Practice Problems A. Miller Spring 97 Math 221

The final exam will be
Monday December 15 at 5:05pm
in 494 Van Hise.

These problems were taken from exams given by
Joel Robbin in 1995-1996.
He will be the one writing your final exam.
For more see http://math.wisc.edu/~robbin/

1.1 Find the constant c which makes g continuous on $(-\infty, \infty)$,

$$g(x) = \begin{cases} x^2 & \text{if } x < 4, \\ cx + 20 & \text{if } x \geq 4. \end{cases}$$

Answer:By the limit laws, $g(x)$ is continuous at any $x \neq 4$.

$$\lim_{x\to 4-} g(x) = \lim_{x\to 4-} x^2 = 16, \qquad \lim_{x\to 4+} g(x) = \lim_{x\to 4+} cx + 20 = 4c + 20.$$

The function $g(x)$ is continuous when these are equal, i.e. when $c = -1$.

1.2 Find $\lim_{t\to -\infty} \frac{t^3 - 1}{t^2 - 1}$

Answer:

$$\lim_{t\to -\infty} \frac{t^3 - 1}{t^2 - 1} = \lim_{t\to -\infty} \frac{t - \frac{1}{t^2}}{1 - \frac{1}{t^2}} = -\infty$$

1.3 Find $\lim_{t\to 3} \frac{t^3 - 1}{t^2 - 1}$

Answer:

$$\lim_{t\to 3} \frac{t^3 - 1}{t^2 - 1} = \frac{3^3 - 1}{3^2 - 1} = \frac{26}{8}$$

1.4 Find $\lim_{t\to 1} \frac{t^3 - 1}{t^2 - 1}$

Answer:

$$\lim_{t\to 1} \frac{t^3 - 1}{t^2 - 1} = \lim_{t\to 1} \frac{(t-1)(t^2 + t + 1)}{(t-1)(t+1)} = \lim_{t\to 1} \frac{(t^2 + t + 1)}{(t+1)} = \frac{3}{2}$$

2.1 Consider the curve

$$y^2 + xy - x^2 = 11.$$

(1) Find the equation of the tangent line to the curve at the point $P(2,3)$. (2) Find $\frac{d^2y}{dx^2}$ at the point $P(2,3)$.

Answer:(1)Differentiate:

$$2y\frac{dy}{dx} + y + x\frac{dy}{dx} - 2x = 0.$$

Evaluate at the point $P(2,3)$:

$$6\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} + 3 + 2\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} - 4 = 0.$$

Solve:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = \frac{1}{8}.$$

This is the slope of the tangent line. The point $P(2,3)$ lies on the tangent line so the equation of the tangent line is

$$(y-3) = \frac{(x-2)}{8}.$$

Answer:(2)Differentiate again:

$$2\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] + \frac{dy}{dx} + \left[\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right] - 2 = 0.$$

Evaluate at $(x,y) = (2,3)$:

$$2\left[\left(\frac{1}{8}\right)^2 + 3\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,3)}\right] + \frac{1}{8} + \left[\frac{1}{8} + 2\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,3)}\right] - 2 = 0.$$

Solve:

$$\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,3)} = \frac{-2\left(\frac{1}{8}\right)^2 - \frac{1}{8} - \frac{1}{8} + 2}{6+2}.$$

2.2 (a) Find the equation for the tangent line to $y = \ln(x)$ at the point $x = 1$.

Answer:(a) The equation for the tangent line is $y = L(x)$ where $L(x)$ is the linear approximation of $y = \ln(x)$ at $x = 1$. Using the formula

$$L(x) = f(a) + f'(a)(x-a)$$

with $f(x) = \ln x$ and $a = 1$ and $f'(x) = 1/x$ we get $f(1) = \ln(1) = 0$, $f'(1) = 1$ and so

$$L(x) = 0 + 1 \cdot (x - 1) = (x - 1).$$

(b) Find the quadratic approximation $Q(x)$ to the function $f(x) = \ln x$ at 1.

