

Exam 2

A. Miller

Spring 99

Math 210

No calculators. No note or index cards.

Show all work. Explain your answers.

In general, little credit will be given for answers alone.

You do not need to simplify fractions or combinatorial expressions.

No partial credit for examples alone.

Name_____

TA: Winston Yang

Circle the time of your discussion section:

Tues 8:50

Tues 9:55

Thurs 8:50

Thurs 9:55

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (10 %) The letters

C

D

G

K

L

O

O

U

are placed in a hat and mixed thoroughly. One at a time they are drawn out and placed on a table (not replaced into the hat). What is the probability that they spell out: GOODLUCK

2. (10 %) Four baskets each contain 10 balls labeled 1 thru 10. A ball is randomly chosen from each basket. What is the probability that at least two of the balls have the same number?

3. (10 %) A box contains 1 red, 1 white, and 2 green balls. A game consists of drawing balls in succession without replacement until a white ball is drawn. Suppose you get 4 points for each green ball and 8 points for each red ball drawn. Let X be the total number of points you get.

What are the possible values of the random variable X ?

What is the probability density function for X ?

What is the expected value, $\mu = E(X)$, of the random variable X ?

4. (10 %) The axioms for a probability, $Pr(E)$ for $E \subseteq S$, are the following:
- (1) $0 \leq Pr(E) \leq 1$ for each $E \subseteq S$,
 - (2) $Pr(S) = 1$, and
 - (3) if E_1 and E_2 are disjoint subsets of S , then $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2)$.

Using these axioms deduce the formula

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

where A and B are any subsets of S .

5. (10 %) Suppose there are 3 women, 2 men, and 5 children in a room. 3 of the 10 are chosen at random. What is the probability that at least 1 of the 3 chosen is a woman if we are given that at least 1 of the 3 chosen is a child?

6. (10 %) A box contains 3 ripe bananas (R) and 4 green bananas (G). A monkey reaches in and grabs a banana at random. Since the box has a very small opening the monkey is unable to see the banana before he chooses it and so is equally likely to pick any of the 7 bananas.

If it is green he puts it back in the box, but if it is ripe he eats it. In either case the monkey reaches in again and randomly grabs another banana from those now in the box.

He does this only two times and he eats only ripe bananas.

(a) Draw a tree to represent this experiment and compute the probabilities of all outcomes.

(b) What is the probability that the monkey eats exactly one banana?

7. (10 %) Max has two coins, one is fair and the other is weighted so that the probability of heads is $\frac{2}{3}$. One of the two coins is selected at random (with equal probability) and then it is flipped two times. If it comes up heads both times, what is the probability that the unfair coin was selected?

8. (10 %) The fly-by-night FBN airline advertises that its flights are on time 90 percent of the time. A business woman takes 6 trips with FBN, each of which consists of 2 flights, i.e., one to her destination and one returning.

Assume that the late-not late flights are a sequence of 12 independent Bernoulli trials.

(a) What is the probability that at least one of the two flights of her first trip is late?

(b) What is the probability that on all of her trips at least one of the two flights is late?

(c) What is the probability that on more than one of her trips both flights are late?

9. (10 %) An experiment consists of rolling a fair die and noting the number that comes up. A random variable X is defined by $X = 1$ if the die comes up 1,2,3,4, or 5 and by $X = 2$ if the die comes up 6.

Find the expected value $\mu = E(X)$ and the variance $\sigma^2 = Var(X)$.

(Hint: $\sigma^2 = E((X - \mu)^2) = E(X^2) - \mu^2$ and the expected value of X^2 is the sum of all $k^2 Pr(X = k)$ for k a possible value of X .)

