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The Mathematics of Interest Rates

The Formulas

There are only two formulas needed to calculate everything in this subject. One is the geometric series formula:

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \text{ for } x \neq 1$$

The other formula needed is for compound interest:

$$Q = P(1+i)^n$$

P is the amount at the beginning of the first time period,
Q is the amount at the end of the last time period,
n is the number of time periods, and
i is the effective interest rate for each time period.

It is proved as follows. To see why it is true for $n = 1$, note that given P dollars, after one time period you will have $P + iP$ dollars. For example given $P = 1000$ if you earn 10 percent interest for one year ($i = .10$), you will have

$$Q = P(1 + i) = P + iP = 1000 + 1000(.10) = 1000 + 100 = 1100.$$

Hence for $n = 1$ we have that $Q = P(1 + i)$. But given 1100 after the second year we would earn an addition 10 percent or 110, and hence we would have

$$1100 + (1100)(.10) = 1100 + 110 = 1210$$

or

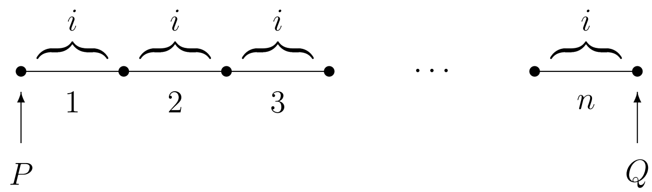
$$Q = [P(1 + i)] + [P(1 + i)](i) = [P(1 + i)](1 + i) = P(1 + i)^2.$$

The general formula is proven by the principle of induction on n. If after n periods the amount P grows to $R = P(1 + i)^n$, then after one more period of time, R will grow to $R(1+i)$. But $R(1+i) = P(1+i)^n(1+i) = P(1+i)^{n+1}$.

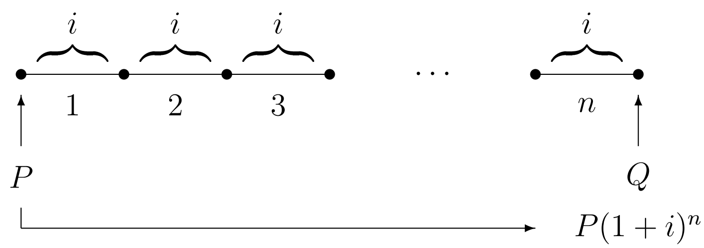
The formula

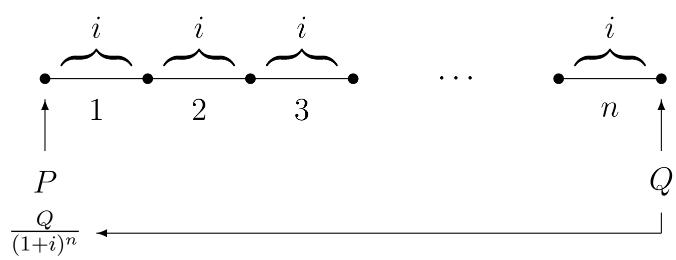
$$Q = P(1+i)^n$$

can be visualized graphically as follows:



We can also graphically illustrate pushing money forward or backward[1] in time as follows:





[1]The formula $P = \frac{Q}{(1+i)^n}$ can also be written $P = Q(1+i)^{-n}$ which makes us think of negative time periods.

Any other formula can be deduced from these two formulas using simple algebra. Recall the algebraic rules for exponents:

$$(x^a)^b = x^{ab}$$

$$x^a x^b = x^{a+b}$$

$$x^{-n} = \frac{1}{x^n}$$

$$x^a y^a = (xy)^a$$

$$x^0 = 1$$

$$x^1 = x$$

Examples and Exercises

Compounding periods
Monthly = 12 times per year
Quarterly = 4 times per year,
Annually = once per year,

For simplicity we assume all months have 30 days, all quarters are exactly 3 months, and all years have exactly 360 days.

Effective interest rate
This is the real interest rate. Annual Percentage Rate (APR) is the effective annual interest rate.

Nominal interest rate
A fictitious[2] interest rate. For example, a credit card which advertises 12 percent nominal interest compounded monthly charges you an effective monthly rate of 1 percent. The effective annual interest rate k satisfies

$$(1 + k) = (1 + .01)^{12} \approx 1.12682503013$$

[2]The word "nominal" means "in name only". The term nominal rate is also used in the context of real interest rates which take into consideration inflation. Here we consider only the effect of compounding.

or $k \approx 12.68$ percent. Truth in lending laws require that the APR be disclosed to the borrower since the nominal rate is deceptively low.

What is the difference between nominal annual rate and the effective annual rate? First of all, nominal rates don't make any sense unless the compounding period is stated. For example if j is the nominal annual interest rate and k is the effective annual rate corresponding to compounding j quarterly, then

$$\left(1 + \frac{j}{4}\right)^4 = (1 + k)$$

If k' is the the effective annual rate corresponding to compounding j monthly, then

$$\left(1 + \frac{j}{12}\right)^{12} = (1 + k')$$

No two of j, k, of k' are equal. The nominal annual rate is never the same as the effective annual rate unless the compounding period is exactly one year.

Example 1.

(a) What is the effective quarterly rate j which corresponds to a nominal annual rate of 8 percent?

(b) What is the effective annual rate k corresponding to a nominal annual rate of 8 percent which is compounded quarterly?

(c) Suppose P=1000 dollars is put in a savings account which earns a nominal annual interest rate of 8 percent compounded quarterly. After 18 months how much is in the account?

