Applications of the maximum principle on Ricci flows

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Abstract

This is an exposition of Chapter 3 of [3].

1 Preliminaries

Let $\mathcal{M}$ denote a closed (compact, without boundary) n-dimensional manifold equipped with an evolving Riemannian metric $g(t)$. In the notes, $A \leq CB$ for some constant $C = C(n)$ means C is independent of A, B, but may depend on n. For each different inequality, C may be different.

Definition 1.1.

1. Raising and lowering indices

   Let $\eta \in \mathcal{T}^*\mathcal{M}$ be a cotangent vector with components η_i. We can *raise the index* by taking $\eta^i = g^{ij}\eta_j$ and obtain a tangent vector $\eta^\# = \eta_i \partial_i$. Similarly, for a tangent vector $X \in \mathcal{T}\mathcal{M}$ with components X^i, we can *lower the index* by taking $X_i = g_{ij}X^j$ and obtain a cotangent vector $X^\flat = X_i dx^i$. We can similarly raise and lower desired indices of general tensors.

2. Inner products

   Let $X, Y \in \mathcal{T}\mathcal{M}$ with components X^i, Y^j, respectively. Then their *inner product* is defined as
   $$< X, Y >= g_{ij}X^iY^j. \tag{1.1}$$
   Let $\eta, \omega \in \mathcal{T}^*\mathcal{M}$. We can define their inner product by
   $$< \eta, \omega > := < \eta^\#, \omega^\# >= g^{ij}\eta_i\omega_j = \eta_i\omega^i. \tag{1.2}$$
   For a general tensor $F, G \in \mathcal{T}^{(k,l)}\mathcal{M}$ with components $F^{i_1,\ldots,i_k}_{j_l,\ldots,j_l}, G^{i_1,\ldots,i_k}_{j_l,\ldots,j_l}$, respectively, we define their inner product to be
   $$< F, G >= F^{i_1\ldots,i_k}_{j_l,\ldots,j_l}, G^{i'_1,\ldots,i'_k}_{j'_l,\ldots,j'_l} \prod_{r,r'=1}^{k} g_{i_r,i'_{r'}} \prod_{s,s'=1}^{l} g^{j_s,j'_{s'}}. \tag{1.3}$$

3. Frobenius norms

   Let $F \in \mathcal{T}^{(k,l)}\mathcal{M}$. The *Frobenius norm*, denoted by $|F|$ is defined as a non-negative function on $\mathcal{M}$ satisfying $|F|^2 =< F, F >$.

4. The $*$-notation

   Let A, B be any tensor fields. Then $A * B$ is a real linear combination of operations of $A \otimes B$ including raising and lowering indices and contractions.

5. Curvature tensors

   Let $R \in \mathcal{T}^{(1,3)}\mathcal{M}$ be a tensor defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.4}$$

   Locally, we can write $R = R_{i,j,k}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l$.

   Define the *(Riemann) curvature tensor* $\mathrm{Rm} \in \mathcal{T}^{(0,4)}\mathcal{M}$ by $\mathrm{Rm} = R^\flat$; that is

$$Rm(X,Y,Z,W) =< R(X,Y)Z, W > . \tag{1.5}$$

   Then locally $\mathrm{Rm} = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$, where $R_{ijkl} = g_{ml} R_{ijk}{}^m$. Conversely, $R_{ijk}{}^m = g^{lm} R_{ijkm}$.

   The *Ricci curvature*, denoted by Ric, is given by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y). \tag{1.6}$$

   Then $\mathrm{Ric} \in \mathcal{T}^{(0,2)}\mathcal{M}$ with components $R_{ij} = R_{kij}{}^k = g^{km} R_{kijm}$.

   The *scalar curvature* is a function S given by

$$R = \mathrm{tr}_g \mathrm{Ric} = R_i{}^i = g^{ij} R_{ij}. \tag{1.7}$$

Remark 1.2. R may be a $(1,3)$-, $(0,4)$-, or $(0,2)$- tensor, or a scalar function, depending on the context.

Then one has the following properties of $*$.

