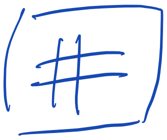


$\partial_i \quad \partial_j$ $\qquad$ $g_{ij} = \langle \partial_i, \partial_j \rangle$

M $\qquad$ $g_{ij}(t)$. given by people.

$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} (\cdots)$

$R_{ijkl} = \cdots$ $\qquad$ will change.

$h \triangleq \partial g_t / \partial t$

Prop 2.3.1. X, Y, Z fixed vector field. but g will change, so ∇ change.

$$\langle \tfrac{\partial}{\partial t} \nabla_X Y, Z \rangle = \tfrac{1}{2} \left[(\nabla_Y h)(X,Z) + (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y) \right]$$

$\triangleq \Pi(X,Y)$

$\frac{\partial}{\partial t}$ not act on g

Proof.

$$LHS = \tfrac{\partial}{\partial t} g(\nabla_X Y, Z) - h(\nabla_X Y, Z)$$

$g = g_{ij}\, dx^i \otimes dx^j$ $\qquad$ g_{ij} — basis

if g_{ij} invariant under t, dx^i, dx^j don't change

$\partial_t g = \underbrace{\partial_t g_{ij}}_{\triangleq h_{ij}} dx^i \otimes dx^j = h$

$\underbrace{h_{ij}}\, dx^i \otimes dx^j$

$$= \tfrac{\partial}{\partial t} [X g(Y,Z) - g(Y, \nabla_X Z)] - h(\nabla_X Y, Z)$$

$\nabla_X g(Y,Z) = 0$ $\quad$ since $\nabla g = 0$

$\qquad \|$

$X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$

$\langle Y, Z \rangle = g(Y,Z)$

$$= \underline{X h(Y,Z)} - h(Y, \nabla_X Z) - g(Y, \tfrac{\partial}{\partial t} \nabla_X Z) \underline{- h(\nabla_X Y, Z)}$$

$\nabla_X h(Y,Z) = \underline{X(h(Y,Z))} \underbrace{- h(\nabla_X Y, Z)} \underline{- h(Y, \nabla_X Z)}$

$$= (\nabla_X h)(Y,Z) - \langle Y, \tfrac{\partial}{\partial t} \nabla_X Z \rangle$$

$$= (\nabla_X h)(Y,Z) - \langle \Pi(X,Z), Y \rangle$$

$\nabla_X Z - \nabla_Z X = [X, Z]$ $\to$ Lie bracket. no relation to t

$\frac{\partial}{\partial t} \nabla_X Z = \frac{\partial}{\partial t} \nabla_Z X$

$$= (\nabla_X h)(Y,Z) - \langle \Pi(Z,X), Y \rangle$$

cycle. $\begin{cases} \langle \Pi(X,Y), Z \rangle + \langle \Pi(Z,X), Y \rangle = (\nabla_X h)(Y,Z) \\ \langle \Pi(Y,Z), X \rangle + \langle \Pi(X,Y), Z \rangle = (\nabla_Y h)(Z,X) \\ \langle \Pi(Z,X), Y \rangle + \langle \Pi(Y,Z), X \rangle = (\nabla_Z h)(X,Y) \end{cases}$

① − ③ $\qquad \langle \Pi(X,Y), Z \rangle \underbrace{- \langle \Pi(Y,Z), X \rangle}_{\text{sub by ②}} = (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y)$

$\qquad \|$

$(\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y) + (\nabla_Y h)(Z,X) - \langle \Pi(X,Y), Z\rangle$

divided by 2 ✓

connection exists. #

Rmk 2.3.2 $g(t)$ $\quad v(t)$ both t-dependent

$$\frac{\partial}{\partial t}\nabla_X V = \underbrace{\Pi(X,V)}_{\dot\nabla_X V} + \nabla_X \frac{dV}{dt}$$

two derivative

Proof. $\nabla_X V = X^i V^j \Gamma_{ij}^k \frac{\partial}{\partial k} + X^i \frac{\partial V^j}{\partial x_i}\partial_j$

$$\partial_t(\nabla_X V) = X^i \underbrace{\frac{\partial V^i}{\partial t}}_{\dot V}\Gamma_{ij}^k \partial_k + X^i V^j \underbrace{\frac{\partial \Gamma_{ij}^k}{\partial t}}_{\dot\nabla}\partial_k + X^i \frac{\partial}{\partial x_i}\left(\frac{\partial V^j}{\partial t}\right)\partial_j$$

$$= \nabla_X\left(\frac{\partial V}{\partial t}\right) + (\partial_t \nabla)_X V$$

$\approx \Pi(X,V)$

$$\nabla_X \frac{\partial V}{\partial t} = X^i \nabla_{\partial_i}\left(\frac{\partial V^j}{\partial t}\partial_j\right) = X^i \frac{\partial V^j}{\partial t}\Gamma_{ij}^k \partial_k + X^i \partial_{x_i}\left(\frac{\partial V^j}{\partial t}\right)\partial_j$$

Rmk 2.3.3. $\partial_t(\nabla Y) = Y * \nabla h$ $\quad$ tr. $\quad$ (*) before. we have

Proof. By Prop 2.3.1

$$\langle Z, \Pi(X,Y)\rangle = \frac{1}{2}\left[(\nabla_Y h)(X,Z) + (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y)\right]$$

Assume $X = x^i\partial_i$ $\quad Y = y^i\partial_i$ $\quad Z = z^i\partial_i$

$\nabla_X Y$ not tensor (nolinear) $\longrightarrow$ $\Pi(X,Y)$ is kill nonlinear term

