Show all work. Simplify your answers. Circle your answer.
No notes, no books, no calculator, no cell phones, no pagers, no electronic devices.

Name____

Hand in to your TA. Circle your TA's name:

Adam Berliner
Ben Ellison
Jon Godshall
E. Alec Johnson
Dan McGinn
Derek Moffitt
Richard Oberlin

Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
Total	50	

Solutions will be posted shortly after the exam: www.math.wisc.edu/~miller/m222

1. (5 pts) Find

$$\frac{dy}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2}$$

as functions of t for the curve

$$x = 1 - \cos(t) \quad y = 1 + \sin(t)$$

Answer:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t)}{\sin(t)} = \text{cotan}(t)$$

letting $u = \frac{dy}{dx}$ then

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du/dt}{dx/dt} = \frac{-\text{cosec}^2(t)}{\sin(t)} = -\text{cosec}^3(t)$$

2. (5 pts) Sketch the region bounded by the equation

$$r^2 = 16\cos(2\theta)$$

which is given in polar coordinates. Find its area.

Answer:

This is a leminscate see page 546. One quarter of it occurs between the angles of 0 and $\frac{\pi}{4} = 45$ degrees and r positive.

$$\text{Area } = 4\int_0^{\pi/4} \frac{r^2}{2} d\theta = 32\int_0^{\pi/4} \cos(2\theta) d\theta = 16\,\sin(2\theta)]_0^{\pi/4} = 16$$

3. (5 pts) Find both of the foci of the ellipse

$$16(x-1)^2 \ + \ 25(y+2)^2 = 400$$

Answer:
The equation is equivalent to:

$$\frac{(x-1)^2}{25} + \frac{(y+2)^2}{16} = 1$$

This is centered at the point $(1,-2)$. If we let $u = x - 1$ and $v = y + 2$ then in the uv coordinates we have

$$\frac{u^2}{25} + \frac{v^2}{16} = 1$$

This has foci at $(\pm 3, 0)$ and is centered at $(0,0)$. Translating back to xy we have foci at $(1 + \pm 3, -2)$ which is $(4,-2)$ and $(-2,-2)$.

4. (5 pts) Transform the equation $xy = 1$ into an equation of u and v by a suitable rotation of the axis so that the cross product term, buv, is zero.

Answer:
$\text{cotan}(2\theta_0) = \frac{A-C}{B} = 0$ so $2\theta_0 = \pi/2$ and $\theta_0 = \pi/4 = 45$ degrees.

$$x = u\cos(\pi/4) - v\sin(\pi/4) = \frac{\sqrt{2}}{2}(u - v)$$

$$y = u\sin(\pi/4) + v\cos(\pi/4) = \frac{\sqrt{2}}{2}(u + v)$$

$$1 = xy = \frac{1}{2}(u^2 - v^2)$$

5. (5 pts) An ellipse is centered at the origin, intercepts the y-axis at $(0, \pm 3)$, and intercepts the x-axis at $(\pm 5, 0)$. Find its eccentricity and its directrix.

Answer:

The focus at $(c, 0)$ satisfies $a^2 = b^2 + c^2$ where $a = 5$ and $b = 3$. Hence $c = 4$. The eccentricity satisfies $ae = c$ so $e = \frac{4}{5}$. The directrix line satisfies $k = \frac{a}{e} = \frac{5}{4/5} = \frac{25}{4}$ so the line is $x = \frac{25}{4}$.

6. (5 pts) Find the equation of the parabola through the point $(6, -5)$ if its vertex is at the origin and its axis of symmetry is the y-axis. Make a sketch.

Answer:
$y = mx^2$ so $-5 = m \cdot 36$ so $y = -\frac{5}{36}x^2$ which opens out downward.

7. (5 pts) Solve the differential equation:

$$y'' + y' - 2y = -10\sin(x)$$

Answer:
$r^2 + 4 - 2 = 0$ so $(r+2)(r-1) = 0$ so $r = -2$ and $r = 1$. The associated homogeneous equation has solution

$$y_H = C_1 e^{-2x} + C_2 e^x$$

To find a particular solution to the nonhomogeneous equation guess

$$y_P = A\sin(x) + B\cos(x)$$

Then

$$y_P'' + y_P' - 2y_P = (-3A - B)\sin(x) + (A - 3B)\cos(x)$$

equating

$$(-3A - B)\sin(x) + (A - 3B)\cos(x) = -10\sin(x)$$

gives $A - 3B = 0$ and $(-3A - B) = -10$ solving gives us $A = 3$ and $B = 1$. Hence the general solution is

$$y = C_1 e^{-2x} + C_2 e^x + 3\sin(x) + \cos(x)$$

8. (5 pts) Solve the following differential equation with initial values:

$$y'' - 4y' + 4y = 0 \text{ where } y = 1 \text{ and } y' = 1 \text{ when } x = 0$$

Answer:
The associated quadratic $(r-2)^2$ gives us the general solution

$$y = C_1 e^{2x} + C_2 x e^{2x}$$

then

$$y' = (2C_1 + C_2)e^{2x} + 2C_2 x e^{2x}$$

setting $y(0) = 1$ and $y'(0) = 1$ gives us $1 = C_1$ and $1 = 2C_1 + C_2$. Solving gives us $C_2 = -1$ and so the unique solution is

$$y = e^{2x} - xe^{2x}$$

9. (5 pts) Solve the following differential equation

$$\frac{dy}{dx} - \frac{y}{x} = xe^x$$

Answer:
Let

$$\mu = e^{\int -\frac{1}{x} dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$\int \mu q = \int \frac{1}{x} x e^x = \int e^x = e^x$$

$$y = \frac{1}{\mu}(\int \mu q + C) = x(e^x + C)$$

10. (5 pts) Find the Maclaurin polynomial of order 2 for $f(x) = \ln(1+x)$. Use it to approximate $\ln(1.2)$. Find an upper bound for the absolute value of the error.

Answer:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \epsilon$$

where $\epsilon = \frac{1}{6}f'''(\zeta)x^3$ for some ζ between x and 0.

$$f(x) = \ln(1+x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

Hence the Maclaurin polynomial is

$$p(x) = x - \frac{1}{2}x^2$$

$$p(.2) = (.2) - \frac{1}{2}(.2)^2 = .18$$

The error is bounded by $\frac{1}{6}M(.2)^3$ where M satisfies

$$0 \leq \frac{2}{(1+\zeta)^3} \leq M$$

for $0 \leq \zeta \leq .2$. This is largest when $\zeta = 0$ so $M = 2$ and the error is bounded by $\frac{1}{3}(.2)^3$.

Another way to obtain error bound is to use alternating series:

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

(by integrating and using $\ln(1+0) = C$). Since $0 < x = .2 < 1$ this is an alternating series:

$$\ln(1+x) = x - \frac{x^2}{2} + \epsilon$$

where

$$0 < \epsilon = \frac{x^3}{3} - \frac{x^4}{4} + \dots \leq \frac{x^3}{3} = \frac{(.2)^3}{3}$$