Harmonic map flow for almost-holomorphic maps

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Harmonic maps

Let \((M, g)\) and \((N, h)\) be compact Riemannian manifolds. The energy density of a map \(u : M \to N\) is given by

\[
e(u) = \frac{1}{2} |du|^2 = \frac{1}{2} g^{ij} h_{\alpha \beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.
\]

The Dirichlet functional is given by

\[
E(u) = \int_{\Sigma} e(u) dV_g.
\]

Harmonic maps are the critical points of \(E(u)\).
Extrinsic formulation: assume $N \subset \mathbb{R}^k$ isometrically (by Nash’s Theorem). Then $u$ is harmonic iff

$$(\Delta u)^T = 0.$$

Intrinsic formulation: $u$ is harmonic iff the tension field

$$\tau(u) = \text{tr}_g \nabla du = 0.$$

In coordinates:

$$\tau(u)^\alpha = g^{ij} \left( \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - (\Gamma^g)_{ij}^k \frac{\partial u^\alpha}{\partial x^k} + (\Gamma^h(u))^{\alpha}_{\beta \gamma} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \right).$$
Harmonic map flow is the downward gradient flow of the Dirichlet energy:

$$\frac{\partial u}{\partial t} = \tau(u).$$

⇒ $E(u(t))$ is decreasing along flow.

**Idea:** Limit as $t \to \infty$ will be a harmonic map (in the initial homotopy class).

Eells-Sampson (1964): *smooth* global existence and convergence if sectional curvature $K^N \leq 0$.  

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Dependence on $\dim(M)$

What if we remove the curvature assumption on $N$?

$\dim(M) = 1$: converges to a geodesic

$\dim(M) = 2$: critical dimension ($E(u)$ is conformally invariant), only Type-II blowup possible

$\dim(M) > 2$: supercritical, expect Type-I blowup ($\lambda(t) = \sqrt{T-t}$).
Critical harmonic map flow \((n = 2)\)

**Theorem (K.-C. Chang, W.-Y. Ding, and R. Ye 1992)**

*Harmonic map flow in dimension two can blow up in finite time!*
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\[
\partial_t \phi(r, t) = \partial_r^2 \phi + \frac{1}{r} \partial_r \phi - \frac{\sin 2\phi}{2r^2}
\]
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Critical harmonic map flow \((n = 2)\)

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\]

\(t = 0.2\)
Critical harmonic map flow ($n = 2$)

**Theorem (K.-C. Chang, W.-Y. Ding, and R. Ye 1992)**

*Harmonic map flow in dimension two can blow up in finite time!*

\[
\partial_t \phi(r, t) = \partial_r^2 \phi + \frac{1}{r} \partial_r \phi - \frac{\sin 2\phi}{2r^2}
\]
Struwe (1985): Global weak solution on $\Sigma \times [0, \infty)$ with finitely many singular times.

For each singular time $0 < T \leq \infty$, there exists a sequence $t_n \uparrow T$, a “body map” $u(T) \in W^{1,2}(\Sigma)$, a finite set $S \subset \Sigma$ and finitely many bubbles (ie. harmonic spheres) $\phi^i : S^2 \to N$, such that

1. **Convergence**: $u(t_n) \to u(T)$ in $C^{\infty}_{loc}(\Sigma \setminus S)$ and weakly in $W^{1,2}(\Sigma)$;
2. **Energy identity**: $\lim_{n \to \infty} E(u(t_n)) = E(u(T)) + \sum E(\phi^i)$;
3. **Connectedness of bubbles**: $\bigcup \phi^i(S^2)$ is connected at each singular point $x \in S$;
4. **“No neck” at infinity**: For $T = \infty$, $u(T)$ is harmonic and connected to the bubbles.

**Remark.** Reduced to study of $v_n : D_1 \to N$ with $\|\tau(v_n)\|_{L^2(D_1)} \to 0$ as $n \to \infty$. 
As $t \uparrow T$:

\[ \Sigma \]
\[ \delta \]
\[ R \lambda \]

(rescale)

\[ v(x) = u(\lambda x) \]

(limit)

Outer Energy Scale

Bubble tree

Body map

Neck?
Question. Suppose $T < \infty$ is a singular time. Is $u(T)$ continuous at the singular points? (I.e. has removable discontinuities.)