Lemma 1.3. *Let A, B be any tensor fields on $\mathcal{M}$. Then*

$$|A * B| \leq C|A||B|, \tag{1.8}$$

for some constant $C = C(n)$.

Lemma 1.4 ((2.1.4) of [3])**.** *Let A, B be any tensor fields on $\mathcal{M}$. Then*

$$\nabla(A * B) = (\nabla A) * B + A * (\nabla B). \tag{1.9}$$

Lemma 1.5 ((2.1.6) of [3])**.** *Let A be any tensor field. Then*

$$\nabla(\Delta A) - \Delta(\nabla A) = (\nabla \mathrm{Rm}) * A + \mathrm{Rm} * (\nabla A). \tag{1.10}$$

Lemma 1.6 ((2.3.3) of [3])**.** *Let A be any tensor field. Then*

$$\frac{\partial}{\partial t}\nabla A - \nabla \frac{\partial}{\partial t} A = A * \nabla h, \tag{1.11}$$

where $h = \frac{\partial}{\partial t} g$.

Throughout the notes, unless otherwise specified, $g(t)$ is a Ricci flow on $\mathcal{M}$ for $t \in [0, T]$; that is, g satisfies the following equation,

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g). \tag{1.12}$$

Let's recall some properties of $g(t)$.

Lemma 1.7.

$$\frac{\partial}{\partial t}\mathrm{Rm} = \Delta \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}. \tag{1.13}$$

Lemma 1.8 (Corollary 2.5.5 of [3])**.** *The scalar curvature R satisfies*

$$\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{n} R^2. \tag{1.14}$$

Lemma 1.9 ((2.5.8) of [3])**.** *Let $V(t) \coloneqq \mathrm{Vol}((\mathcal{M}, g(t)))$. Then*

$$\frac{dV}{dt} = -\int R dV. \tag{1.15}$$

2 The weak maximum principle

Theorem 2.1 (Weak maximum principle)**.**
Let $F : \mathbb{R} \times [0, T] \longrightarrow \mathbb{R}$ be a smooth function. Suppose $u \in C^\infty(\mathcal{M} \times [0, T], \mathbb{R})$ solves

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + F(u, t). \tag{2.1}$$

Suppose further that $\phi : [0, T] \longrightarrow \mathbb{R}$ solves

$$\begin{cases} \frac{d\phi}{dt} & = & F(\phi(t), t), \\ \phi(0) & = & \alpha \in \mathbb{R}. \end{cases} \tag{2.2}$$

If $u(\cdot, 0) \leq \alpha$, then $u(\cdot, t) \leq \phi(t)$ for all $t \in [0, T]$.

Proof. Let $\varepsilon \geq 0$. Consider an ε-perturbation of (2.2)

$$\begin{cases} \frac{d\phi_\varepsilon}{dt} & = & F(\phi_\varepsilon(t), t) + \varepsilon, \\ \phi_\varepsilon(0) & = & \alpha + \varepsilon \in \mathbb{R}. \end{cases} \tag{2.3}$$

Then one can find some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a solution ϕ_ε to (2.3) on $[0, T]$. Thus, it suffices to show that $u(\cdot, t) < \phi_\varepsilon(t)$ for all $t \in [0, T]$ and for all $\varepsilon \in (0, \varepsilon_0)$.

Suppose, otherwise, there exists some $\varepsilon \in (0, \varepsilon_0)$, $t \in (0, T])$, and $x_t \in \mathcal{M}$ such that $u(x_t, t) > \phi_\varepsilon(t)$. Let $t_0 \geq 0$ denotes the infimum of such t. Then one can find a sequence $\{t_i\} \searrow t_0$ and corresponding $\{x_{t_i}\}$ such that $u(x_{t_i}, t_i) \geq \phi_\varepsilon(t_i)$ for each i. By compactness of $\mathcal{M}$, one can find a subsequence of $\{x_{t_i}\}$ that converge to some $x \in \mathcal{M}$ such that $u(x, t_0) \geq \phi_\varepsilon(t_0)$. On the other hand, by definition of t_0, one has $u(x, s) < \phi_\varepsilon(s)$ for all $s \in (0, t_0)$. Thus, $u(x, t_0) = \phi_\varepsilon(t_0)$, and $\frac{\partial u}{\partial t}(x, t_0) - \phi'_\varepsilon(t_0) \geq 0$.