Answer:(b) Using the formula

$$Q(x) = f(a) + f'(a)(x - 1) + \frac{f''(a)(x - a)^2}{2}$$

with $f(x) = \ln(x)$. we get $f'(x) = 1/x$, $f''(x) = -1/x^2$ so $f''(1) = -1$ and hence

$$Q(x) = (x - 1) - \frac{(x - 1)^2}{2}.$$

(c) Estimate $\ln(1.1)$ without a calculator.

Answer:(c) $\ln(1.1) \approx Q(1.1) = 0.1 - (0.1)^2/2 = 0.095$.

2.3 Find $\dfrac{d}{dx} 3^{\sin x}$.

Answer:$\ln(3)\cos(x)3^{\sin x}$

2.4 Find $\dfrac{d}{dx} e^{\sin^{-1}(x)}$.

Answer:$\frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}$

2.5 Find $\dfrac{d}{dx} F(e^x)$. where the derivative of the function F is $F'(u) = \sin(u^2)$.

Answer:$e^x \sin(e^{2x})$

2.6 Find $f'(x)$ and $f''(x)$ if $f(x) = \sin(x^3 - 2)$.

Answer:

$$f'(x) = \left(\cos(x^3 - 2)\right) 3x^2, \qquad f''(x) = -\left(\sin(x^3 - 2)\right)\left(3x^2\right)^2 + \left(\cos(x^3 - 2)\right) 6x.$$

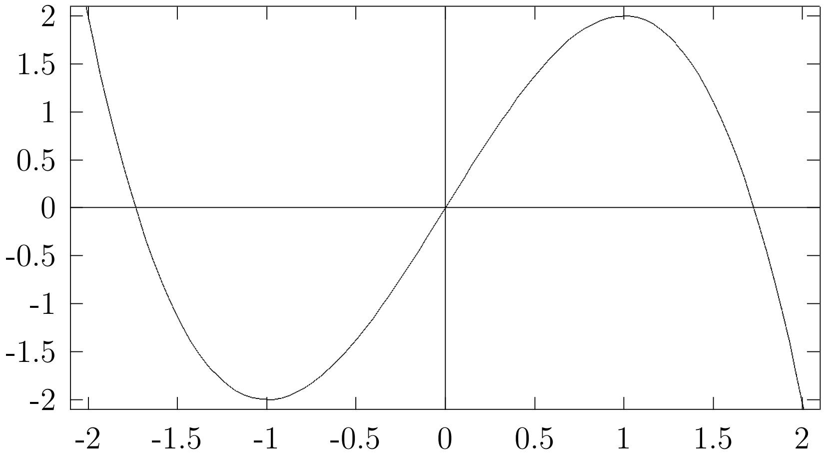
2.7 Find $g'(3)$ if $h'(9) = 17$ and $g(x) = h(x^2)$.

Answer:

$$g'(x) = h'(x^2)2x, \qquad g'(3) = h'(9)6 = 102.$$

2.8 Consider the function $y = f(x)$ whose graph is shown below. Match the expression in the left column with the correct corresponding value in the right column.

$f'(0)$	-6
$f'(0.9)$	0
$f'(1)$	3
$f'(1.732)$	0.6



Answer:For positive x, the slope of the tangent line is decreasing as x increases, so $f'(0) > f'(0.9) > f'(1) > f'(1.732)$. The only possibility is $f'(0) = 3$, $f'(0.9) = 0.6$, $f'(1) = 0$, $f'(1.732) = -6$.

2.9 Prove the product rule $(fg)' = f'g + fg'$ using the definition of the derivative, high school algebra, and the appropriate limit laws. Justify each step.

3.1 Find $\lim_{x \to \infty} \frac{\sin(3x)}{x}$

Answer:Since $-1 \leq \sin(3x) \leq 1$ for all x we have

$$-\frac{1}{x} \leq \frac{\sin(3x)}{x} \leq \frac{1}{x}$$

for $x > 0$. Hence $\lim_{x \to \infty} \frac{\sin(3x)}{x} = 0$ by the Squeeze theorem.

