Instructions

No notes, no books, no calculators, no cell phones, no pagers, no electronic devices of any kind.

Show all of your work. Circle your answer.

Name____________________________________

Circle your section number.
Hand in to your TA.

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Section                     Start
Number                      Time
 421 Cheng, Jingrui         07:45
 422 Cheng, Jingrui         08:50
 424 Lynch, John            09:55
 429 Lynch, John            01:20
 426 Powers, Michael        11:00
 427 Powers, Michael        12:05
 430 Soule, Michelle        01:20
 431 Soule, Michelle        02:25
 432 Charles, Zachary       07:45
 433 Charles, Zachary       02:25
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Problem	Points	Score
1	8	
2	6	
3	8	
4	8	
5	8	
6	8	
7	8	
8	9	
9	8	
10	8	
11	8	
12	8	
13	5	
Total	100	

Solutions will be posted shortly after the exam:
www.math.wisc.edu/∼miller

1. (8 pts) Prove the Fundamental Theorem Part 2: If f is a continuous function on $[a, b]$ and $F' = f$ is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

2. (6 pts) Find, simplify, and place your answer in the blank.

(a) $\sum_{k=3}^{5} k(k-1) \ =$ ________

(b) $\sum_{k=1}^{4} \frac{6}{k} \ =$ ________

(c) $\sum_{k=0}^{3} k\cos(k\frac{\pi}{2}) \ =$ ________

3. (8 pts) Show that the equation $\sin x = 1 - 2x$ has exactly one solution x. First show that it has at least one solution and then show that it cannot have more than one.

4. (8 pts)

$$f(x) = \frac{x}{1 + x^2}$$

Find intervals where it is increasing or decreasing. Find local maxima and minima. Find inflection points and intervals where it is concave up or down. Find limits at ∞ and $-\infty$. Sketch graph.

5. (8 pts) Suppose

$$f(x) = \begin{cases} ax^3 + bx^2 & \text{if } x < 1 \\ x^2 + 1 & \text{if } x \geq 1 \end{cases}$$

Find a and b so that f is differentiable everywhere.

6. (8 pts) A 20 inch piece of wire is cut into two pieces. One piece is bent into a square and the other is bent into a triangle which is similar to the right-angle triangle with sides 3, 4, and 5. What is the minimal total area enclosed by both figures?

7. (8 pts) Compute the following limits or say the limit is not defined:

(a) $\lim_{x\to 1} \frac{\sin \pi x}{x^2+x-2}$

(b) $\lim_{x\to 0} \quad \frac{1}{\sin(x)} - \frac{1}{x}$

8. (9 pts) Find an antiderivative $F(x)$ for each of the following functions $f(x)$.

(a) $f(x) = x + 2$

(b) $f(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}$

(c) $f(x) = \sin x - \cos 2x$

9. (8 pts) Recall the $\epsilon - \delta$ definition of the Riemann integral: $\int_a^b f(x)dx = I$ iff for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$ with $x_k - x_{k-1} = \Delta x_k \leq \delta$ and sample points c_k with $x_{k-1} \leq c_k \leq x_k$ for $k = 1 \ldots n$ we have

$$\left| f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n - I \right| \leq \epsilon$$

Define

$$f(x) = \begin{cases} 5 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

(a) Draw the graph of f. What is $I = \int_0^2 f(x)dx$?

(b) Using the $\epsilon - \delta$ definition of the integral show that f is Riemann integrable, i.e., given $\epsilon > 0$ what should $\delta > 0$ be? Explain.

10. (8 pts) Find

$$\int_{-e^2}^{-e} \frac{1}{3x}\,dx$$

11. (8 pts) Compute

$$I = \int_0^2 3x^2(1+x^3)^2\,dx$$

in two different ways:

(a) Expand $(1+x^3)^2$, multiply with $3x^2$, and integrate each term.

(b) Use the substitution $u = 1 + x^3$.

12. (8 pts) Suppose

$$f(x) = \int_0^{\sin(x)} e^{t^2} \, dt$$

Find $f'(x)$.

13. (5 pts) Matching problem. Fill in the blanks.

(a) Intermediate Value Theorem ______

(b) Extreme Value Theorem ______

(c) Fermat's Theorem ______

(d) Rolle's Theorem ______

(e) Mean Value Theorem ______

1. For any $a < b$ and f differentiable on (a, b) and continuous on $[a, b]$ there exists c with $a < c < b$ and
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2. For any $a < b$ and f differentiable on (a, b), continuous on $[a, b]$, and $f(a) = f(b) = 0$ there exists c with $a < c < b$ and
$$f'(c) = 0$$

3. If f is differentiable at a and for some $\epsilon > 0$ $f(a) \geq f(x)$ for any x in $(a - \epsilon, a + \epsilon)$, then $f'(a) = 0$.

4. For any $a < b$ and f continuous on $[a, b]$ there exists c and d in $[a, b]$ such that for any x in $[a.b]$
$$f(c) \leq f(x) \leq f(d)$$

5. For any $a < b$ and f continuous on $[a, b]$ and any v between $f(a)$ and $f(b)$, there exists c in $[a, b]$ with
$$f(c) = v$$

Answers

2. (a) 38 (b) 12.5 (c) -2

3. Let $f(x) = \sin(x) - 1 + 2x$. Then $f(0) = -1$ and $f(10) = \sin(10) - 1 + 20 \geq 20$. So by the Intermediate Value Theorem there is some x between 0 and 10 where $f(x) = 0$. Now $f'(x) = \cos(x) + 2 \geq 1$ for every x and so has no zeros. But Rolle's Theorem implies that if f has two zeros, that in between them is a zero of f'. Hence f has at most one zero. Since $f(x) = 0$ iff $\sin x = 1 - 2x$ the result follows.

4. Local max at $x = 1$. Local min at $x = -1$. (These are global max and min.) There are inflection points at $x = \sqrt{3}$, $x = -\sqrt{3}$ and $x = 0$. It is concave down $(-\infty, -\sqrt{3})$, up $(-\sqrt{3}, 0)$, down $(0, \sqrt{3})$, up $(\sqrt{3}, \infty)$. The limit at ∞ is 0^+ and the limit at $-\infty$ is 0^-.

5. $a = -2$ and $b = 4$

6. Minimal area 10.

7. (a) $-\frac{\pi}{3}$ (b) 0

8. (a) $F(x) = \frac{x^2}{2} + 2x$
(b) $F(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{20}$
(c) $F(x) = -\cos(x) - \frac{1}{2}\sin(2x)$

9. (a) $I = 0$.
(b) $\delta = \frac{\epsilon}{10}$ works. At most two of the c_k can be 1. So $|\sum_{k=1}^{n} f(c_k)\Delta x_k - I|$ is at most $5\Delta x_k + 5\Delta x_{k+1}$ which is at most 10δ.

10. $-\frac{1}{3}$

11. $242\frac{2}{3}$

12. $f'(x) = e^{\sin^2(x)}\cos(x)$.