Numbers and functions

Purpose of this project

To get a deeper understanding of numbers and functions. To get used to working in groups in a math class.

$\sqrt{2}$ is not a rational number!

In the notes, it is stated that $\sqrt{2}$ is *not* a rational number. This may not seem like a big deal to you, but legend has it that the first person to discover this was murdered for the discovery since it was so disconcerting to the Pythagoreans, see
`http://en.wikipedia.org/wiki/Square_root_of_2`.

Let's see if we can find an "easy" proof of the fact that $\sqrt{2}$ is not rational. The proof has several steps, but essentially relies on the following fact, which should be intuitively clear, and we state without proof (this result will be proved in Math XXX).

Fact: For any integer $n \in \{2, 3, 4, \dots\}$, there exists a unique prime factorization of n. That is, there exists prime numbers $p_1, \dots, p_m$, and positive integers $a_1, \dots, a_m$ such that

$$n = p_1^{a_1} \cdots p_m^{a_m},$$

and these choices of p_i and a_i are unique.

As a few examples, we have:

- $10 = 2 \times 5$.
- $21 = 3 \times 7$.
- $171,955 = 17^3 \times 7 \times 5$.

We now try to find out why $\sqrt{2}$ is not a rational number, and begin by doing something strange: we suppose that $\sqrt{2}$ *is* a rational number! That is, we suppose there are positive integers m and n for which

$$\left(\frac{m}{n}\right)^2 = 2. \tag{1}$$

We further assume that m and n have no primes in common in there unique factorizations. Note that for any rational number we can always choose a unique pair m, n for if they do have a common factor, we may simply cancel it from both without changing the ratio. For example $11/2$ would be the choice over, for example, $33/6$. Our goal is now to show that these assumptions lead to a contradiction (implying the assumptions were incompatible in the first place).

First note that equation (1) implies that

$$m^2 = 2n^2.$$

Now we simply observe that since the right hand side of the above equation has a factor of two, the left hand side must be also. Thus, m must have a factor of 2 in its prime decomposition. Writing $m' = m/2$, we can now conclude that

$$(2m')^2 = 2n^2 \implies 2(m')^2 = n^2.$$

However, we are now able to conclude that n has a 2 in its prime factorization. Since we already showed that m is divisible by 2, we have reached a contradiction with our assumption that m and n had no factors in common. Thus, the proof is complete.

Exercise: prove that $\sqrt{3}$ is not a rational number.

Report Instructions: Using complete sentences, write a few paragraphs summarizing the project. Incorporate your solution to the exercise above. Be sure to include all pertinent information from the project itself. That is, a reader should be able to sit down with only your report and be able to understand what you are doing, and the conclusions you have drawn.