3.2 Find $\lim_{x \to 0} \frac{\sin(x^2)}{x}$

Answer:

$$\lim_{x\to 0}\frac{\sin(x^2)}{x} = \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot x = \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot \lim_{x\to 0} x = \lim_{u\to 0}\frac{\sin(u)}{u}\cdot \lim_{x\to 0} x = 1\cdot 0 = 0.$$

3.3 Find $\lim_{x\to\pi/3}\frac{\sin(x)-\sin(\pi/3))}{x-\pi/3}$

Answer:Let $f(x)=\sin(x)$ and $a=\pi/3$. Then $f'(x)=\cos(x)$ and

$$\lim_{x\to\pi/3}\frac{\sin(x)-\sin(\pi/3)}{x-\pi/3} = \lim_{x\to a}\frac{f(x)-f(a))}{x-a} = f'(a)=\cos(\pi/3)=\frac{1}{2}.$$

3.4 Find $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$. Justify your steps.

Answer:Using the formula

$$e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m \qquad (\heartsuit)$$

implies

$$e = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n/2} = \left(\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right)^{1/2}.$$

so squaring gives:

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$$

The wording of the question leaves some doubt as to whether the professor wants a proof of ($\heartsuit$). To be on the safe side we'll provide it:

$$\ln\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right) = \lim_{m\to\infty}\ln\left(1+\frac{1}{m}\right)^m =$$

$$\lim_{m\to\infty}\frac{\ln\left(1+\frac{1}{m}\right)-\ln(1)}{1/m} = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}.$$

The limit on the right is the derivative of the $\ln(x)$ evaluated at $x=1$. This is $\ln'(1)=1/1=1$ so

$$\ln\left(\lim_{m\to\infty} 1+\frac{1}{m}\right)^m = 1 \qquad \text{so} \qquad \lim_{m\to\infty} 1+\frac{1}{m} = e.$$

You can also use l'Hospital's rule.

3.5 Suppose g is the inverse function to $f(x)=x^5+x+1$. Find $g'(1)$ and $g'(3)$

Answer:$x = g(y) \iff y = f(x)$. Since $f(0) = 1$ we have $g(1) = 0$. Since $f(1) = 3$ we have $g(3) = 1$. Now $g(f(x)) = 1$ so by the chain rule

$$g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{5x^4 + 1}.$$

When $x = 0$ this gives

$$g'(1) = g'(f(0)) = \frac{1}{5 \cdot 0^4 + 1} = 1$$

When $x = 1$ this gives

$$g'(3) = g'(f(1)) = \frac{1}{5 \cdot 1^4 + 1} = \frac{1}{6}.$$

3.6 (a) If $f(x) = x^{\ln x}$ what is $f'(x)$?

Answer: (a) $x = e^{\ln x}$ so $x^{\ln x} = e^{(\ln x)^2}$. Hence by the Chain Rule

$$f'(x) = \frac{d}{dx}e^{(\ln x)^2} = e^{(\ln x)^2}\frac{d}{dx}(\ln x)^2 = e^{(\ln x)^2}2(\ln x)\frac{d}{dx}\ln x = \frac{e^{(\ln x)^2}2(\ln x)}{x}.$$

(b) Express $\frac{du}{dt}$ in terms of t and u if $u = t\ln(t + u)$.

Answer:(b) Use implicit differentiation:

$$\frac{du}{dt} = \ln(t + u) + \frac{t}{t + u} \cdot \left(1 + \frac{du}{dt}\right).$$

Solve for du/dt:

$$\left(1 - \frac{t}{t + u}\right)\frac{du}{dt} = \ln(t + u) + \frac{t}{t + u};$$

so

$$\frac{du}{dt} = \left(1 - \frac{t}{t + u}\right)^{-1}\left(\ln(t + u) + \frac{t}{t + u}\right).$$

3.7 (a) Find $\lim_{t\to 0} \frac{\sin t}{t^3 - t}$.

Answer: (a) By l'Hospital's Rule:

$$\lim_{t\to 0} \frac{\sin t}{t^3 - t} = \lim_{t\to 0} \frac{\cos t}{3t^2 - 1} = -1.$$

(b) Find $\lim_{x\to\infty} \ln(2x+1) - \ln(x)$.