Answer

(a) To find the effective rate from the nominal we just divide, so

$$j = .08/4 = .02$$

For example, the effective monthly rate would be

$$\frac{.08}{12} \approx .00666$$

(b) $(1 + k) = (1 + j)^4$ or

$$k = (1.02)^4 - 1 \approx .08243216$$

This means that $P(1+k)^n$ which is the amount we would have after n years is exactly the same as $P(1+j)^{4n}$, i.e., amount after $4n$ quarters. This is true since by choosing k so that $(1+k) = (1+j)^4$ we have that:

$$P(1+k)^n = P((1+j)^4)^n = P(1+j)^{4n}.$$

Similarly if the nominal annual rate of 8 percent is compounded monthly, then the effective annual rate l must satisfy $(1+l) = (1+\frac{.08}{12})^{12}$ or

$$l = \left(1 + \frac{.08}{12}\right)^{12} - 1 \approx .08299950681$$

In general, if the nominal annual rate i is compounded n time periods per year, then the effective rate for each time period would be i/n and the effectively annual rate k would satisfy:

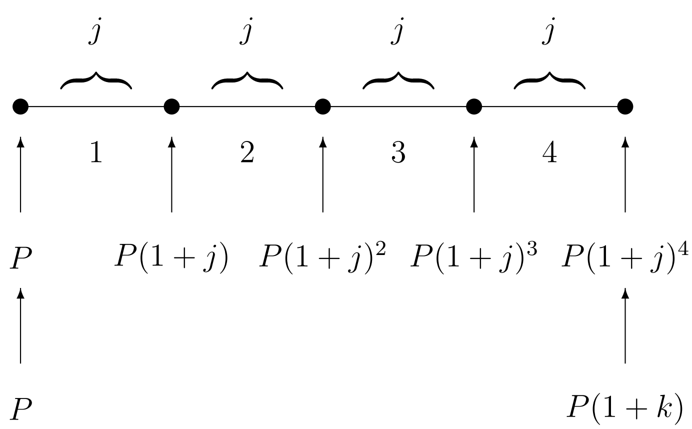
$$(1+k) = (1+\frac{i}{n})^n$$

For example, if $n = 360$ and $i = .08$, then $k \approx .08327743993$.

(c) 18 months is a year and a half or 6 quarters. Hence the answer is $1000(1.02)^6 \approx 1126$. Note that this is exactly the same as

$$1000(1+k)^{\frac{3}{2}} = 1000((1+j)^4)^{\frac{3}{2}} = 1000(1+j)^6.$$

Effective rates work correctly for fractions of a time period.



Exercise 1-1.

What effective monthly rate corresponds to annual nominal rate of 12 percent which is compounded monthly?

Exercise 1-2.

What is the nominal annual rate compounded quarterly which corresponds to an effective annual rate of 8 percent?

Exercise 1-3.

Harry buys a house for 100,000 dollars and sells it three years later for 150,000. What is the effective annual yield on his investment, i.e., what effective annual interest rate would a certificate of deposit need to have so that after 3 years, a deposit of 100,000 would compound to 150,000 dollars?

Example 2.

Harry bought 100 shares in the Inca Lost Gold Mine Corporation (ILGM) for 90 dollars per share on Nov 15, 2000. On April 1, 2001 the stock price rises to 120 dollars per share and the company makes a two for one split so that each share is now worth 60. On Nov 15, 2002 rumors hit the market that there is no lost Inca Gold and the stock price of ILGM plunges to 10 dollars per share. However, the company finds that tourists will pay a lot of money to be taken out into the Yucatan Jungle to search for the Inca Lost Gold Mine. The price of each share of ILGM rises steadily to 80 on Nov 15, 2004 at which time Harry sells all his shares of ILGM stock.

Did Harry lose money or make money on his investment? What was the effective annual yield (or loss) on his investment?

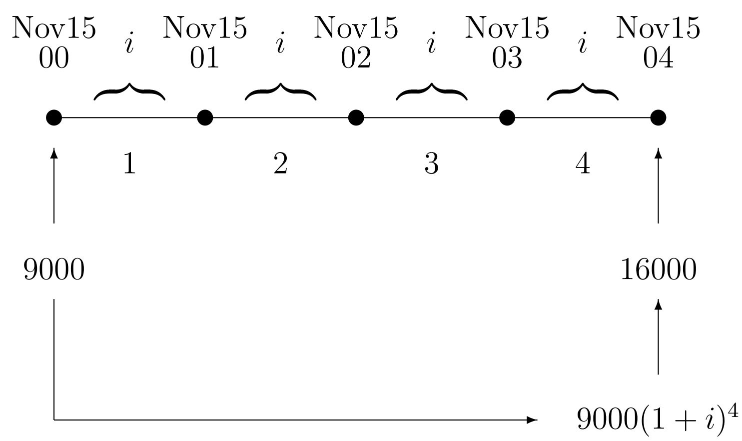
Answer

Harry made money. On Nov 15, 2000 he paid 9000 dollars for his 100 shares. The stock split 2 for 1 which means that each stock holder doubles his number of shares. On Nov 15, 2004 he sold his 200 shares for 16000. The yield is the effective annual interest rate which when compounded would result in the same amount of gain. His annual yield i satisfies:

$$16000 = 9000(1+i)^4.$$

Hence

$$i = (16/9)^{1/4} - 1 \approx .1547.$$



As far as I know there is no mathematical difference between

yield = rate of return = interest rate

The four year effective rate of return k would satisfy

$$9000(1+k) = 16000$$

since in this case there would be one four year period of time. Hence if we set $P = 9000$ and $Q = 16000$ then

$$P(1+k) = Q$$

$$1 + k = \frac{Q}{P}$$

$$k = \frac{Q}{P} - 1 = \frac{Q-P}{P}$$

$Q - P = 7000$ is the gain and $P = 9000$ is the amount invested. Hence

$$k = 7/9 = .7777\ldots \approx 78 \text{ percent}$$

The nominal annual rate of return j would satisfy

$$4j = k.$$

Hence $j \approx 19\frac{1}{2}$ percent. The effective annual rate of return i should satisfy

$$(1+i)^4 = (1+k)$$

Which is the same i as above, about $15\frac{1}{2}$ percent.

A stockbroker that advised you to buy and sell ILGM stock as above would be exaggerating your rate of return by telling you that you got $19\frac{1}{2}$ percent per year. If you put 9000 dollars into a certificate of deposit earning 15.47 percent per year you would have 16000 after 4 years. Nominal rates are deceptive.

