

1.1. A square sheet of tin 12 inches on a side is used to make an open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner in order that the box shall have as large a volume as possible?

Let x be the length of the side of the small square. Give your answer in the form $x = \dots$

1.2. Solve the following differential equation:

$$\frac{dy}{dx} = 3x^2y^2$$

Give your answer in the form $y = \dots$

1.3. Evaluate the following indefinite integral:

$$\int \sin^3(3x) \cos(3x) dx$$

1.4. Establish the formula

$$\sum_{k=1}^n (2k-1) = 1 + 3 + 5 + \dots + (2n-1) = n^2$$

for every positive integer n , by showing

(a) the formula is correct for $n = 1$, and

(b) if true for n , the formula is also true for $n + 1$.

1.5. Find the area A between the curve $y = 9 - x^2$ and the x -axis. Give your answer in the form $A = \dots$

1.6. Prove the following part of the fundamental theorem of calculus:

Suppose $f(x)$ is continuous on the interval $[a, b]$, $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with the $F'(x) = f(x)$ for all x in (a, b) . Let n be a large integer.

(a) Define $\Delta x =$

(b) Define $x_k =$

(c) Show there exists c_k with $x_{k-1} \leq c_k \leq x_k$ for $k = 1, 2, \dots, n$ and such that

$$F(b) - F(a) = \sum_{k=1}^n f(c_k) \Delta x$$

2.1. The area bounded by the curves $y = x^3$, $y = 0$, and $x = c > 0$ has area 1. What is c ?

2.2. The base of a certain solid is the region bounded by the curves $y = x^2$ and $y = 1$. Each plane section of the solid cut out by a plane perpendicular to the x -axis is a square with one edge of the square in the base of the solid. Find the volume of the solid.

2.3. The area bounded by the curves $y = \sqrt{x}$ and $y = x$ is rotated about the y -axis. Find the volume.

2.4. Find the length of the curve $x^2 = 4y^3$ from $(0, 0)$ to $(2, 1)$.

2.5. Find the surface area of the surface generated by rotating about the line $y = 1$ the arc of the curve $y = \frac{x^3}{3} + 1$ between $x = 0$ and $x = 1$.

2.6. Prove the following part of the fundamental theorem of calculus:

Suppose $f(x)$ is continuous on the interval $[a, b]$, $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with the $F'(x) = f(x)$ for all x in (a, b) . Let n be a large integer.

Define $\Delta x = \frac{b-a}{n}$.

Define $x_k = a + k\Delta x$ for $k = 0, 1, \dots, n$.

Show there exists c_k with $x_{k-1} \leq c_k \leq x_k$ for $k = 1, 2, \dots, n$ and such that

$$F(b) - F(a) = \sum_{k=1}^n f(c_k) \Delta x$$

3.1. Find the value of

$$\cos(12^\circ) \cos(-18^\circ) + \sin(18^\circ) \sin(-12^\circ)$$

and simplify it.

3.2. Find $\sin^2(\frac{\theta}{2})$ given that $\cos(\theta) = \frac{4}{5}$.

3.3. Find all θ which solve the equation:

$$2 \sin(\theta) \cos(\theta) + 2 \sin(\theta) + \cos(\theta) + 1 = 0$$

3.4. Suppose $z = \tan(\theta) \sec(\theta)$.

Find $\frac{dz}{d\theta}$.

Simplify and express in terms of $\sec(\theta)$ only.

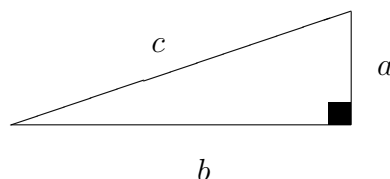
3.5. Find the value of

$$\int_0^2 \frac{dx}{1 + \frac{x^2}{4}}$$

and simplify.

3.6. Prove Pythagorus's Theorem. In a right triangle with hypotenuse c and sides a and b , we have that

$$a^2 + b^2 = c^2$$



4.1. Establish the formula

$$\sum_{k=1}^n 2k = 2 + 4 + \cdots + 2n = n + n^2$$

for every positive integer n , by showing

(a) the formula is correct for $n = 1$, and

(b) if true for n , the formula is also true for $n + 1$.

