

Quillen's Constant

D. Quillen. Determinants of Cauchy-Riemann operators over a Riemann surface

Darboux & Koenigs, Geom. of H-surfs, §6.5.5

Today: construct a metric (on a natural $\bar{\partial}$ -line bundle on a Riemann surface) & compute its curvature.

Applications: Darboux theory, alg. geom., QFT
Friedman & Morgan, Bismut, Gilkey, "anomalies"
completes GUT picture. Enrico

Σ - cpt Riem surface. $E \rightarrow \Sigma$ hol. v.b. w/ metric

$$\bar{\partial}_A : \Omega^{0,0}(E) \rightarrow \Omega^{0,1}(E)$$

$$\text{locally: } \bar{\partial}_A = d\bar{z}(\bar{\partial}_z + \lambda(z))$$

$A_z = \{\bar{\partial}_A\} =$ affine space modelled on $\Omega^{0,1}(\text{End}_\mathbb{C} E)$.

($A =$ Chern connection of $\bar{\partial}_A$.)
 $\bar{\partial}_A = d\bar{z}(\bar{\partial}_z - \lambda^*(z))$

$\Rightarrow$ trivially has a holomorphic structure
 $\{d\bar{z}(\bar{\partial}_z + w d_1(z) + (1-w)d_2(z))\}$
 $(w \in \mathbb{C})$ is a deformation.
may go terribly wrong.

Define the determinant line bundle

$L_E \rightarrow A_E$ with fibers

$$\Lambda^{\text{max}}(\ker \bar{\partial}_A)^* \otimes \Lambda^{\text{max}}(\text{Im } \bar{\partial}_A)$$

where $\bar{\partial}_A$.

Recall: this is a smooth (holomorphic) line bundle over A_E . ($\mathbb{C}^*$ -equivariant.)

more generally: V, W Banach spaces

$D_t: V \rightarrow W$ Fredholm family: D_t bdd w/f.l. ker does
etc in operator norm topology.

$$L_t = 1^{\max} \ker D_t \oplus (1^{\max} \operatorname{coker} D_t)^*$$

let $\psi: \mathbb{C}^N \rightarrow W$ s.t.

$$D \oplus \psi: V \oplus \mathbb{C}^N \rightarrow W$$

Define $L = (1^{\max} \ker(D \oplus \psi)) \oplus (1^N \mathbb{C}^N)^*$

is a subbundle of $V \oplus \mathbb{C}^N$ of rank N . (easy to see).

Exact seq for each t :

$$0 \rightarrow \ker D_t \rightarrow \ker(D_t \oplus \psi_t) \xrightarrow{\pi_2} \mathbb{C}^N \rightarrow \operatorname{coker} D_t \rightarrow 0$$

$L \simeq L_t$ by the following

Lemma: (V_i, d_i) finite exact sequence $\Rightarrow$ canonical

isomorphism $\bigotimes 1^{\max} V_i = \bigotimes 1^{\max} V_i \cdot 0$

Obv. for s.c.s.

$$\begin{array}{ccccccc} V_0 & \xrightarrow{d_0} & V_1 & \xrightarrow{d_1} & V_2 & \xrightarrow{d_2} & V_3 \\ & \searrow \downarrow \text{coker } d_0 & & \searrow \downarrow \text{coker } d_1 & & \searrow \downarrow \text{coker } d_2 & \\ & 0 & & 0 & & 0 & \end{array}$$

$$\wedge V_i = 1^{\max} \operatorname{coker} d_i \oplus 1^{\max} \operatorname{coker} d_{i+1} \Rightarrow \bigotimes_{\text{odd}} V_i = \bigotimes 1^{\max} \operatorname{coker} d_i = \bigotimes_{\text{even}} V_i$$

Notice: if $\ker D_t = 0$ then $\exists$ canonical

isomorphism section $\sigma: A_{\mathbb{Z}} \rightarrow L_{\mathbb{Z}}$

given by

$$\sigma = \begin{cases} 1 & D \text{ invertible} \\ 0 & D \text{ not invertible} \end{cases}$$

more seriously: in the above construction

$$\sigma(A_t) := \det(\pi_2|_{\ker(D_{A_t} \oplus \psi)})$$

Clearly vanishes on non-invertible ("jumping") locus.
 TO DEFINE A METRIC on L_E , suppose to define the length of γ :

$$D(A) = |\sigma(A)|^2$$

To be smooth,

should vanish to second order on jumping locus.