Moreover, we observe that $u(x, t_0) = \max_{\mathcal{M}} u(\cdot, t_0)$. Otherwise, there exists some $x' \in \mathcal{M}$ satisfying $u(x', t_0) > u(x_0, t_0) = \phi_\varepsilon(t_0)$. By continuity of u, ϕ_ε, this violates the definition of t_0. Then, one has $\Delta u(x, t_0) \leq 0$, and $\nabla u(x, t_0) = 0$, which implies

$$\begin{array}{rcl} 0 & \geq & \frac{\partial u}{\partial t}(x, t_0) - \Delta u(x, t_0) - F(u(x, t_0), t_0) \\ & \geq & \phi'_\varepsilon(t_0) - F(\phi_\varepsilon(x, t_0), t_0) \\ & = & \varepsilon > 0. \end{array} \tag{2.4}$$

Contradiction arises. □

By reversing the inequalities, one has the following minimu principle.

Corollary 2.2 (Weak minimum principle)**.** *Theorem 2.1 holds with all the $\leq$ replaced by $\geq$.*

Remark 2.3. The strong maximum principle, where one has $u(\cdot, t) < \phi(t)$ for all $t \in (0, T]$, unless $u(x, t) = \phi(t)$ for all $(x, t) \in \mathcal{M} \times [0, T]$ is true under the same assumptions in Theorem 2.1.

3 Basic control on the evolution of curvature

In this section, we will apply the maximum and minimum principles to obtain some control on how R and Rm evolve.

3.1 Lower bounds of the scalar curvature

Theorem 3.1. *If the scalar curvature satisfies $R(\cdot, t = 0) \geq \alpha \in \mathbb{R}$ on $\mathcal{M}$, then for all $t \in [0, T]$,*

$$R \geq \frac{\alpha}{1 - (\frac{2\alpha}{n})t}. \tag{3.1}$$

Proof. Let $u \equiv R$, $F(r, t) \equiv \frac{2}{n} r^2$, and $\phi(t) = \frac{\alpha}{1 - (\frac{2\alpha}{n})t}$. By Lemma 1.8, one has

$$\frac{\partial u}{\partial t} \geq \Delta u + F(u, t). \tag{3.2}$$

On the other hand,

$$\left\{ \begin{array}{rcl} \frac{d\phi}{dt} & = & \frac{2\alpha^2}{n(1 - (\frac{2\alpha}{n})t)^2} = F(\phi(t), t). \\ \phi(0) & = & \alpha. \end{array} \right. \tag{3.3}$$

Applying Corollary 2.2, we obtain (3.1). □

There are several obvious corollaries of the theorem. We pick the least obvious one to prove here.

Corollary 3.2. *(Corollary 3.2.5) Suppose $g(t)$ is a Ricci flow on $\mathcal{M}$ for $t \in (0,T]$. Then $R \geq -\frac{n}{2\alpha}$.*

Proof. We observe that (1.12) is translation invariant with respect to time t. Let $t_0 \in (0,T)$. Consider the metric $g_{t_0} = g(\cdot + t_0)$ on $\mathcal{M}$. Then it is also a Ricci flow, but for $t \in [0, T - t_0]$. The corresponding scalar curvature is $R_{t_0}(x,t) = R(x, t+t_0)$ for $(x,t) \in \mathcal{M} \times [0, T-t_0]$. Let $\alpha = \inf_{\mathcal{M}} R(\cdot, t_0)$. By compactness of $\mathcal{M}$ and continuity of R, one has $\alpha \in \mathbb{R}$. Now we apply Theorem 3.1 to g_{t_0} to obtain

$$R(\cdot, t) \geq \frac{\alpha}{1 - (\frac{2\alpha}{n})(t - t_0)} = \frac{1}{\frac{1}{\alpha} - \frac{2(t-t_0)}{n}} \tag{3.4}$$

for $t \in [t_0, T]$. We finish the proof by taking $t_0 \searrow 0$ and $\alpha \searrow -\infty$. □

