Varieties, reduced, $\pi$-proj.

Schemes, $\pi$-proj. $C$ (unless $C$ toric).

Intersection Theory (ground rules)

$X$ smooth variety.

$\mathbb{Z}$algebraic $\subset \mathbb{Z}^{r}$ Homological.

$\langle \pi \cdot [\Pi_{t}(X \times C)] \rangle$ homologous as $\pi$-cycles.

Examples: a countable lattice of finite subvarieties of $X$.

Finest equivalence to have intersecion products. (Chow ring

$\text{Ch}^{d}(X) = \mathbb{Z}_{\text{nat}}^{d}$ \hspace{1cm} $\text{Ad}^{d}(X) = \mathbb{Z}_{\text{alg}}^{d}$

$\text{Ch}^{d} = \text{Ch}^{n-d}$ \hspace{1cm} $\text{Ad} = \text{Ad}^{n-d}$

$\text{Ch}_{0} \supset \text{Ad} = \text{group of degree 0 } \pi$-cycles (points).

Have flat pullback, finite pushforward.

$\Rightarrow \exists \mathbb{P}_{\pi} \subset \text{Ch}^{k}(X \times Y)$ gives a correspondence

$\left[ \mathbb{P}_{\pi} \right] : \text{Ch}^{k}(X) \xrightarrow{h_{\pi \cdot}} \text{Ch}^{k}(Y)$

$\left[ \mathbb{P}_{\pi} \right] \cdot \left[ \mathbb{P}_{\pi \cdot} \right] = \left[ \mathbb{P}_{\pi \cdot} \right] \cdot \left[ \mathbb{P}_{\pi} \right]$

$\mathbb{P}_{\pi} \cdot \mathbb{P}_{\pi} = \mathbb{P}_{\pi} \cdot \mathbb{P}_{\pi}$

$\mathbb{P}_{\pi} \cdot \mathbb{P}_{\pi} = \mathbb{P}_{\pi} \cdot \mathbb{P}_{\pi}$
Today: $X$ smooth projective surface over $\mathbb{C}$.

$q = H^1(X, \mathbb{C}_x) = H^0(X, \mathbb{C}_x^*)$, $P_g = H^2(X, \mathcal{O}_x) = H^0(X, \mathbb{C}_x^*)$.

Recall the Albanese variety is higher-dim Jacobian: (Voisin, Beauville).

$$A_l(x) = \frac{H^0(X, \mathbb{C}_x^*)}{\mathbb{C}_x^*}$$

well-defined due to closedness of 1-forms / Stokes (i.e. $\int_C w = 0$).

Abelian variety with polarization coming from any product form $\eta, \eta^* \mapsto \tilde{\eta} \eta^* \omega \omega$ on $H^0(X, \mathbb{C}_x^*)$.

/Hodge Theory/Albanese

Fixing a basepoint $p_0 \in X$, define the Albanese map

$$\phi: X \longrightarrow \bigoplus_{p_0} \mathbb{C}$$

$$\phi^*(x) = \sum_{p_0} \psi_p^* \rightarrow \bigoplus_{p_0} \psi_p^*$$

Prop: \(\phi\)

1. $A_l$ is functorial $X \rightarrow Y$.
2. $\text{Im} \phi$ is not connected in a proper submanifold.
3. $\phi: X \rightarrow A_l(X)$ is surjective (look on tangent spaces, use (0)).
4. $\phi$ factors $CH_0(X) \rightarrow A_l(X)$.

To prove (3): Prop: (intuitive/Fulton 1.6.3)

Rational equivalence is generated by $A - B = f(0) - f(\infty)$ \(f: \mathbb{P}^1 \rightarrow \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2 \times \mathbb{C}^2 \)
Example 1: $E \times E$, elliptic curve

Let $\sigma(t, e) = (-t, e + t)$

Let $E' = E/\{1, i\}$, elliptic

$X = E \times E/\{1, i\}$, smooth since $\sigma$ free.

Have map $f : X \to E'$ with $f^{-1} = F$.

\[
\begin{align*}
\Gamma(X, \mathcal{L}_X) & \cong \Gamma(E \times E, \mathcal{L}_{E \times E})/b/c \\
\text{Kunneth} & \cong \left[\Gamma(E, \mathcal{L}_1) \otimes \Gamma(E, \mathcal{L}_1)\right]^{\ell_2, \ell_2}.
\end{align*}
\]

\[
\cong \Gamma(E, \mathcal{L}_1)^{\ell_2, \ell_2} \cong \mathbb{C}.
\]

$\Rightarrow \mathcal{O}(X) = 1 = \dim \text{Alb}(x)$.

