Final Exam A. Miller Fall 98 Math 210

Show all work. Explain your answers.

Name____

Circle the time of your TA section:

Tues 8:50 Tues 9:55 Thurs 8:50 Thurs 9:55

Problem	Points	Score
1	15 %	
2	15 %	
3	20 %	
4	30 %	
5	20 %	
Total	100%	

1. A box contains 5 red, 3 white, and 7 blue balls. An experiment consists of drawing balls in succession **without replacement** and noting the color of each ball selected until either a red ball or a blue ball is drawn.

(a) Draw a tree diagram for this experiment.

(b) List the elements of the sample space, i.e., the possible outcomes.

(c) What is the probability of the outcome White,White,Red?

2. Four fair dice are rolled. Find the probability that at least one of the numbers is a six.

3. An aquarium contains 7 fish:
3 of the fish weigh 200 grams each,
2 weigh 150 grams each, and the remaining
2 weigh 100 grams each.

A sample of 2 fish is selected at random. A random variable Y is defined by associating with each outcome the total weight in grams of the two fish.

(a) What are the possible values of Y?

(b) What is the density function of Y?

(c) What is the expected value of Y, i.e., $E(Y)$?

4. Find all solutions of the system of equations:

$$\begin{aligned} 2x - 5y &= 0 \\ x - 3y - z &= -1 \\ -x + 2y - z &= -1 \end{aligned}$$

5. Find the maximum and minimum values of the function $z = 2x + 3y$ on the feasible set defined by

$$\begin{array}{lcr} x - y & \geq & -2 \\ x - 4y & \leq & 4 \\ x + y & \leq & 5 \end{array}$$

6. A Markov chain has the transition matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

   (a) Determine whether this Markov chain is regular and explain why.

   (b) If it is regular, find the vector of stable probabilities.

   (c) If it is regular, in which state is the system most likely to be in after many transitions have taken place?

7. Joel purchased a tract of Texas land on January 1 1998 for \$8000. If he sells it on April 1 2000 for \$10000, what is his effective annual yield on this investment?

(You may assume the the period of time from January 1 to April 1 is exactly one quarter of a year.)

8. Max owes Tamara \$6000 payable in 3 years and \$3000 payable in 10 years. The two loans are to be consolidated into one loan payable in 5 years. What is the amount Max should pay Tamara in 5 years if the effective annual interest rate is 8% ?

9. Tamara plans to borrow \$1000 on January 1 1999 and she can do so at a rate of 8% per year compounded monthly. She can afford payments of \$180 per month. Each payment is to be made at the beginning of the month, starting with the first payment on the closing day of the loan, January 1 1999.

(a) What is the fewest number of payments she can make to repay the loan? (Assume each payment is 180 except possibly the last.)

(b) In order to come out even, the last payment she makes will be less than \$180. How much will it be?

(c) Suppose instead she decides to make 10 payments. How much should each payment be?

10. TV Manny of the **Get it Cheap** electronics store says

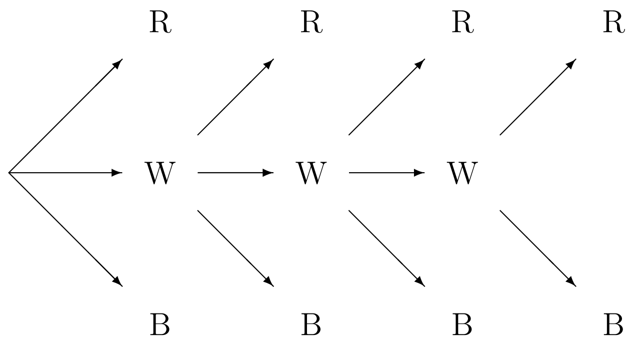
> "Buy your 57 inch big screen color TV now with nothing down and no payments until January 1 2000. Then make just 18 easy monthly payments of 55 dollars each."

If you make such an arrangement with Manny on July 1 1999 when you buy the TV, Manny will give your contract to a financing agency which immediately pays Manny for his TV. The financing agency uses an interest rate of 8% compounded monthly. How much will the financing agency give Manny on the day of the sale, July 1 1999?

(Your first payment is on January 1 2000, next on February 1 2000, and so on for 18 payments.)

Answers

1.



(a)
(b) R,B,WR,WB,WWR,WWB,WWWR,WWWB
(c) $\frac{3}{15} \times \frac{2}{14} \times \frac{5}{13} = .010989$

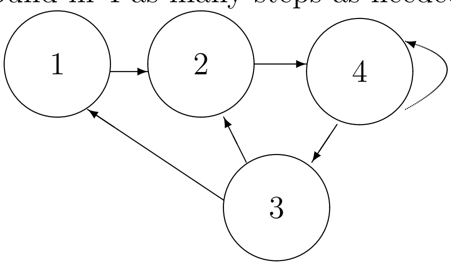
2. $1 - (\frac{5}{6})^4 = .51774691$

3.
$Pr(Y = 400) = \frac{3}{21}$
$Pr(Y = 350) = \frac{6}{21}$
$Pr(Y = 300) = \frac{7}{21}$
$Pr(Y = 250) = \frac{4}{21}$
$Pr(Y = 200) = \frac{1}{21}$
$E(Y) = \frac{6600}{21} = 314.28571$

4. $x = 5 - 5z$ $y = 2 - 2z$ z arbitrary.