Corollary 3.3. *If $\alpha := \inf_{\mathcal{M}} R(\cdot, t=0) < 0$, then $\frac{V(t)}{(1+\frac{2(-\alpha)}{n}t)^{\frac{n}{2}}}$ is weakly decreasing, and in particular,*

$$V(t) \leq V(0)\left(1 + \frac{2(-\alpha)}{n}t\right)^{\frac{n}{2}}. \tag{3.5}$$

Proof.

$$\begin{aligned} \frac{d}{dt}\ln\left[\frac{V(t)}{(1+\frac{2(-\alpha)}{n}t)^{\frac{n}{2}}}\right] &= \frac{d}{dt}\left[\ln V - \frac{n}{2}\ln\left(1 + \frac{2(-\alpha)}{n}t\right)\right] \\ &= \frac{1}{V}\frac{dV}{dt} + \frac{\alpha}{1+\frac{2(-\alpha)}{n}t} \\ &= -\frac{1}{V}\int R dV + \frac{\alpha}{1+\frac{2(-\alpha)}{n}t} \\ &< -\inf_{\mathcal{M}} R(t) + \frac{\alpha}{1+\frac{2(-\alpha)}{n}t} \\ &\leq 0. \end{aligned} \tag{3.6}$$

The last equality is by Lemma 1.9, and the last inequality by Theorem 3.1. □

Remark 3.4.

1. If $T = \infty$, then
$$\overline{V} := \lim_{t\to\infty} \frac{V(t)}{(1 + \frac{2(-\alpha)}{n}t)^{\frac{n}{2}}} \tag{3.7}$$
exists.

2. [1] considers a local version of the Ricci flow: if $\overline{V} > 0$, then the open ball is becoming hyperbolic. If $\overline{V} = 0$, then it behaves like a graph manifold.

3.2 Upper bounds for $|\mathrm{Rm}|$

Proposition 3.5.

$$\frac{\partial}{\partial t}|\mathrm{Rm}|^2 \leq \Delta|\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + C|\mathrm{Rm}|^3, \tag{3.8}$$

where $C = C(n)$ is a constant.

Proof. Expanding the components of $|\mathrm{Rm}|^2$, one has

$$|\mathrm{Rm}|^2 = |\mathrm{Rm}|^2 = g^{ij}g^{kl}g^{ab}g^{cd}R_{ikac}R_{jlbd}. \tag{3.9}$$

Let $h \coloneqq \frac{\partial g}{\partial t}$. We first observe that $\frac{\partial}{\partial t}g^{ij} = -h^{ij}$. (Warning: h^{ij} is obtained by raising the indices instead of taking the inverse of h.) This can be seen by differentiating the constant matrix $\delta_i{}^j = g_{ik}g^{kj}$ to obtain

$$\begin{aligned} 0 &= \tfrac{\partial}{\partial t}(g_{ik}g^{kj}) \\ &= h_{ik}g^{kj} + g_{ik}\tfrac{\partial}{\partial t}g^{kj} \\ &= h^{pq}g_{ip}g_{kq}g^{kj} + g_{ik}\tfrac{\partial}{\partial t}g^{kj} \\ &= h^{pj}g_{ip} + g_{ik}\tfrac{\partial}{\partial t}g^{kj} \\ &= g_{ik}(h^{kj} + \tfrac{\partial}{\partial t}g^{kj}). \end{aligned} \tag{3.10}$$

The observation is valid because g is non-degenerate. By (1.12), we then have

$$\frac{\partial}{\partial t}g^{ij} = 2R^{ij} \tag{3.11}$$

$\frac{\partial}{\partial t}g^{ij} = 2R^{ij}$.