If yes, is $u(T)$ connected to the bubble tree? (I.e. $\lim_{x \to p} u(x, T) \in \bigcup \phi_i(S^2)$.)

- J. Qing (2003): $u(T)$ is asymptotically radial at singularity:

$$\partial_\theta u(x, T) \to 0 \text{ as } x \to p.$$ 

- Topping (2004): if $E(u(t))$ is Hölder continuous in $t$, then $u(x, T)$ is Hölder continuous in $x$.

- Topping (2004): NO in general!

  Note: counterexample has fastest possible Type-II blowup rate:

$$\lambda(t) = o \left( (T - t)^{\frac{1}{2} + \varepsilon} \right)$$

  for each $\varepsilon > 0$.

- **Topping’s conjecture:** If $N$ is real-analytic, then $u(T)$ is continuous.
The Kähler context

When $N$ has a complex structure, we have the decomposition

$$du^\mathbb{C} = \begin{pmatrix} \partial u & \bar{\partial} u \\ \bar{\partial} u & \partial u \end{pmatrix} : \underbrace{T^{1,0} \Sigma \oplus T^{0,1} \Sigma}_{\Sigma} \longrightarrow \underbrace{u^* T^{1,0} N \oplus u^* T^{0,1} N}_{u^* TN^\mathbb{C}}.$$ 

Hence

$$e(u) = |\partial u|^2 + |\bar{\partial} u|^2 =: e_\partial(u) + e_{\bar{\partial}}(u).$$

If $N$ is Kähler,

$$E_\partial(u) - E_{\bar{\partial}}(u) = \int_{\Sigma} u^* \omega_N =: \kappa$$

is an invariant of the homotopy class of $u$. Therefore

$$E(u) = E_\partial(u) + E_{\bar{\partial}}(u) = \kappa + 2E_{\bar{\partial}}(u).$$

$\Rightarrow$ Maps with energy close to $\kappa$ are almost holomorphic in the energy sense.

Preserved by harmonic map flow!

**Remark.** Strong analogy to 4d Yang-Mills

$$F_A = F^+_A + F^-_A.$$

Waldron (2014): 4d YM flow exists for all time if $\|F^+_A(0)\|_{L^2} \leq \varepsilon_0$.

2016: true in full generality.
2. Main results
Main results - 1

**Theorem 1** [C. Song-Waldron (2020)]

Let $\Sigma$ be a compact Riemann surface and $u : \Sigma \times [0, T) \to N$ a classical solution of HM flow. Suppose

- $N$ is compact Kähler with nonnegative holomorphic bisectional curvature
- $E_\bar{\partial}(u(0)) < \varepsilon_0(N)$.

Then for $0 < \tau \leq t < T$, we have the bounds:

- $\|\bar{\partial}u(t)\|_{L^\infty(\Sigma)} \leq C$
- $\|S(u(t))\|_{L^2(\Sigma)} \leq C$.

Here $S(u) = \langle du \otimes du \rangle - \frac{1}{2}|du|^2 g$ is the **stress-energy tensor** of $u$.

**Remark.** By the Generalized Frankel Conjecture [Mori (1979), Siu-Yau (1980), Mok (1988)], $N$ is biholomorphic to $\mathbb{CP}^n$ or a Hermitian symmetric space; but the metric $h$ need not be symmetric.

*E.g.* $h = \text{any metric on } S^2 \text{ with } R \geq 0.$
**Theorem 2** [C. Song-Waldron (2020)]

Let $\Sigma$ be a compact Riemann surface and $N$ any compact Riemannian manifold. Suppose $u : \Sigma \times [0, T) \rightarrow N$ is a HM flow with uniform $L^q$-bound, $q > 1$, on stress energy tensor:

$$\|S(u(t))\|_{L^q(\Sigma)} \leq C.$$

Then we have:

- **Improved blowup rate:** $\lambda(t) = O \left( (T - t)^{q/2} \right), \quad t \nearrow T$

