

Joel Robbin- Calculus 221 - Final Exam (Two Hours)

Monday December 16 1996

I In each of the following, find dy/dx .

(a) $y = \sin x$.

Answer: $\frac{dy}{dx} = \cos x$.

(b) $y = (\sin x)^{-1}$.

Answer: $\frac{dy}{dx} = \frac{-\cos x}{(\sin x)^2}$.

(c) $y = \sin(x^{-1})$.

Answer: $\frac{dy}{dx} = (\cos(x^{-1}))(-x^{-2})$

(d) $y = \sin^{-1}(x)$.

Answer: $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

II Evaluate the integral.

(a) $\int \frac{\cos x \, dx}{\sin x}$.

(b) $\int_1^2 x \sin(x^2) \, dx$.

(c) $\int_{-1}^2 |x^3| \, dx$

III Evaluate the limit.

(a) $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1}$.

(b) $\lim_{x \rightarrow 0} \frac{3^{\sin x} - 1}{\sin x}$.

(c) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \cos\left(3 + \frac{4i}{n}\right) \frac{4}{n}$

IV (a) Find an equation for the tangent line to the curve $y^2 + x \cos(y) = \pi^2 - 3$ at the point $(x, y) = (3, \pi)$.

Answer: $y - \pi = \frac{x - 3}{2\pi}$.

(b) A function $f(x)$ satisfies the identity $f(x)^2 + x \cos(f(x)) = \pi^2 - 3$ and moreover $f(3) = \pi$. Find the linearization $L(x)$ of $f(x)$ at 3. Hint: Note the similarity to part (a).

Answer: $L(x) = \pi + \frac{x - 3}{2\pi}$.

V (a) Find the point on the curve $y = \sqrt{x}$, $0 \leq x \leq 4$ which is closest to $(2, 0)$.

(b) Find the point on the curve $y = \sqrt{x}$, $0 \leq x \leq 4$ which is farthest from $(2, 0)$.

VI The count in a bacteria culture was 400 after 2 hours and 25,600 after 8 hours.

(a) Give a formula for the count after t hours.

Answer: The general formula is $N = N_0 e^{kt}$. We are given $t = 2 \implies N = 400$ and $t = 8 \implies N = 25,600$; i.e.

$$400 = N_0 e^{2k}, \quad 25,600 = N_0 e^{8k}.$$

Dividing gives $64 = e^{6k}$ so (as $2^6 = 64$) $e^k = 2$ and $N_0 = 400/2^2 = 100$. Thus

$$N = 100 \cdot 2^t.$$

(b) How long does it take for the count to double?

Answer: From $N = 100 \cdot 2^t$ it follows that the count doubles every hour.

VII Let $f(x) = \frac{x}{16 + x^3}$

(a) Find the absolute maximum and the absolute minimum of $f(x)$ on the interval $1 \leq x \leq 4$,

Answer: The derivative

$$f'(x) = \frac{(16 + x^3) - x(3x^2)}{(16 + x^3)^2} = \frac{16 - 2x^3}{(16 + x^3)^2}$$

vanishes at $x = 2$. Moreover $f(1) = 1/17$, $f(2) = 1/12$, and $f(4) = 1/20$. Therefore the maximum value is $1/12$ and the minimum is $1/20$.

(b) Find positive numbers A and B such that

$$0 < A \leq \int_1^4 \frac{x dx}{16 + x^3} \leq B.$$

You must justify your answer.

Answer: If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a).$$

Therefore

$$\frac{1}{20}(4 - 1) \leq \int_1^4 \frac{x dx}{16 + x^3} \leq \frac{1}{12}(4 - 1).$$

VIII A particle moves along a straight line so that its velocity at time t is

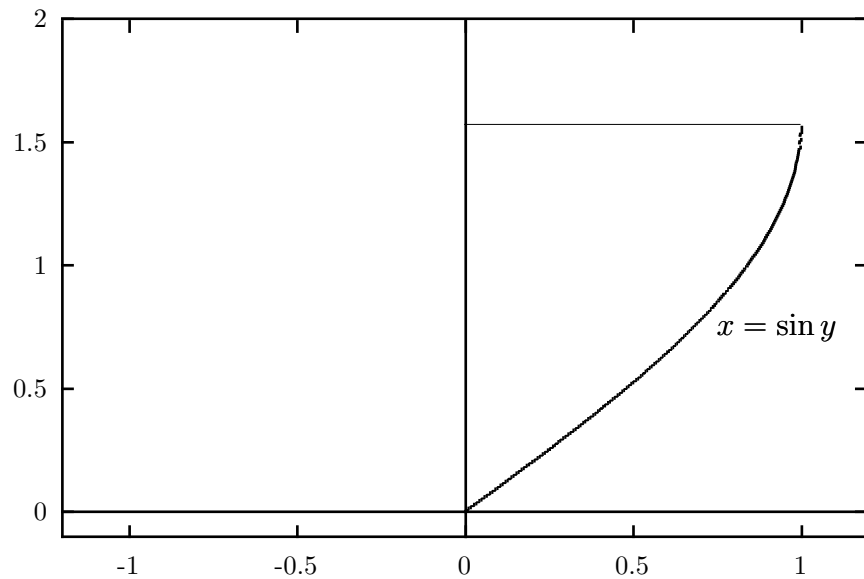
$$v = \frac{ds}{dt} = t^3 - 4t = (t^2 - 4)t$$

where t is the time in minutes.

(a) Write an integral which gives the displacement (the net change in position) during the first five minutes.

(b) Write an integral which gives the total distance travelled during the first five minutes.

IX A tank is constructed by rotating the area bounded by the y -axis, the line $y = \pi/2$ and the curve $x = \sin y$ about the y -axis. (Note: *Not* about the x -axis.) Assume that the unit of length is feet.



- (a) Write a definite integral which gives the volume of this tank. You need not evaluate the integral.
- (b) The tank is being filled with water at the rate of 7 cubic feet per minute. How fast is the water level h increasing when $h = \pi/3$ feet? Hint: What is the volume of the water when the water level is h ?