Answer: (b) $\ln(2x+1) - \ln(x) = \ln\left(\frac{2x+1}{x}\right) = \ln\left(2 + \frac{1}{x}\right)$. Hence

$$\lim_{x\to\infty} \ln(2x+1) - \ln(x) = \ln\left(\lim_{x\to\infty} 2 + \frac{1}{x}\right) = \ln(2).$$

3.8 Find $\lim_{x\to 3} \frac{2^x - 1}{3^x - 1}$

Answer: $\frac{7}{26}$

3.9 Find $\lim_{x\to 0} \frac{2^x - 1}{3^x - 1}$

Answer: $\frac{\ln(2)}{\ln(3)}$

3.10 Find $\lim_{x\to\infty} \frac{2^x - 1}{3^x - 1}$

Answer: 0

3.11 Find $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2+1}\,dt$

Answer: e

3.12 Find $\lim_{x\to\infty} \left(\frac{3+x}{x}\right)^x$.

Answer: e^3

3.13 Find $\lim_{x\to 0.6} \sin(\cos^{-1} x)$.

Answer: .8

3.14 Find the equation for the tangent line to $y = e^{x-2y}$ at the point $(x, y) = (2, 1)$.

Answer: $3(y-1) = x - 2$

3.15 The population of the country of Slobia grows exponentially.

(a) If its population in the year 1980 was 1,980,000 and its population in the year 1990 was 1,990,000, what will be its population in the year 2000?

Answer: (a) $1980000e^{2\ln(\frac{199}{198})}$

(b) How long will it take the population to double? (Your answer may be expressed in terms of exponentials and natural logarithms.)

Answer: (b) The population doubles every $10\frac{\ln(2)}{\ln\frac{199}{198}} years.$

3.16 Find $f'(x)$ if $f(x) = \dfrac{e^{5x}}{1+e^{2x}}$.

Answer:$f'(x) = \dfrac{5e^{5x}(1+e^{2x}) - e^{5x} \cdot 2e^{2x}}{(1+e^{2x})^2}$.

3.17 Find $\dfrac{d\theta}{dt}$ when $\theta = \sin^{-1}(t^2)$.

Answer:Let $y = t^2$ so $\theta = \sin^{-1}(y)$. Then

$$\frac{d\theta}{dt} = \frac{d\theta}{dy}\frac{dy}{dt} = \frac{1}{\sqrt{1-y^2}} \cdot 2t = \frac{2t}{\sqrt{1-t^4}}.$$

3.18 After 3 days a sample of radon-222 decayed to 58% of its original amount.
(a) What is the half life of radon-222?

Answer: (a) Let Y be the amount of radon-222 at time t and Y_0 be the amount when $t = 0$. Thus $Y = Y_0 e^{ct}$ for all time t. When $t = 3$ we have $Y = 0.58Y_0$. Hence $0.58Y_0 = Y_0 e^{c3}$. Apply ln to find c: $c = \dfrac{\ln(0.58)}{3}$. The half life is the time τ when $Y = 0.5Y_0$ i.e. $0.5Y_0 = Y_0 e^{c\tau}$. Apply ln to find τ: $\tau = \dfrac{\ln(0.5)}{c} = \dfrac{3\ln(0.5)}{\ln(0.58)}$.
(b) How long would it take for the sample to decrease to 10% of its original amount?

Answer: (b) The desired time t satisfies $0.10Y_0 = Y_0 e^{ct}$. To find t cancel $Y + 0$ and apply ln : $t = \dfrac{\ln(0.10)}{c} = \dfrac{3\ln(0.10)}{\ln(0.58)}$.

3.19 (a) Find the inverse function $g(y)$ to the function $f(x) = \sqrt{9+x}$.

Answer: (a) For $y \geq 0$ we have $y = \sqrt{9+x} \iff y^2 = 9 + x \iff x = y^2 - 9$. Thus $g(y) = f^{-1}(y) = y^2 - 9$.
(b) What is the domain of $f(x)$?

Answer:(b) When a function is given by a formula, the domain is the set of x for which the formula is meaningful, *unless the contrary is asserted.* The domain of f is $x \geq -9$.
(c) What is the domain of $g(y)$?

Answer: (c) The formula defining $g(y)$ is meaningful for all y, but g is defined to be the inverse function to f. Hence $domain(g) = range(f)$. Thus $domain(g)$ is the set of all $y \geq 0$.