$\dot\nabla_a Y$

derivative of Y along a, choose component on b

inner product is linear

$$\langle Z, \Pi(X,Y)\rangle = g_{bc}\, X^a \underbrace{z^c}_{\partial_c} \underbrace{(\Pi(a,Y))^b}_{\partial_b}$$

$$= \frac{1}{2}\left[y^a (\nabla_a \underset{\text{2-tensor}}{h})_{ij}\, x^i z^j + x^a(\nabla_a h)_{ij}\, y^i z^j - z^a(\nabla_a h)_{ij}\, x^i y^j\right]$$

$\forall X, Z$ $\quad$ derivative is also 2-tensor

↳ disappear $\quad X = (1,0,\cdots 0)$ $\quad Y = (0,1,\cdots)$

$$\text{So } g_{bc}(\Pi(a,Y))^b = \frac{1}{2}\left[Y^b(\nabla_b h)_{ac} + Y^b(\nabla_a h)_{bc} - Y^b(\nabla_c h)_{ab}\right]$$

$$(\Pi(a,Y))^d = \underbrace{\frac{1}{2} g^{dc}\left[Y^b(\nabla_b h)_{ac}\right.} + Y^b(\nabla_a h)_{bc} - \left.Y^b(\nabla_c h)_{ab}\right]$$

$i \to a,\ j \to c$

$= Y * \nabla h$

$g^{dc}\, Y^b\, (\nabla_b h)_{ac}$ $\quad$ tr

$$= (Y^b\partial_b) \otimes (\nabla h)_{ijk}\, \overset{b}{dx^i} \otimes \overset{a}{dx^j} \otimes \overset{c}{dx^k}$$

tr

If $\omega \in \Gamma(T^*M)$ 1-form independent of t

$\underbrace{\left(\frac{\partial}{\partial t} \nabla_X \omega\right)(Y)} = \frac{\partial}{\partial t}\left(X(\omega(Y))\right) - \underbrace{\omega\left(\frac{\partial}{\partial t} \nabla_X Y\right)}$ ✓

$\downarrow$ independent of t

$\|$

$\partial_t X(\omega(Y))$

$-\omega(\partial_t \nabla_X Y)$

$dx^b(z^a \partial_a) = z^b$

$\omega = \omega_b dx^b$

- $\partial_t \nabla \omega = \omega * \nabla h$

by $X = x^a \partial_a$ $\quad Y = y^b \partial_b$

by linearity $x^a Y^b \underbrace{\left(\frac{\partial}{\partial t} \nabla_a \omega\right)_b} = -x^a \omega_b \left(\frac{\partial}{\partial t} \nabla_a Y\right)^b = -x^a \omega_b (Y * \nabla h)^b_a$

$\frac{\partial}{\partial t}\omega$. still 1-form

$\uparrow$

$\left(\frac{\partial}{\partial t} \nabla_a \omega\right)_i dx^i (Y^b \partial_b)$

$i=b$

X ~~ $\quad Y = (0 \cdots 0, 1, 0 \cdots 0)$

$\partial_i \cdots \otimes \cdots$

$\nabla_X (A \otimes B) = (\nabla_X A) \otimes B + A \otimes (\nabla_X B)$ (property.)

$\frac{\partial}{\partial t} \nabla A = A * \nabla h$

$A = A^{1 \cdots k}_{\cdots} \quad \omega_1 \otimes \cdots \otimes \omega_k \otimes X_1 \cdots$

$\omega * \nabla h$

2.3.3. if $A = A(t)$, $\quad \partial_t(\nabla A) - \nabla(\partial_t A) = A * \nabla h$

Prop 2.3.4 (curvature) $\quad \frac{\partial}{\partial t}\left(R(X,Y)W\right) = (\nabla_Y \pi)(X,W) - (\nabla_X \pi)(Y,W)$

$R \sim g$ $\quad\quad \|$

$\nabla^2_{Y,X} W = \nabla_Y \nabla_X W - \nabla_{\nabla_Y X} W$ $\quad\quad LHS = \frac{\partial}{\partial t}\left[(\nabla_Y \nabla)(X,W) - (\nabla_X \nabla)(Y,W)\right]$

Proof. $R(X,Y)W = \nabla_Y \nabla_X W - \nabla_X \nabla_Y W + \nabla_{[X,Y]} W$

$\frac{\partial}{\partial t}[R(X,Y)W] = \frac{\partial}{\partial t}\nabla_Y \nabla_X W + \underline{\nabla_Y \frac{\partial}{\partial t} \nabla_X W} - \underline{\frac{\partial}{\partial t}\nabla_X \nabla_Y W} - \nabla_X \frac{\partial}{\partial t} \nabla_Y W + \frac{\partial}{\partial t} \nabla_{[X,Y]} W$

formula.

e.g. $(\nabla_Y \frac{\partial}{\partial t} \nabla)(X,W) = \underline{\nabla_Y\left(\frac{\partial}{\partial t} \nabla_X W\right)} - \frac{\partial}{\partial t} \nabla(\nabla_Y X, W) - \underline{\frac{\partial}{\partial t} \nabla(X, \nabla_Y W)}$

tensor drivative $\quad\quad \frac{\partial}{\partial t} \nabla_X \nabla_Y W$

$(\nabla_Y \pi)(X,W)$. $\quad\quad X(g(Y,Z)) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

substituation $\Rightarrow$ $LHS = (\nabla_Y \frac{\partial}{\partial t}\nabla)(X,W) + \frac{\partial}{\partial t}\nabla(\nabla_Y X, W) - (\nabla_X \frac{\partial}{\partial t}\nabla)(Y,W) - \frac{\partial}{\partial t}\nabla(\nabla_X Y, W)$