Exercise 2-1.

Bill buys 200 shares of INTEL for 45 dollars per share on July 1, 2005 one month latter on Aug 1, 2005 he sells 100 shares for 47 dollars per share and 100 shares for 46 per share. What is the annual effective yield on this investment?

Exercise 2-2.

The Wisconsin Badgers Turnip Company decides to raise funds for expansion by issuing zero coupon notes which are payable in 3 years. The notes sell for 2000 dollars and are redeemable 3 years latter for 2300 dollars. If such a note is purchased and held to maturity, what is the effective annual percentage yield to the investor?

Exercise 2-3.

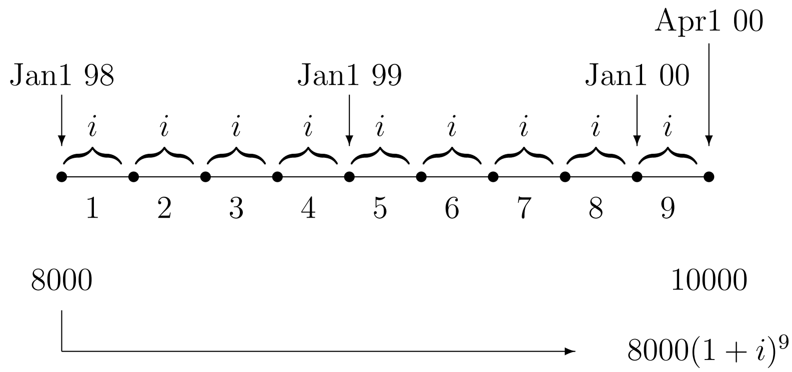
The Wisconsin Badgers Rutabaga Company decides to raise funds for expansion by issuing zero coupon notes which are payable in 4 years. The notes are redeemable 4 years latter for 1200 dollars. If effective annual percentage yield to the investor is 4 and 2/3 percent, what is the purchase price?

Example 3.

Joel purchased a tract of Texas land on January 1 1998 for 8000 dollars. If he sells it on April 1 2000 for 10000, what is his effective annual yield on this investment?

(You may assume the the period of time from January 1 to April 1 is exactly one quarter of a year.)

Answer



If i is the rate per quarter, then

$$10000 = 8000(1+i)^9$$

since exactly 9 quarters will have passed by. Hence

$$i = (\frac{5}{4})^{\frac{1}{9}} - 1 \approx .0251036$$

The effective annual yield j satisfies $(1+j) = (1+i)^4$. and so

$$j = (1+i)^4 - 1 \approx .10425943$$

Equivalently if $(1+j) = (1+i)^4$, then $(1+j)^{2\frac{1}{4}} = (1+i)^9$. This is close to (but not the same as) the **nominal** annual interest $k = 4i = .1004144$. Nominal interest rates are commonly used in the United States, but I am told that the rest of the world has more sense.

Exercise 3-1.

Tom buys two gold rings on April 1 2008. For one he pays 10000 dollars and for the other he pays 2000. Exactly, six months latter on October 1 2008 he is able to sell both rings for 15000. What was the annual rate of return on this investment?

Exercise 3-2.

Dick buys a rental house in Austin, Texas in 1980 for 55000 dollars. In 2008 he sells it for 165000. What was his annual yield on this investment?

Example 4.

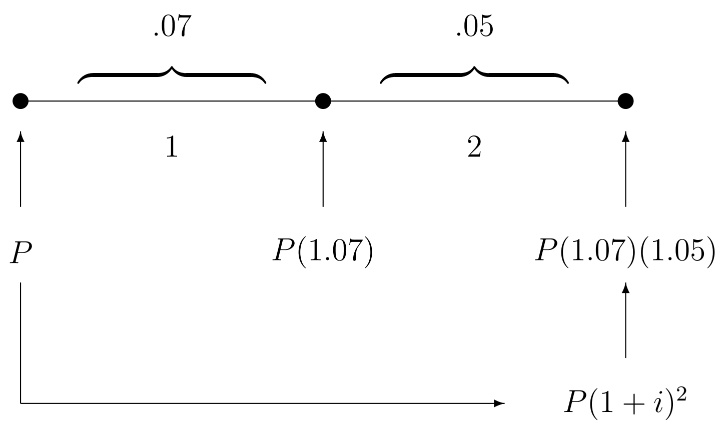
SuzieQ made an investment of P dollars on Pork Bellies on the Chicago Mercantile Exchange. The first year she got 7 percent return on her investment. She sold the Pork Bellies and bought Frozen Pork Bellies Futures with the proceeds. She got 5 percent return the second year. What was the annual effective yield on her investment?

Answer

After one year she had $P(1.07)$. At the end of the second year she had $P(1.07)(1.05)$. The effective annual yield is that interest rate i which satisfies $P(1.07)(1.05) = P(1+i)^2$. Hence $(1+i)^2 = (1.07)(1.05)$ or

$$i = \sqrt{(1.07)(1.05)} - 1 \approx .0599528$$

This number is close to, but not equal to, the average rate of 6 percent[3].



Exercise 4-1.

Suppose SuzieQ made (as in the example) 7 percent the first year but she **lost** 3 percent the second year. What was the annual effective yield on her two year investment?

[3]It doesn't look like much difference but if P is 10 billion dollars then the difference between $P(1.07)(1.05)$ and $P(1.06)^2$ is one million dollars.

Exercise 4-2.

Suppose an investment earned 3 percent the first year, 7 percent the second year, and 5 percent the third year. What was the annual effective yield on this three year investment?

Exercise 4-3.

Suppose George invested 1000 dollars and earned a return of 5 percent the first year. He then added an additional 2000 dollars as well as the 1000 plus the first years interest and earned 7 percent the second year. What was the annual effective yield on this investment?

Example 5.

You borrowed 5000 dollars from Manny the Loan Shark. For the first month Manny charges you 10 percent (monthly) interest. For the second month, if you have not paid Manny back, he charges 15 percent on the amount you then owe him. After two months you must pay Manny back or face the consequences. Anticipating that the Hospital Bill will be 6250 dollars, do you pay Manny back or what?