3.20 Find $\sin^{-1}(\sin(3))$ exactly. Justify your reasoning.

Answer: For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ we have

$$\theta = \sin^{-1}(\sin(3)) \iff \sin(\theta) = \sin(3).$$

Also $\sin(\pi - 3) = \sin(3)$ and $-\frac{\pi}{2} \leq \pi - 3 \leq \frac{\pi}{2}$. Therefore $\sin^{-1}(\sin(3)) = \pi - 3$. (The horizontal line $y = \sin(3)$ intersects the curve $y = \sin(\theta)$ infinitely often.)

4.1 An athlete is at point A on a bank of a straight river, 3 km wide and wants to reach B, 8 km downstream on the opposite bank, as quickly as possible. He will row to a point X, x km downstream and on the side opposite A and then run along the shore to B. If he rows at 6 km/hour and runs at 8 km/hour, what should x be?

Answer:See Text Page 297 Example 4.

4.2 The second derivative of a function $f(x)$ satisfies

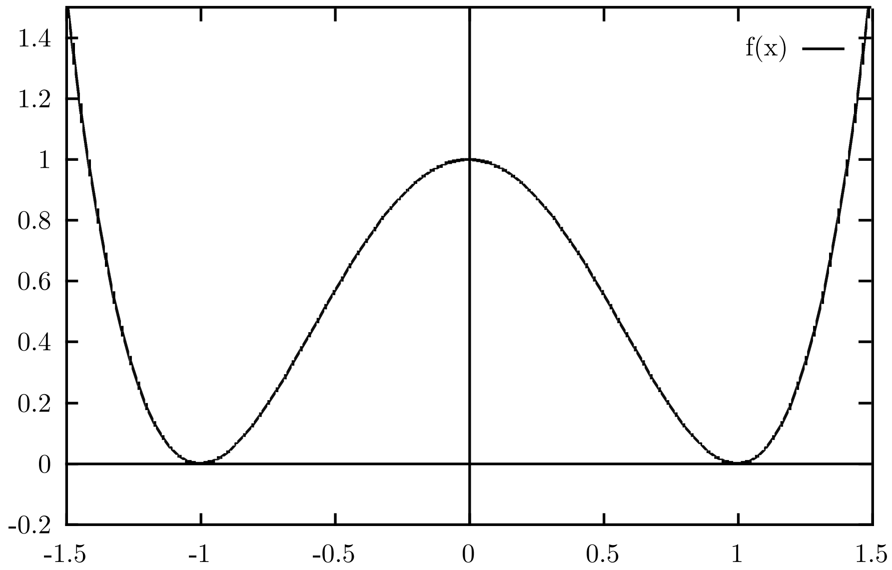
$$f''(x) = 12x^2 - 4.$$

Moreover, $f'(0) = 0$ and $f(1) = 0$. (a) Find the function $f(x)$.

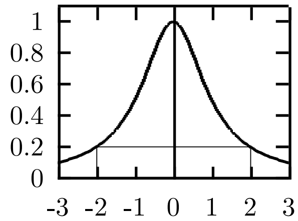
Answer: (a) $f'(x) = 4x^3 - 4x + C_1$ so $0 = f'(0) = C_1$ so $f'(x) = 4x^3 - 4x$. $f(x) = x^4 - 2x^2 + C_2$ so $0 = f(1) = 1^4 - 2 \cdot 1^2 + C_2$ so $f(x) = x^4 - 2x^2 + 1$.
(b) Draw a graph of $f(x)$. Indicate all asymptotes (if any), local maxima and minima, inflection points, intervals where f is increasing, intervals where f is concave up like a cup.

Answer: (b)

x	$f(x)$	$f'(x)$	$f''(x)$	
$-\infty$	∞			no hor asymp
$x < -1$		$-$	$+$	$\downarrow$, CU
-1	0	0	$+$	local min
$-1 < x < -1/\sqrt{3}$		$+$	$+$	$\uparrow$, CU
$-1/\sqrt{3}$		$+$	0	inflection
$-1/\sqrt{3} < x < 0$		$+$	$-$	$\uparrow$, CD
0	1	0	$-$	local max
$0 < x < 1/\sqrt{3}$		$-$	$-$	$\downarrow$, CD
$1/\sqrt{3}$		$-$	0	inflection
$1/\sqrt{3} < x < 1$		$-$	$+$	$\downarrow$, CU
1	0	0	$+$	local min
$1 < x$		$+$	$+$	$\uparrow$, CU
∞	∞			no hor asymp



4.3 A rectangle with one edge on the x axis lies beneath the curve $y = \frac{1}{1+x^2}$. What dimensions maximize its area?



