Crash course in Riemannian geometry

1. Metrics
2. Tensors
3. Covariant derivs
4. Riemann curvature tensor.

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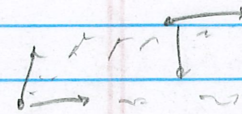
1. M = smooth mfd.

A Riemannian metric, g, is a smoothly varying inner product on the tangent spaces of M.

$v, w \in T_xM \rightsquigarrow g_x(v,w) \in \mathbb{R}$



↳ pos. def. symmetric bilinear.

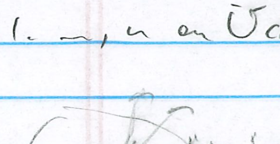
$X, Y \in \mathfrak{X}(M) \rightsquigarrow g(X,Y) \in C^\infty(M)$.

$f, h \in C^\infty(M)$

$g(fX + hY, Z) = f\,g(X,Z) + h\,g(Y,Z)$.

Local notation: fix coords x^i $i=1,\dots,n$ on $U \subset M$

$\left\{\partial_i = \frac{\partial}{\partial x^i}\right\}_{i=1}^n$ local frame for $TM\big|_U$



Let $g_{ij} = g(\partial_i, \partial_j)$.

= local components: $(g_{ij}(x))$ = matrix-valued function.

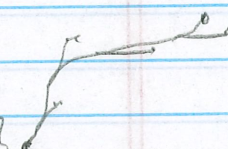
Einstein summation convention

$X = \cancel{\sum_i} X^i \partial_i , \quad Y = \cancel{\sum_j} Y^j \partial_j \in \mathfrak{X}(M)$.

$$
\begin{aligned}
g(X,Y) &= g(X^i\partial_i , Y^j \partial_j) \\
&= X^i Y^j g(\partial_i , \partial_j) \\
&= X^i Y^j g_{ij} .
\end{aligned}
$$

From Riem. metric get length, distance & volume

$$\ell(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\, dt$$



$$d_g(p,q) = \operatorname{Inf} \{ \ell(\gamma) \mid \gamma \text{ connects } p \text{ and } q \}.$$

Metric on M in usual sense!

$$dV_g \overset{loc}{=} \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$$

$\rightarrow$ oriented basis.

Riemannian volume form.

(also on k-dim'l submfds.)

2. A covariant q-tensor $T \in T^{(0,q)}$ is a smoothly varying multilinear function on $\overbrace{T_xM \times \cdots \times T_xM}^{q \text{ times}}$.

Given $X_1, \ldots, X_q \in \mathcal{X}(M)$,

$$T(X_1, \ldots, X_q) \in C^\infty(M)$$

Defines C^∞-linear function

$$T : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \longrightarrow C^\infty(M).$$

$$T(fX + gY, X_2, \ldots, X_q) = fT(X, X_2, \ldots, X_q) + gT(Y, X_2, \ldots, X_q).$$

Tensor product:

$T \in T^{(0,q)}, S \in T^{(0,\ell)} \longmapsto S \otimes T \in T^{(0,q+\ell)}$

$(S \otimes T)(X_1, \ldots, X_q, X_{q+1}, \ldots, X_\ell) = S(\quad)$

Locally: $x^i \rightsquigarrow \frac{\partial}{\partial x^i}$.

$$T_{i_1 \cdots i_q} = T(\partial_{i_1}, \ldots, \partial_{i_q}), \quad i_1, \ldots, i_q = 1, \ldots, n.$$

n^q functions!

Local components. Determine T uniquely by:

$X_1 = X_1^i \partial_i$ etc.

$$T(X_1, \ldots, X_q) = X_1^{i_1} \cdots X_q^{i_q} T_{i_1 \cdots i_q}.$$

Different coords? $\{\tilde{x}^i\}_{i=1}^n$

$$\tilde{T}_{j_1 \cdots j_q} = T\left(\frac{\partial}{\partial \tilde{x}^{j_1}}, \cdots, \frac{\partial}{\partial \tilde{x}^{j_q}}\right)$$

$$= T\left(\frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial x^{i_q}}{\partial \tilde{x}^{j_q}} \frac{\partial}{\partial x^{i_q}}\right)$$

$$= \underbrace{\frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{i_q}}{\partial \tilde{x}^{j_q}}}\, T_{i_1 \cdots i_q} \, .$$

Transf. law for covariant tensor components.

$\rightsquigarrow$ Anything that "transforms as a tensor" gives a well-def'd fcl on $\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)$.

$\rightsquigarrow T^{(0,q)} =$ sections of $\underbrace{T^*M \otimes \cdots \otimes T^*M}_{q \text{ times}}$.

Ex: • $g \in T^{(0,2)}$, Riem. metric

• $\Omega^q(M) \subset T^{(0,q)}$, alternating tensors $=$ differential q-forms!

Mixed tensors: $T^{(p,q)} =$ smoothly varying multilinear fcns on $\underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{q \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{p \text{ times}}$.

Ex: $\mathfrak{X}(M) = T^{(1,0)}M$.

$X \rightsquigarrow$ fcn on diff'l forms:

$$\alpha \longmapsto \alpha(X) \in C^\infty(M)$$

$X \overset{\text{loc}}{=} X^i \partial_i$, $\quad \alpha \overset{\text{loc}}{=} \alpha_j dx^j$

$$\begin{aligned} \alpha(X) &= \alpha_j dx^j (X^i \partial_i) \\ &= \alpha_j X^i dx^j(\partial_i) \\ &= \alpha_j X^i \delta^j{}_i \end{aligned}$$

Reason for Einstein convention! $\boxed{\alpha(X) = \alpha_i X^i}$ $\quad \tilde{\alpha}_k = \alpha_i \frac{\partial x^i}{\partial \tilde{x}^k}$.

