Strict type-II blowup in harmonic map flow

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Harmonic maps

Let (M, g) and (N, h) be compact Riemannian manifolds.

The *energy density* of a map $u : M \to N$ is given by

$$e(u) = \frac{1}{2}|du|^2 = \frac{1}{2} g^{ij} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$

The *Dirichlet functional* is

$$E(u) = \int_M e(u) dV_g.$$

Harmonic maps are the critical points of $E(u)$.

Harmonic maps, cont'd

Extrinsic formulation. Assume $N \subset \mathbb{R}^k$ isometrically (by Nash's Theorem). Then u is harmonic iff

$$(\Delta u)^T = 0.$$

Intrinsic formulation. A map u is harmonic iff the *tension field*

$$\mathcal{T}(u) = \mathrm{tr}_g \nabla du = 0$$

vanishes. In coordinates:

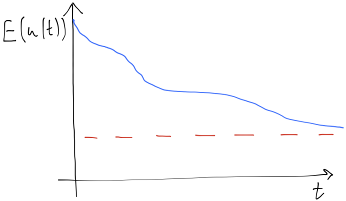
$$\mathcal{T}(u)^\alpha = g^{ij} \left(\frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - (\Gamma^g)^k_{ij} \frac{\partial u^\alpha}{\partial x^k} + \left(\Gamma^h(u) \right)^\alpha_{\beta\gamma} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \right).$$

Harmonic map flow

Harmonic map flow is the downward gradient flow of the Dirichlet energy:

$$\frac{\partial u}{\partial t} = \mathcal{T}(u).$$

$\Rightarrow E(u(t))$ is decreasing along flow.



Idea: Limit as $t \to \infty$ will be a harmonic map (in the initial homotopy class).

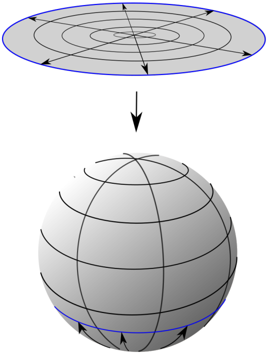
Eells-Sampson (1964): if the sectional curvature $K^N \leq 0$, this actually works!

But....

Critical harmonic map flow ($n = 2$)

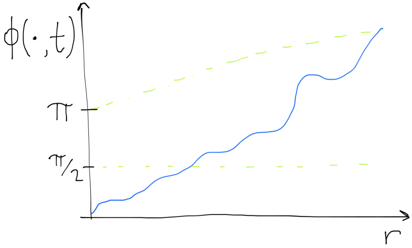
Theorem (K.-C. Chang, W.-Y. Ding, and R. Ye 1992)

Harmonic map flow with $M = D^2$, $N = S^2$ can blow up in finite time.



$t = 0$

$$\partial_t \phi(r, t) = \partial_r^2 \phi + \frac{1}{r} \partial_r \phi - \frac{\sin 2\phi}{2r^2}$$



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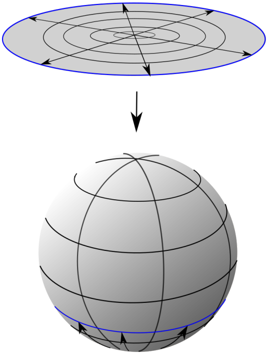
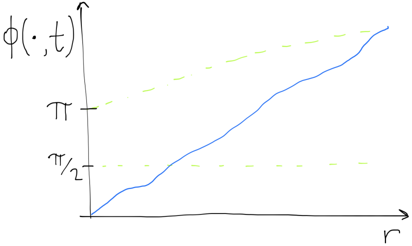


$t = .1$

$$\partial_t \phi(r,t) = \partial_r^2 \phi + \frac{1}{r}\partial_r \phi - \frac{\sin 2\phi}{2r^2}$$



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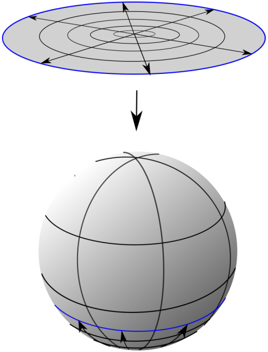
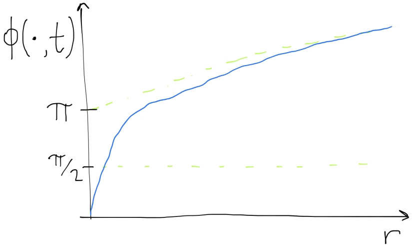