Answer

Note that 10 percent of 5000 is 500 and 15 percent of 5000 is 750, for a total of $5000 + 500 + 750 = 6250$. But this does not take into consideration compounding.

After one month you owe Manny

$$5000(1.10) = 5500$$

After the second month you owe Manny

$$5000(1.10)(1.15) = 5500(1.15) = 6325$$

So you can save 75 dollars by going it tough with Manny. Especially since maybe you can borrow the 6250 to pay your Hospital Bill. Probably not from Manny.

Exercise 5-1.

The I & S Blot Income Tax Preparation Company will give you an instant refund. If you have them prepare your tax return they will give you your refund immediately instead of having to wait for the IRS to send you a check. For this service they will charge you 20 percent of the amount that the IRS is going to refund to you. Then your refund check from the IRS will be sent directly to I & S Blot. Assuming your refund check[4] would have been sent to you in 6 weeks what annual effective interest does this correspond to?

Example 6.

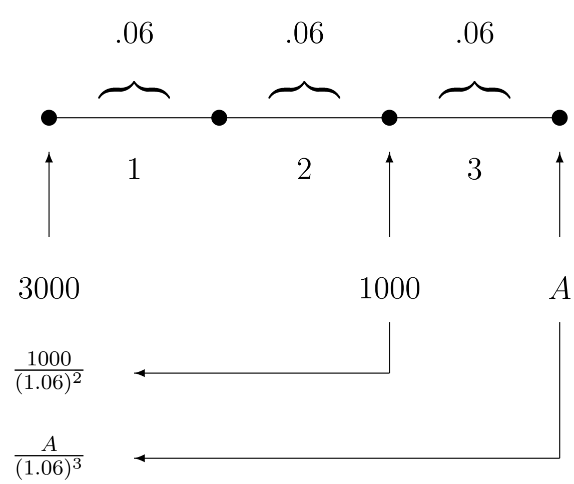
Sam buys a car from his sister Sally for 3000 dollars. He agrees to pay her 1000 two years from now and the remaining amount, A dollars, three years from now. Assuming an effective annual interest rate of 6 percent, how much is A?

Answer

$$3000 = \frac{1000}{(1.06)^2} + \frac{A}{(1.06)^3}$$

Hence

$$A = (1.06)^3 \left(3000 - \frac{1000}{(1.06)^2} \right) \approx 2513.048$$



Exercise 6-1.

[4]Would you have been better off borrowing the money from Manny?

Sam buys a desktop computer from his other sister Sue for 1000 dollars. He agrees to pay her 500 now and B dollars in 18 months. Assuming an effective annual interest rate of 6 percent, how much is B?

Exercise 6-2.

George buys 100 shares of Amazon stock for 100 dollars per share on Jan 1, 2005. One June 1, 2005 he sells 50 Amazon shares for 90 per share. On July 1, 2005, he sells his remaining 50 Amazon shares for 150 dollars per share. What is the effectively monthly yield on his investment? (You will need a calculator to find a numerical approximation for i.)

Example 7.

Max owes Tamara 6000 dollars payable in 3 years and 3000 dollars payable in 10 years. The two loans are to be consolidated into one loan payable in 5 years. What is the amount Max should pay Tamara in 5 years if the effective annual interest rate is 8 percent?

Answer

Max will pay Tamara 6000 dollars exactly three years from now. We use the formula

$$Q = P(1+i)^n \quad \text{or} \quad P = \frac{Q}{(1+i)^n}$$

where P is the present value, Q is the future value, n is the number of time periods, and i is the interest rate per period. Thus the present value of the 6000 is $\frac{6000}{(1.08)^3}$ and similarly the present value of the 3000 is $\frac{3000}{(1.08)^{10}}$. If A is the amount Max will pay Tamara in 5 years, then we want its present value to match the sum of these two present values:

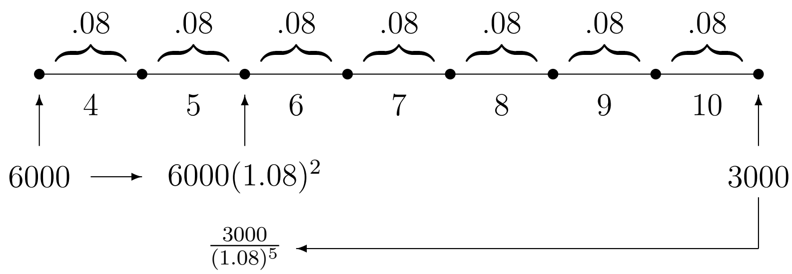
$$\frac{A}{(1.08)^5} = \frac{6000}{(1.08)^3} + \frac{3000}{(1.08)^{10}}$$

or

$$A = 6000(1.08)^2 + \frac{3000}{(1.08)^5}$$

This makes $A \approx 9040$.

An equivalent way to work this problem is to move the 6000 forward 2 years and the 3000 backward five years.



Exercise 7-1.

If this Max-Tamara loan is consolidated into one loan payable in 4 years, what is the amount Max should pay in 4 years?

Exercise 7-2.

Suppose this Max-Tamara loan is consolidated into two equal payments of p dollars, one at the end of 4 years and one at the end of 5 years. How much should p be?

Example 8.

Mrs Jones wants to buy her son Max an IPOD for a present. She plans to pay for it by making two deposits in her savings account. The first deposit she makes on Nov 1, 2003 and the second on Dec 1, 2003. The second deposit will be twice as large as the first. On Feb 1, 2004 she pays for the 300 dollars IPOD by withdrawing the money from her account. Her savings account earns 1/2 percent a month in interest. What is the size of the smaller deposit?