Differentiating (3.9) on both sides, and using (3.11) we obtain

$$\begin{aligned} &\tfrac{\partial}{\partial t}|\mathrm{Rm}|^2 \\ = \;& 2(R^{ij}g^{kl}g^{ab}g^{cd} + g^{ij}R^{kl}g^{ab}g^{cd} + g^{ij}g^{kl}R^{ab}g^{cd} \\ &+ g^{ij}g^{kl}g^{ab}R^{cd})R_{ikac}R_{jlbd} + 2 < \mathrm{Rm}, \tfrac{\partial}{\partial t}\mathrm{Rm} > \end{aligned} \tag{3.12}$$

By Cauchy–Schwarz inequality, the first term is bounded above by $|\mathrm{Rm}|^3$, which the second term, by Lemma 1.7, becomes

$$< \mathrm{Rm}, \tfrac{\partial}{\partial t}\mathrm{Rm} > \quad = \quad < \mathrm{Rm}, \Delta \mathrm{Rm} > + < \mathrm{Rm}, \mathrm{Rm} * \mathrm{Rm} > \tag{3.13}$$

Since $|\mathrm{Rm} * \mathrm{Rm}| \leq C|\mathrm{Rm}|^2$ for some constant $C = C(n)$, we can once more apply Cauchy–Schwarz inequality to bound the second term by $|\mathrm{Rm}|^3$ up to some constant.

We also observe that $d|\mathrm{Rm}|^2 = 2 < \mathrm{Rm}, \nabla \mathrm{Rm} >$ (the computation is tedious, but think of the case where Rm is replaced by a vector function F, and ∇F is the gradients of each component of F, then this boils down to the product rule). Expanding $\Delta|\mathrm{Rm}|^2$ by definition, we have

$$\begin{aligned} \Delta|\mathrm{Rm}|^2 &= \mathrm{tr}_{12}\nabla^2(|\mathrm{Rm}|^2) \\ &= \mathrm{tr}_{12}\nabla(d|\mathrm{Rm}|^2) \\ &= 2\mathrm{tr}_{12}\nabla(< \mathrm{Rm}, \nabla \mathrm{Rm} >) \\ &= 2|\nabla \mathrm{Rm}|^2 + 2 < \mathrm{Rm}, \Delta \mathrm{Rm} > . \end{aligned} \tag{3.14}$$

Again, the last equality is a generalization of that for functions to tensors. Then the proposition is proven. □

Theorem 3.6. *Suppose* $|\mathrm{Rm}| \leq M$ *at* $t = 0$. *Then for all* $t \in [0, T]$,

$$|\mathrm{Rm}| \leq \frac{M}{1 - \frac{1}{2}CMt}, \tag{3.15}$$

where C *is the same constant as in Proposition 3.5.*

Proof. By Proposition 3.5, we have

$$\frac{\partial}{\partial t}|\mathrm{Rm}|^2 \leq \Delta|\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + C|\mathrm{Rm}|^3 \leq \Delta|\mathrm{Rm}|^2 + C|\mathrm{Rm}|^3. \tag{3.16}$$

Let $u = |\mathrm{Rm}|^2$ and $F(r,t) = Cr^{\frac{3}{2}}$. Then the above implies $\frac{\partial}{\partial t}u \leq \Delta u + F(u,t)$, and $u(\cdot,t) \leq M^2$. Let $\phi(t) = (M^{-1} - \frac{1}{2}Ct)^{-2}$. Then

$$\begin{cases} \frac{\partial}{\partial t}\phi(t) &= \frac{C}{(M^{-1} - \frac{1}{2}Ct)^3}. = F(\phi(t), t), \\ \phi(0) &= M^2 \end{cases} \tag{3.17}$$

Now we can apply Theorem 2.1 to obtain (3.15). □

Remark 3.7.

1. By scaling invariance with respect to time t, we may assume $M = 1$. From Theorem 3.6, we know that $|\mathrm{Rm}|$ won't reach twice it's initial data until after $t = \frac{1}{C}$. Therefore, growth of $|\mathrm{Rm}|$ is bounded.