$t = .2$

$$\partial_t \phi(r, t) = \partial_r^2 \phi + \frac{1}{r}\partial_r \phi - \frac{\sin 2\phi}{2r^2}$$



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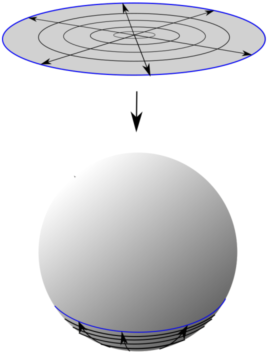
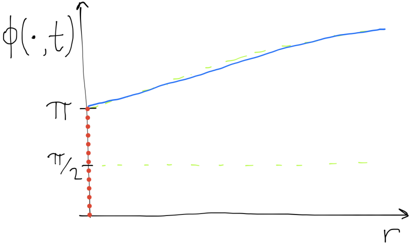


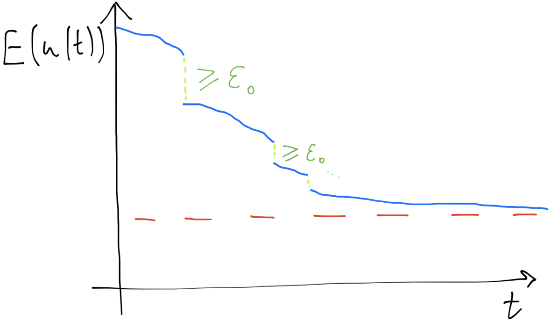
$t = .3$

$$\partial_t \phi(r, t) = \partial_r^2 \phi + \frac{1}{r}\partial_r \phi - \frac{\sin 2\phi}{2r^2}$$



General picture in dimension two

Struwe (1985): Global weak solution on $\Sigma \times [0, \infty)$ with finitely many singular times.



For any singular time $T < \infty$, the **body map**

$$u(T) = \lim_{t \nearrow T} u(t)$$

exists weakly in $W^{1,2}$ and smoothly away from the singular set.

Continuity of the body map

Note that $W^{1,2} \not\subset C^0$ in 2D, so it is possible that $u(T)$ may be *discontinuous*.

Topping (2004) constructs a pathological metric on target $N = T^2 \times S^2$ such that for an inital map

$$u(0) : D^2 \to \{x_0\} \times S^2 \subset N,$$

the body map $u(T)$ has an essential singularity at the first singular time.

Topping's conjecture (also from 2004): If N is *real-analytic*, then $u(T)$ must be continuous.

Related question: **How fast** does the blowup occur?

Energy scale

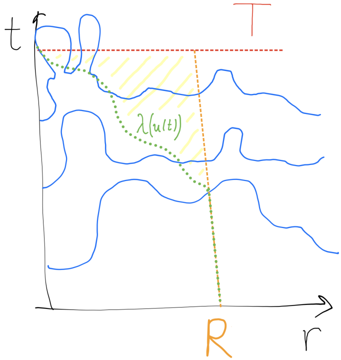
Definition

Let $\varepsilon, R > 0$ and $p \in M$.

Let $u : \Sigma \to N$ be a $W^{1,2}$ map.

The *(outer) energy scale* $\lambda(u) = \lambda_{\varepsilon,R,p}(u)$ is the minimal number $0 \leq \lambda \leq R$ such that

$$\sup_{\lambda < r < R} E\left(u, U^r_{r/2}(p)\right) < \varepsilon.$$



Lemma

For $R > 0$ sufficiently small, we have

$$\lambda_{\varepsilon,R,p}(u(t)) \to 0 \quad (t \nearrow T).$$

Moreover, $\lambda_{\varepsilon,R,p}(u(t)) \equiv 0$ for t near $T \Leftrightarrow (p, T)$ is a smooth point.

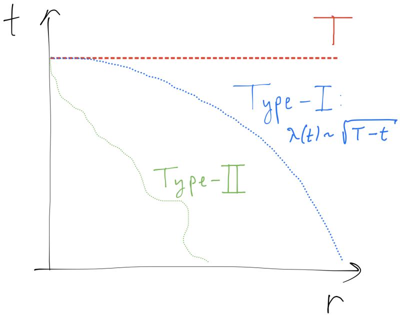
Type-I versus type-II blowup

Definition

Suppose $u(t)$ blows up at (p, T). We say the blowup is *type-I* if

$$\sup |du(t)| \leq \frac{C}{T - t}.$$

It is called type-II otherwise.