$\frac{\partial}{\partial t}\nabla(\nabla_Y X, W) = \frac{\partial}{\partial t}\nabla_{\nabla_Y X} W$, $\quad - \frac{\partial}{\partial t}\nabla(\nabla_X Y, W) = \frac{\partial}{\partial t}\nabla_{\nabla_X Y} W$, $\quad + \frac{\partial}{\partial t}\nabla_{[X,Y]} W$

$$= (\nabla_Y \tfrac{\partial}{\partial t}\nabla)(X,W) - (\nabla_X \tfrac{\partial}{\partial t}\nabla)(Y,W) + \tfrac{\partial}{\partial t}(\underbrace{\nabla_Y X - \nabla_X Y + [X,Y]}_{=0}, W)$$

$$= \nabla_Y \pi(X,W) - \nabla_X \pi(Y,W).$$

#

Prop 2.3.5 full curvature tensor $\to Rm$. $\quad h$

$$\frac{\partial}{\partial t} Rm(X,Y,W,Z) = \frac{1}{2}\left[h(R(X,Y)W,Z) - h(R(X,Y)Z,W)\right]$$
$$+ \frac{1}{2}\left[\nabla^2_{Y,W} h(X,Z) - \nabla^2_{X,W} h(Y,Z) + \nabla^2_{X,Z} h(Y,W) - \nabla^2_{Y,Z} h(X,W)\right]$$

Proof. choose normal coor. $\boxed{\text{near } p}$

i.e. $g_{ij} = \delta_{ij}$ $\quad \Gamma_{ij}^k = 0$ $\quad$ so $\nabla X|_p = \nabla Y|_p = \cdots = 0$ $\quad \nabla_X Y|_p = 0$

$$\frac{\partial}{\partial t} Rm(X,Y,W,Z) := \frac{\partial}{\partial t}\langle R(X,Y)W, Z\rangle = h(R(X,Y)W,Z) + \langle \partial_t(R(X,Y)W), Z\rangle$$

$$\overset{2.3.4}{=} h(R(X,Y)W,Z) + \langle (\nabla_Y \pi)(X,W) - (\nabla_X \pi)(Y,W), Z\rangle$$

proof of ! LC-∇

property of R. m. $\quad Y\langle \pi(X,W), Z\rangle = \boxed{\langle \nabla_Y(\pi(X,W)), Z\rangle} + \underbrace{\langle \pi(X,W), \nabla_Y Z\rangle}_{=0}$ near p

chain rule

$\langle (\nabla_Y \pi)(X,W), Z\rangle$

$$= \langle \nabla_Y(\pi(X,W)), Z\rangle - \langle \pi(\underbrace{\nabla_Y X}_{=0}, W), Z\rangle - \langle \pi(X, \underbrace{\nabla_Y W}_{=0}), Z\rangle$$

$$= \langle \nabla_Y(\pi(X,W)), Z\rangle$$

So $\langle (\nabla_Y \pi)(X,W), Z\rangle = Y\langle \pi(X,W), Z\rangle$

by prop 2.3.1. $\quad \langle \pi(X,Y), Z\rangle = \frac{1}{2}[(\nabla_Y h)(X,Z) \cdots$

$$= \frac{1}{2} Y[(\nabla_W h)(X,Z) + (\nabla_X h)(W,Z) - (\nabla_Z h)(X,W)]$$

$$(\nabla_Y(\nabla_W h))(X,Z) \overset{\text{formula}}{=} \underbrace{Y[(\nabla_W h)(X,Z)]}_{} - (\nabla_W h)(\nabla_Y X, Z) - (\nabla_W h)(X, \nabla_Y Z) \quad \checkmark$$

$$= \frac{1}{2}[(\nabla_Y \nabla_W h)(X,Z) + (\nabla_Y \nabla_X h)(W,Z) - (\nabla_Y \nabla_Z h)(X,W)]$$

$$\nabla^2_{Y,W} h = \nabla_Y \nabla_W h - \nabla_{\underbrace{\nabla_Y W}_{=0}} h$$

$$= \frac{1}{2}\left[(\nabla^2_{Y,W}h)(X,Z) + (\nabla^2_{Y,X}h)(W,Z) - (\nabla^2_{Y,Z}h)(X,W)\right]$$

$$\partial_t \langle R(X,Y)W, Z\rangle = h(R(X,Y)W,Z) + \frac{1}{2}\left[(\nabla^2_{Y,W}h)(X,Z) - (\nabla^2_{X,W}h)(Y,Z) + \cdots + \cdots\right]$$

change one

(2.3.5): $R(\cdots)(h(W,Z))$

$$-(\nabla^2_{X,Y}h)(W,Z) + (\nabla^2_{Y,X}h)(W,Z) = [R(X,Y)h](W,Z)$$ Ricci identity (2.1.2)

$$= -h(R(X,Y)W,Z) - h(W,R(X,Y),Z)$$

finally get prop 2.3.5.

#

Prop 2.3.6 $\alpha(t) \in \Gamma(\otimes^2 T^*M)$ $\quad \partial_t(\mathrm{tr}\,\alpha) = -\langle h,\alpha\rangle + \mathrm{tr}\frac{\partial\alpha}{\partial t}$

inner product on M can extend to tensor field. $\quad$ if $h = \partial_t g = h_{ij}\,dx^i\otimes dx^j$

$\alpha = \alpha_{ab}\,dx^a\otimes dx^b$

$\langle h,\alpha\rangle = h_{ij}\,\alpha_{ab}\,g^{ia}g^{jb}$

Proof. $\alpha = \alpha_{ij}\,dx^i\otimes dx^j$

$\partial_t g_{ij} = h_{ij} \Rightarrow \partial_t g^{ij} = -h^{ij} \triangleq -h_{kl}\,g^{ik}g^{jl}$

ago. $\partial_t(\mathrm{tr}\,\alpha) = \partial_t(g^{ij}\alpha_{ij})$

$$= \partial_t g^{ij}\alpha_{ij} + g^{ij}\partial_t\alpha_{ij} = -h^{ij}\alpha_{ij} + g^{ij}\frac{\partial\alpha_{ij}}{\partial t}$$

$$= -\langle h,\alpha\rangle + \mathrm{tr}\left(\frac{\partial\alpha}{\partial t}\right)$$