5. maximum $13\frac{1}{2}$ at $(\frac{3}{2}, \frac{7}{2})$, minimum -14 at $(-4, -2)$.

6. (a) Regular, because any state can be connected to another state and there is a state (State 4) in which we can wait around in as long as we need to. It is possible to connect any state to 4 in 0,1, or 2 steps and 4 to any other in 0,1,or 2 steps. Hence any two states A and B can be connected in exactly 4 steps by first going from A to 4, wait around in 4 as many steps as needed, and then go from 4 to B.



(b) $[\frac{1}{9}\ \frac{2}{9}\ \frac{2}{9}\ \frac{4}{9}]$
(c) State 4.

7. If i is the rate per quarter, then

$$10000 = 8000(1+i)^9$$

since exactly 9 quarters will have passed by. Hence

$$i = (\frac{5}{4})^{\frac{1}{9}} - 1 = .0251036$$

The effective annual yield j satisfies $(1+j) = (1+i)^4$. and so

$$j = (1+i)^4 - 1 = .10425943$$

Equivalently if $(1+j) = (1+i)^4$, then $(1+j)^{2\frac{1}{4}} = (1+i)^9$. This is close to (but not the same as) the **nominal** annual interest $k = 4j = .1004144$. Nominal interest rates are commonly used in the United States, but I am told that the rest of the world has more sense.

8. Max will pay Tamara \$6000 exactly three years from now. We use the formula

$$F = P(1+i)^n \quad \text{or} \quad P = \frac{F}{(1+i)^n}$$

where P is the present value, F is the future value, n is the number of time periods, and i is the interest rate per period. Thus the present value of the 6000 is $\frac{6000}{(1.08)^3}$ and similarly the present value of the 3000 is $\frac{3000}{(1.08)^{10}}$. If A is the amount Max will pay Tamara in 5 years, then we want its present value to match the sum of these two present values:

$$\frac{A}{(1.08)^5} = \frac{6000}{(1.08)^3} + \frac{3000}{(1.08)^{10}}$$

This makes $A = 9040$.

9. It is clear that $5 \times 180 = 900$ is too few payments and $6 \times 180 = 1080$ is too many, since this would be around 8% in just six months. The present value of the first five payments is

$$PV = p + \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \frac{p}{(1+i)^4}$$

where $p = 180$ and $i = \frac{.08}{12}$. Using the formula for geometric series,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

we get that

$$PV = p + \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \frac{p}{(1+i)^4} = p\left(\frac{1-x^5}{1-x}\right)$$

where $x = \frac{1}{1+i}$. ($PV = 888$) The sixth and final payment of q dollars, which is to be determined, is made on June 1 1999 and has a value on January 1 of

$$\frac{q}{(1+i)^5}$$

Hence we must pick q so that

$$1000 = p + \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \frac{p}{(1+i)^4} + \frac{q}{(1+i)^5}$$

or

$$1000 = PV + \frac{q}{(1+i)^5}$$

Thus the last payment on June 1 1999 will be

$$q = (1000 - PV)(1+i)^5 = 116$$

(c) If she makes 10 payments then we must pick p so that

$$1000 = p\left(\frac{1-x^{10}}{1-x}\right) \text{ where } x = \frac{1}{1+\frac{.08}{12}}$$

Then $p = 103$.

10. Let $i = \frac{.08}{12}$ be our monthly interest and let $p = 55$ be the amount of each payment. Our first payment is made six months in the future and therefor has a present value of $\frac{p}{(1+i)^6}$. The second has a present value of $\frac{p}{(1+i)^7}$, and so on until the 18^{th} payment which has present value $\frac{p}{(1+i)^{23}}$. Thus the total present value of these payments is

$$\frac{p}{(1+i)^6} + \frac{p}{(1+i)^7} + \cdots + \frac{p}{(1+i)^{23}}$$

This equals

$$\frac{p}{(1+i)^6}\left(1 + \frac{p}{(1+i)} + \cdots + \frac{p}{(1+i)^{17}}\right)$$

which equals by the geometric series formula

$$\frac{p}{(1+i)^6}\left(\frac{1-x^{18}}{1-x}\right) \text{ where } x = \frac{1}{1+i} = \frac{1}{1+\frac{.08}{12}}$$

This is about \$900.