Show all work. Simplify your answers. Circle your answer.
No notes, no books, no calculator, no cell phones, no pagers, no electronic devices.

Name______________________________________

Hand in to your TA. Circle your TA's name:

Adam Berliner
Ben Ellison
Jon Godshall
E. Alec Johnson
Dan McGinn
Derek Moffitt
Richard Oberlin

| Problem | Points | Score |
|---|---|---|
| 1 | 4 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 5 | |
| 6 | 5 | |
| 7 | 5 | |
| 8 | 5 | |
| 9 | 5 | |
| 10 | 6 | |
| Total | 50 | |

Solutions will be posted shortly after the exam: www.math.wisc.edu/~miller/m222

1. (4 pts) Find the following integral:

$$\int \frac{e^{\arctan(x)}}{1+x^2}\ dx$$

Answer:
Substitute $u = \arctan(x)$ and $du = \frac{1}{1+x^2}dx$ getting

$$\int \frac{e^{\arctan(x)}}{1+x^2}\ dx = \int e^u\ du = e^u + C = e^{\arctan(x)} + C$$

2. (5 pts) In the power series for the following function

$$(1+x)^{\frac{5}{2}} = \sum_{n=0}^{\infty} a_n x^n$$

what is $a_3$?

Answer:

$f(x) = (1+x)^{\frac{5}{2}}$

$f'(x) = \frac{5}{2}(1+x)^{\frac{3}{2}}$

$f''(x) = \frac{15}{4}(1+x)^{\frac{1}{2}}$

$f'''(x) = \frac{15}{8}(1+x)^{-\frac{1}{2}}$

$f'''(0) = \frac{15}{8}$

$a_3 = \frac{f'''(0)}{3!} = (\frac{15}{8})(\frac{1}{6}) = \frac{5}{16}$

3. (5 pts) Starting with the geometric series, show how to obtain the power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

Answer:

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

substituting $-x$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

differentiating both sides
$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 + \cdots$

multiplying by $-1$
$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

4. (5 pts) Determine all $x$ such that the following series converges:

$$\frac{3^2x^2}{3} + \frac{3^3x^3}{4} + \frac{3^4x^4}{5} + \frac{3^5x^5}{6} + \cdots$$

Answer:

$\sum_{n=2}^{\infty} \frac{3^nx^n}{n+1}$. Apply the ratio test

$$\frac{|\frac{3^{n+1}x^{n+1}}{n+2}|}{|\frac{3^nx^n}{n+1}|} = |x| \cdot \frac{3^{n+1}}{3^n} \cdot \frac{n+1}{n+2} = |x| \cdot 3 \cdot \frac{n+1}{n+2} \to |x| \cdot 3$$

So the series converges for $|x| \cdot 3 < 1$ or equivalently $|x| < \frac{1}{3}$ and diverges for $|x| > \frac{1}{3}$. For the two end points, $x = \frac{1}{3}$ and $x = -\frac{1}{3}$, the series is

$$\sum_{n=2}^{\infty} \frac{3^nx^n}{n+1} = \sum_{n=2}^{\infty} \frac{3^n(\frac{1}{3})^n}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n+1}$$

which is a tail of the harmonic series and so diverges, and

$$\sum_{n=2}^{\infty} \frac{3^nx^n}{n+1} = \sum_{n=2}^{\infty} \frac{3^n(-\frac{1}{3})^n}{n+1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1}$$

which is an alternating series and so converges. Hence this power series converges for exactly those $x$ which satisfy $-\frac{1}{3} \leq x < \frac{1}{3}$.

5. (5 pts) Determine whether the following series is absolutely convergent, conditionally convergent, or divergent and explain why.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$$

Answer:

$$\frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \frac{n^2}{n^2+1} \to 1 \text{ as } n \to \infty$$

The harmonic series $\sum \frac{1}{n}$ diverges, so by the limit comparison test $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges. The original series converges by the alternating series test and so the series is conditionally convergent.