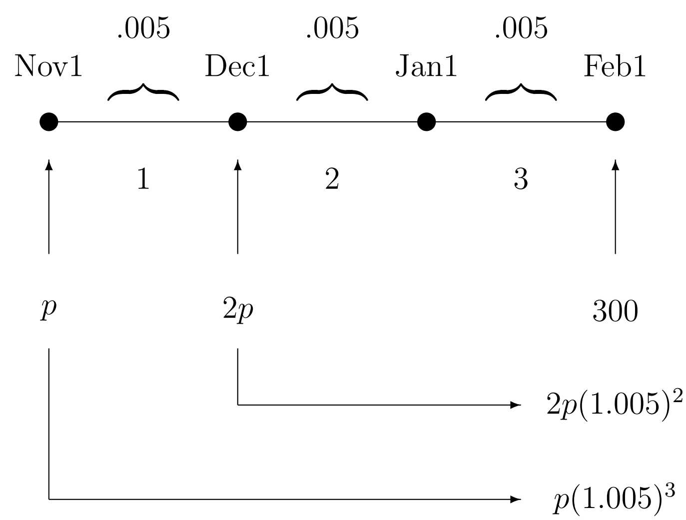
Answer

Let $i = .005$ and let p be the amount of the first deposit. The amount of the second deposit is $2p$. The amount in the savings account on Feb 1 is $p(1+i)^3 + 2p(1+i)^2$. Hence

$$300 = p(1+i)^3 + 2p(1+i)^2$$

and so

$$p = \frac{300}{(1.005)^3 + 2(1.005)^2} \approx 98.84$$



Exercise 8-1.

On January 1, 1998 you deposited 3000 dollars in a savings account which pays 4 percent nominal annual interest compounded quarterly. On July 1, 1998 you withdrew 1000. On July 1,1999 how much will you have in this account?

Annuity

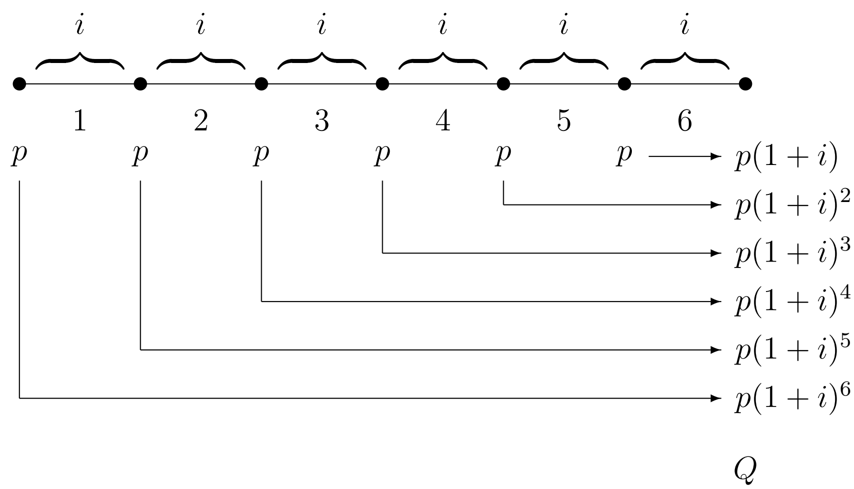
Periodic payments, p, with each payment at the end of the each of n time periods.

Annuity due

Same as an annuity but payments are made at the beginning of each time period. Other variations on a simple annuity are variable payment annuities. The payment may increase or decrease depending on either a set percentage or an amount depending on (government) inflation rates or the prime interest rate. Variable length annuities might terminate with death or death of a spouse.

Example 9.

You decide to put $p = 100$ dollars in a bank account at the beginning of each month for 6 months. If you earn an effective $i = 1$ percent monthly interest how much will you have in the bank at the end of 6 months?



We see from the picture that the amount Q we have in the bank at the end of sixth month is

$$Q = p(1+i) + p(1+i)^2 + p(1+i)^3 + p(1+i)^4 + p(1+i)^5 + p(1+i)^6$$

If we let $x = (1+i)$ and factor out the px we get:

$$Q = p(x + x^2 + x^3 + x^4 + x^5 + x^6) = px(1 + x^2 + x^3 + x^4 + x^5).$$

Using the formula for the geometric series (see Appendix) we can simplify Q further and get

$$Q = px\left(\frac{1 - x^6}{1 - x}\right).$$

Since $x = (1+i) = (1.01)$ and $p = 100$ we can use a computer to get that $Q \approx 621.35$ dollars.

If you do the same thing for 30 years or 360 months, then

$$Q_0 = px\left(\frac{1 - x^{360}}{1 - x}\right) \approx 352,991.38$$

Exercise 9-1.

Suppose you put 100 dollars in the bank at the end of each month for six months at 1 percent monthly interest. How much would you have at the end of the six months?

Exercise 9-2.

You wish to save money for your child's college education. You contribute 2000 each year to a savings account which pays 8 percent annual effective interest. Your first contribution is made on January 1, 2000, the second on January 1, 2001, and so on, making the same contribution each January. You make your last contribution on January 1, 2009. At that time, how much will your child have in this savings account?

Exercise 9-3.

Suppose you put 150 dollars in the bank at the beginning of each month for 12 months. Suppose for the first six months you earned 1 percent monthly interest but for last six months you earned 2 percent monthly interest. How much money would you have in the bank at the end of the twelve th month?

Exercise 9-4.

Suppose the Tortoise and the Hare have a savings contest. The Hare puts 1000 dollars in the bank for one year at an effective annual interest rate of k. The Tortoise puts 100 dollars in the bank at the end of each month for twelve months at an effective monthly rate i which is equivalent to the effective annual rate k. At the end of 12 months, the contest is a tie, i.e., both have the same amount in the bank. What[5] is i?

