

OVERVIEW OF “GIT FROM THE DIFFERENTIAL GEOMETRIC PERSPECTIVE” BY GEORGOULAS-ROBBIN-SALAMON

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Zooming out. The book’s Holy Trinity is:

- (i) Weights
- (ii) Stability (of G^c orbits)
- (iii) μ -stability

The moment-weight inequality relates μ -stability and weights (in one direction). The Hilbert-Mumford criterion relates weights and stability. The Kempf-Ness Theorem relates stability and μ -stability.

The point of the book is to get the full relationship between weights and μ -stability (both moment-weight and Hilbert-Mumford) by direct analysis of the moment map and a related functional. By the Kempf-Ness Theorem, which they also prove in the book, you get the relationship between weights and stability, which is traditional GIT.

Recall setup.

- A Lie group G acts on a Kähler manifold X , preserving the symplectic form ω , the complex structure J , and the metric g .
- This can be upgraded to an action by G^c , which only preserves J . This action is called linear if $X \subset \mathbb{P}(V)$ for a G -rep V and the action on X is just the restriction of the action on V .

Definition 0.1. $v \in V$ is:

- unstable if $\overline{G^c(v)} \ni 0$
- semistable if $\overline{G^c(v)} \not\ni 0$
- polystable if $G^c(v) = \overline{G^c(v)}$
- stable if $\overline{G^c(v)} \not\ni 0$ and $G_v^c = \text{Fix}(v)$ is discrete.

The GIT quotient is

$$X//G^c = X^{ss}/G^c,$$

where you further identify semistable points whose orbit closures intersect. Sean sketched for us the argument that this is a projective variety.

Example 0.2. $G = \text{U}(1)$ and $G^c = \mathbb{C}^*$. Let $V = \mathbb{C}^2$, so $\mathbb{P}(V) = \mathbb{CP}^1 \cong S^2$, and allow G to act by

$$[x : y] \mapsto [tx : t^{-1}y].$$

Orbits in V under this action are hyperbolas, the origin, and the axes.

- Orbits of points $[x : c/x]$ are stable.
- Orbits of $\{[0 : y]\}$, $\{[x : 0]\}$, $\{[0 : 0]\}$ are unstable.

Letting G and G^c act on $X = \mathbb{P}(V)$, there’s only one (semi)stable orbit, and we have $X//\mathbb{C}^* = \{\text{pt}\}$. ◇

Consider the moment map $\mu : X \rightarrow \mathfrak{g}$, which satisfies

$$\langle d\mu(x)(-), \xi \rangle = \omega(v_\xi(x), -) = \omega(L_x \xi, -).$$

In the case of a linear action, you can obtain μ by restricting from $\mathbb{P}(V)$.

Lemma 0.3 (8.2 in G-R-S). A moment map for $G \subset \mathrm{U}(n)$ acting on $\mathbb{P}(V)$ is given by

$$\langle \mu(x), \xi \rangle = \hbar \frac{\langle v, i\xi v \rangle}{|v|^2},$$

where $x = [v]$ and we give $\mathbb{P}(V)$ the symplectic form $\sqrt{2\hbar}\omega_{FS}$.

Proof. Let $x = [v]$ and $\hat{x} = [\hat{v}]$ where $|\hat{v}| = \sqrt{2\hbar}$ and $\langle v, \hat{v} \rangle = 0$. The infinitesimal action of $\xi \in \mathfrak{u}(n)$ on x is $L_x \xi = [\xi \cdot v]$. We have

$$\omega_x(L_x \xi, \hat{x}) = \langle i\xi v, \hat{v} \rangle.$$

Choose a path $\mathbb{R} \rightarrow V$, $t \mapsto v(t)$ with $v(0) = v$ and $v'(0) = \hat{v}$, and let $x(t) = [v(t)]$. We have

$$\langle d\mu(x)\hat{x}, \xi \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle \mu(x(t)), \xi \rangle = \left. \frac{d}{dt} \right|_{t=0} \hbar \frac{\langle i\xi v(t), v(t) \rangle}{|v(t)|^2} \equiv 2\hbar = \left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \langle i\xi v(t), v(t) \rangle = \langle i\xi v, \hat{v} \rangle = \omega_x(L_x \xi, \hat{x}).$$

Consequently, it's a moment map. $\square$

Remark 0.4. You can always add a central element to a moment map, so for actions by reductive but non-semisimple groups (such as $\mathrm{U}(1)$), it's not unique. But when you're comparing to AG (as we are), you always want to look at the particular moment map in the previous lemma.