Since $x \to \text{Alb}(x)$,

$\Rightarrow \text{connected}$.

Since $F : E \hookrightarrow \text{map of curves}$

$\Rightarrow \text{Alb}(x) \cong E'$.

Lemma 1: $\text{An}(X)$ is divisible.

Proof: Generated by $C \times X, \Gamma \Rightarrow [\Gamma] : A_0(C) \to \text{An}(X)$

Lemma 2: Let

$T(X) = \ker (A_0(X) \to \text{Alb}(X))$.

Also divisible.
Claim: \( A_0(x) \equiv E' \equiv A_1b(x) \), i.e. \( T(x) = 0 \).

Proof:

\[
\begin{array}{ccc}
F \times E & \longrightarrow & E \\
\downarrow \pi & & \downarrow \chi \\
x & \longrightarrow & E'
\end{array}
\]

Let \( x \in T(x) \), and write

\[
\pi^{-1}(x) = \sum \mathcal{E}_r : \{(q, p); (-q, p_1 + q)\}
\]

where \( \mathcal{E}_r = 0 \) if \( c \in \mathcal{A}_0(x) \)

and

\[
\mathcal{E}_r : \text{Im}(p) = 0 \quad \text{in} \quad E' \implies \sum \mathcal{E}_r : p = 0 \quad \text{in} \quad E.
\]

Using Abel's theorem on each component of \( F \times E \),

\( E \) is its own Jacobian, same for \( E' \), we get

\[
\begin{align*}
2(q, p) & \sim 2(q, p + q) \quad \text{on} \quad F \times E \\
(q, p) + (-q, p) & \sim 2(0, p)
\end{align*}
\]

\[
\implies 2 \times 2 \times E \sim \sum \mathcal{E}_r : \{(q, p); (-q, p_1 + q)\} \approx 4(0, p)
\]

\[
= (0, x) \times E \cdot \left[ F \times 2 \cdot \sum \mathcal{E}_r : (p, q) \right]
\]

\[
\sim 0
\]

\[
\text{intersection preserved}, \\
\text{rational equivalence}, \\
(\text{Abel's theorem})
\]

Since \( \pi_{F_0}(2 \times 2) = 4 \times 2 \sim 0 \)

\[
\implies 2 \sim 0 \quad \text{since} \quad T(x) \text{ divisible}
\]

\( \square \)
Example 2: Godeaux Surface:
\[ Y : T_3 + \cdots + T_n + T_5 = 0 \text{ in } \mathbb{P}^2 \]
\[ \pi \downarrow \]
\[ X = \pi / D_5 = \langle (T_1, \omega T_1, \omega^2 T_2, \omega^3 T_3) \rangle, \omega^5 = 1. \]

Get \( p_g = 0, q = 0 \). (\( Y \sim X \) etale, use Noether)
\[ l_g^y = 4, q = 0, \omega^2; \omega^3; \omega^4 = 5 \Rightarrow q(x) = 0 \]
\[ x(y) = 1 + y = 5 \Rightarrow x(x) = 1 = 4^2 = 0. \]

Since it etale,
\[ 5 A_0(x) = \pi^* \pi^* A_0(x), \text{ so } \text{is suff. to show } = 0. \]

Since \( T_5^* \) surjective, suff. to show
\[ T_5^* : A_0(x) \rightarrow A_0(Y) = 0. \]

Consider \( e_i \in \text{Aut}(Y), i = 1, 2, 3, \) s.t.
\[ e_i \cdot T_j = \left\{ \begin{array}{ll}
T_j & i = j \\
0 & i \neq j.
\end{array} \right. \]

Gives \( \text{Aut}(Y) \) rep. in \( \mathbb{Z} [B/5^{\mathbb{Z}}] \rightarrow R \in \text{End}(A_0(Y)). \)

Here, \( \pi^* \pi^* = 1 + \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 + \cdots \), so we must show this is zero. There are lots of relations in \( R \):
the quotient of \( Y \) by \( \langle (e_1, e_2) \rangle \) is rational, so have well-defined morphism from \( \mathbb{P}^2 \), \[ [0, \epsilon_1 T_1, \epsilon_2 T_2, T_3, \epsilon_3 T_3 T_4] \]
a rational surface has \( A_0 = 0 \) so we conclude
\[ 0 = \Sigma (e_i \epsilon_i \epsilon_2 \epsilon_3) \in R \]
Also \( \Sigma \epsilon_i^k = 0 = \Sigma (e_i \epsilon_i \epsilon_2 \epsilon_3)^k, \) \( b/d \) quotients are rational.