6. (5 pts) Use the integral test to determine whether the following series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$$

Answer:
$\int \frac{1}{x(\ln(x))^2} dx = \int \frac{1}{u^2}\, du = \frac{-1}{u}$ after substituting $u = \ln(x)$ and $du = \frac{1}{x} dx$. Hence

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2} dx = -\left(\frac{1}{\ln(x)}\right]_2^{\infty} = \frac{1}{\ln(2)} - \frac{1}{\ln(\infty)} = \frac{1}{\ln(2)}$$

which is finite. Since the function is positive and eventually decreasing the associated series converges by the integral test.

7. (5 pts) Find

$$\int \tan^4(x)\; dx$$

Answer:

Using $1 + \tan^2(x) = \sec^2(x)$ we have that

$$\int \tan^4(x)\; dx = \int (\sec^2(x) - 1)\tan^2(x)\; dx = \int \sec(x)^2 \tan^2(x)\; dx - \int \tan^2(x)\; dx$$

The first integral is $\frac{\tan^3(x)}{3}$ after substituting $u = \tan(x)$ and $du = \sec^2(x)dx$. The second integral is

$$\int \tan^2(x)\; dx = \int \sec^2(x) - 1\; dx = \tan(x) - x$$

giving us that answer

$$\frac{\tan^3(x)}{3} - \tan(x) + x + C$$

8. (5 pts) Find

$$\int \sqrt{1-x^2}\, dx$$

Answer:

Substituting $x = \sin(\theta)$ and $dx = \cos(\theta)d\theta$ gives us

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta = \int \cos^2(\theta)\, d\theta$$

using the formula $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and so

$$\cos(2A) = \cos^2(A) - \sin^2(A) = \cos^2(A) - (1-\cos^2(A)) = 2\cos^2(A) - 1$$

and hence $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$ we have that

$$\int \cos^2(\theta)\, d\theta = \int \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

Since $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$ we have that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Since $x = \sin(\theta)$ and $\cos(\theta) = \sqrt{1-\sin^2(\theta)} = \sqrt{1-x^2}$ we have that the answer is

$$\frac{1}{2}\arcsin(x) + \frac{1}{2}(x\sqrt{1-x^2}) + C$$

9. (5 pts) Find

$$\int \frac{2x^2-1}{x^2(x-1)}\,dx$$

Answer:

Let

$$\frac{2x^2-1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)}$$

cross multiplying and equating numerators gives us

$$2x^2-1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (-A+B)x - B$$

so $-B=-1$, $A+C=2$, $-A+B=0$. Solving these gives $A=B=C=1$ and therefore

$$\int \frac{2x^2-1}{x^2(x-1)}\,dx = \int \frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x-1)}\,dx = \ln|x| - \frac{1}{x} + \ln|x-1| + C$$

10. (6 pts) Evaluate or show that it diverges:

$$\int_2^\infty \frac{\ln(x)}{x^2}\,dx$$

Answer:

Using integration by parts let $u = \ln(x)$ and $dv = \frac{1}{x^2}\,dx$ we have that $du = \frac{1}{x}\,dx$ and $v = -\frac{1}{x}$ and so

$$\int \frac{\ln(x)}{x^2}\,dx = -\frac{1}{x}\ln(x) + \int \frac{1}{x^2}dx = -\frac{1}{x}\ln(x) - \frac{1}{x} + C$$

Hence

$$\int_2^\infty \frac{\ln(x)}{x^2}\,dx = -\frac{1}{x}\ln(x) - \frac{1}{x}\Big]_2^\infty$$

$$\lim_{b\to\infty}\frac{\ln(b)}{b} = \lim_{b\to\infty}\frac{\frac{1}{b}}{1} = 0$$

by L'Hopitol's Rule, so our integral converges to

$$\frac{1}{2}\ln(2) + \frac{1}{2}$$