10. (10 %) In table below the first column is a and the second column is $\Pr(0 < Z < a)$ (rounded) where Z has the standard normal distribution.

a	$\Pr(0 < Z < a)$
0.0	.00
0.1	.04
0.2	.08
0.3	.12
0.4	.16
0.5	.19
0.6	.23
0.7	.26
0.8	.29
0.9	.32
1.0	.34
1.1	.36
1.2	.38
1.3	.40
1.4	.42
1.5	.43
1.6	.45
1.7	.46
1.8	.46
1.9	.47
2.0	.48
2.1	.48
2.2	.49
2.3	.49
2.4	.49
2.5	.49
2.6	.50
2.7	.50
2.8	.50
2.9	.50
3.0	.50

Using this table find:

(a) $\Pr(0 < Z < 2.3)$

(b) $\Pr(1.2 < Z)$

(c) $\Pr(-1 < Z < 1)$

(d) X is normally distributed with mean $\mu = 60$ and standard deviation $\sigma = 5$. Find $\Pr(X < 67.5)$

Answers

1. If we put subscripts on the 0's, i.e., GO_1O_2DLUCK , then there would be $8!$ permutations of these letters and each of these permutations is equally likely to occur. Exactly two, GO_1O_2DLUCK and GO_2O_1DLUCK , spell out GOODLUCK when we drop the subscripts. Hence the probability is $\frac{2}{8!}$.

2. This is similar to the birthday problem. If instead we count the complementary event, E' , i.e., no two balls have the same number, then out of $10 \times 10 \times 10 \times 10$ possible choices there are $10 \times 9 \times 8 \times 7$ choices with no number repeated, so

$$Pr(E') = \frac{10 \times 9 \times 8 \times 7}{10 \times 10 \times 10 \times 10}$$

and $Pr(E) = 1 - Pr(E')$ so the answer is

$$1 - \frac{10 \times 9 \times 8 \times 7}{10 \times 10 \times 10 \times 10}$$

We can also compute the probability of E' using conditional probability. If we let $B_1 = S$, and B_2 be the event that the first two balls have different numbers, B_3 the event that the first 3 balls have different numbers and B_4 the event that all 4 balls have different numbers. Then $B_4 = E'$ and $B_4 = B_1 \cap B_2 \cap B_3 \cap B_4$. By conditional probability:

$$Pr(B_1 \cap B_2 \cap B_3 \cap B_4) = Pr(B_1) \times Pr(B_2|B_1) \times Pr(B_3|(B_2 \cap B_1)) \times Pr(B_4|(B_3 \cap B_2 \cap B_1))$$

and this product is

$$1 \times \frac{9}{10} \times \frac{8}{10} \times \frac{7}{10}$$

3. We make the following chart:

Outcome	Probability	Probability	Value of X
W	$1/4$	$3/12$	0
RW	$1/4 \times 1/3$	$1/12$	8
GW	$2/4 \times 1/3$	$2/12$	4
RGW	$1/4 \times 2/3 \times 1/2$	$1/12$	12
GRW	$2/4 \times 1/3 \times 1/2$	$1/12$	12
GGW	$2/4 \times 1/3 \times 1/2$	$1/12$	8
RGGW	$1/4 \times 2/3 \times 1/2$	$1/12$	16
GRGW	$2/4 \times 1/3 \times 1/2$	$1/12$	16
GGRW	$1/4 \times 2/3 \times 1/2$	$1/12$	16

We see from the chart that the possible values of X are 0, 4, 8, 12, and 16. The density function for X is the first two columns of the chart below:

k=Value of X	$Pr(X = k)$	$k \times Pr(X = k)$
0	3/12	0
4	2/12	8/12
8	2/12	16/12
12	2/12	24/12
16	3/12	48/12

The expected value of X is obtained by adding the numbers $k \times Pr(X = k)$ for all possible values k . Summing the third column gives us $E(X) = (8 + 16 + 24 + 48)/12$.