Now recall the Kempf-Ness functional, defined by

$$\Phi_x : G^c/G \rightarrow \mathbb{R}, \quad d\Phi_x(g)\hat{g} - \langle \mu(g^{-1}x), \mathrm{im}(g^{-1}\hat{g}) \rangle.$$

The main properties (described by Nianzi) are:

- Φ_x has the same gradient trajectories as $f = \frac{1}{2}|\mu|^2$. In other words, if $x(t) = g^{-1}(t)x_0$ solves $\dot{x} = -\nabla f(x)$, then $\dot{g} = -\nabla \Phi_{x_0}(g(t))$. However, $\dot{x} = \nabla f(x)$ always converges (by Anuk's talk), whereas $g(t)$ may not. The convergence or non-convergence of $g(t)$ is what detects stability.
- Along a gradient trajectory, we have

$$\frac{d}{dt} \Phi_{x_0}(g(t)) = -|\mu(g^{-1}(t)x_0)|^2 \leq -|\mu(x_\infty)|^2.$$

Among other things, this tells you immediately that if $\mu(x_\infty) \neq 0$, then Φ_{x_0} is unbounded below!

- The critical sets of Φ_x have $\mu = 0$ and are nondegenerate (i.e. Φ_x is a Morse-Bott function, unlike $f = \frac{1}{2}|\mu|^2$).
- Φ_x is convex along geodesics in $M = G^c/G$ (which has non-positive sectional curvature).

We went through great pains to prove existence of the Kempf-Ness functional; but in the case of the above moment map on $\mathbb{P}(V)$, it is easy to write down.

Lemma 0.5 (Lemma 8.3 in G-R-S). For the above moment map on $\mathbb{P}(V)$, $x = [v]$, we have

$$\Phi_x(g) = \hbar(\log |g^{-1}v| - \log |v|).$$

Proof. We have $G = \mathrm{U}(n)$ and $\Phi_x(G) = 0$ and

$$\begin{aligned} d\Phi_x(g)\hat{g} &= \hbar \frac{\langle g^{-1}v, -g^{-1}\hat{g}gv \rangle}{|g^{-1}v|^2} \\ &= -\hbar \frac{\langle g^{-1}v, i \mathrm{im}(g^{-1}\hat{g}) \cdot g^{-1}v \rangle}{|g^{-1}v|^2} \\ &= -\langle \mu(g^{-1}x), \mathrm{im}(g^{-1}\hat{g}) \rangle. \end{aligned}$$

□

Notice that g minimizes $\Phi_x(g)$ if and only if g minimizes $|g^{-1}v|$, if and only if $\mu(g^{-1}x) = 0$. Consequently, if you’re trying to minimize the Kempf-Ness functional/find critical points of the moment map in this case, you’re just trying to minimize the norm! This was their original setup.

Example 0.6. Let $X = \mathbb{CP}^1$ with the usual action. Let $\xi = i \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. The moment map is

$$\left\langle \mu \begin{pmatrix} x \\ y \end{pmatrix}, \xi \right\rangle = \frac{\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ -y \end{pmatrix} \right\rangle}{|x|^2 + |y|^2} = \frac{|x|^2 - |y|^2}{|x|^2 + |y|^2}.$$

Hence $\mu^{-1}(0) = \{[e^{i\theta} : e^{-i\theta}]\}$ (the equator). We also have

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} (e^{i\xi}) = \log \frac{|e^{-i\xi}x|^2 + |e^{i\xi}y|^2}{|x|^2 + |y|^2}.$$

This should agree with the expression that Nianzi wrote down. ◇

Definition 0.7. A point $x = [v] \in \mathbb{P}(V)$ is

- μ -unstable if $\overline{G^c} \cap \mu^{-1}(0) = \emptyset$
- μ -semistable if $\overline{G^c(v)} \cap \mu^{-1}(0) \neq \emptyset$
- μ -polystable if $G^c(v) \cap \mu^{-1}(0) \neq \emptyset$
- μ -stable if $G^c(v) \cap \mu^{-1}(0) \neq \emptyset$ and $G^c x$ is discrete.

