y = x$ again, but since $x = e^t$ can only be positive we only get the part in the first quadrant. At $t = -\infty$ we start at the origin, as $t \to +\infty$ both x and y go to $+\infty$.

(466) The graph of $y = \ln x$, or $x = e^y$ (same thing), traversed in the upwards direction.

(467) The part of the graph of $y = 1/x$ which is in the first quadrant, traversed from left to right.

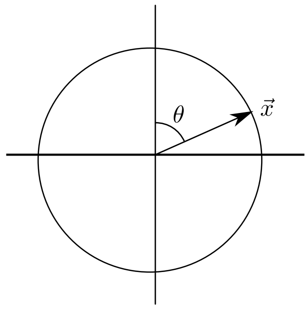
(468) The standard parabola $y = x^2$, from left to right.

(469) The graph $x = \sin y$. This is the usual Sine graph, but on its side.

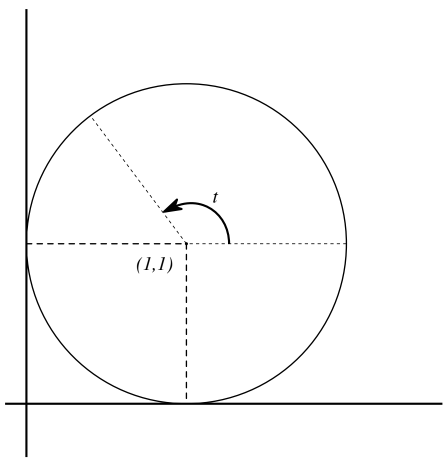


(470) We remember that $\cos 2\alpha = 1 - 2\sin^2\alpha$, so that $\vec{x}(t)$ traces out a part of the parabola $y = 1 - x^2$. Looking at $x(t) = \sin t$ we see $\vec{x}(t)$ goes back and forth on the part of the parabola $y = 1 - 2x^2$ between $x = -1$ and $x = +1$.

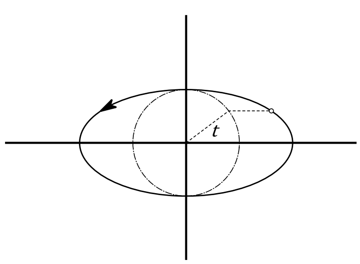
(471) The unit circle, traversed ***clockwise***, 25 times every 2π time units. Note that the angle $\theta = 25t$ is measured from the y-axis instead of from the x-axis.



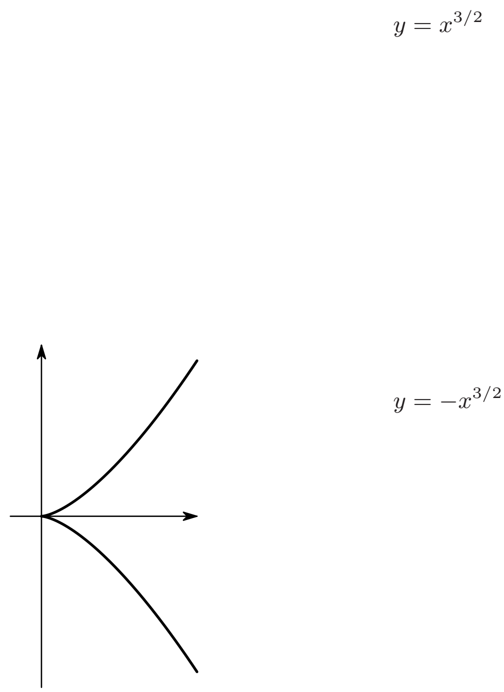
(472) Circle with radius 1 and center $(1, 1)$ (it touches the x and y axes). Traversed infinitely often in counterclockwise fashion.



(473) Without the 2 this would be the standard unit circle (dashed curve below). Multiplying the x component by 2 stretches this circle to an ellipse. So $\vec{\boldsymbol{x}}(t)$ traces out an ellipse, infinitely often, counterclockwise.



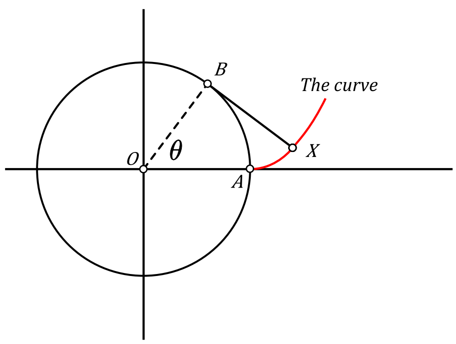
(474) For each $y = t^3$ there is exactly one t, namely, $t = y^{1/3}$. So the curve is a graph (with x as a function of y instead of the other way around). It is the graph of $x = y^{2/3} = \sqrt[3]{y^2}$.



The curve is called ***Neil's parabola***.

(475) If θ is the angle through which the wheel has turned, then $\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} \theta - a\sin\theta \\ 1 - a\cos\theta \end{pmatrix}$.