4. I did this twice in lecture. We can write $A \cup B$ as the disjoint union

$$A \cup B = A \cup (A' \cap B)$$

By the axiom (3) we have that

$$Pr(A \cup B) = Pr(A \cup (A' \cap B)) = Pr(A) + Pr(A' \cap B)$$

The event B can be written as a disjoint union:

$$B = (A \cap B) \cup (A' \cap B)$$

and so by axiom (3)

$$Pr(B) = Pr(A \cap B) + Pr(A' \cap B)$$

and so

$$Pr(A' \cap B) = Pr(B) - Pr(A \cap B)$$

Substituting this into our first formula we have

$$Pr(A \cup B) = Pr(A) + Pr(A' \cap B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

5. Let S be the set of all groups of three chosen from these 10 people. Let W be the set of all groups of three where at least one woman is chosen, and C be the set of all groups with at least one child. Hence

$$Pr(\text{at least one child is chosen}) = \frac{|C|}{|S|}$$

The conditional probability we are asked for is:

$$Pr(\text{at least one woman} \mid \text{at least one child}) = \frac{|W \cap C|/|S|}{|C|/|S|} = \frac{|W \cap C|}{|C|}$$

$|C'|$ is the set of groups of three which includes no child. There are $\binom{5}{3}$ ways to choose 3 nonchildren, hence

$$|C'| = \binom{5}{3}$$

There $\binom{10}{3}$ ways to choose 3 people from 10 so $|S| = \binom{10}{3}$ and so

$$|C| = \binom{10}{3} - \binom{5}{3}$$

To calculate $|W \cap C|$ break it into the three disjoint sets:

B_1 : elements of S consisting of one child, one woman, one man.

B_2 : elements of S consisting of one child, two women.

B_3 : elements of S consisting of two children, one woman.

Since $W \cap C$ is the disjoint union of B_1, B_2, B_3 we compute these three sizes and add the results.

$|B_1|$: there are $5 \times 3 \times 2$ groups of 3 like this, i.e., choose one child from 5, one woman from 3, and one man from 2.

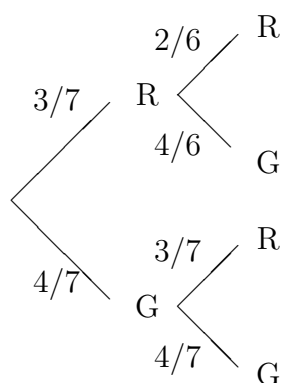
$|B_2|$: there are $5 \times \binom{3}{2}$ groups of 3 like this.

$|B_3|$: there are $\binom{5}{2} \times 3$ groups of 3 like this. Hence the answer is:

$$\frac{5 \times 3 \times 2 + 5 \times \binom{3}{2} + \binom{5}{2} \times 3}{\binom{10}{3} - \binom{5}{3}}$$

6. (a) The probability of each outcome is obtained by multiplying along the branch.

Outcome	Probability
RR	$\frac{3}{7} \times \frac{2}{6}$
RG	$\frac{3}{7} \times \frac{4}{6}$
GR	$\frac{4}{7} \times \frac{3}{7}$
GG	$\frac{4}{7} \times \frac{4}{7}$



He eats exactly one banana is the event $\{RG, GR\}$, which has probability

$$\left(\frac{3}{7} \times \frac{4}{6}\right) + \left(\frac{4}{7} \times \frac{3}{7}\right)$$

7. Let F denote the event that the fair coin is chosen and let U denote the complementary event that the unfair coin is chosen. Let HH denote the event that two heads occur when either coin is chosen. We are told that

$$Pr(U) = Pr(F) = \frac{1}{2}$$

$$Pr(HH|F) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$Pr(HH|U) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

and are asked what is $Pr(U|HH)$? We know by the definition of conditional probability that

$$Pr(U|HH) = \frac{Pr(U \cap HH)}{Pr(HH)}$$

To compute this note that HH is split into the disjoint sets

$$HH = (U \cap HH) \cup (F \cap HH)$$

since the events U and F are complementary. Thus

$$Pr(HH) = Pr(U \cap HH) + Pr(F \cap HH)$$

Hence

$$Pr(U|HH) = \frac{Pr(U \cap HH)}{Pr(HH)} = \frac{Pr(U \cap HH)}{Pr(U \cap HH) + Pr(F \cap HH)} = \frac{1}{1 + \frac{Pr(F \cap HH)}{Pr(U \cap HH)}}$$

