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M221
Fall 2013

Exercises given in lecture on the day in parantheses.

The $\epsilon - \delta$ game.

$\lim_{x \to a} f(x) = L$ iff Hero has a winning strategy in the following game:
Devil plays: $\epsilon > 0$
Hero plays: $\delta > 0$
Devil plays: x such that $0 < |x - a| \leq \delta$

Pay-Off: Hero wins iff $|f(x) - L| \leq \epsilon$

1. (09-18 Wed) Consider $\lim_{x \to 3} x^2 = 9$.
(a) Write out the $\epsilon - \delta$ game for this limit.
(b) Show that $\delta = \sqrt{\epsilon}$ is not a winning strategy for Hero because if Devil plays $\epsilon = 1$ and Hero plays $\delta = 1 = \sqrt{1}$ then Devil wins with $x = 4$.
(c) More generally for any $\epsilon > 0$ put $\delta = \sqrt{\epsilon}$. Find a real number x which is a win for the Devil: $|x - 3| \leq \delta$ and $|x^2 - 9| > \epsilon$.

2. (09-18 Wed) Find all pairs of real numbers a, b such that

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

3. (09-20 Fri) Find a polynomial $q(x)$ such that

$$x^5 - 1 = (x-1)q(x)$$

4. (09-20 Fri) Find

$$\lim_{x - 1} \frac{x^5 - 1}{x^3 - 1}$$

5. (09-23 Mon) Find

$$(a) \quad \lim_{\theta \to 0} \frac{\sin(\sin(\theta))}{\sin(\theta)} \qquad (b) \quad \lim_{\theta \to 0} \frac{\sin(\sin(\theta))}{\theta}$$

6. (09-25 Wed) Give a direct proof from the definition of derivative that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

The definition of derivative is given on the first page of Chapter 4. An example of a direct proof is Example 2.1 page 59 where a direct proof is given that $\frac{d}{dx}\left(x^2\right) = 2x$. This problem is the same as 4.5 : 2.

7. (09-25 Wed) Give a direct proof from the definition of derivative that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

This is the same as 4.5 : 5.

8. (09-27 Fri) $(f\,g)' \neq f'\,g'$? Let $f(x) = x$ and $g(x) = \frac{1}{1-x}$.
Calculate f', g', and $(fg)'$. Verify $(f\,g)' \neq f'\,g'$. Or are they equal?

9. (09-27 Fri) Let $f(x) = \frac{x}{1-x}$ and $g(x) = x$. Is it true that

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$$

10. (09-27 Fri) Suppose $p(x)$ and $q(x)$ are polynomials which are not constant. Show that $(p\,q)' \neq p'\,q'$.

11. (10-02 Wed) Find the derivative of $\sin(\sin(\sin\theta))$.

12. (10-02 Wed) Find the derivative of $\sin^3\theta$. Recall that

$$\sin^3\theta =^{def} (\sin\theta)^3$$

13. (10-02 Wed) Similarly recall that

$$\sin^{-1}\theta =^{def} \frac{1}{\sin\theta}$$

which is not the arcsine. The difference is the multiplicative inverse and the inverse with respect to composition of functions. Can the two coincide?

Find a function f with domain and range the non-zero real numbers such that if $g(x) =^{def} \frac{1}{f(x)}$ for all non-zero x, then $f(g(x)) = x = g(f(x))$ for all non-zero x.

14. (10-14 Mon) Use the intermediate value theorem to show that

$$f(x) = 2x^2 + x - \frac{5}{x}$$

has a zero between 1 and 2.

15. (10-14 Mon) Let $f(x) = x\cos(x)$. Use Rolle's Theorem to show that for some a with $0 < a < \frac{\pi}{2}$ we have that $f'(a) = 0$.

16. (10-14 Mon) Suppose f is a twice differentiable function on the whole real line. Suppose f has at least three zeros. Prove that f'' must have at least one zero.

17. (10-14 Mon) Use the Mean-value Theorem to find c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

where $f(x) = \frac{1}{x}$ and $[a, b] = [1, 2]$.

18. (10-14 Mon) Let $f(x) = x|x|$. Is f differentiable? If so, find its derivative. Is f twice differentiable? If so, find f''.

19. (10-18 Fri) Suppose

$$f(\theta) = \sin(\pi \sin(\theta)).$$

Find all critical points, determine all intervals where it is increasing, and determine all intervals where it is decreasing. (Recall that critical points are the same as stationary points.)

20. (10-18 Fri) Suppose

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ ax^3 + bx^2 + cx + d & \text{if } 0 < x < 1 \\ x^3 + 1 & \text{if } x \geq 1 \end{cases}$$

Find a, b, c, d so that f is differentiable everwhere.

21. (10-23 Wed) Let

$$x = \frac{1-t^2}{1+t^2} \qquad y = \frac{2t}{1+t^2} \qquad \text{for } -\infty < t < \infty$$

Show that this parameterizes the unit circle except for the point $(-1, 0)$. Find $\frac{dx}{dt}$, $\frac{dy}{dt}$, and

$$\frac{ds}{dt} =^{def} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

for each of the points $(1, 0)$, $(0, 1)$, and $(-\frac{3}{5}, \frac{4}{5})$.

Trig Problem: If we reparameterize by substituting

$$t = \tan\left(\frac{\theta}{2}\right)$$

then show that $x = \cos(\theta)$ and $y = \sin(\theta)$.