\[ \text{In contrast to Fano, surface itself, which is general type, so } f_g = p_g = q. \]
Example 2 cont'd: Since $A_0(4)$ is divisible, we can argue in $R \otimes \mathbb{Q} = \text{quotient of } \mathbb{Q} \otimes \mathbb{Z}/5 \otimes \mathbb{Q}$, which is f.d. domain, so $R \otimes \mathbb{Q}$ has an embedding to $\overline{\mathbb{Q}}$. Determined by images of $e_1, e_2, e_3 \mapsto \omega_1, \omega_2, \omega_3 \in \overline{\mathbb{Q}}$ s.t. $\omega_1 \omega_2 \omega_3 = 1$. To show $\prod x_i = 0$, can just show it goes to zero under any map $R \otimes \mathbb{Q} \rightarrow \overline{\mathbb{Q}}$. If not, then need $\omega_1 \omega_2 \omega_3 = 1$ (so image of $\prod x_i = 5$.)

But for the identities in $R$ to be respected, also need $\omega_i \neq 1$, $\omega_i \omega_j \neq 1$. (So that $x = 0$.)

These are impossible.

$\Rightarrow \prod x_i \neq 0$.

Example 3: Fano surface: $Pg > 0$

$S = \{ \text{Lines on cubic } cP^3 \}$. So 0-cycles are lines.

Block shows:

$$\text{Pic } S \otimes \text{Pic }^0(5) \rightarrow A_0(5)$$

$$\text{Pic } S \otimes \text{Pic }^0(5) \rightarrow T(5).$$

So intersections are surjective!!

(Alternatively: takes correspondence $t \in T \rightarrow \text{lines through }$)

$$CH_0(\mathcal{X}) \rightarrow CH_0(5)$$

Used to get real equivalence!
Mumford's Theorem

Assume $p_g > 0$. Then the following equivalent statements,

1. $\exists n \in \mathbb{N}$ s.t. $A \in \mathcal{A}$ with $\lambda(A^n) \geq n$,

2. $\exists n \in \mathbb{N}$ s.t. the natural map

   $$S^n X \times S^n X \to \mathcal{A}_0(X)$$

   $$(A, B) \mapsto A - B$$ is surjective,

3. $\exists n \in \mathbb{N}$ s.t. $A \in \mathcal{A}$, $A \in S^n X$ is effective,

4. $\exists C \subset X$ s.t. $J(C) \to \mathcal{A}_0(X)$

All fail.

Proof: (Mumford) Focus on (3): let $A \in S^n X$ be distinct, generic.

Let $\omega^{(n)}$ be the holomorphic form on $S^n X$.

$\omega^{(n)} = \omega^{\otimes n}$, where $\omega$ is a non-degenerate, symplectic form on $X \\
\Rightarrow \omega^{(n)}$ is generic, nonzero, i.e., symplectic form.

Step 1: For a family of rationally equivalent eff. cycles $d : S \to S^n X$, with some distinct, have $d \cdot \omega^{(n)} = 0$.

Proof idea: basic family of ant. equiv. is just $P^1 \to S^n X$.

$\Rightarrow \omega$ must vanish (no forms on $P^1$)

(need whole family of rational equivalences)

$\Rightarrow \text{codim} \geq \frac{1}{2} \dim S^n X = n$.

So, basically, the image of $S^n X \to \mathcal{A}_0(X)$ is $\geq n$. Dimension.
For Fano surface, $\Lambda^\perp H^1(s, O_s) \cong H^2(s, O_s)$.

$H^0$ is tangent space to $A_1(s) = \text{Pic}^0(s)$.

We also just showed that intersection is surjective!

$$\text{Pic}^0(s) \otimes \text{Pic}^0(s) \longrightarrow T(s)$$

Tangent space $\leftarrow$ "fibering space" $

H^1(s, O_s) \otimes H^1(s, O_s) \longrightarrow H^2(s, O_s)$

Lecture 6.
Appendix: Slick Proof of Mumford’s.

We work in characteristic $p$ and take $\mathcal{O}_{\text{et}}(x, Q_p)$ transcendental instead of $\mathcal{O}_{\text{et}}(x, Q_p)$.

We stay over $\mathbb{C}$ except for Lemma, where $k = \mathbb{C}$.