4.2. Find all θ which solve the equation:

$$2 \sin(\theta) \cos(\theta) - 2 \sin(\theta) - \cos(\theta) + 1 = 0$$

4.3. Find x if $\log_x(3x) = \frac{2}{3}$

4.4. Solve the logarithmic equation for x .

$$\ln(x) + \ln(x + 2) = \ln(3)$$

4.5. Find $\int t 2^{t^2} dt$.

4.6. Find the derivative of $y = 10^{\log_2(x)}$

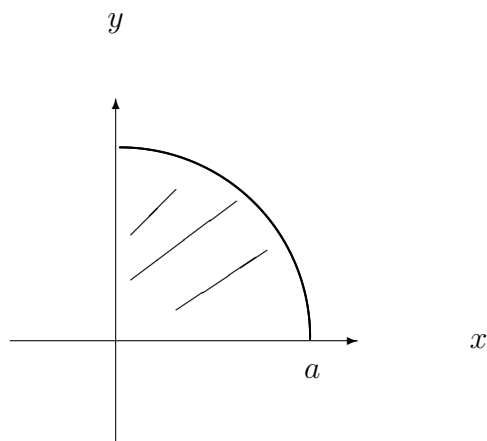
5.1. Sketch the ellipse with equation:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Label all four points where the ellipse intersects the axis.

Find, draw, and label the focal points.

5.2. Find the center of mass of a thin homogeneous plate covering the first quadrant of the disk bounded by the circle $x^2 + y^2 = a^2$.

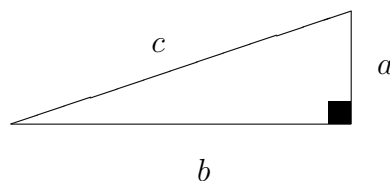


5.3. The amount of \$4000 is invested at the interest rate of 6% compounded monthly. How much will you have after 3 years and 4 months?

5.4. A roasted Turkey is taken from an oven when its temperature has reached 170 degrees and placed on a table in a room where the temperature is 70 degrees. If the temperature of the Turkey is 150 degrees after 10 minutes, when will its temperature be 140 degrees?

5.5. Prove Pythagorus's Theorem: In a right triangle with hypotenuse c and sides a and b , we have that

$$a^2 + b^2 = c^2$$



5.6. (A) (Do either problem 6A or problem 6B. **Do not do both.**)

Prove the following part of the fundamental theorem of calculus:

Suppose $f(x)$ is continuous on the interval $[a, b]$, $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with the $F'(x) = f(x)$ for all x in (a, b) . Let n be a large integer.

Define $\Delta x = \frac{b-a}{n}$.

Define $x_k = a + k\Delta x$ for $k = 0, 1, \dots, n$.

Show there exists c_k with $x_{k-1} \leq c_k \leq x_k$ for $k = 1, 2, \dots, n$ and such that

$$F(b) - F(a) = \sum_{k=1}^n f(c_k) \Delta x$$

5.6. (B) (Do either problem 6A or problem 6B. **Do not do both.**)

Prove the following version of the fundamental theorem of calculus:

Suppose $f(x)$ is a continuous function on the interval $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$. For any $x \in [a, b]$ let $A(x)$ be the area under f and above the interval $[a, x]$ on the x -axis.

Prove

$$\frac{dA}{dx} = f(x)$$

Answers

1.1. The volume of the box is $V = x(12 - 2x)^2$ and is a maximum on $[0, 12]$ when $x = 2$.

1.2. $y = \frac{-1}{x^3+C}$ Remark: there is also the constant solution $y \equiv 0$.

1.3. $\int \sin^3(3x) \cos(3x) dx = \frac{\sin^4(3x)}{12} + C$

1.4. (a) $\sum_{k=1}^1 (2k - 1) = 2 \cdot 1 - 1 = 1 = 1^2$

(b) $\sum_{k=1}^{n+1} (2k - 1) = \sum_{k=1}^n (2k - 1) + (2(n + 1) - 1) = n^2 + (2(n + 1) - 1) = n^2 + 2n + 1 = (n + 1)^2$

1.5. $A = 36$

1.6. (a) $\Delta x = \frac{b-a}{n}$

(b) $x_k = a + k\Delta x$

(c) Using the Mean Value Theorem we have that for every k there exists c_k with $x_{k-1} < c_k < x_k$ and

$$F'(c_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}$$

Noting that $F' = f$ and $\Delta x = x_k - x_{k-1}$ we have that

$$f(c_k) = F'(c_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = \frac{F(x_k) - F(x_{k-1})}{\Delta x}$$

and hence

$$f(c_k)\Delta x = F(x_k) - F(x_{k-1})$$

Hence,

$$\begin{aligned} \sum_{k=1}^n f(c_k)\Delta x &= \sum_{k=1}^n F(x_k) - F(x_{k-1}) \\ &= (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + (F(x_3) - F(x_2)) + \cdots + (F(x_n) - F(x_{n-1})) \end{aligned}$$