- **Hölder continuity of body map:** $u(T) \in C^{1 - \frac{1}{q}}(\Sigma)$

- **Given any sequence** of times $t_n \nearrow T$, $u(t_n)$ sub-converge in the bubble-tree sense, with $u(T)$ connected to the bubble tree (i.e. $\lim_{x \to p} u(x, T) \in \bigcup \phi_i(S^2)$).
**Main results - III**

**Corollary** [C. Song-Waldron (2020)]

Let $u$ and $N$ be as in **Theorem 1**. Then we have:

- **Blowup rate:** $\lambda(t) = O(T - t), \quad t \nearrow T$
- **Hölder continuity:** $u(T) \in C^\alpha(\Sigma) \quad \forall \ 0 < \alpha < 1$
- $u(T)$ is connected to the bubble tree.

**Remark.** Compare (a) with blowup rate in rotationally symmetric case (Chang-Ding-Ye example).

Angenent, Hulshof, and Matano (2009): $\lambda(t) = o(T - t)$.

Raphael and Schweyer (2013) prove

$$\lambda(t) \sim \kappa \frac{|T - t|}{(\log |T - t|)^2} \quad (1)$$

for generic (rotationally symmetric) almost-holomorphic initial data, as predicted by Van den Berg, Hulshof, and King (2003).

Davila, Del Pino, and Wei (2020) construct many non-symmetric examples with blowup rate (1).
3. Sketch of proofs

\[ E_\delta(u) < \varepsilon_0 \quad \text{Thm 1} \quad \rightarrow \quad \text{Bound on } S \quad \text{Thm 2a} \quad \rightarrow \quad \text{Improved blowup rate} \]

\[ K^N_{\text{hol.bi.}} \geq 0 \quad \rightarrow \quad \downarrow \quad \text{Thm 2b} \quad \downarrow \quad \text{Decay of } |du| \quad \text{in neck region.} \]
Stress-energy tensor

Full variational formula

\[ \delta E_g(u) = - \int \left( \langle \tau(u), \delta u \rangle + \frac{1}{2} \langle S, \delta g \rangle \right) dV_g. \]

Stress-energy tensor

\[ S(u) = \langle du \otimes du \rangle - \frac{1}{2} |du|^2 g \in \text{Sym}^2 T^* M \]

satisfies

\[ \text{div } S = \langle \tau(u), du \rangle. \]

How to use in parabolic context?
Role of stress-energy in harmonic map flow

Take another divergence

$$\text{div}^2 S = \langle \nabla \tau(u), du \rangle + |\tau(u)|^2.$$ 

Under HM flow

$$\partial_t e(u) = \langle \nabla \tau(u), du \rangle.$$ 

⇒ “pointwise” energy identity:

$$\partial_t e(u) = -|\tau(u)|^2 + \text{div}^2 S.$$ 

Integrating, we recover the usual global energy identity:

$$E(u(t_2)) + \int_{t_1}^{t_2} \int_\Sigma |\tau(u)|^2 dV dt = E(u(t_1)).$$

Integrating against a cutoff function $\varphi$, we get a refined local energy inequality:

$$E \left( u(t_2), B_{\frac{R}{2}} \right) \leq E \left( u(t_1), B_R \right) + \int_{t_1}^{t_2} \int_\Sigma \langle \nabla^2 \varphi, S \rangle dV dt.$$
In polar coordinates, we have

\[ S = \frac{1}{2} \left( |u_r|^2 - \frac{1}{r^2} |u_\theta|^2 \right) (dr^2 - r^2 d\theta^2) + 2 \langle u_r, u_\theta \rangle dr d\theta, \]

Recall the Hopf differential

\[ \Phi(u) \overset{\text{loc}}{=} 2 \langle u_z, u_{\bar{z}} \rangle dz \otimes dz. \]

These are related by

\[ S(u) = 2 \text{Re} \Phi(u). \]

In particular,

\[ |S(u)| \leq 2|\Phi(u)| \leq 4|\partial u||\bar{\partial} u|. \]
Bochner formula [Schoen-Yau (1978)] for (harmonic) maps $\Sigma \to N$:

$$-\Delta e_\bar{\partial}(u) = -\langle \nabla \tau(u), \bar{\partial} u \rangle - 2|\nabla \bar{\partial} u| - 2R^\Sigma e_\bar{\partial}(u) + q_1(u) + q_2(u),$$

where $q_1(u) \leq Ce_\bar{\partial}(u)e_\bar{\partial}(u)$ and $q_2(u) \leq C(e_\bar{\partial}(u))^2$.