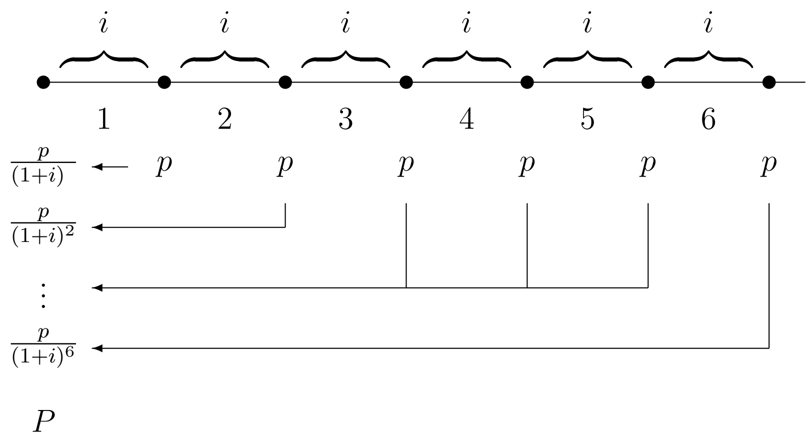
Example 10.

You borrow 10,000 dollars from your Uncle Bill and agree to pay him back in six yearly payment of p dollars starting at the end of the first year and ending at the end of the sixth year. If Uncle Bill charges you 8 percent effective annual interest, what should each payment p be?

Answer

[5] You will need a calculator to find a numerical approximation for the root of this polynomial.

Let $i = 08$. Looked at from today the value of p dollars one year now would be $\frac{p}{1+i}$, i.e., the amount which when multiplied by $(1+i)$ gives p. Similar, the current or present value of p dollars two years from now is $\frac{p}{(1+i)^2}$.



Hence the value of the loan $P = 10000$ would satisfy

$$P = \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \frac{p}{(1+i)^4} + \frac{p}{(1+i)^5} + \frac{p}{(1+i)^6}$$

Factoring out p and setting $x = \frac{1}{1+i}$ gives us

$$P = p(x + x^2 + x^3 + x^4 + x^5 + x^6)$$

Factoring out x and using the geometric series formula gives:

$$P = px(1 + x^2 + x^3 + x^4 + x^5) = px\left(\frac{1-x^6}{1-x}\right).$$

Solving for p gives us:

$$p = \frac{P(1-x)}{x(1-x^6)}$$

Recall that $P = 10000$ and $x = \frac{1}{1+i}$ and $i = .08$, so $p \approx 2163.15$.

Exercise 10-1.

On January 1, 2000 you buy a home for 150,000 dollars. After paying a down payment of 30,000 dollars the remaining amount is borrowed from a bank at 7 percent nominal annual interest compounded monthly for 30 years. The first monthly payment is paid on January 1, 2000. What is the amount of each monthly payment?

Exercise 10-2.

A mortgage at a fixed monthly effective interest rate of one-half of one percent is to be paid back in 15 years in monthly deposits of 1500 dollars to be paid at the beginning of each month starting on the day the loan is made (closing day).

(a) What is the amount borrowed?

(b) After exactly five years from the day of closing the house is sold. What is the amount which must be paid to the lender to settle the loan? (Hint: Find the value of the remaining payments.)

Exercise 10-3.

Jebediah is buying a new car. He has 3000 dollars in cash and can borrow the rest of the money needed to buy the car from his Credit Union at a nominal annual interest rate of 12 percent compounded monthly. He purchases an 11,000 dollar car by putting down the cash and financing the rest for 8 equal monthly payments, each made at the beginning of the month, and starting on the day he buys the car. What is the amount of each payment?

Exercise 10-4.

Jeremiah is buying a new car. He has 2000 dollars in cash and can borrow the rest of the money needed to buy the car from his Credit Union at a nominal annual interest rate of 12 percent compounded monthly. He purchases the car by putting down the cash and financing the rest for 6 equal monthly payments of 1025 dollars, each made at the beginning of the month, and starting on the day he buys the car. What is the selling price of the car?

Example 11.

A car is advertised in the newspaper as follows: You receive a cash bonus B of 3275 dollars together with your car (i.e., the reverse of a down payment). In return for which you must pay 30 monthly payments p of 398 each. The first payment is to be made exactly six months after you receive the car and cash bonus. If the monthly interest rate for car loans is $i = .004825$ per month, what is the cost C of the car?

Answer

This is the same as asking what would you expect to pay for the car in cash on the day of the sale, instead of the bonus plus payments deal being offered. The buyer gets the bonus B and the car or equivalently the cash value of the car C. The seller gets the present values of each of the 30 payments. In a fair deal each one gets the same, otherwise they will not be willing to make the exchange. Hence

$$B + C = \frac{p}{(1+i)^6} + \frac{p}{(1+i)^7} + \cdots + \frac{p}{(1+i)^{35}}$$

So

$$C = \frac{p}{(1+i)^6} + \frac{p}{(1+i)^7} + \cdots + \frac{p}{(1+i)^{35}} - B$$

Using the geometric series formula this can also be written:

$$C = \frac{p}{(1+i)^6}\left(\frac{1 - \frac{1}{(1+i)^{30}}}{1 - \frac{1}{(1+i)}}\right) - B$$

where $B = 3275$, $p = 398$, and $i = .004825$. Using a computer we get that $C \approx 7552.48$.

Exercise 11-1.
TV Manny of the **Get it Cheap** electronics store says

> "Buy your 57 inch big screen color TV now with nothing down and no payments until January 1 2000. Then make just 18 easy monthly payments of 55 dollars each."

If you make such an arrangement with Manny on July 1 1999 when you buy the TV, Manny will give your contract to a financing agency which immediately pays Manny for his TV. The financing agency uses a nominal annual interest rate of 8 percent compounded monthly. How much will the financing agency give Manny on the day of the sale, July 1 1999?

(Your first payment is on January 1 2000, next on February 1 2000, and so on for 18 payments.)

Example 12.

Tamara plans to borrow 1000 dollars on January 1 1999 and she can do so at a nominal annual rate of 8 percent per year compounded monthly. She can afford payments of 180 per month. Each payment is to be made at the beginning of the month, starting with the first payment on the closing day of the loan, January 1 1999.

1. What is the fewest number of payments she can make to repay the loan? (Assume each payment is 180 except possibly the last.)
2. In order to come out even, the last payment she makes will be less than 180. How much will it be?
3. Suppose instead she decides to make 10 equal payments. How much should each payment be?

