Show all work. Simplify your answers. Circle your answer.

You may use a cheat sheet: one $8\frac{1}{2} \times 11$ sheet of paper with anything written on both sides.

No books, no calculator, no cell phones, no pagers, no electronic devices.

Name____________________________________

Hand in to your TA. Circle your TA's name:

Adam Berliner
Ben Ellison
Jon Godshall
E. Alec Johnson
Dan McGinn
Derek Moffitt
Richard Oberlin

| Problem | Points | Score |
|---|---|---|
| 1 | 8 | |
| 2 | 8 | |
| 3 | 8 | |
| 4 | 8 | |
| 5 | 8 | |
| 6 | 8 | |
| 7 | 8 | |
| 8 | 8 | |
| 9 | 8 | |
| 10 | 8 | |
| 11 | 8 | |
| 12 | 8 | |
| 13 | 4 | |
| Total | 100 | |

Solutions will be posted shortly after the exam: www.math.wisc.edu/~miller/m222

1. (8 pts) Find the general solution of the differential equation:

$$y'' - 4y = e^x$$

2. (8 pts) Find the polar coordinates of the point with Cartesian coordinates:

$$(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$$

3. (8 pts) Find a vector that has the same direction as $3\mathbf{i} - 4\mathbf{j}$ but twice its length.

4. (8 pts) Find the $\cos(\theta)$ where $\theta$ is the angle between the vectors $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

5. (8 pts) Find

$$\int e^x \cos(x)\, dx$$

6. (8 pts) Evaluate this improper integral or show that it diverges:

$$\int_1^e \frac{dx}{x\ln(x)}$$

7. (8 pts) Determine whether the following series converges or diverges and indicate which test you use.

$$\frac{1}{2\sqrt{3}} + \frac{1}{3\sqrt{4}} + \frac{1}{4\sqrt{5}} + \dots$$

8. (8 pts) Find the power series $\sum_{n=0}^{\infty} a_n x^n$ for the function

$$f(x) = \frac{1}{1 - 4x^2}$$

and determine the interval where it converges.

9. (8 pts) In the plane a curve $\mathbf{r}(t)$ satisfies

$$\mathbf{r}''(t) = \mathbf{i} \ + \ e^t \ \mathbf{j}$$

$$\mathbf{r}'(0) = 2\mathbf{i} \ + \mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{i} \ + \mathbf{j}$$

Find the curve $\mathbf{r}$.

10. (8 pts) Find the equation, $ax + by + cz = d$, of the plane that contains the intersecting lines:

$$\mathbf{L}_1(t) = (1,1,1) \ + \ t(1,-2,1)$$

$$\mathbf{L}_2(t) = (2,-1,2) \ + \ t(-1,1,1)$$

11. (8 pts) Find the curvature $\kappa$ for the curve

$$\mathbf{r}(t) = \cos(t)\,\mathbf{i} \;\; + \;\; \sin(t)\,\mathbf{j} \;\; + \;\; t\,\mathbf{k}$$

as a function of $t$.

12. (8 pts) A tangent line to the ellipse $x^2 + 2y^2 = 1$ intersects the $y$-axis at $(0, 2)$. Find the $y$-coordinate of the point of tangency.

13. (4 pts) Use this page for scratch paper or for any comments you would like to make about the course.

(You will get four points whether or not you write anything - I just wanted the exam to add up to 100.)

Answers

1. $y = -\frac{1}{3}e^x + C_1 e^{2x} + C_2 e^{-2x}$.

2. $r = \sqrt{3}$, $\theta = \frac{3}{4}\pi$

3. $6\mathbf{i} - 8\mathbf{j}$.

4. $\cos(\theta) = \frac{2}{\sqrt{18}} = \frac{\sqrt{2}}{3}$

5. $\frac{1}{2}(e^x \cos(x) + e^x \sin(x)) + C$

6. It diverges $\ln(\ln(x))]_1^e = \infty$.

7. It converges by comparison to the convergent series $\sum \frac{1}{n^{3/2}}$.

8. $\sum 4^n x^{2n}$. Converges for $-\frac{1}{2} < x < \frac{1}{2}$.

9. $\mathbf{r}(t) = (\frac{t^2}{2} + 2t + 1)\mathbf{i} \quad + \quad e^t\mathbf{j}$

10. $(-1, 1, 1) \times (1, -2, 1) = (3, 2, 1)$ so equation is
$3x+2y+z = d$. Plug in $(1, 1, 1)$ to find out $d = 6$. Hence equation is $3x+2y+z = 6$.

11. $\kappa = \frac{1}{2}$

12. Let $(x, y)$ be the point of tangency. Then the tangent line has slope

$$\frac{y-2}{x}$$

By implicit differentiation of $x^2 + 2y^2 = 1$ we get that

$$x + 2y\frac{dy}{dx} = 0 \text{ hence } \frac{dy}{dx} = \frac{x}{-2y}$$

Equating gives

$$\frac{y-2}{x} = \frac{x}{-2y} \qquad (-2y)(y-2) = x^2 = 1 - 2y^2$$

$$4y = 1 \qquad y = \frac{1}{4}$$