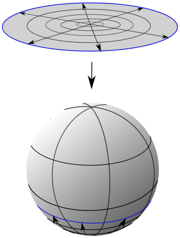


Theorem (Struwe)

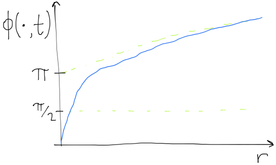
If $\dim(M) = 2$*, all blowups are type-II. More precisely, we have*

$$\lambda(t) = o(T - t)^{\frac{1}{2}}.$$

Blowup rate in Chang-Ding-Ye example



$$\partial_t \phi(r,t) = \partial_r^2 \phi + \frac{1}{r}\partial_r \phi - \frac{\sin 2\phi}{2r^2}$$



- Van den Berg, Hulshof, and King (SIAM J. Anal., 2003) predict

$$\lambda(t) \sim \kappa \frac{|T-t|}{(\log|T-t|)^2}. \tag{1}$$

- Angenent, Hulshof, and Matano (SIAM J. Anal., 2009) prove $\lambda(t) = o(T-t)$.
- Raphael and Schweyer (CPAM, 2013) prove (1) for generic rotationally symmetric initial data close to ground state.

  Also prove $u(T) \in W^{2,2}$, hence C^α for each $\alpha < 1$.
- Davila, Del Pino, and Wei (Inventiones, 2020) construct many non-symmetric examples with blowup rate (1).

Note: Body map is continuous in all cases.

Harmonic map flow for almost-holomorphic maps

Theorem 1 (C. Song-Waldron (2020))

Let Σ be a compact Riemann surface and $u : \Sigma \times [0, T) \to N$ a classical solution of harmonic map flow. Suppose

- *N is compact Kähler with nonnegative holomorphic bisectional curvature (e.g. $\mathbb{CP}^n$)*
- $E_{\bar{\partial}}(u(0)) < \varepsilon_0(N)$.

Then:

a. $\lambda(t) = O\left(T - t\right), \quad t \nearrow T$
b. $u(T) \in C^{\alpha}(\Sigma)$ *for* $0 < \alpha < 1$
c. *No neck between $u(T)$ and the bubble tree.*

Note: This result applies to the rotationally symmetric blowups on the last slide.

$\Rightarrow$ geometric proof of Raphael and Schweyer's continuity result.

Strict type-II blowup

Definition

A finite-time singularity of harmonic map flow at (p, T) is called "strict type-II" if for $R > 0$ sufficiently small, the energy scale satisfies

$$\lambda_{\varepsilon,R,p}(u(t)) = O(T - t)^{\frac{1+\alpha}{2}} \tag{2}$$

for some $0 < \alpha \leq 1$.

Note: Topping proved that his counterexample blows up with rate

$$\lambda(t) \gtrsim (T - t)^{\frac{1}{2}+\varepsilon}$$

for all $\varepsilon > 0$. So this is *not* strictly type-II!

Theorem 2 (Waldron 2021)

The body map $u(T)$ at a strict type-II blowup is $C^{\frac{\alpha}{3}}$.

Stress-energy tensor

Full variational formula

$$\delta E_g(u) = -\int \left(\langle \mathcal{T}(u), \delta u \rangle + \frac{1}{2} \langle S, \delta g \rangle \right) dV_g .$$

Stress-energy tensor

$$S(u) = \langle du \otimes du \rangle - \frac{1}{2} |du|^2 g \in \mathrm{Sym}^2 T^* M$$

satisfies

$$\operatorname{div} S = \langle \mathcal{T}(u), du \rangle.$$

The stress-energy tensor plays a dual role in harmonic map flow.

Role of stress-energy tensor in Theorem 1(a)

Take another divergence

$$\mathrm{div}^2 S = \langle \nabla \mathcal{T}(u), du \rangle + |\mathcal{T}(u)|^2.$$

Under HM flow

$$\partial_t e(u) = \langle \nabla \mathcal{T}(u), du \rangle.$$

$\Rightarrow$ "pointwise" energy identity:

$$\partial_t e(u) = -|\mathcal{T}(u)|^2 + \mathrm{div}^2 S.$$

Integrating over M, we recover the usual global energy identity:

$$E(u(t_2)) + \int_{t_1}^{t_2} \int_\Sigma |\mathcal{T}(u)|^2 dV dt = E(u(t_1)).$$

Integrating against a cutoff function φ, we get a *refined* local energy inequality:

$$E\left(u(t_2), B_{\frac{R}{2}}\right) \leq E\left(u(t_1), B_R\right) + \int_{t_1}^{t_2} \int_\Sigma \langle \nabla^2 \varphi, S \rangle dV dt.$$

Control of $S \Rightarrow$ control on blowup rate.