What vanishes to second order when $\bar{\tau}_A$ stops being invertible? An eigenvalue of the Hessian

$$\Delta_A = \bar{\tau}_A^* \bar{\tau}_A !$$

So formally we want to write

$$D(A) = \det \Delta_A = \prod \lambda$$

↳ wildly divergent.

Let $\zeta_A(s) = \sum \lambda^{-s}$. Hol. for $\operatorname{Re}(s) > 1$

$$\Rightarrow \zeta_A'(s) = -\sum \lambda^{-s} \log \lambda$$

$$\text{Formally } \zeta'(0) = -\sum \log \lambda$$

$$\Rightarrow e^{-\zeta'(0)} = e^{\sum \log \lambda} = \prod \lambda$$

analytic continuation.

So we define $D(A) := e^{-\zeta_A'(0)}$ (index 0 case.)

To define for arb. index / show smoothness:

let $c > 0$,

$$U_c = \{A \mid c \neq \text{spec } \Delta_A\}$$

$$\text{let } \mathcal{H}_c^+ \subset C_c^\infty(\Sigma)$$

= span of eigenspaces of $\bar{\partial}_A^* \bar{\partial}_A$ of e-val $0 < \lambda < c$

$\mathcal{H}_c^+ \rightarrow U_c$: f.d. $\mathbb{C}$ -vector bundle.

$$\text{Similarly: } \mathcal{H}_c^- \subset \Omega_c^{0,1}(\Sigma)$$

= span of eig - - of $\bar{\partial}_A \bar{\partial}_A^*$ - $0 < \lambda < c$

$$\text{Kerl Kerl } \Psi_\lambda \mapsto \bar{\partial}_A \Psi_\lambda$$

gives a canonical isomorphism b/w negative eigenspaces.

$$\text{So: } L_\Sigma \underset{\text{can}}{\cong} \bigwedge^{\text{max}} \mathcal{H}_c^- \otimes (\bigwedge^{\text{max}} \mathcal{H}_c^+)^*$$

Define $| \cdot |_c$ using L^2 metrics on $\mathcal{H}_c^-, \mathcal{H}_c^+$.

The Quillen metric is defined by

$$| \cdot |^2 = \left(\prod_{\lambda > c} \lambda \right) | \cdot |_c^2$$

↳ zeta-fun regularized.

Notice: well-defined. $c' > c$ Ψ_λ , $c < \lambda < c'$

$$\Rightarrow | \cdot |_{c'}^2 = \lambda^{-1} | \cdot |_c^2$$

Due to $\| \bar{\partial}_A \Psi_\lambda \|^2 = \langle \bar{\partial}_A \Psi_\lambda, \bar{\partial}_A \Psi_\lambda \rangle = \langle \Delta_A \Psi_\lambda, \Psi_\lambda \rangle$

Remark: Curvature R is $\Omega_{\mathbb{C}(A, \Psi)} = \frac{1}{2} (F + \langle R, \cdot \rangle) \cdot \lambda \|\Psi_\lambda\|^2 = \lambda$

Assume ^{wlog} $\text{index}(\bar{\gamma}) = 0$ (@ hold w/ opposite ends).

Curvature of wt. v. l. : $\bar{\gamma} \log |s|^2$, s -hol. section (can vanish!)

$\Rightarrow$ Curv. of Quillen metric: $-\int'' \bar{s}' \log D(A)$

$$\begin{aligned} D(A) &= |\sigma(A)|^2 \\ &= \exp(-\bar{\gamma}_A'(0)) \end{aligned} \quad = \boxed{-\int'' \bar{s}' \bar{\gamma}_A'(0)}$$

~~Maximal extension of $\bar{\gamma}$ -form~~ Kernels

$\Delta = \Delta_A$ on $\mathcal{O}$ -forms $\exp(-t\Delta)$ = heat operator on Σ .
 $\leadsto$ heat kernel $h_t(x, y)$.