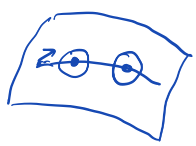
#

Rmk 2.3.8 $\mathrm{Ric}(X,Y) \triangleq \langle \mathrm{Ric}(X), Y\rangle \qquad \mathrm{Ric}(X) \triangleq [\mathrm{Ric}(X,\cdot)]^\#$

$\mathcal{L}_X Y = [X,Y]$

flow of X. $\begin{cases} \frac{d}{dt}P(t,P_0) = X(P(t,P_0)) \\ P(0,P_0) = P_0 \end{cases}$ $\qquad (\mathcal{L}_X f)(P) = \lim\limits_{t\to 0}\frac{f(P(t,P_0)) - f(P_0)}{t}$

$\mathcal{L}_X f = \nabla_X f$



different tangent space

assume X's local flow is $T_X^t : M \to M$, this is a local diffeomorphism $\qquad T_X^0$ identity map

$d_{P_0}T_X^t : T_{P_0}M \to T_{T_X^t(P_0)}M$ $\quad$ isomorphism. $\Rightarrow \exists$ inverse. $\quad$ can define lim...

pull back. $(T_X^t)^*_{P_0} : T^*_{T_X^t(P)}M \to T_P^*M$

so can define $\mathcal{L}_X\omega \qquad \omega\in T^*M$

For tensor $h_{ij}^{ab}\, dx^i \otimes dx^j \otimes \partial_a \otimes \partial_b$ define $\mathcal{L}_Y$ s.t. satisfy $\mathcal{L}_Y(S\otimes T) = \mathcal{L}_Y S \otimes T + S \otimes (\mathcal{L}_Y T)$

How to compute:

$(\mathcal{L}_X T)(Y_1 \cdots Y_n) = \mathcal{L}_X(T(Y_1, \cdots Y_n)) - T(\mathcal{L}_X Y_1, \cdots) \cdots$ similar to $(\nabla_X T)(Y_1 \cdots Y_n)$

property. $[\mathcal{L}_X, d] = 0$ $\quad d(h_{ij}\, dx^i \otimes dx^j) = \frac{\partial h_{ij}}{\partial x^a} dx^a \otimes dx^i \otimes dx^j$

$(\mathcal{L}_X Y)(f) = \mathcal{L}_X(df(Y)) = df(\mathcal{L}_X Y) = df([X,Y]) = [X,Y]Y$

Prop 2.3.7 $\frac{\partial}{\partial t} Ric = -\frac{1}{2}\Delta_L h - \frac{1}{2}\mathcal{L}_{(\delta G(h))^\#}\, g$

$$\partial_t [Ric(X,W)] = \partial_t[tr\, Rm(X,\cdot,W,\cdot)] \overset{\text{prop 2.3.6}}{=} -\langle h, Rm(X,\cdot,W,\cdot)\rangle + tr(\underbrace{\partial_t Rm(X,\cdot,W,\cdot)})$$

last term computed before. $\partial_t Rm(X,Y,W,Z) = \frac{1}{2}[h(R(X,Y)W,Z) - h(R(X,Y)Z,W)] + \frac{1}{2}[\nabla^2_{Y,W} h(X,Z)$
$- \nabla^2_{X,W} h(Y,Z) + \underbrace{\nabla^2_{X,Z} h(Y,W)} - \nabla^2_{Y,Z} h(X,W).]$

(2.3.5). $-\nabla^2_{X,Y} h(W,Z) + \nabla^2_{Y,X} h(W,Z) = -h(R(X,Y)W,Z) - h(R(X,Y),Z,W)$

NTU $\quad tr\, \nabla^2_{X,\cdot} h(\cdot, W) = -(\nabla \delta h)(X,W)$ $\quad$ (no proof).

LHS $= g^{ij} \nabla^2_{X,\partial_i} h(\partial_j, W)$

$\delta(T) = -tr_{12}(\nabla T)$ $\quad$ again $\nabla_X g = 0$

RHS $= -[\nabla_X(\delta h)]W = [\nabla_X(tr_{12} \nabla h)]W = [\nabla_X(g^{ij} \nabla h(\partial_i, \partial_j, \cdot)]W$

$= g^{ij}[\nabla_X \nabla_{\partial_i} h(\partial_j, \cdot) - \nabla_{\nabla_X \partial_i} h(\partial_j, \cdot) - \nabla_{\partial_i} h(\nabla_X \partial_j, \cdot)]W$

$= g^{ij} \nabla^2_{X,\partial_i} h(\partial_j, W) - g^{ij} \nabla_{\partial_i} h(\underbrace{\nabla_X \partial_j}, W)$

equal.

disappear by using normal coor.