Theorem 1. Suppose that

- N is compact *Kähler* with $K^N_{hol.bi.} \geq 0$
- $E_{\bar\partial}(u(0)) < \varepsilon_0$.

Then at a finite-time singularity

a. $\lambda(t) = O(T - t)$.

Proof:

1. $E_{\bar\partial}(0) < \varepsilon_0 \Rightarrow E_{\bar\partial}(t) < \varepsilon_0 \ \forall t > 0$, since $E = \kappa + 2E_{\bar\partial}$ and $E(t)$ is decreasing
2. $\sup E_{\bar\partial}(u(t)) < \varepsilon_0 \Rightarrow \|\bar\partial u(t)\|_{L^\infty} \leq C$ by ε-regularity argument
3. $\|S(u(t))\|_{L^2} \leq C\|\bar\partial u\|_{L^\infty}\|\partial u\|_{L^2} \leq C$, since $S(u) = \mathrm{Re}\langle\bar\partial u, \partial u\rangle$
4. $\|S(u(t))\|_{L^p(\Sigma)} \leq C \Rightarrow \lambda(t) = O(T - t)^{\frac{p}{2}}$ by refined energy identity
5. Apply with $p = 2$. □

Contracting $\operatorname{div} S = \langle \mathcal{T}(u), du \rangle$ with the radial vector field $\vec{r}$, one obtains

$$\operatorname{div}(\vec{r} \lrcorner\, S) = \langle \mathcal{T}(u), \vec{r} \lrcorner\, du \rangle. \tag{3}$$

In polar coordinates, we have

$$S = \frac{1}{2} \left(|u_r|^2 - \frac{1}{r^2} |u_\theta|^2 \right) (dr^2 - r^2 d\theta^2) + 2 \langle u_r, u_\theta \rangle dr d\theta.$$

Integrating (3) over a disk D_r and applying the divergence theorem, we obtain

$$\int_{S^1_r} \left(r^2 |u_r|^2 - |u_\theta|^2 \right) \, d\theta = \int_{D_r} \langle \mathcal{T}(u), r\, u_r \rangle \, dV. \tag{4}$$

$\Rightarrow$ basic control over the difference between angular and radial components of du (familiar trick from harmonic maps).

Theorem 2. Suppose the blowup is strictly type-II, *i.e.*

$$\lambda(t) = O(T - t)^{\frac{1+\alpha}{2}}.$$

Then $u(T)$ is $C^{\frac{\alpha}{3}}$.

Proof. For $u : D_1(p) \times [0, T) \to N$, define the *angular energy*

$$f(r,t) := \left(\int_{S^1_r} |u_\theta(r,\theta,t)|^2 d\theta \right)^{\frac{1}{2}}.$$

Direct computation gives

$$\partial_t f - \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1-\eta}{r^2} \right) f \leq 0,$$

where $\eta = C_N \sup r^2 |du|^2$.

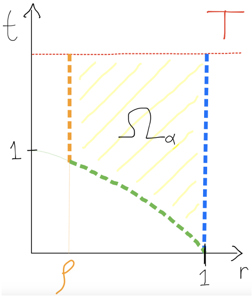
Supersolution for angular energy

Given $T > 1$ and $0 < \rho < 1$, let

$$\Omega_\alpha = \left\{ (r,t) \in [\rho, 1] \times [0, T) \mid r \geq (1-t)_+^{(1+\alpha)/2} \right\}.$$

