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A project report for Math 221 Lab: **Numbers and functions**

In this project, we learned that both $\sqrt{2}$ and $\sqrt{3}$ are not rational numbers. We first recall that a number, x, is a rational number if there are two integers, m and n say, for which

$$x = \frac{m}{n}.$$

For example, the following are all clearly rational:

$$\frac{11}{3}, \quad , -\frac{43}{17}, \quad \frac{33}{9}.$$

Also note that the first and third of the above numbers are actually the same, and both are equal to 11/3. This brings us to another point brought up in this project, that every rational number can be written in a unique way by reducing the fraction as much as possible.

While it is very easy to give examples of rational numbers, it is apparently very difficult to show that any specific number is not rational. The major mathematical goal of this project was to prove that neither $\sqrt{2}$ nor $\sqrt{3}$ were rational. The project began by walking us through the proof that $\sqrt{2}$ is not rational, and we will reproduce this proof here. To do so, we state another important fact given in the project:

Fact: For any integer $n \in \{2, 3, 4, \dots\}$, there exists a unique prime factorization of n. That is, there exists prime numbers $p_1, \dots, p_m$, and positive integers $a_1, \dots, a_m$ such that

$$n = p_1^{a_1} \cdots p_m^{a_m},$$

and these choices of p_i and a_i are unique.

We are now ready to repeat the proof of the fact that $\sqrt{2}$ is not rational:

Step 1. We began by assuming that $\sqrt{2}$ *is* a rational number. The hope is that such an assumption would lead to a contradiction with something we know to be true. This would imply that $\sqrt{2}$ must not be rational.

Step 2. We suppose there <u>are</u> positive integers m and n for which

$$\left(\frac{m}{n}\right)^2 = 2,$$

and we further assumed that m and n have been reduced as much as possible.

Step 3. Rearranging the equation above, we have that

$$m^2 = 2n^2,$$

which implies that m must be an even number. Hence, we must have that $m = 2m'$ for some other integer m'. Thus, we must conclude that

$$(2m')^2 = 2n^2.$$

or

$$4(m')^2 = 2n^2,$$

which says that (after factoring a 2 from each side),

$$2(m')^2 = n^2.$$

However, this says that n must also be even, since there is a 2 in its prime factorization.

Step 4. We can now conclude that $\sqrt{2}$ must not be rational because the assumption that it was rational forced us to conclude that both m and n were divisible by 2. However, we specifically chose them so that m/n was in its reduced form. If they were both even, then we would have already divided out by 2! Thus, the proof is complete.

We are now ready to use a similar argument to prove that $\sqrt{3}$ is not rational. The steps are the same as above.

Step 1. We began by assuming that $\sqrt{3}$ *is* a rational number. As in the case of $\sqrt{2}$ above, the hope is that such an assumption will lead to a contradiction with something we know to be true.

Step 2. We suppose there are positive integers m and n for which

$$\left(\frac{m}{n}\right)^2 = 3,$$

and we further assumed that m and n have been reduced as much as possible.

Step 3. Rearranging the equation above, we have that

$$m^2 = 3n^2,$$

Since m is an integer, and a three appears on the right hand side, which is the square of m, it must be that m is divisible by 3. Thus, we have that $m = 3m'$ for some other integer m', and we may conclude that

$$(3m')^2 = 3n^2.$$

or

$$9(m')^2 = 3n^2,$$

which says that,

$$3(m')^2 = n^2.$$

However, this says that n must also be divisible by 3, and we reach the same contradiction as in the case of $\sqrt{2}$. Thus, the proof is complete.