Answer

It is clear that $5 \times 180 = 900$ is too few payments and $6 \times 180 = 1080$ is too many, since this would be around 8 percent in just six months. The present value of the first five payments is

$$PV = p + \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \frac{p}{(1+i)^4}$$

where $p = 180$ and $i = \frac{.08}{12}$. Using the formula for geometric series,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

we get that

$$PV = p + \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \frac{p}{(1+i)^4} = p\left(\frac{1-x^5}{1-x}\right)$$

where $x = \frac{1}{1+i}$. ($PV = 888$) The sixth and final payment of q dollars, which is to be determined, is made on June 1 1999 and has a value on January 1 of

$$\frac{q}{(1+i)^5}$$

Hence we must pick q so that

$$1000 = p + \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \frac{p}{(1+i)^4} + \frac{q}{(1+i)^5}$$

or

$$1000 = PV + \frac{q}{(1+i)^5}$$

Thus the last payment on June 1 1999 will be

$$q = (1000 - PV)(1+i)^5 = 116$$

(c) If she makes 10 payments then we must pick p so that

$$1000 = p\left(\frac{1-x^{10}}{1-x}\right) \text{ where } x = \frac{1}{1+\frac{.08}{12}}$$

Then $p = 103$.

Exercise 12-1.

The effectively monthly interest rate on your credit card is 1 percent and the credit card compounds monthly. You will charge your card for two purchases, one will be made two months from today (on Jan 15) for 1200 dollars and the other will be made four months from now (on Mar 15) for 400. After the first purchase you must pay the credit card a minimum of 100 dollars each month starting with Feb 15. After making 3 minimum payments of 100 you decide to pay off the entire amount on the fourth payment on May 15.

How much is the fourth and final payment?

Principal of a loan

The amount owed at any point in time is called the principal of the loan. Each payment goes in part to paying down or reducing the principal and in part to interest. At the beginning most of the payment goes toward interest, since a lot is owed. The final payments consist mostly of reduction of principle.

Example 13.

Jerome borrows money to buy a car. He agrees to make 48 monthly payments of 500 dollars at the end of each month at an effective monthly interest rate of one percent. Just after making the 40th payment Jerome sells the car to Mary.

(a) How much does Jerome owe the lender when he sells the car?

(b) The last four payments he makes are in the calendar year 2008. The part of those four payments which is interest in dollars is deductible from his income tax. How much is it?

Answer

(a) What Jerome owes the lender is less than the remaining eight payments or $8 \times p = 8 \times 500 = 4000$, since they are made in the future. The 41st payment is due at the end of the month and so its present value is $\frac{p}{(1+i)}$. The 42nd payment is due in two months so its present value is $\frac{p}{(1+i)^2}$, and so on until the 48th payment. Hence the amount, A, that he owes is

$$A = \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \cdots \frac{p}{(1+i)^8}$$

where $i = .01$ and $p = 500$. Using the geometric series formula and a computer we get that $A \approx 3825.84$.

(b) The last four payments he makes are the 37, 38, 39, and 40th ones. By a similar analysis the amount, B, that he owes just after making the 36th payment is the value at that time of the remaining 12 payments:

$$B = \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3} + \cdots \frac{p}{(1+i)^{12}}$$

Thus the four payments made in 2008 have reduced his principal by the amount

$$B - A = \frac{p}{(1+i)^9} + \frac{p}{(1+i)^{10}} + \frac{p}{(1+i)^{11}} + \frac{p}{(1+i)^{12}}$$

The amount of interest in dollars that he paid in 2008 is

$$4p - (B - A) \approx 198.30$$

since everything that he paid, $4p$, is either interest or reduction in principal, $(B - A)$.

Exercise 13-1.

Max buys a house for 210,000 dollars and puts 20 percent down and borrows the rest from the credit union at a fixed interest rate, 30 year, loan at 7 percent effectively annual interest. If he makes a monthly payment of p dollars at the beginning of each month starting on the closing day of the loan, what is p?

Exercise 13-2.

Suppose Max borrowed the money on March 1, 2000. What part of the 10 payments that he made in the year 2000 is reduction of principal and what part is interest in dollars?

Exercise 13-3.

After a 10 years, Max decides to refinance his loan, to get a 20 year loan at 4 percent effective annual interest. What is his new monthly payment?

Exercise 13-4.

A loan is paid back in three annual payments of p dollars made at the end of each year. The effective annual interest rate is i.

(a) What is the principal or amount borrowed B? Assume that it is borrowed at the beginning of the first year. (Express B in terms of p and i.)

(b) What is the total amount of interest I (in dollars) paid thru the course of the loan? (Express I in terms of B, p, and i.)

(c) Each payment p in a loan is part interest and part reduction in the principal. What part I_1, I_2, I_3 of each of the three payments is interest?

Exercise 13-5.

Bubba-Billy-Bob (BBB) buys a house in Wilmood Subdivision in Austin, TX. The developer Big-Bill-Wilmood (BBW) offers BBB a deal. For the first 5 years of BBB's mortgage the monthly payments will be interest only. After 5 years the payment will increase and be amortized for 25 years. This is designed to attract younger home buyers whose incomes are low but will increase as they get older.

(a) BBB buys his house for 55,000 dollars. He is required to make a 20 percent down payment. How much does he borrow?

(b) The effective annual interest rate that BBW offers is $8\frac{3}{4}$ percent. What is the corresponding effective monthly rate i? What is the amount of each monthly payment p for the first 5 years? (Keep in mind the these payments are interest only.)

(c) What is the amount of each of the remaining payments?

Exercise 13-6.

Mortgage lenders sometimes require that homeowners escrow their property taxes. This means that in addition to the monthly loan amount p they pay another amount r. This makes their monthly home payment $p+r$. Each year the 12 payments of r go into an Escrow Account that the bank uses to pay the home-owner's property tax when it comes due.