**Lemma A1.** $X$ smooth over $k$, $Y$ any $k$-variety.

Write $K = k(Y)$. Then

$$\text{CH}^i(X \times_k k) \cong \lim_{x \to \text{open}} \text{CH}^i(X \times_k U).$$

**Proof:** First remark that as sets,

$$(X \times_k k) = \bigcap (X \times_k U).$$

Can check using functor of points if $k = \lim U$. For codimensions choose transcendence basis. For $x \in U$, have filtered direct system

$$\text{CH}^*(X \times_k k) \cong \text{Coker} \left( \bigoplus_{x \in X \times_k k} \mathbb{Z} \rightarrow \bigoplus_{x \in X \times_k k} \mathbb{Z} \right).$$

Can check using functor of points if $k = \lim U$. For codimensions choose transcendence basis. For $y \in U$, have filtered direct system

$$\text{CH}^*(X \times_k U) \cong \text{Coker} \left( \bigoplus_{y \in Y \times_k U} \mathbb{Z} \rightarrow \bigoplus_{y \in Y \times_k U} \mathbb{Z} \right).$$

Hence, cohomolgy commutes with direct limit, which is true now.

(Notice: for $x \to y$, have $k = k(x) \rightarrow k(y)$)

$$\Rightarrow k(y) = k(x).$$
Lemma A3. Let $k \to k' \to k''$ be any extensions of $k$. Then the kernel of $d: CH^2(X_k) \to CH^2(X_{k''})$ is torsion.

Proof: First assume $[k': k] < \infty$. Then this is factored

by pushforward (i.e., norm) map $\Rightarrow d \circ \text{id} = d \circ \text{norm}$

Can take limits to get for any algebraic extension.

Since we just want to show kernel is torsion, can take

$k = k', \text{alg. closed}; \quad k \to k' \to k''$ and we’ll show kernel of

$0$ is torsion, so

$\quad \text{true for composition, so true of } \square$.

We can write $k''$ as a limit of finite extensions $k''_1$,

for which we have $K'' = K''_1$ for some $k$-variety $Y$. By L1, we then have

$\text{CH}^2(X \times K''_1) = \lim_{\to} \text{CH}^2(X \times U)$.

But since $k''$ is now alg. closed, $U$ has a $K''$ rational

point $p$, and we may factor the identity

$\text{id}_{k''} : \text{CH}^2(X_{k''}) \to \text{CH}^2(X_{k''}) \to \text{CH}^2(X \times p)$,

showing that the first map is injective. So in fact

$\text{CH}^2(X_k) \to \text{CH}^2(X_{k''})$ is as well.

Taking the limit of $k'' \to k'$, get the claim. $\square$

i.e., ignoring torsion, get $X$ defined over $\mathbb{Q}$,

$\text{CH}^2(X_{\mathbb{Q}}) \to \text{CH}^2(X_{\mathbb{Q}}) \to \text{CH}^2(X_{\mathbb{Q}})$. 
Proposition: \((\text{Block-Strum unwrapping decomposition of } \Delta)\)

Proof: Let \(k \in \mathcal{C}\) be any local subfield of \(\mathcal{D}\). Let \(k = k(X)\) be the smallest \(k\)-extension of \(k(X)\) that \(\mathcal{D}\) contains. Assume \(A(X)\) is finite-dimensional, i.e., exist one \(d\) such that \( \mathcal{C} \subseteq A(X) \). Then there exist such \( \mathcal{C} \subseteq A(X) \). The result follows.

\[ \begin{align*}
\Delta & \ni \Delta^3 \rightarrow (x^2)(x^2) \\
\Delta & \ni \Delta^4 \rightarrow (x^2)(x^2) \\
\Delta & \ni \Delta^5 \rightarrow (x^2)(x^2) \\
\Delta & \ni \Delta^6 \rightarrow (x^2)(x^2) \\
\Delta & \ni \Delta^7 \rightarrow (x^2)(x^2) \\
\end{align*} \]
Bloch's conjecture: $g=0 \Rightarrow T(X) = 0$.

Holds for non-general-type, (Voisin's book), and for Catanese surface, like one we did.

Can make descending filtration if $F^\cdot = \text{CH}_0(x) \geq \text{A}_0(x) \geq T(x)$?

$\Rightarrow \text{gr}^\cdot F^\cdot (\text{CH}_0(x)) = \mathbb{Z} \oplus \text{Alb}(x) \oplus T(x)$

Adjusted an earlier correspondence on $X \mapsto \text{X}^!$.

Conjecture: For surface, $\text{gr}^\cdot F^\cdot$ depends only on homology class of $F^\cdot$.

$\Rightarrow$ Bloch's (similar to appendix argument, components of $A$ + moving).

Meta-conjecture was dumb.