= fund. sol. of $\frac{\partial f}{\partial t} = -\Delta f$

$$\text{On } \mathbb{R}^2: \frac{1}{4\pi t} \exp\left(-\frac{|x|^2}{4t}\right)$$

let $\rho = \text{injrad}(\Sigma)$, $d(x, y) < \rho$

$P(x, y): E_x \rightarrow E_y$ = parallel transport via A along minimal geodesic x to y .

$$\text{Define } h_t^\#(x, y) = \begin{cases} \frac{1}{4\pi t} \exp\left(-\frac{d(x, y)^2}{4t}\right) P(x, y) & d(x, y) < \rho \\ 0 & d(x, y) > \rho \end{cases}$$

$$\Rightarrow h_t(x, y) = h_t^\#(x, y) + E_t(x, y), \quad E_t \text{ bdd.}$$

$$\Rightarrow T_n(\exp(-t\Delta)) = \sum_{\lambda} e^{-\lambda t} = \int_{\Sigma} t_n h_t(x, x) d\mu_x$$

(Recall: $h_t(x, y) = \sum_{\lambda} \psi_{\lambda}(x) \psi_{\lambda}^*(y) e^{-\lambda t}$, $\int |\psi_{\lambda}(x)|^2 d\mu_x = 1$)

$$\Rightarrow \sum_{\lambda} 1 \cdot e^{-\lambda t} = \sum_{\lambda} \left(\int |\psi_{\lambda}(x)|^2 d\mu_x \right) e^{-\lambda t}$$

$$= \int \left(\sum_{\lambda} |\psi_{\lambda}(x)|^2 e^{-\lambda t} \right) d\mu_x = \int t_n h_t(x, x) d\mu_x$$

$$\Rightarrow T_n(\exp(-t\Delta)) = \frac{n \text{Vol}(\Sigma)}{4\pi t} + O(1)$$

(Now to do)

In fact

$$\sim \sum_{i=-1}^{\infty} a_i t^i, \quad t \rightarrow 0.$$

Use DO
to find
fla!

$$f(t) \sim \sum A_i$$

$$\left(f(t) - \sum_{i=0}^N A_i t^i = O(A_{N+1}) \text{ as } t \rightarrow 0 \right).$$

Let

$$\bar{\partial}_A^{-1} \xrightarrow{\text{kernel}} L(w, z) = \left(\frac{i \cdot \text{Id}}{2\pi(w-z)} + f(z, w) \right) dz$$

(1,1)-form.

$$\phi(z) \in \mathcal{H}^{E,1}$$

$$\Rightarrow \int_{z \in \Sigma} L(w, z) \phi(z) = \bar{\partial}_A^{-1} \phi \in \mathcal{H}^{E,0}$$

intrinsic.

$\Gamma \in \Omega^{1,0}(\text{End } E)$
 $\hookrightarrow$ its anti-sym.

By construction,

$$\bar{\partial}_A^{-1}$$

is holomorphic in A .

$$\left(\frac{\partial}{\partial A} \bar{\partial}_A^{-1} \right) = 0.$$

Zeta - function regularization.

Lemma: Assume that $\psi \in C^N[0,1]$ has an asymptotic expansion

$$\psi(t) \sim \sum_{i=-\infty}^{\infty} a_i t^i \quad \text{as } t \rightarrow 0.$$

Then $f(s) = \int_0^1 \psi(t) t^{s-1} dt$ (e.g. $n=0$: Taylor expansion)

admits a meromorphic expansion w/ simple poles at $\{n, n-1, n-2, \dots\}$ with residues $-a_i$ at $-i$.

Proof: $-\infty < N < \infty$

$$\psi(t) = \sum_{i=-N}^N a_i t^i + E_N(t), \quad |E_N(t)| \leq C \cdot t^{N+1}.$$

$$\Rightarrow f(s) = \int_0^1 \sum_{i=-N}^N a_i t^i t^{s-1} dt + \int_0^1 E_N(t) t^{s-1} dt$$

$\Rightarrow$ for $s > -N-1$ wh. for $N+1+s-1 > -1$
 $s > -N-1$

$$f(s) = \sum_{i=-N}^N a_i \int_0^1 t^{s+i-1} dt + h(s)$$

$$= \sum_{i=-N}^N a_i \frac{1}{s+i} + h(s), \quad \text{works for } N \text{ and,}$$