$q_1(u) \leq 0 \ \forall \ u \iff N$ has nonnegative holomorphic bisectional curvature.

Under HM flow, we obtain the “split” Bochner inequalities:

$$\begin{cases} 
\frac{1}{2} (\partial_t - \Delta) e_\bar{\partial}(u) \leq -|\nabla \bar{\partial} u|^2 - R^\Sigma e_\bar{\partial}(u) + C_N e_\bar{\partial}(u)^2 \\
\frac{1}{2} (\partial_t - \Delta) e_\partial(u) \leq -|\nabla \partial u|^2 - R^\Sigma e_\partial(u) + C_N e_\partial(u)^2.
\end{cases}$$

Q. Liu and Y. Yang (2010) generalize Topping’s rigidity theorem from $N = S^2_{std}$ to $N =$ cpt. Kähler with $K_{\text{hol.bi.}}^N \geq 0$. 
Proof of Theorem 1

**Theorem 1.** Suppose that

- $N$ is compact *Kähler* with $K_{\text{hol.bi.}}^N \geq 0$
- $E_\bar{\partial}(u(0)) < \varepsilon_0$.

Then

$$\|S(u(t))\|_{L^2(\Sigma)} \leq C,$$  
$0 < \tau \leq t < T.$

**Proof:**

1. $E_\bar{\partial}(u(0)) < \varepsilon_0 \Rightarrow E_\bar{\partial}(u(t)) < \varepsilon_0 \forall t > 0$, since $E = \kappa + 2E_\bar{\partial}$ and $E$ is decreasing.
2. $K_{\text{hol.bi.}}^N \geq 0 \Rightarrow$ split Bochner formula for $|\bar{\partial}u|$
3. $E_\bar{\partial}(u(t)) < \varepsilon_0 \Rightarrow \|\bar{\partial}u(t)\|_{L^\infty} \leq C$ by $\varepsilon$-regularity argument (from split Bochner f’la)
4. $\|S(u(t))\|_{L^2} \leq C\|\bar{\partial}u\|_{L^\infty} \|\partial u\|_{L^2} \leq C.$
Proof of Theorem 2a

**Theorem 2.** Given a uniform $L^q$-bound, $q > 1$, on stress energy tensor $\|S(u(t))\|_{L^q(\Sigma)} \leq C$, $\forall 0 \leq t < T$ we have:

- Improved blowup rate:
  \[ \lambda(t) = O\left((T - t)^{q/2}\right), \quad t \uparrow T. \]

- Hölder continuity: $u(T) \in C^{1 - \frac{1}{q}}(\Sigma)$
Energy scale

**Definition.** Let $\varepsilon, \rho > 0$ and $p \in M$.

For any $W^{1,2}$ map $u : \Sigma \to N$, the (outer) energy scale $\lambda(u) = \lambda_{\varepsilon, \rho, p}(u)$ is the minimal number, $\lambda \geq 0$, such that

$$
\sup_{\lambda < r < \rho} E(u, U_{r/2}^r(p)) < \varepsilon.
$$
Control of energy scale

Let $\varphi$ be a cutoff for $B_{\frac{R}{2}} \subset B_R$ with $|\nabla^2 \varphi| \leq \frac{C}{R^2}$.

