

How manifolds evolve(d)

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Stony Brook University

June 21, 2018

Ten-minute lecture @ New York Hungarian Scientific Society. Thanks to Dorottya Nagy-Szakal for the invitation!

I. What is a manifold?

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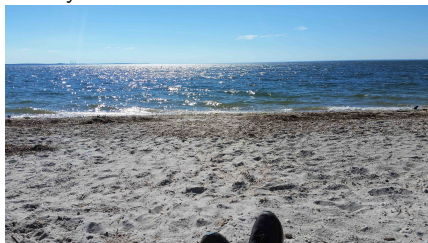
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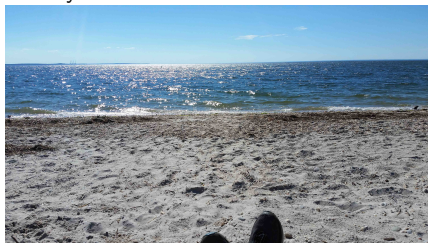
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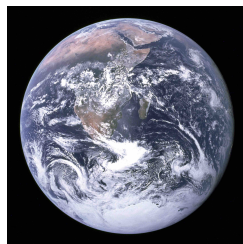
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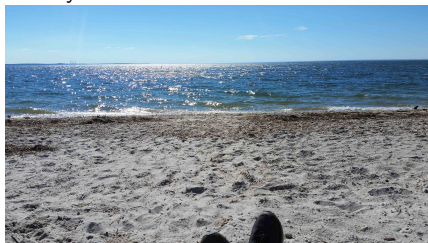
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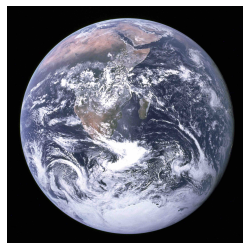
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Turned out to be a *closed* (i.e. compact) manifold.

Classification: $n = 1$.

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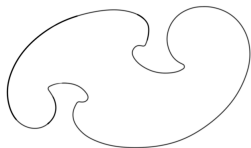
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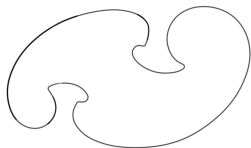


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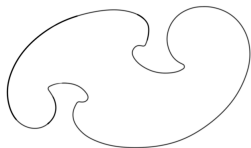
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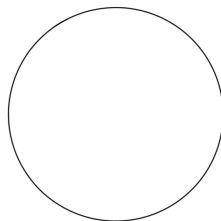
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$\cong$

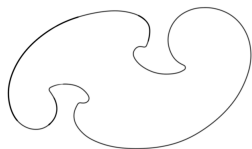


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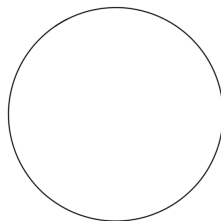
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$n = 1$:



$\approx$



The circle S^1 is the only closed 1-manifold.

Classification: $n = 2$. (Surfaces)

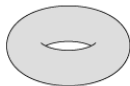


Sphere

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Sphere

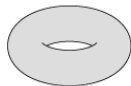


Torus

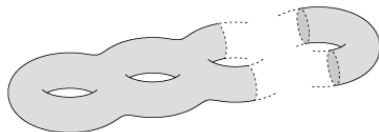
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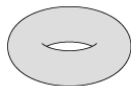


g -holed torus

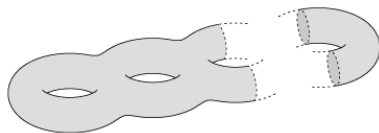
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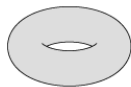
Genus:

= # of "holes"

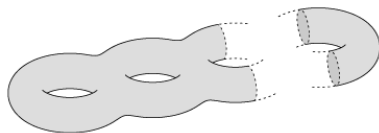
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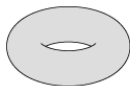
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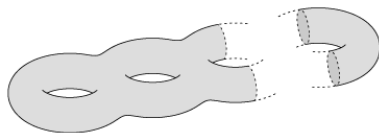
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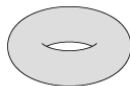
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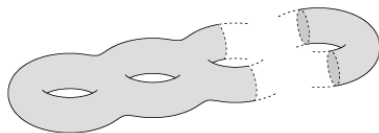
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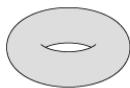
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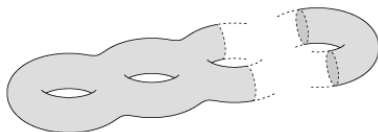
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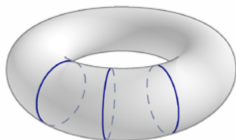
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Formally, a “hole” is detected by an *incontractible closed loop* on the surface, i.e. a nontrivial element of the fundamental group.



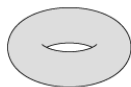
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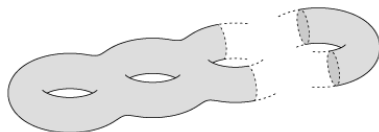
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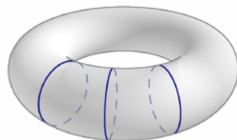
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Theorem (Classification of surfaces, Möbius et al.)

Closed, orientable surfaces are classified by their genus, i.e. the number of holes.

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Both have genus $g = 1$.

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If the universe is finite (closed), which manifold is it?

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Caillerie, S., Lachieze-Rey, M., Luminet, J. P., Lehoucq, R., Riazuelo, A., and Weeks, J. (2007). A new analysis of the Poincaré dodecahedral space model. *Astronomy and Astrophysics*, 476(2), 691-696.

The Poincaré conjecture



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Conjecture (Poincaré, 1904)

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One of the 7 Millennium prize problems (\$1m each).

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A **geometric evolution equation** is a “continuous algorithm” (partial differential equation) which deforms the shape a manifold over time, to find its “best” (least curved) form.

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(Image: Henry Segerman)

Hamilton's program



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In 1982, Richard Hamilton proposed to use his “Ricci flow”

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on a 3-dimensional manifold, with *surgery*, to solve the Poincaré conjecture.

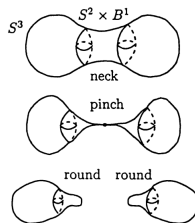
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Köszönöm szépen!