(478) Here's the picture:



The arc AB has length θ, and we are told the line segment BX has the same length. From this you get

$$\vec{x}(\theta) = \begin{pmatrix} \cos\theta + \theta\sin\theta \\ \sin\theta - \theta\cos\theta \end{pmatrix}$$

This curve is called the ***evolute of the circle.***

(483.i) $\vec{x}(0) = \vec{a}$, $\vec{x}(1) = \vec{c}$ so the curve goes from A to C as t increases from $t = 0$ to $t = 1$. $\vec{x}'(0) = 2(\vec{b} - \vec{a})$ so the tangent at $t = 0$ is parallel to the edge AB, and pointing from A to B. $\vec{x}'(1) = 2(\vec{c} - \vec{b})$ so the tangent at $t = 1$ is parallel to the edge BC, and pointing from B to C. For an animation of the curve in this problem visit Wikipedia at
http://en.wikipedia.org/wiki/File:Bezier_2_big.gif

(483.ii) At $t = 1/2$. If you didn't get this, you can still get partial credit by checking that this answer is correct.

(485.i) Horizontal tangents: $t = 1/4$; Vertical tangents: $t = 0$; Directions: SouthEast $-\infty < t < 1/4$, NorthEast $1/4 < t < 0$, NorthWest $0 < t < \infty$.

(485.ii) This vector function is 2π periodic, so we only look at what happens for $0 \le t \le 2\pi$ (or you could take $-\pi \le t \le \pi$, or any other interval of length 2π).

Horizontal tangents: $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; Vertical tangents: $t = \frac{\pi}{2}, \frac{3\pi}{2}$;

Directions: NE $0 < t < \frac{\pi}{4}$, SE $\frac{\pi}{4} < t < \frac{\pi}{2}$, SW $\frac{\pi}{2} < t < \frac{3\pi}{4}$, NW $\frac{3\pi}{4} < t < \frac{5\pi}{4}$, SW $\frac{5\pi}{4} < t < \frac{3\pi}{2}$, SE $\frac{3\pi}{2} < t < \frac{7\pi}{4}$, NE $\frac{7\pi}{4} < t < 2\pi$.

The curve traced out is a figure eight on its side, i.e. the symbol for infinity "∞".

(485.iii) Very similar to the previous problem. In fact both this vector function and the one from the previous problem trace out exactly the same curve. They just assign different values of the parameter t to points on the curve.

(485.iv) Horizontal points: $t = \pm\sqrt{a}$; Vertical points: $t = 0$; . Directions: SE $-\infty < t < -\sqrt{a}$, NE $-\sqrt{a} < t < 0$, NW $0 < t < \sqrt{a}$, SW $\sqrt{a} < t < \infty$.

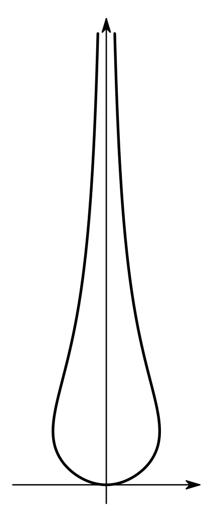
The curve looks like a "fish" (with some imagination.)

(485.v) No horizontal points; Vertical point: $t = 0$. Directions: NE $-\infty < t < 0$, NW $0 < t < \infty$.

(485.vi) This one has lots of horizontal and vertical tangents. If you replace the numbers 2 and 3 by other integers you get curves called "***Lissajous figures***". Get a graphing calculator/program and draw some. Or go to http://en.wikipedia.org/wiki/Lissajous_curve
.

(485.vii) Horizontal point: $t = 0$; Vertical points: $t = \pm 1$; . Directions: SW $-\infty < t < -1$, SE $-1 < t < 0$, NE $0 < t < 1$, NW $1 < t < \infty$.

It sort of looks like this



(But this is really the graph of $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t/(1+t^4) \\ t^2 \end{pmatrix}$.)

(485.ix) This vector function traces out the right half of the parabola $y = 2(x-1)^2$ (i.e. the part with $x \geq 1$), going from right to left for $-\infty < t < 0$, and then back up again, from left to right for $0 < t < \infty$.

(490) Mercury's year is approximately 88 days.

Pluto is 3671 million miles from the sun.

(491) 539 miles.

(492) 27 thousand miles.

(493) 302942 to one.

(494) 6778 miles.

An odd number of orbits does not work. The KMart space engineers knew that six orbits was too few but did not realize that 7 orbits is worse than six. The KMart7 satelite was tragically misnamed.

(495) $A = \frac{1}{2}\int_0^T r^2 \frac{d\theta}{dt} dt = \frac{1}{2}\int_0^T \beta \, dt = \frac{1}{2} \, T \, \beta$

(496) $a = \frac{D+d}{2}$ and $b = \sqrt{dD}$.

(497) The furthest distance from sun is 3313 million miles. Its max and min speeds are 1086 and 17 million miles per year.