Recall the refined local energy identity

$$E \left( u(t_2), B_{\frac{R}{2}} \right) \leq E \left( u(t_1), B_R \right) + \int_{t_1}^{t_2} \int_{\Sigma} \langle \nabla^2 \varphi, S \rangle dV dt.$$

Applying Hölder to the last term, we obtain

$$E \left( u(t_2), B_{\frac{R}{2}} \right) \leq E \left( u(t_1), B_R \right) + C(t_2 - t_1) \|\nabla^{(2)} \varphi\|_{L^q} \frac{q-1}{q} \sup_t \|S(u(t))\|_{L^q}.$$

We have

$$\|\nabla^{(2)} \varphi\|_{L^q} \frac{q-1}{q} \leq \left[ \int_{B_R} \left( \frac{C}{R^2} \right)^{\frac{q}{q-1}} dV \right]^{\frac{q-1}{q}} \leq CR \left(2 - \frac{2q}{q-1}\right)^{\frac{q-1}{q}} = CR^{\frac{2q}{q-1}}.$$

Therefore

$$E \left( u(t_2), B_{\frac{R}{2}} \right) \leq E \left( u(t_1), B_R \right) + C \frac{t_2 - t_1}{R^2} \sup_t \|S(u(t))\|_{L^q}.$$

$\Rightarrow$ Need $(t_2 - t_1)^{\frac{q}{2}} \geq R$ for energy scale to decrease.
Proof of Theorem 2b

**Theorem 2.** Given

\[ \|S(u(t))\|_{L^q(\Sigma)} \leq C, \quad \forall 0 \leq t < T \]

we have

1. \( \lambda(t) = O( (T - t)^{q/2} ) \)
2. \( u(T) \in C^{1 - \frac{1}{q}}(\Sigma) \)

\[ E(\bar{\partial})(u) < \varepsilon_0 \quad \text{Thm 1} \quad \rightarrow \quad \text{Bound on } S \quad \text{Thm 2a} \quad \rightarrow \quad \text{Improved blowup rate} \]

\[ K_{\text{hol.bi.}}^N \geq 0 \quad \rightarrow \quad \text{Decay of } |du| \text{ in neck region.} \]
Evolution of angular energy

For $u : D_1(p) \times [0, T) \to N$, define the angular energy $f(r, t) := \left( \int_{\{r\} \times S^1} |u_\theta(r, \theta, t)|^2 \, d\theta \right)^{\frac{1}{2}}$.

Direct computation gives

$$\partial_t f - \left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{1 - \eta}{r^2} \right) f \leq 0,$$

where $\eta = C_N \sup r^2|du|^2$. 

Given $T > 1$ and $0 < R < 1$, let

$$
\Omega_q = \left\{(r, t) \in [R, 1] \times [0, T) \mid r \geq (1 - t)^{q/2}\right\}.
$$

Let $\nu = \sqrt{1 - \eta}$, and choose $\frac{1}{q} \leq \mu < \nu$.

**Lemma.** Suppose $f$ satisfies (2) on $\Omega_q$, with $f \leq \sqrt{\epsilon}$ on the parabolic boundary of $\Omega_q$. Then

$$
f(r, t) \leq C\sqrt{\epsilon} \left( \left( \frac{R}{r} \right)^\nu + r^\mu \right)
$$

for $R \leq r \leq 1$ and $1 \leq t < T$.

**Proof.** Supersolution of (2) on $\Omega_q$:

$$
\left( \frac{R}{r} \right)^\nu + \frac{(1 - t)_+ + r^{2\nu}}{r^\nu} \frac{q\nu/2}{r^\nu} + \frac{\nu + 1}{\nu^2 - \mu^2} r^\mu.
$$
Decay estimate in neck region

**Theorem.** Suppose $\|S(u(t))\|_{L^q}$ is uniformly bounded. Then for $\rho > 0$ sufficiently small, the decay estimate

$$r|du(x, t)| \leq C_q \sqrt{\varepsilon} \left( \frac{\lambda_{\varepsilon, \rho, p}(t)}{r} + \frac{r}{\rho} \right)^{1 - \frac{1}{q}}$$

(3)

holds for $2\lambda_{\varepsilon, \rho, p}(t) \leq |x| \leq \rho/2$ and $T - \rho^2 \leq t < T$.

**Proof.** Theorem 2a + lemma $\Rightarrow$ decay estimate on angular energy.

Since

$$S_{rr} = \frac{1}{2} \left( |u_r|^2 - \frac{1}{r^2} |u_\theta|^2 \right),$$

decay estimate on angular energy $+ L^q$-bound on $S \Rightarrow$ decay estimate on radial energy. $\square$

Hölder continuity and connectedness follow directly from (3).
Thank you!