$(\partial_1 \cdots \partial_n)\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn}\end{pmatrix}\begin{pmatrix}\partial_1 \\ \vdots \\ \partial_n\end{pmatrix}$

$\partial_i g_{ij} \partial_j$

$g^{ij} \partial_i \partial_j$

similarly $\quad tr\, \nabla^2_{W,\cdot} h(X,\cdot) = -(\nabla \delta h)(W,X)$

also, second $\quad tr\, \nabla^2_{X,W} h(\cdot,\cdot) = \nabla^2_{X,W}(tr\, h)$

|| commutative

$\nabla^2_{X,W}\, tr\, h(\cdot,\cdot) = Hess(tr\, h)(X,W)$

$Hess(f) = \nabla(df)$ and $(\nabla(df))(X,W) = [\nabla_X(df)]W = X[df(W)] - df(\nabla_X W) = X(W(f)) - (\nabla_X W)(f)$
$= \nabla^2_{X,W} f$

✓

defn of Laplacian $\Rightarrow$ $tr\, \nabla^2_{\cdot,\cdot} h(X,W) = (\Delta h)(X,W)$ $\quad$ since $tr\, h$ is a function.

$\#.$ $\cdot g_{ij}$ under general basis $\cdot e_i$ or. b.

since $Ric(X) \triangleq (Ric(X,\cdot))^{\#}$
$= Ric(X, e_i) e_i$

(2.3.12) in book. $tr\, h(R(X,\cdot)W,\cdot) = \langle Rm(X,\cdot,W,\cdot), h\rangle$

Now prove this. use orthonormal frame $\{e_i\}$

$$h(X, Ric(W))$$

$$tr\, h(X,\cdot) \otimes Ric(W,\cdot) = g^{ij}\, \underbrace{h(X,e_i)\, Ric(W,e_j)} = h(X,e_i)\, \underbrace{Ric(W,e_i)}_{\text{since this is number}} = h(X, Ric(W,e_i)e_i)$$

So $h(X, Ric(W))$

$$= \langle h(X,\cdot), Ric(W,\cdot)\rangle = tr\, h(X,\cdot)\otimes Ric(W,\cdot) = h(X, Ric(W,e_i)e_i)$$
$$= -tr\, h(R(W,\cdot)\cdot, X) = tr\, h(R(X,\cdot)W,\cdot)$$
$$= \langle Rm(X,\cdot,W,\cdot), h\rangle$$

Combine: $\partial_t Ric(X,W) = -\frac{1}{2} tr[h(R(X,\cdot)W,\cdot) + h(R(X,\cdot)\cdot, W) + h(R(W,\cdot)X,\cdot) + h(R(W,\cdot)\cdot, X)]$
$-\frac{1}{2}[(\nabla\delta h)(X,W) + Hess(tr\, h)(X,W)$
$+ (\nabla\delta h)(W,X) + (\Delta h)(X,W)]$

$h(R(X,\cdot)W,\cdot)$

Use (2.3.12)

$$\partial_t Ric(X,W) = -tr\, h(R(X,\cdot)W,\cdot) + \frac{1}{2}[h(W, Ric(X)) + h(X, Ric(W))]$$

by (2.3.1) $h(X, Ric(W)) = -tr\, h(R(W,\cdot)\cdot, X)$

$$-\frac{1}{2}[\mathcal{L}_{(\delta h)^{\#}} g(X,W) + Hess(tr\, h)(X,W) + (\Delta h)(X,W)]$$

(2.3.8) $\mathcal{L}_{(\delta h)^{\#}} g(X,W) = (\nabla\delta h)(X,W) + (\nabla\delta h)(W,X)$

Now since $\mathcal{L}_{(\delta G(h))^{\#}} g = \mathcal{L}_{(\delta h)^{\#}} g + Hess(tr\, h)$

So

$$\partial_t Ric(X,W) = \underbrace{-tr\, h(R(X,\cdot)W,\cdot) + \frac{1}{2}[h(W,Ric(X)) + h(X,Ric(W))] - \frac{1}{2}(\Delta h)(X,W)}_{\text{by defn, } = -\frac{1}{2}\Delta_L h} - \frac{1}{2}\mathcal{L}_{(\delta G(h))^{\#}} g$$

Δ_L. Lichnerowicz Laplacian $\#$

Prop 2.3.9 $\frac{\partial}{\partial t} R = -\langle Ric, h\rangle + \delta^2 h - \Delta(tr\, h)$

Proof. $\partial_t R = \partial_t(tr\, Ric) = -\langle h, Ric\rangle + tr(\partial_t Ric)$

from prop 2.3.7, $\partial_t Ric = -\frac{1}{2}\Delta_L h - \frac{1}{2}\mathcal{L}_{(\delta G(h))^{\#}} g$

(2.3.10) $\mathcal{L}_{(\delta G(h))^{\#}} g = \mathcal{L}_{(\delta h)^{\#}} g + Hess(tr\, h)$

$$\Rightarrow \partial_t R = -\langle h, Ric\rangle - \frac{1}{2} tr(\Delta_L h) - \frac{1}{2} tr(\mathcal{L}_{(\delta G(h))^{\#}} g)$$
$$= -\langle h, Ric\rangle - \frac{1}{2} tr(\Delta_L h) - \frac{1}{2} tr(\mathcal{L}_{(\delta h)^{\#}} g) - \frac{1}{2} tr[Hess(tr\, h)]$$

where $tr(\Delta_L h) = tr(\Delta h) - \langle h, Ric\rangle - \langle h, Ric\rangle + 2\langle h, Ric\rangle = \Delta(tr\, h)$

separately (2.3.11) (2.3.12)

$tr[h(\cdot, Ric(\cdot))]$

(2.3.8) $\mathcal{L}_{\omega^{\#}} g(X,V) = \nabla\omega(X,V) + \nabla\omega(V,X)$

$tr[\mathcal{L}_{(\delta h)^\#} g] = tr[(\nabla \delta h)(\cdot,\cdot) + (\nabla \delta h)(\cdot,\cdot)] = 2\,tr(\nabla \delta h) = -2\delta^2 h$

since $\delta(T) = -tr_{12}(\nabla T)$

Combine $\frac{\partial}{\partial t} R = -\langle h, Ric\rangle + \delta^2 h - \Delta(tr h)$

