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Proof of the Fundamental Theorem of Calculus

Definition of the Riemann integral. If f is a function defined on an interval $[a, b]$, then we say that

$$\int_a^b f(x)dx = I,$$

iff for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$ with $x_k - x_{k-1} = \Delta x_k < \delta$ and and sample points c_k with $x_{k-1} \leq c_k \leq x_k$ we have that

$$\left| f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n - I \right| < \epsilon$$

These are called Riemann sums. In summation notation we write

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n$$

Expressing the integral as a kind of limit, we could write

$$\int_a^b f(x)dx =^{def} \text{Limit}_{\max\{\Delta x_k : k=1\ldots n\}\to 0} \ \sum_{k=1}^{n} f(c_k)\Delta x_k$$

We assume without proof that this limit exists for any f continuous on $[a, b]$. The following is perhaps the simplest version of this limit. Given any positive integer n let

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad c_k = a + k\Delta x \text{ for } k = 1, \ldots, n$$

Then

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

The numerical analysists call this the right-hand rectangle rule.

Lemma 1 (Mean-Value Theorem for Integrals) If f is continuous on $[a, b]$, then for some c in $[a, b]$

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\ dx$$

proof:

Let m be the minimum of f on $[a, b]$ and M its maximum. Then since $m \leq f(x) \leq M$ for any x in $[a, b]$ we have for any Riemann sum that

$$\sum_{k=1}^{n} m\Delta x_k \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq \sum_{k=1}^{n} M\Delta x_k$$

Now $\sum_{k=1}^{n} m\Delta x_k = m\sum_{k=1}^{n} \Delta x_k = m(b-a)$ and similarly for M, hence

$$m(b-a) \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq M(b-a)$$

Passing to the limit gives that

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$$

and dividing thru by $(b-a)$ gives that

$$m \leq \frac{1}{b-a} \int_a^b f(x)\ dx \leq M$$

By the intermediate value theorem f assumes all values between its minimum m and maximum M so such a c must exist.
QED

Lemma 2 (Additivity property for area under a curve) If $a < b < c$ and f is continuous on $[a, c]$, then

$$\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx$$

proof:

We can see it as an additive property for areas in the case f is positive. The area under the curve between a and b plus the area under the curve between b and c is the area under the curve between a and c.

More formally note that taking two sufficiently fine Riemann sums for $[a, b]$ and $[b, c]$ (the second renumbered to be from $n+1$ to $n+m$) gives:

$$\sum_{k=1}^{n} f(c_k)\Delta x_k \approx \int_a^b f(x)\,dx \quad \text{and} \quad \sum_{k=n+1}^{n+m} f(c_k)\Delta x_k \approx \int_b^c f(x)\,dx$$

but

$$\int_a^c f(x)\,dx \approx \sum_{k=1}^{n+m} f(c_k)\Delta x_k = \sum_{k=1}^{n} f(c_k)\Delta x_k + \sum_{k=n+1}^{n+m} f(c_k)\Delta x_k$$

And so passing to the limit gives the result.
QED

Fundamental Theorem of Calculus Part I
Suppose f is continuous on $[a, b]$ and

$$G(x) =^{def} \int_a^x f(t)\,dt$$

Then G is differentiable and $G'(x) = f(x)$.

proof:

$$G'(x) = \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x}$$

Assuming for simplicity that $a < x < x + \Delta x < b$ and note that

$$G(x+\Delta x) - G(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+\Delta x} f(t)\,dt$$

where we have used Lemma 2. By the Mean-Value Theorem for integrals, Lemma 1, we may find x^* between x and $x + \Delta x$ with

$$\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t) = f(x^*)$$

Hence

$$\frac{G(x+\Delta x)-G(x)}{\Delta x} = f(x^*)$$

and since x^* is between x and $x+\Delta x$ and f is continuous, we have

$$\lim_{\Delta x \to 0^+} \frac{G(x+\Delta x)-G(x)}{\Delta x} = \lim_{\Delta x \to 0^+} f(x^*) \;=\; f(x)$$

For $\Delta x \to 0^-$ the proof is similar and for $x = a$ or $x = b$ the result holds for the one-sided derivative of G.
QED

Fundamental Theorem of Calculus Part II
Suppose f is continuous on $[a, b]$ and F is any antiderivative of f, i.e., $F' = f$. Then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

proof:

Since G is also an antiderivative of f it follows that $F' - G' = f - f$ is identically zero. The ordinary Mean-Value Theorem implies that $F - G$ must be a constant function, say $F(x) = G(x) + C$ for all x.

Note that $G(b) = \int_a^b f(t)\,dt$ and $G(a) = \int_a^a f(t)\,dt = 0$ and so

$$F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) = \int_a^b f(t)\,dt$$

QED