+ unique continuation. $\square$

Example: $T(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$

$$= \int_0^1 e^{-t} t^{s-1} dt + \int_1^{\infty} e^{-t} t^{s-1} dt$$

main part of sample
has poles at $\{0, -1, -2, \dots\}$

w/ residues $\{1, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{24}, \dots\}$

Meromorphic ext'n of ζ -functions

Observe: $\int_0^{\infty} e^{-\lambda t} t^{s-1} dt \cdot \lambda^s = \lambda^{-s} T(s)$

$$\Rightarrow \lambda^{-s} = \frac{1}{T(s)} \int_0^{\infty} e^{-\lambda t} t^{s-1} dt$$

$\Re(s) > 1 \Rightarrow \zeta_A(s) = \sum_{\lambda \in A} \lambda^{-s} = \frac{1}{T(s)} \sum \int_0^{\infty} e^{-\lambda t} t^{s-1} dt$

$$\left(\zeta_A(s) \right) = \frac{1}{T(s)} \int_0^{\infty} T_A(\exp(-tA)) t^{s-1} dt$$

$$= \frac{1}{T(s)} \left(\int_0^1 T_A(\exp(-tA)) t^{s-1} dt + \int_1^{\infty} T_A(\exp(-tA)) t^{s-1} dt \right)$$

$\hookrightarrow a \cdot t^{-1} + a_0 + \text{smooth}$

Lemma

$\begin{matrix} \nearrow \\ Ce^{-\lambda t} \\ \text{by gl. bound} \end{matrix} \quad t \gg 1$

$$= \frac{1}{T(s)} \cdot (\text{mer. w/ simple poles at } \{1, 0, -1, \dots\})$$

"dim $\mathbb{Z}^n/2$ "

poles cancel
except at

$\zeta_A(s) = \text{mer. on } \mathbb{C} \text{ with simple pole at } s=1!$

Curvature calculation.

~~FORMALLY~~ $\delta' = \text{hol. part.}$ ~~WIKI = ...~~

$$\delta' \bar{\partial}_A = \delta' (\partial \bar{z} (\bar{z}_i + \alpha)) = \partial \bar{z} \delta' \alpha$$

$$\delta' \bar{\partial}_A^* = \delta' (\omega \cdot \partial \bar{z} (\bar{z}_i - \alpha^*)) = \omega \cdot \partial \bar{z} \delta' \alpha^* = 0.$$

So: FORMALLY: $\delta' \log \det \Delta$

where:

$$= \text{Tr}(\Delta^{-1} \delta' \Delta)$$

$$= \text{Tr}(\Delta^{-1} \bar{\partial}_A^* \delta' A)$$

$$= \text{Tr}(\bar{\partial}_A^{-1} (\delta' A)) =$$

$$\Delta^{-1} \bar{\partial}_A^* \bar{\partial}_A = \text{id.}$$

$$\rightarrow \Delta^{-1} \bar{\partial}_A^* = \bar{\partial}_A^{-1}$$

$$\text{AND } \bar{\partial}_A^* = \Delta \bar{\partial}_A^{-1}$$

But $\bar{\partial}_A$ varies hol'ly with A , so $\bar{\partial}_A^{-1}$ does too! (see fls.)
 $\Rightarrow \int \delta' \log \det A = \int \text{Tr}(\bar{\partial}_A^{-1} \delta' A) = 0.$

False! $\bar{\partial}$ -Regularization of " $\det \Delta$ " introduces non-holomorphic stuff. "Anomaly."

Genuine calculation of curvature.

$$\delta' \text{Tr}(\exp(-t\Delta)) = -t \text{Tr}(\exp(-t\Delta) \delta' \Delta)$$

$$\text{Re}(s) \gg 1. \quad \delta' \zeta(s) = -\frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(\exp(-t\Delta) \delta' \Delta) t^s dt$$

$$(\text{from before: } \delta' \Delta = \bar{\partial}_A^* \delta' A = \Delta \bar{\partial}_A^{-1} \delta' A)$$

$$= -\frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(\exp(-t\Delta) \Delta \bar{\partial}_A^{-1} \delta' A) t^s dt$$

$$= -\frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{dt} \left[\text{Tr}(\exp(-t\Delta) \bar{\partial}_A^{-1} \delta' A) \right] t^s dt$$

"We will not pause to justify"

but by parts:

$$= \frac{-s}{\Gamma(s)} \int_0^\infty \text{Tr}(\exp(-t\Delta) \bar{\partial}_A^{-1} \delta' A) t^{s-1} dt$$