2. In contrast, there is no such bound for the scalar curvature. Indeed for $\mathcal{M}(t = 0) = S^2 \times \mathbb{H}^2$, the product of a 2-sphere and a hyperbolic surface, the initial scalar curvature $R(t = 0) = 0$, but for $t > 0$, $R \neq 0$. Then we can scale so that R blows up at $t = 0$.

4 Global curvature derivative estimates

Theorem 4.1. *Suppose that* $M > 0$ *and* $T = \frac{1}{M}$. *Then for all* $k \in \mathbb{N}$, *there exists* $C = C(n,k)$ *such that the following statement is true. If* $|\mathrm{Rm}| \leq M$ *on* $\mathcal{M} \times [0,T]$, *then for all* $t \in [0,T]$,

$$|\nabla^k \mathrm{Rm}| \leq \frac{CM}{t^{\frac{k}{2}}}. \tag{4.1}$$

Proof. We prove the theorem by induction on k.

Let's first consider $k = 1$. Let $u(x,t) = t|\nabla \mathrm{Rm}|^2 + \alpha|\mathrm{Rm}|^2$, where $\alpha \in \mathbb{R}$ is to be determined. Then

$$\frac{\partial}{\partial t}u = |\nabla \mathrm{Rm}|^2 + t\frac{\partial}{\partial t}|\nabla \mathrm{Rm}|^2 + \alpha\frac{\partial}{\partial t}|\mathrm{Rm}|^2. \tag{4.2}$$

We may run the similar argument as in the proof of Proposition 3.5 to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}|\nabla \mathrm{Rm}|^2 \\
= \; & 2(R^{ij}g^{kl}g^{ab}g^{cd} + g^{ij}R^{kl}g^{ab}g^{cd} + g^{ij}g^{kl}R^{ab}g^{cd} \\
&+ g^{ij}g^{kl}g^{ab}R^{cd})(\nabla \mathrm{Rm})_{ikac}(\nabla \mathrm{Rm})_{jlbd} + 2 < \nabla \mathrm{Rm}, \frac{\partial}{\partial t}\nabla \mathrm{Rm} >
\end{aligned} \tag{4.3}$$

Applying Cauchy–Schwarz inequality, we obtain $|\mathrm{Rm}||\nabla \mathrm{Rm}|^2$ as an upper bound for the first term. Now it remains to bound $< \nabla \mathrm{Rm}, \frac{\partial}{\partial t}\nabla \mathrm{Rm} >$.

we observe that

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla \mathrm{Rm} &= \nabla \frac{\partial}{\partial t}\mathrm{Rm} + \mathrm{Rm} * \nabla \mathrm{Rm} \\
&= \nabla(\Delta \mathrm{Rm}) + \nabla(\mathrm{Rm} * \mathrm{Rm}) + \mathrm{Rm} * (\nabla \mathrm{Rm}) \\
&= \Delta(\nabla \mathrm{Rm}) + \mathrm{Rm} * (\nabla \mathrm{Rm}),
\end{aligned} \tag{4.4}$$

where the first equality is due to Lemma 1.6, the second due to Lemma 1.7, and the third due to Lemmas 1.4 and 1.5. Then,

$$< \nabla \mathrm{Rm}, \frac{\partial}{\partial t}\nabla \mathrm{Rm} >=< \nabla \mathrm{Rm}, \Delta(\nabla \mathrm{Rm}) > + < \nabla \mathrm{Rm}, \mathrm{Rm} * (\nabla \mathrm{Rm}) > . \tag{4.5}$$

Replacing Rm with ∇Rm in (3.14), we obtain

$$2 < \nabla \mathrm{Rm}, \Delta(\nabla \mathrm{Rm}) >= \Delta|\nabla \mathrm{Rm}|^2 - 2|\nabla^2 \mathrm{Rm}|^2. \tag{4.6}$$

By Lemma 1.3 and Cauchy–Schwarz inequality, we have

$$| < \nabla \mathrm{Rm}, \mathrm{Rm} * (\nabla \mathrm{Rm}) > | \leq C|\mathrm{Rm}||\nabla \mathrm{Rm}|^2. \tag{4.7}$$