But again by the definition of conditional probability

$$Pr(HH|F) = \frac{Pr(F \cap HH)}{Pr(F)}$$

and so

$$Pr(F \cap HH) = Pr(F) \times Pr(HH|F) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

and similarly

$$Pr(U \cap HH) = Pr(U) \times Pr(HH|U) = \frac{1}{2} \times \frac{4}{9} = \frac{2}{9}$$

Hence

$$Pr(U|HH) = \frac{1}{1 + \frac{1/8}{2/9}}$$

Of course, this is just another way to write Bayes formula:

$$Pr(U|HH) = \frac{Pr(U)Pr(HH|U)}{Pr(U)Pr(HH|U) + Pr(F)Pr(HH|F)} = \frac{\frac{2}{9}}{\frac{2}{9} + \frac{1}{8}}$$

8. (a) The probability that both flights are on time is $.9 \times .9 = .81$ Hence the probability that at least one of the two is late is $1 - .81 = .19$

(b) Call a trip bad if at least one of the two flights is late. By part (a) the probability of a bad trip is .19 Hence the probability that all 6 trips are bad is

$$.19 \times .19 \times .19 \times .19 \times .19 \times .19 = (.19)^6$$

(c) Call a trip superbad if both flights are late. The probability of a superbad flight is $.1 \times .1 = .01$. Let $E_0, E_1, \dots, E_6$ be the events defined by

E_0 is the event that there are no superbad trips out of six,

E_1 is the event that there is exactly 1 superbad trips out of six,

E_2 is the event that there are exactly 2 superbad trips out of six, and so on up to E_6 exactly 6 superbad trips.

What we are asked for is

$$Pr(E_2 \cup E_3 \cup \dots \cup E_6)$$

Since the E_k partition our probability space this is exactly the same as

$$1 - Pr(E_0) - Pr(E_1)$$

We have that $Pr(E_0) = (.99)^6$ and $Pr(E_1) = 6 \times (.01) \times (.99)^5$ We get the six here because there are 6 choices for which trip is superbad. Thus the answer is

$$1 - (.99)^6 - 6 \times (.01) \times (.99)^5$$

9. $Pr(X = 1) = \frac{5}{6}$ and $Pr(X = 2) = \frac{1}{6}$.

$$\mu = E(X) = 1 \times Pr(X = 1) + 2 \times Pr(X = 2) = \frac{5}{6} + \frac{2}{6} = \frac{7}{6}$$

$$E(X^2) = 1^2 \times Pr(X = 1) + 2^2 \times Pr(X = 2) = \frac{5}{6} + \frac{4}{6} = \frac{10}{6}$$

$$Var(X) = E(X^2) - \mu^2 = \frac{10}{6} - \left(\frac{7}{6}\right)^2$$

10. (a) .49 (b) .12 (c) .68 (This is the only value I remember by heart. The probability that a normal distribution is within one standard deviation of its mean is 68%.)

(d) If X is a random variable with a normal distribution and mean μ and standard deviation σ , then $\frac{X-\mu}{\sigma} = Z$ is the standard normal distribution. (This can be taken to be its definition.) The inequality

$$X < 67.5$$

is equivalent to

$$X - 60 < 67.5 - 60 = 7.5$$

and then by dividing by 5 we get

$$\frac{X - 60}{5} < \frac{7.5}{5} = 1.5$$

It follows that

$$Pr(X < 67.5) = Pr\left(\frac{X - 60}{5} < 1.5\right) = Pr(Z < 1.5)$$

the last can be read off from the chart as

$$Pr(Z < 1.5) = Pr(Z < 0) + Pr(0 < Z < 1.5) = .5 + .43 = .93$$