$\frac{s}{\Gamma(s)}$ has a double zero at $s=0$.

$$\zeta_0 \int \delta' \bar{\partial}_A^{-1}(0)$$

$$= -\text{Res}_0 \int_0^\infty \text{Tr}(\exp(-t\Delta) \bar{\partial}_A^{-1} (\delta' A)) t^{s-1} dt$$

LEMMA.

$$= -\lim_{t \rightarrow 0} \text{Tr}(\exp(-t\Delta) \bar{\partial}_A^{-1} (\delta' A))$$

provided it exists. (At $t=0$, formal answer.)

Explicitly: let $\delta' A = \tilde{x} - \tilde{x}^*$

$$\text{Tr}(\exp(-t\Delta) \bar{\partial}_A^{-1} \tilde{x})$$

$$= \int \tau_t(z) dz$$

operator trace!
→ mult. by 2!

$$\text{where } \tau_t(z) = \int_{\Sigma} \text{tr}(h_t(x, w) L(w, z) \tilde{x}(z)) d\mu_w.$$

Note that $\xi_t = h_t - h_t^\#$ is bounded and $\rightarrow 0$ eff. diag. while $L(z, w)$

$$\text{is integrable: } \int \frac{1}{|z-w|} + f(w, z) d\mu_w dz$$

$$\leq C \left(\int \frac{1}{r} \cdot r dr + O(1) \right) \leq C.$$

11

So $\lim_{t \rightarrow 0} \int \ln \varepsilon_t(z, w) L(w, z) \tilde{z}(z) d\mu_w \rightarrow 0$
unif. in z .

$\Rightarrow$ may replace h_t by $h_t^\#$:

$$\tau_t^\#(z) = \int_{\Sigma} \ln(h_t^\#(z, w) L(w, z) \tilde{z}(z)) d\mu_w$$

Notice that $\int_{|z-w|=r} e^{-|z-w|^2} \frac{1}{w-z} d\theta = \frac{1}{r} \int_0^{2\pi} e^{-r^2} d\theta = 0.$

$\therefore \lim_{t \rightarrow 0} \tau_t^\#(z) =: \tau(z)$ exists uniformly in z .
Cancellation!

Recall we have shown $\langle \delta' \log D, \tilde{z} \rangle = \int_{\Sigma} \tau(z)$

Consider another variation $b = \tilde{\beta} - \tilde{\beta}^*$.

$$\langle \delta'' \log D, \tilde{z}, \tilde{\beta} \rangle = \int_{\Sigma} \tau''(z)$$

where $\tau''(z) = \lim_{t \rightarrow 0} \int_{\Sigma} \ln(\delta'' h_t^\#(z, w) L(w, z) \tilde{z}(z)) d\mu_w$

Since all but $h_t^\#$ depend only on A .

Recall $h_t^\# = h_t(z, w) P_A(z, w).$

$\hookrightarrow$ only A -dependence.



$$p(z, w) = 1 - (\bar{z} - \bar{w}) \alpha(w) + (z - w) \alpha^*(w)$$

$$\Rightarrow \delta'' p(z, w) = (z - w) \tilde{\beta}^*(w) + O(|z - w|^2)$$

$$\begin{aligned} \delta'' \tau(z) &= \lim_{t \rightarrow 0} \int h_t(z, w) (z - w) t_n \tilde{\beta}^*(w) \cdot \left(\frac{i}{2\pi} \frac{1}{z - w} + f(z, w) \right) \frac{dw}{dw} \\ &= \lim_{t \rightarrow 0} \frac{i}{2\pi} \int h_t(z, w) \left(\tilde{\beta}^* + O(|z - w|) \wedge \tilde{\alpha} \right) \end{aligned}$$

$$\delta'' \tau(z) = \frac{i}{2\pi} T_n(\tilde{\beta}^*(z) \wedge \tilde{\alpha}(z))$$

$$\Rightarrow \int_{\Sigma} \delta'' \tau(z) = \frac{i}{2\pi} \int_{\Sigma} t_n(\tilde{\beta}^* \wedge \tilde{\alpha})$$

□

$$= -\frac{i}{4\pi} \int_{\Sigma} t_n(a \wedge b).$$