Thus,

$$\frac{\partial}{\partial t}|\nabla \mathrm{Rm}|^2 \leq \Delta|\nabla \mathrm{Rm}|^2 - 2|\nabla^2 \mathrm{Rm}|^2 + C|\mathrm{Rm}||\nabla \mathrm{Rm}|^2. \tag{4.8}$$

Together with Proposition 3.5, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}u &\leq |\nabla \mathrm{Rm}|^2 + t(\Delta|\nabla \mathrm{Rm}|^2 + C|\mathrm{Rm}||\nabla \mathrm{Rm}|^2) \\
&\quad + \alpha(\Delta|\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + C|\mathrm{Rm}|^3) \\
&= \Delta u + |\nabla \mathrm{Rm}|^2(1 + Ct|\mathrm{Rm}| - 2\alpha) + C\alpha|\mathrm{Rm}|^3.
\end{aligned} \tag{4.9}$$

Since $|\mathrm{Rm}| \leq M$, $t \leq \frac{1}{M}$, we can take $\alpha \geq \frac{1+C}{2}$ so that the second term is non-positive. Let $F(r,t) = C\alpha M^3$. Then u satisfies (2.1).

Let $\phi(t) = \alpha M^2 + C\alpha t M^3$. Then it is easy to verify that ϕ solves (2.2) with initial data $\phi(0) = \alpha M^2 \geq u(\cdot, 0)$. Then we can apply Theorem 2.1 to obtain

$$u \leq C(M^2 + tM^3) \leq CM^2. \tag{4.10}$$

because $t \leq \frac{1}{M}$. Expanding the definition of u gives $t|\nabla \mathrm{Rm}|^2 \leq CM^2$, which implies $|\nabla \mathrm{Rm}| \leq \frac{CM}{\sqrt{t}}$.

For $k > 1$, we consider $u = t^{\frac{k+1}{2}}|\nabla^k \mathrm{Rm}|^2 + \alpha|\nabla^{k-1}\mathrm{Rm}|^2$, and use the induction hypothesis and Lemmas 1.6 and 1.5 to obtain (4.1). The key to killing the second term in (4.9) is the non-positive term in (4.8). □

Corollary 4.2. *Under the same assumptions of Theorem 4.1, for* $j, k \in \{0\} \cup \mathbb{N}$*, there exists a constant* $C = C(j, k, n)$ *such that the following statement holds. If* $|\mathrm{Rm}| \leq M$ *on* $\mathcal{M} \times [0, T = \frac{1}{M}]$*, then for all* $t \in [0, T]$*,*

$$|\frac{\partial^j}{\partial t^j} \nabla^k \mathrm{Rm}| \leq \frac{CM}{t^{j+\frac{k}{2}}}. \tag{4.11}$$

Proof. By scaling invariance, one may assume $t = 1$ and $M \leq 1$. Then we mainly use Lemmas 1.6 and 1.5 to prove the statement. □

Theorem 4.3 ([2])**.** *Suppose that* $g(t)$ *is a Ricci flow defined on an arbitrary manifold* U *without boundary for* $t \in [0, T]$*.* g *is not necessarily complete. Suppose further that* $|\mathrm{Rm}| \leq M$ *on* $U \times [0, T]$*, and that* $\overline{B_{g(0)}(p, r)} \subset U$ *for some* $p \in U, r > 0$*, where* $B_{g(0)}(p, r)$ *is the geodesic ball with respect to* $g(0)$*, centered at* p*, of radius* r*. Then*

$$|\nabla \mathrm{Rm}(p, T)|^2 \leq C(n) M^2 (\frac{1}{r^2} + \frac{1}{T} + M). \tag{4.12}$$

References

[1] Perelman, Grisha. *Ricci flow with surgery on three manifolds* arXiv preprint math.DG/0303109 (2003).

[2] Shi, Wan-Xiong. *Deforming the metric on complete Riemannian manifolds.* Journal of Differential Geometry 30 (1989), no. 1, 223-301.

[3] Topping, Peter M. *Lectures on the Ricci Flow.* London Mathematical Society Lecture Note Series 325 (2006).