Theorem 0.8 (μ -stability theorem; 7.2 in G-R-S). It suffices to check μ -stability at $x_\infty = \lim_{t \rightarrow \infty} x(t)$ where $\dot{x}(t) = -\nabla f(x)$, and $x(0) = x_0$. Also, x is μ -stable iff $G_{x_\infty}^c$ is discrete.

For the last statement, note that G_x^c being discrete implies that $\ker L_x = 0$. We know that x_∞ is a critical point of f , so $L_{x_\infty} \mu(x_\infty) = 0$. Hence, $\mu(x_\infty) = 0$.

Theorem 0.9 (Kempf-Ness Theorem; 7.3 and 8.5 in G-R-S). For the above moment map, we have

- (i) x is unstable $\iff \mu$ -unstable $\iff \Phi_x$ is unbounded below
- (ii) x is semistable $\iff \mu$ -semistable $\iff \Phi_x$ is bounded below
- (iii) x is polystable $\iff \mu$ -polystable $\iff \Phi_x$ has a critical point
- (iv) x is stable $\iff \mu$ -stable $\iff \Phi_x$ is bounded below and proper.

Corollary 0.10. $X//G^c \cong \mu^{-1}(0)/G$ as sets

Proof of the special case $G = \mathrm{U}(1)$.¹

Decompose V as a direct sum

$$V = \bigoplus_{r \in \Delta \subset \mathbb{Z}} V_r$$

where

$$V_r = \{w \in V : t(w) = t^r \cdot w\}.$$

Let

$$t(v) = \sum_{r < 0, r \in \Delta} t^{-|r|} v_r + v_0 + \sum_{r > 0, r \in \Delta} t^r v_r.$$

Stability of v is trivial to check:

¹I’ve seen this case referred to as the “Kempf-Ness theorem,” for reasons you’ll see, this might offend Kempf and Ness.

- v is unstable iff $v_r = 0$ for all $r \leq 0$ or $v_r = 0$ for all $r \geq 0$
- v is semistable but not polystable if $v_0 \neq 0$ and $v_r \neq 0$ for some $r > 0$ but $v_r = 0$ for all $r < 0$ (or vice-versa)
- v is polystable but not stable if $v_0 \neq 0$ but $v_r = 0$ for all $r \neq 0$
- v is stable if $v_{r_1} \neq 0$ for some $r_1 < 0$ and $v_{r_2} \neq 0$ for some $r_2 > 0$.

According to the above Lemmas, our preferred moment map is

$$\mu = \frac{1}{|v|^2} \sum_r r |v_r|^2,$$

for which the Kempf-Ness functional is

$$\Phi_{[v]} = \log \frac{\sum t^r |v_r|^2}{|v|^2}.$$

So the stated properties follow by inspection. □

Mumford/G-R-S define weights for general actions, which goes as follows:

Definition 0.11. For $\xi \in \mathfrak{g} \setminus \{0\}$ and $x \in X$, we define

$$w_\mu(x, \xi) = \lim_{t \rightarrow \infty} \frac{\langle \mu(e^{it\xi}x), \xi \rangle}{|\xi|^2},$$

Note that in the $U(1)$ case, by looking at the formula for the moment map, we just have $w_\mu(x, i) = r_{max}$ and $w_\mu(x, -i) = -r_{min}$. So our proof of Kempf-Ness for the case $G = U(1)$ amounted to the following.

Theorem 0.12 (Punchline of G-R-S). A point x is (μ) -stable if and only if $w_\mu(x, \xi) > 0$ for all $\xi \in \mathfrak{g} \setminus \{0\}$.

The $(\Rightarrow)$ direction is called the (strict) moment-weight inequality; the $(\Leftarrow)$ direction is called the Hilbert-Mumford criterion.

For the case of general G , this is all nontrivial and based on the wonderful properties of the above functionals. The following table lists their best-known infinite-dimensional analogues:

Us	Hermitian-Yang-Mills	CscK
μ	$\frac{1}{2} \int F_A ^2 dV$	Calabi functional
Φ	Donaldson functional	Mabuchi functional

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