This is a telescoping sum in which everything cancels except $-F(x_0)$ and $F(x_n)$

$$= F(x_n) - F(x_0) = F(b) - F(a)$$

Noting that $x_0 = a$ and $x_n = b$ finishes the proof.

$$\sum_{k=1}^n f(c_k)\Delta x = F(b) - F(a)$$

2.1. $c = \sqrt{2}$

2.2. $\frac{16}{15}$

2.3. $\frac{2\pi}{15}$

2.4. $\frac{2}{27}(10^{3/2} - 1)$

2.5. $\frac{\pi}{9}(2^{3/2} - 1)$

2.6. See Exam 1 problem 6.

3.1. $\cos(30^\circ) = \frac{\sqrt{3}}{2}$

3.2. $\frac{1}{10}$

3.3.

$\theta = \pi + 2\pi n$ or

$\theta = -\frac{\pi}{6} + 2\pi n$ or

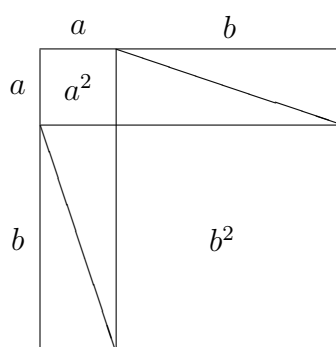
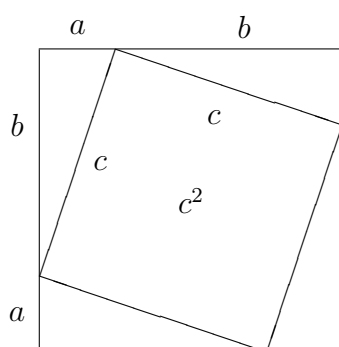
$\theta = -\frac{5\pi}{6} + 2\pi n$

for $n = 0, 1, -1, 2, -2, \dots$

3.4. $2\sec^3(\theta) - \sec(\theta)$

3.5. $\frac{\pi}{2}$

3.6. After removing the triangles the remaining areas must be equal. (I would have shaded the triangles but my computer program wasn't up to it.)



4.1. See Exam 1 problem 4.

(a) $\sum_{k=1}^1 2k = 2 = 1^2 + 1$

(b) $\sum_{k=1}^{n+1} 2k = \sum_{k=1}^n 2k + 2(n+1) = n^2 + n + 2(n+1) = n^2 + 3n + 2$
 and $(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = n^2 + 3n + 2$ and so

$$\sum_{k=1}^{n+1} 2k = (n+1)^2 + (n+1)$$

4.2. See Exam 3 problem 3. $(2 \sin \theta - 1)(\cos \theta - 1) = 0$

$\theta = 2\pi n$ or

$\theta = \frac{\pi}{6} + 2\pi n$ or

$\theta = \frac{5\pi}{6} + 2\pi n$

for $n = 0, 1, -1, 2, -2, \dots$

4.3. $x = \frac{1}{27}$

4.4. $x = 1$

4.5. $\frac{1}{2 \ln(2)} 2^{t^2} + C$

4.6. $\frac{\log_2(10)}{x} 10^{\log_2(x)}$ or $\frac{\ln 10}{\ln 2} \frac{1}{x} e^{\frac{\ln 10}{\ln 2} (\ln x)}$

5.1. The focal points are at $(\pm 4, 0)$. The ellipse intersects the axes at $(\pm 5, 0)$ and $(0, \pm 3)$.

5.2. $(\frac{4}{3\pi}a, \frac{4}{3\pi}a)$

5.3. $4000(1 + \frac{.06}{12})^{40} = 4000(1.005)^{40} = 4883.1769$ dollars

5.4. $10^{\frac{\ln(.7)}{\ln(.8)}} = 15.984103$ minutes

5.5. See Exam 3 problem 6.

5.6. See Exam 1 problem 6 for 6A. For 6B see pages 196-198 of your calculus book.