$-\frac{1}{2} tr \Delta_L h = -\frac{1}{2}\Delta(tr h)$

$-\frac{1}{2} tr\, Hess(tr h) = -\frac{1}{2} tr[tr_{12} \nabla^2_{(\cdot,\cdot)} h(1,2)]$

$= -\frac{1}{2}\Delta(tr h)$

Prop 2.3.10 $\omega(t) \in \Gamma(T^*M)$, $\frac{\partial}{\partial t}\delta\omega = \delta \frac{\partial\omega}{\partial t} + \langle h, \nabla\omega\rangle - \langle \delta G(h), \omega\rangle$

Proof. Divergence Thm $\Rightarrow$ fix 1-form α , $\int (\delta\alpha)\, dV = 0$.

since $\delta: (-1)^{n(p+1)+1} * d *$ on p-form so $(\delta\alpha)dV = (\pm * d * \alpha)\, dV$

$= \pm d(*\alpha)$

So we can Integral by Parts . $f: M \to \mathbb{R}$

$\delta(f\alpha) = -\langle df, \alpha\rangle + f(\delta\alpha)$ $\quad f\alpha$ is also 1-form $\quad \int \delta(f\alpha) dV = 0$

$\Rightarrow$ (2.3.19) $\int \langle df, \alpha\rangle\, dV = \int f(\delta\alpha)\, dV$

diff: $\partial_t \int \langle df, \omega\rangle\, dV = \partial_t \int f(\delta\omega)\, dV$

$\partial_t g_{ij} = h_{ij} \Rightarrow \partial_t g^{ij} = -h^{ij}$

LHS $= \int -h(df, \omega)\, dV + \int \langle df, \partial_t \omega\rangle dV + \int \langle df, \omega\rangle \frac{1}{2}(tr h)\, dV$

$\partial_t g^{ij} \frac{\partial f}{\partial x_i}\alpha$

equal

assume prop 2.3.12

$\partial_t dV = \frac{1}{2}(tr h)\, dV$

RHS $= \partial_t \int f(\delta\omega)\, dV$

$= \int \partial_t(\delta\omega) f\, dV + \int (\delta\omega) f \frac{tr h}{2}\, dV$

NTU $\quad \delta(f\omega) = -\langle df, \omega\rangle + f(\delta\omega)$ (2.3.18)

$\int \langle df, \omega\rangle dV = \int f(\delta\omega)\, dV$ (2.3.19)

So $\int \frac{\partial}{\partial t}(\delta\omega) f\, dV = -\int h(df, \omega)\, dV + \int \langle df, \frac{\partial\omega}{\partial t}\rangle\, dV - \int [(\delta\omega) f - \langle df, \omega\rangle] \frac{1}{2}(tr h)\, dV$

move

$\int \frac{\partial}{\partial t}(\delta\omega) f\, dV + \int h(df,\omega)\, dV - \int \langle df, \frac{\partial\omega}{\partial t}\rangle\, dV + \int [(\delta\omega) f - \langle df, \omega\rangle] \frac{1}{2}(tr h)\, dV = 0$

(2.3.18)

$= \int \frac{\partial}{\partial t}(\delta\omega) f\, dV + \int \langle df, h(\omega,\cdot)\rangle\, dV - \int f(\delta \partial_t \omega)\, dV + \int [\delta(f\omega)] \frac{1}{2}(tr h)\, dV$

(2.3.19) $\quad (\delta\omega) f + \delta(f\omega) - \cancel{f(\delta\omega)}$

$df = \frac{\partial f}{\partial x_i} dx^i \quad \omega = \omega_i dx^i$

$h(df, \omega) = h^{ij} \frac{\partial f}{\partial x_i} \omega_j$

$h(\omega, \cdot) = \omega_i h(dx^i, \cdot) = \omega_i h^i_j dx^j$

$\langle df, h(\omega,\cdot)\rangle = \langle \frac{\partial f}{\partial x_i} dx^i, \omega_i h^i_j dx^j\rangle = g^{ij} \frac{\partial f}{\partial x_i} \omega_k h^k_j = \frac{\partial f}{\partial x_i} \omega_k h^{ki}$

formal adjoint

$= \int \partial_t(\delta\omega) f\, dV + \int f \delta(h(\omega,\cdot))\, dV - \int f(\delta \frac{\partial\omega}{\partial t})\, dV + \int \langle d(\frac{tr h}{2}), \omega f\rangle\, dV$

formal adjoint

$= \int [\partial_t(\delta\omega) + (\delta h, \omega) - \langle h, \nabla\omega\rangle - \delta\frac{\partial\omega}{\partial t} + \langle d(\frac{tr h}{2}), \omega\rangle] f\, dV$

Now prove $\delta(h(\omega,\cdot)) = \langle \delta h, \omega\rangle - \langle h, \nabla\omega\rangle$

T is k-form

$$(\delta T)_{a_2\cdots a_k} = -(tr_{12}\nabla T)(\overset{a_1}{\overset{\shortparallel}{\cdot}}, \overset{a_1}{\overset{\shortparallel}{\cdot}}, a_2, \cdots a_k) = -tr(\nabla_{a_1} T)(a_1, a_2, \cdots a_k) = -g^{ij}(\nabla_i T)(\partial_j\, a_2 \cdots a_k)$$
$$= -\nabla^{a_1} T_{a_1 a_2 \cdots a_k}$$