The reason is that if the property owner fails to pay property tax the county can seize the property, sell it, recover the property taxes, fees, and penalties. If there is any money left, the bank gets it. Since the bank doesn't want to be left holding the bag, it's to everyone's interest to be sure this doesn't happen. The bank may or not pay interest on the Escrow Account. If the interest rates are high, it will probably have to, in order to compete.

Bubba-Billy-Bob bought his aforementioned home in 1979. In the year 1992 (because of over-payments in earlier years) the escrow account holds 150 dollars on January 1, 1992. His bank gives Escrow Accounts an effective monthly interest rate i of one-half of a percent, $i = .005$.

The Travis County Texas Tax Collector requires that the 1992 property tax be paid on BBB's house in two payments:

The first of 1300 to be made on January 1, 1993 and

The second of 1200 to be made on July 1, 1993.

What is the amount r which should be paid into the Escrow Account by BBB on Jan 1, Feb 1, ..., Dec 1 in the year 1992 so that the Escrow Account can be used to make the two tax payments?

Exercise 13-7.

There is an algorithmic way of computing the principal of a loan. Suppose that the principal or amount owed at the beginning of a time period is A. Suppose that the effective interest rate for each time period is i and a payment p is made at the end of the time period. And let B be the amount owed at the end of the time period.

(a) Show that $B = A(1+i) - p$.

(b) Show that Ai is the amount of interest in dollars paid during this time period.

(c) Show that the following algorithm will compute the principal and amount of interest for each payment of a loan for A dollars, to be paid back, in p dollars at the end of each of n time periods, each with an effective interest rate of i.

```
Do
   print A*i, A
   A:=A*(1+i)-p
Loop until A<0
```

Example 14.

Arnie wishes to retire soon. He would like a yearly income at the beginning of each year for n years, e.g. $n = 20$. He would like it to be p dollars, e.g. $p = 40,000$, but adjusted for inflation, which he estimates to be k, e.g., $k = .02$ or 2 percent. This means he would like to receive $p(1 + k)$ at the beginning of the second year, $p(1+k)^2$ at the beginning of the third year, and so on for n years. Arnie guesses that he will be able to invest his retirement nest egg at an effective annual interest rate of i, e.g. $i = .05$ or 5 percent. How much cash does Arnie need on the day he retires?

Answer

He will need p on the first day of the year he retires, $p(1 + k)$ on the first day of the second year, $p(1 + k)^2$ on the first day of the third year, and so on, until $p(1 + k)^{n-1}$ on the first day of n^{th} year. Moving these payments back to the first day of the first year of retirement, we get that the amount he will need is:

$$A = p + \frac{p(1 + k)}{(1 + i)} + \frac{p(1 + k)^2}{(1 + i)^2} + \cdots + \frac{p(1 + k)^{n-1}}{(1 + i)^{n-1}}$$

Factor out the p and get

$$A = p\left(1 + \left(\frac{1 + k}{1 + i}\right) + \left(\frac{1 + k}{1 + i}\right)^2 + \cdots + \left(\frac{1 + k}{1 + i}\right)^{n-1}\right)$$

Substituting $x = \left(\frac{1+k}{1+i}\right)$ we get

$$A = p(1 + x + x^2 + \cdots x^{n-1}) = p\left(\frac{1 - x^n}{1 - x}\right)$$

Putting in $p = 40000$, $n = 20$, $i = .05$, $k = .02$, gives $A \approx 615946.88$.

Exercise 14-1.

Assume that he estimates that inflation is 5 percent and his annual yield on investments is 2 percent. Then how much will Arnie need?

Exercise 14-2.

What if he desires a monthly income of 4000 dollars for 20 years. Then how much will Arnie need?

Appendix A

Here is a proof of the geometric series formula:

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \text{ (for } x \neq 1)$$

Proof
Put

$$S = 1 + x + x^2 + \cdots + x^n$$

Then

$$xS = x(1 + x + x^2 + \cdots + x^n) = x + x^2 + x^3 \cdots + x^n + x^{n+1}$$

and so

$$\begin{aligned} S &= 1 + x + x^2 + \cdots + x^n & (1) \\ xS &= \quad\;\; x + x^2 + \cdots + x^n + x^{n+1} & (2) \\ S - xS &= 1 \qquad\qquad\qquad\qquad\; - x^{n+1} & (3) \end{aligned}$$

The equation (3) is gotten by subtracting equation (2) from equation (1). Hence

$$(1 - x)S = 1 - x^{n+1}$$

or

$$S = \frac{1 - x^{n+1}}{1 - x}.$$

QED

Answers

1-1. .01

1-2. $\approx .0777$

1-3. $\approx .1447$

2-1. $\approx .4821$

2-2. ≈ 4.8 percent

2-3. ≈ 1000 dollars

3-1. 5.25 percent

3-2. $\approx .0400$

4-1. $\approx .0188$

4-2. $\approx .0499$

4-3. $\approx .0648$

5-1. ≈ 5.92 percent

6-1. ≈ 546

6-2. $.0329 < i < .0330$

7-1. ≈ 8371

7-2. ≈ 4346

8-1. ≈ 2144 dollars

9-1. ≈ 615

9-2. ≈ 28973 dollars

9-3. ≈ 2015

9-4. $.0292 < i < .0293$

10-1. ≈ 794 dollars

10-2. (a) ≈ 178644 (b) ≈ 135786

10-3. ≈ 1035 dollars

10-4. ≈ 8000 dollars

11-1. ≈ 899.60

12-1. ≈ 1351

13-1. ≈ 1087

13-2. ≈ 271 and 10599

13-3. ≈ 1442

13-4. (a) $B = \frac{p}{(1+i)} + \frac{p}{(1+i)^2} + \frac{p}{(1+i)^3}$

13-4. (b) $3p = B + I$ or $I = 3p - B$

13-4. (c) $I_1 = p - \frac{p}{(1+i)^3} \quad I_2 = p - \frac{p}{(1+i)^2} \quad I_3 = p - \frac{p}{(1+i)}$.

13-5. (a) ≈ 44000 (b) ≈ 309 (c) ≈ 349

13-6. ≈ 199

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