Answer:2 by 1/2

5.1 Find $\int \sqrt{x}\, dx$.

Answer:$\frac{2}{3}x^{\frac{3}{2}} + C$

5.2 Find $\int_2^5 (x-1)(x+2)\, dx$.

Answer:$(\frac{125}{3} + \frac{25}{2} - 10) - (\frac{8}{3} + \frac{4}{2} - 4)$.

5.3 Find $\int_{\ln(\pi/4)}^{\ln(\pi/2)} \Big(\sin(e^x)\Big)^2 e^x\, dx$.

Answer:$\int_{\pi/4}^{\pi/2} \sin^2(u) du = \int_{\pi/4}^{\pi/2} \frac{1+\cos(2u)}{2} du = \frac{\pi}{8} + \frac{1}{4}$

5.4 (a) Evaluate $\int_3^4 \sqrt{x-1}\, dx$. (b) Evaluate $\int_2^3 \frac{\cos(1/t)}{t^2}\, dt$.

Answer:

$$\int_3^4 \sqrt{x-1}\, dx = \frac{2}{3}(x-1)^{3/2}\Big|_3^4 = \frac{2}{3} \cdot 3^{3/2} - \frac{2}{3} \cdot 2^{3/2}$$

Let $u = 1/t$ so $du = -dt/t^2$ and $t = 2 \to u = 1/2$, $t = 3 \to u = 1/3$. Thus

$$\int_2^3 \frac{\cos(1/t)}{t^2}\,dt = \int_{1/2}^{1/3} -\cos u\,du = -\sin(1/3) + \sin(1/2).$$

5.5 (a) Evaluate $\frac{d}{dx}\int_{x^2}^{x} \sin(t^4)\,dt$. (b) Evaluate $\int_{x^2}^{x} \frac{d}{dt}\sin(t^4)\,dt$

Answer:If $F'(t) = \sin(t^4)$ then

$$\frac{d}{dx}\int_{x^2}^{x} \sin(t^4)\,dt = \frac{d}{dx}\Big(F(x) - F(x^2)\Big) = F'(x) - F'(x^2)2x = \sin(x^4) - \sin(x^8)2x.$$

If $F(t) = \sin(t^4)$ then $\frac{d}{dt}\sin(t^4) = F'(t)$ so

$$\int_{x^2}^{x} \frac{d}{dt}\sin(t^4)\,dt = F(x) - F(x^2) = \sin(x^4) - \sin(x^8).$$

5.6 Evaluate $\lim_{h\to 0} \frac{1}{h}\int_3^{3+h} x^3\sqrt{x^2-1}\,dx$.

Answer:The key step in the proof of the Fundamental Theorem is

$$\lim_{h\to 0}\int_a^{a+h} f(x)\,dx = f(a).$$

Hence

$$\lim_{h\to 0} \frac{1}{h}\int_3^{3+h} x^3\sqrt{x^2-1}\,dx = 3^3\sqrt{3^2-1}.$$

5.7 (a) The numbers $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ divide the closed interval $[a, b]$ into n tiny intervals of equal length. Give a formula for x_i, the right endpoint of the i-th interval.

Answer: (a) $x_i = a + \frac{i(b-a)}{n}$. Note that $\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}$

(b) Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n}\left(5 + \frac{3i}{n}\right)^7 \frac{3}{n}$. Hint: This is a Riemann sum where $x_i^* = x_i$.

Answer: (b) If $a = 5$ and $b = 8$ then $x_i = 5 + \frac{3i}{n}$ and $\Delta x_i = 3/n$. If $f(x) = x^7$ then

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(5 + \frac{3i}{n}\right)^7 \frac{3}{n} = \int_a^b f(x)\,dx = \int_5^8 x^7\,dx = \left.\frac{x^8}{8}\right|_5^8 = \frac{8^8}{8} - \frac{5^8}{8}.$$