$\omega = \omega_i\, dx^i$

$h(\omega,\cdot) = \omega_i h^i_j\, dx^j = h^b_a \omega_b\, dx^a$ $\qquad dx^j(\partial_b) = \delta^j_b$

$$\delta(h,\omega,\cdot)) = -tr_{12}\nabla[\omega_i h^i_j dx^j] = -g^{ab}[\nabla_a(\omega_i h^i_j dx^j)]\partial_b = -g^{ab}[\partial_a(\omega_i h^i_b) - \omega_j h^j_i \Gamma^i_{ab}]$$
$$= -g^{ab}[\partial_a\omega_i h^i_b + \omega_i \partial_a h^i_b - \omega_i h^i_j \Gamma^j_{ab}]$$

提,降指标;

Lebiniz rule

$$\text{So } -\delta(h(\omega,\cdot)) = \nabla^a[h(\omega,\cdot)]_a = \nabla(h^b_a\omega_b) = (\nabla^a h^b_a)\omega_b + h^b_a \nabla^a\omega_b$$
$$= g^{bc}(\nabla^a h_{ab})\omega_c + g^{ac}g^{bd}\, h_{ab}\nabla_c\omega_d$$

defn of δ
$$= \langle -\delta h, \omega\rangle + \langle h, \nabla\omega\rangle$$

Since $\forall f$, Original $\Rightarrow$ $\partial_t(\delta\omega) = \delta\frac{\partial\omega}{\partial t} + \langle h, \nabla\omega\rangle - \langle\delta h, \omega\rangle - \langle d(\frac{tr h}{2}), \omega\rangle$

$$= \delta\frac{\partial\omega}{\partial t} + \langle h, \nabla\omega\rangle - \langle\delta G(h), \omega\rangle.$$

defn of tr

#

Short time existence for Ricci flow

Prop 2.3.11 $T \in \Gamma(Sym^2 T^*M)$ independent of t, then

$$\left(\frac{\partial}{\partial t}\delta G(T)\right) z = -T((\delta G(h))^{\#}, z) + \langle h, \nabla T(\cdot,\cdot,z) - \tfrac12 \nabla_z T\rangle$$

Prop. 2.3.12 $dV = *1 = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$

$$\frac{\partial}{\partial t} dV = \frac{1}{2}(\operatorname{tr} h)\, dV$$

generally, $\frac{d}{dt}\det(A(t)) = \det(A(t)) \cdot \operatorname{tr}\left(A(t)^{-1} \frac{dA(t)}{dt}\right)$

Proof. Since $\frac{d}{dt} \ln \det(A(t)) = \operatorname{tr}\left(A(t)^{-1} \frac{dA(t)}{dt}\right)$

$$\Rightarrow \frac{d}{dt} \ln \det(g_{ij}) = \operatorname{tr}\left(g^{ij} \frac{dg_{ij}}{dt}\right) = \operatorname{tr}(g^{ij} h_{ij})$$

$$= (\det(g_{ij}))^{-1} \cdot \frac{d}{dt}\det(g_{ij}) =$$

$$\Rightarrow \frac{\partial}{\partial t} dV = \frac{1}{2}\sqrt{\det(g_{ij})} \cdot \operatorname{tr}(g^{ij} h_{ij}) = \frac{1}{2}(\operatorname{tr} h)\, dV.$$

#

Laplacian of Curvature tensor. $\rightsquigarrow$ second derivative of Ricci tensor

Defn. Tensor $B \in \Gamma(\otimes^4 T^*M)$, $B(X,Y,W,Z) = \langle Rm(X,\cdot,Y,\cdot), Rm(W,\cdot,Z,\cdot)\rangle$

since $Rm(X,Y,W,Z) = Rm(W,Z,X,Y)$

So $B(X,Y,W,Z) = B(W,Z,X,Y) = B(Y,X,Z,W)$

Proof. Use the second Bianchi identity $\nabla_i R_{jkla} + \nabla_j R_{kila} + \nabla_k R_{ijla} = 0$ $\quad (\nabla_i R_{jkla} = \nabla_{\partial x_i} Rm(\partial x^j, \partial x^k, \partial x^l, \partial x^a))$

(2.4.1) under normal basis $\underbrace{\nabla_i \nabla_i}_{=\Delta \text{ since } g^{ii}=1} R_{jkla} + \nabla_i \nabla_j R_{kila} + \nabla_i \nabla_k R_{ijla} = 0$

Ricci (2.1.2) $-(\nabla^2_{X,Y} A)(W,Z,\cdots) + (\nabla^2_{Y,X} A)(W,Z,\cdots) = -A(R(X,Y),W,Z,\cdots) - A(W, R(X,Y)Z,\cdots) - \cdots$

So $\nabla_i \nabla_j \underbrace{R_{kila}}_{= Rm(\partial_k,\partial_i,\partial_l,\partial_a)} - \nabla_j \nabla_i R_{kila}$ $\quad (X,Y = i,j)$

$= -Rm(R(\partial_j \partial_i)\partial_k, \cdots) - Rm(\partial_k, R(\partial_j \partial_i)\partial_i, \partial_l, \partial_a) \cdots$ (normal coor.)