22. (10-25 Fri) Let

$$x = a + r\cos(t) \qquad y = b + r\sin(t) \qquad 0 \leq t \leq 2\pi$$

(a) Show that this is a parameterization of the circle of radius r center (a, b).
(b) Show that the curvature $\frac{d\alpha}{ds}$ is constant: $\frac{1}{r}$.

23. (10-28 Mon) Given $0 < a < b$ show that there is a c with $a < c < b$ such that

$$\frac{b^3 - a^3}{b^2 - a^2} = \frac{3}{2}c$$

24. (10-28 Mon) Given $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$ show that there is a c with $a < c < b$ such that

$$\frac{\cos(b) - \cos(a)}{\sin(a) - \sin(b)} = \tan(c)$$

25. (10-28 Mon) Find

$$\lim_{x \to 0} \quad \frac{1}{\sin(x)} - \frac{1}{x}$$

26. (11-06 Wed) The Brouwer fixed point theorem for the unit interval says that if f is a continuous map taking $[0,1]$ into $[0,1]$ then $f(x) = x$ for some x with $0 \leq x \leq 1$. Show that the intermediate value theorem implies the Brouwer fixed point theorem for the unit interval.

Hint: consider $g(x) = f(x) - x$.

27. (11-06 Wed) Suppose f is continuous on $[a,b]$ and $c_1, c_2, \ldots, c_n$ are in $[a,b]$. Show that there exists c in $[a,b]$ such that

$$f(c) = \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n}$$

Hint: show that the average is between the min and the max.

28. (11-06 Wed) Find

(a) $\sum_{k=3}^{7} k(k+1)$

(b) $\sum_{k=1}^{5} \frac{1}{k}$

(c) $\sum_{k=0}^{3} k \sin(k\frac{\pi}{2})$

29. (11-06 Wed) Suppose $\sum_{k=1}^{n}(2a_k + b_k) = 14$ and $\sum_{k=1}^{n} a_k = 12$. Then what is $\sum_{k=1}^{n} 2b_k$?

30. (11-06 Wed) Suppose $\sum_{k=1}^{n}(2a_k + 1) = 3017$. Then what is $\sum_{k=0}^{9} 9a_{k+1}$?

31. (11-06 Wed) Suppose $\sum_{k=1}^{n} 2a_k = 7$ and $\sum_{k=1}^{2n} 3a_k = 17$. Then what is $\sum_{k=n+1}^{2n} 5a_k$?

32. (11-08 Fri) Define

$$f(x) = \begin{cases} 3 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

Draw the graph of f. What is $I = \int_0^2 f(x)dx$? Using the $\epsilon - \delta$ definition of limit show that f is Riemann integrable, i.e., given $\epsilon > 0$ what should $\delta > 0$ be?

33. (11-08 Fri) Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

(a) For the partition $0 = x_0 < x_1 < \cdots < x_5 = 1$ of $[0, 1]$

$$0 < \frac{1}{16} < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1$$

and choice of right-hand points $c_k = x_k$ what is

$$\sum_{k=1}^{5} f(c_k)\Delta x_k$$

(b) In terms of n what is $\sum_{k=1}^{n} f(c_k)\Delta x_k$ for the partition

$$0 < \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}} < \cdots < \frac{1}{4} < \frac{1}{2} < 1$$

(c) Show that $\int_0^1 f(x)dx$ is ∞.

34. (11-13 Wed) Given that

$$\int_1^3 f(x)dx = 20 \text{ and } \int_3^2 2f(x)dx = 12$$

what is

$$\int_1^2 3f(x)dx$$

35. (11-13 Wed) Find

$$\int_0^\pi f(\sin(x))\cos(x)dx$$

where

$$f(u) = \left(\frac{u^3+1}{u^2+u+1}\right)e^u\sqrt{u^5+1}$$

36. (11-13 Wed) Let f be the function from exercise 32:

$$f(x) = \begin{cases} 3 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

Show that f has no antiderivative, i.e., there is no F with $F'(x) = f(x)$ for every x.

37. (11-13 Wed) Define f by

$$f(x) = \begin{cases} 2x-1 & \text{if } x \geq 1 \\ 1 & \text{if } x < 1 \end{cases}$$

(a) Show that f is continuous.

(b) Draw the graph of f for $0 \leq x \leq 2$. Draw the area under the graph f and above the x-axis for x in $[0, 2]$.

(c) Using the area formula for rectangles and triangles show that

$$\int_0^2 f(x)dx = 3$$

(d) Let $F(x)$ be the antiderivative of f:

$$F(x) = \begin{cases} x^2 - x & \text{if } x \geq 1 \\ x & \text{if } x < 1 \end{cases}$$

(e) Using the Fundamental Theorem of Calculus note that

$$\int_0^2 f(x)dx = F(2) - F(0) = (2^2 - 2) - (0) = 2$$

(f) Is the Fundamental Theorem wrong? Does $3 = 2$? What is this?

38. (11-13 Wed) The derivative of $F(x) = -\frac{1}{x}$ is $f(x) = \frac{1}{x^2}$ so by the Fundamental Theorem

$$\int_{-1}^1 \frac{dx}{x^2} = F(1) - F(-1) = (-1) - (1) = -2$$

But $\frac{1}{x^2} > 0$ so how can the integral be negative?