Let $\nu = \sqrt{1-\eta}$, and choose μ with

$$\frac{1}{1+\alpha} \leq \mu < \nu.$$



Lemma 1

Suppose $f \leq \eta$ on the parabolic boundary of Ω_α. Then

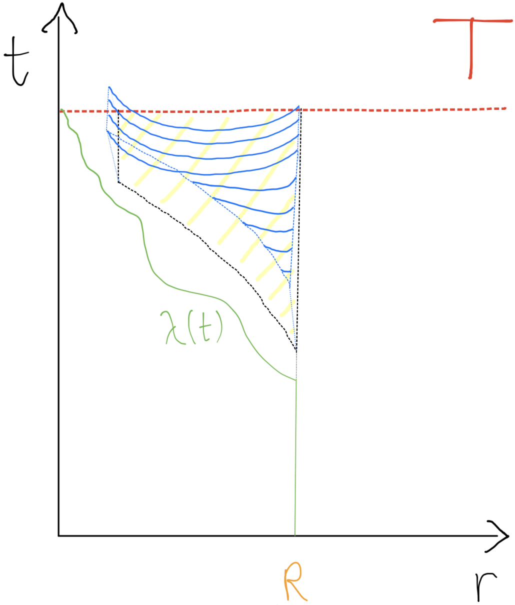
$$f(r,t) \leq C\eta \left(\left(\frac{\rho}{r} \right)^\nu + r^{\min\left[\mu, (1+\alpha)\nu^2 - \nu\right]} \right)$$

for $\rho \leq r \leq 1$ and $1 \leq t < T$.

Proof. Supersolution on Ω_α :

$$\left(\frac{\rho}{r} \right)^\nu + \frac{\left((1-t)_+ + r^{2\nu} \right)^{(1+\alpha)\nu/2}}{r^\nu} + \frac{\nu+1}{\nu^2 - \mu^2} r^\mu. \quad \square$$

Decay of angular energy



Estimate on radial energy

Define

$$g = g(u; r, t) := \sqrt{\int_{S^1_r} r^2 |u_r(r, \theta, t)|^2 d\theta}.$$

Under the flow, this satisfies

$$\left(\partial_t - \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1-\eta}{r^2}\right)\right)\left(\frac{g}{r}\right) \leq \frac{6f}{r^3}. \qquad (5)$$

$\Rightarrow$ weaker decay than f.

Construct inner boundary kernel $G_\rho(r, t)$ such that

$$v_1(r, t) = \int_0^t \psi(\tau) G_\rho(r, t - \tau)\, d\tau$$

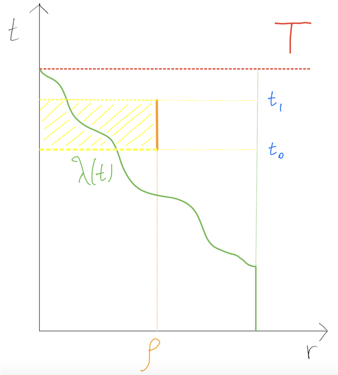
solves (5) with $v_1(\rho, t) = \psi(t)$.

Proposition 3

For $2\rho \leq r \leq 1$ and $t \geq 0$, we have

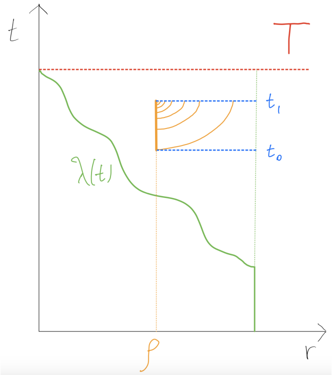
$$|v_1(r, t)| \leq Ce^{-(r-\rho)^2/5t} \left(\frac{\rho^{2\nu}}{r^{2\nu+1}}\right) \sqrt{\int_0^t \psi^2(\tau) d\tau}.$$

Decay of radial energy



Step 1. Use stress-energy identity (4) to bound $\int_{t_0}^{t_1} \left(f^2(\rho,t) - g^2(\rho,t)\right)\,dt \Rightarrow$ bound on $\int_{t_0}^{t_1} g^2(\rho,t)\,dt$ (since f decays).

Decay of radial energy



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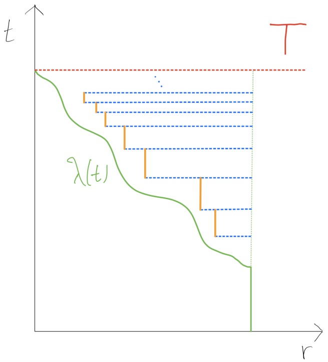
Step 2. Use Proposition 3 to convert this time-integral bound to spatial decay of $g(r, t_1)$.

Decay of radial energy



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Step 3. Bootstrap. □

Thank you!