$= -Rm(R_{jik}^c \partial_c, \partial_i \partial_l \partial_a) - \cdots$ $\quad R_{jik}^c = \underbrace{g^{cl}}_{=\delta_{cl}} R_{jikl} = R_{jikc}$

$= -R_{jikc} R_{cila} - \cdots$

$R_{ij} = g^{ab} R_{iajb} \Rightarrow R_{jiic} = -R_{jici} = -R_{jc}$

sub $\Rightarrow$ $\nabla_i \nabla_j R_{kila} - \nabla_j \nabla_i R_{kila}$

$= R_{jikc} R_{ilca} + R_{jikc} R_{lcia} + \underline{R_{jc} R_{kcla}} - R_{jilc} R_{kica} - R_{jiac} R_{kilc}$ $\quad R_{jikc} R_{elac} = B_{jkla}$

$= R_{jc} R_{kcla} + B_{jkla} - B_{jkal} + B_{jlka} - B_{jakl}$

Now $\nabla_i \nabla_j \Rightarrow \Delta$

so $\nabla_b R_{laki} + \nabla_l R_{abki} + \nabla_a R_{blki} = 0$

muti g^{bi} and take trace $\quad g^{bi} \nabla_b R_{kila} + \nabla_l R_{ak} - \nabla_a R_{lk} = 0$

$\underbrace{\nabla_b R_{laki}} = \nabla_l R_{abki}$ $\quad g^{bi} \nabla_l R_{abki} = \nabla_l g^{bi} R_{abki}$ $\quad \nabla g = 0$

apply ∇_j: $\nabla_j \underbrace{\nabla_i}_{\text{since } g^{bi} = \delta^{bi}} R_{kila} = \nabla_j \nabla_a R_{lk} - \nabla_j \nabla_l R_{ak}$

sub into $\longleftarrow$ similarly can do for i,k

So $\Delta R_{jkla} = -\nabla_j \nabla_a R_{lk} + \underbrace{\nabla_j \nabla_l}_{\nabla^2_{j,l} \,=\, \nabla_j \nabla_l + \underbrace{\nabla_{\nabla_j l}}_{=0}} R_{ak} + \nabla_k \nabla_a R_{ij} - \nabla_k \nabla_l R_{aj} - Ric \cdots$

#

Evolution $(h = -2Ric)$. Ricci flow

$\partial_t Rm$ we did before.

curvature tensor Rm evolves under heat equation

Prop 2.5.1 = 2.4.1 + 2.3.5

Rmk 2.5.2 $\partial_t Rm = \Delta Rm + Rm * Rm$

since Rm. $Ric(R(X,Y)Z, W)$

$\downarrow$ term like $Ric(R_{ijk}^{l} X^i Y^j Z^k \partial_l, W^a \partial_a)$

$R_{ijk}^{l} X^i Y^j Z^k W^a \underbrace{R_{la}}_{Rm}$

$\downarrow$

$g^{la} R_{ijka}$ $\forall X, Y, Z, W$

$\downarrow$

$Rm * Rm$

Prop 2.5.3 $\frac{\partial}{\partial t} Ric = \Delta_L (Ric)$ since $h = -2Ric$

Since P 2.3.7 $\partial_t Ric = -\frac{1}{2}\Delta_L h - \frac{1}{2}\cancel{\mathcal{L}_{(\delta G(h))^{\#}} g}$

also (2.1.9) $\delta G(Ric) = \delta Ric + \frac{1}{2} dR = 0$

Prop 2.5.4 Under Ricci flow, $\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2$

$\begin{cases} \text{P 2.3.9} & \partial_t R = -\langle Ric, \underset{-2Ric}{h}\rangle + \underset{-\Delta R}{\delta^2 h} - \Delta(\underset{tr\, Ric = R}{tr\, h}) \\ (2.1.9) & \delta G(Ric) = 0 \\ & \qquad = \delta Ric + \frac{1}{2} dR \end{cases}$

$\delta^2 Ric = -\frac{1}{2}\delta(dR)$

$= \frac{1}{2}\nabla^a (dR)_a$ → 第a个分量

$dR = \boxed{\frac{\partial R}{\partial x_a}} dx^a$.

$= \frac{1}{2}\nabla^a(\nabla_a R)$

$= \frac{1}{2}\Delta R$

$$\frac{\partial}{\partial t}\mathrm{Rm}(X,Y,W,Z) = (\Delta \mathrm{Rm})(X,Y,W,Z)$$
$$\begin{aligned}&- \mathrm{Ric}(R(X,Y)W,Z) + \mathrm{Ric}(R(X,Y)Z,W)\\ &- \mathrm{Ric}(R(W,Z)X,Y) + \mathrm{Ric}(R(W,Z)Y,X)\\ &+ 2(B(X,Y,W,Z) - B(X,Y,Z,W)\\ &+ B(X,W,Y,Z) - B(X,Z,Y,W)).\end{aligned} \tag{2.5.1}$$

$\overset{\circ}{Ric} = Ric - \frac{R}{n} g \qquad tr(\overset{\circ}{Ric}) = 0$

trace free

$|Ric|^2 = |\overset{\circ}{Ric}|^2 + \frac{R^2}{n^2}|g|^2 \geq 0 + \frac{R^2}{n} \qquad |g|^2 = g^{ij}g_{ij} = \sum_{i=1}^{n} \delta_i^i = n$

$\Rightarrow \partial_t R \geq \Delta R + \frac{2}{n} R^2$

Prop 2.5.6 $f(t): M \to \mathbb{R}$, $\frac{\partial}{\partial t}\Delta f = \Delta \frac{\partial f}{\partial t} + 2\langle Ric, Hess(f)\rangle$

Proof. P 2.3.10 : $\partial_t(\delta\omega) = \delta\frac{\partial\omega}{\partial t} + \langle h, \nabla\omega\rangle - \langle \delta G(h), \omega\rangle$

$\nabla(-, df)$

$\Delta = -\delta d$

take $\omega = -df$

$\delta\omega = -\delta df = \Delta f$

So $\partial_t(\Delta f) = \Delta(\frac{\partial f}{\partial t}) + 2\langle Ric, \nabla(df)\rangle + 2\langle \delta G(Ric), df\rangle$

$\nabla(df) = Hess(f)$ $\qquad \delta G(Ric) = 0$ (2.1.9)

#

Prop 2.3.12 $\Rightarrow$ $\frac{\partial}{\partial t} dV = -R\, dV$ ✓

direct