Check in local coords: $X = X^i \frac{\partial}{\partial x^i} = X^i \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} \Rightarrow \tilde{X}^j = X^i \frac{\partial \tilde{x}^j}{\partial x^i}$

Can use this duality to contract indices:

$$\mathrm{tr}_{m,n} : T^{(p,q)} \longrightarrow T^{(p-1,q-1)}$$

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \longmapsto T^{i_1 \cdots \overset{m\text{th}}{i} \cdots i_p}{}_{j_1 \cdots \underset{n\text{th}}{i} \cdots j_q}$$

Can also use a metric g to raise & lower indices:

$$(\ )^\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$$

$$X^\flat = g(X, -).$$

$$(X^i) \longmapsto g_{ij} X^j.$$

isomorphism. Let $(\ )^{\#} : \Omega^1(M) \to \mathfrak{X}(M)$ be the inverse. $\alpha \mapsto \alpha^{\#}$.

Local f'la: let (g^{ij}) = inverse matrix of (g_{ij})

$$(\alpha^{\#})^i = g^{ij}\alpha_j .$$

Can do on any tensor:

$$(\ )^{\#}_{M} : T^{(p,q)} \longrightarrow T^{(p+1,q-1)}$$

$$T^{i_1 \dots i_p}{}_{j_1 \dots j_q} \mapsto g^{i_{p+1} l}\, T^{i_1 \dots i_p}{}_{j_1 \dots l \dots j_q}$$

$$=: (T^{\#})^{i_1 \dots i_p \ \ i_{p+1}}{}_{j_1 \dots \ \ \dots j_q}$$

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3. How to take the derivative of a tensor?

<u>Special case</u>: $C^\infty(M)$

- 0-tensor = C^∞ fn.

$X \qquad f \rightsquigarrow X(f) \in C^\infty(M)$.

$$f \rightsquigarrow df \in T^{(0,1)}$$

$$X \mapsto X(f) \quad !$$

<u>tensorial</u> b/c $(gX + hY)(f) = gX(f) + hY(f)$ ✓.

- d extends to <u>alternating</u> covariant tensors

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$\hat{T}^{(0,k)} \qquad \hat{T}^{(0,k+1)}$$

$$d\omega(X_0, X_1, \dots, X_k) = \sum (-1)^i X_i(\omega(X_0, \dots \hat{X}_i, \dots, X_k)) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots \hat{X}_i \dots \hat{X}_j, X_k)$$

- Lie bracket of v.-fields:

$X, Y \in \mathcal{X}(M)$ $\quad X \circ Y : f \mapsto X(Y(f))$ not derivn.

but $[X, Y]$ is.

$$\overset{\shortparallel}{\mathcal{L}_X Y}.$$

<u>Extends</u>:

$$\mathcal{L}_X(T \otimes S) = \mathcal{L}_X T \otimes S + T \otimes \mathcal{L}_X S$$

$$(\mathcal{L}_X T)^{i_1 \dots i_p}{}_{j_1 \dots j_q} = X^k \partial_k T^{i_1 \dots i_p}{}_{j_1 \dots j_q}$$

$$- \partial_k X^{i_1} T^{k i_2 \dots i_p}{}_{j_1 \dots j_q} - \dots - \partial_k X^{i_p} T^{i_1 \dots k}{}_{\dots}$$

$$+ \partial_{j_1} X^k T^{i_1 \dots i_p}{}_{k j_2 \dots j_q} + \dots + \partial_{j_q} X^k T^{\dots i_p}{}_{j_1 \dots k}.$$

$\mathcal{L}_{fX} T \neq f \mathcal{L}_X T \Rightarrow$ not determined at p by $X(p)$.

Can fix this but not (yet) in a unique way.

<u>Defn</u>. a <u>covariant derivative</u> is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$X\ ,\ Y \longmapsto \nabla_X Y$$ satisfying:

(1) For constants $c_1, c_2 \in \mathbb{R}$,

$$\nabla_X (c_1 Y_1 + c_2 Y_2) = c_1 \nabla_X Y_1 + c_2 \nabla_X Y_2$$

(2) For $f_1, f_2 \in C^\infty(M)$, have

$$\nabla_{(f_1 X_1 + f_2 X_2)} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$$

(3) For $g \in C^\infty(M)$, have:

$$\nabla_X (gY) = X(g) Y + g \nabla_X Y. \quad \text{(Leibniz)}$$

<u>Examples</u>: directional deriv. on $\mathbb{R}^n$: $\nabla^0_X Y = (X^i \partial_i Y^j) \partial_j$. induced conn. on $M \subset \mathbb{R}^n$: $\pi^{\top} \nabla^0_X$

<u>Locally</u>: Let $A^k_{ij} \in C^\infty(U)$ be determined by

$\partial_i = \frac{\partial}{\partial x^i}$ $$\nabla_{\partial_i} \partial_j = A^k_{ij} \partial_k .$$

Then for $X = X^i \partial_i$, $Y = Y^j \partial_j$, have:

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i \partial_i} (Y^j \partial_j) \\ &\overset{(2)}{=} X^i \nabla_{\partial_i} (Y^j \partial_j) \\ &= X^i \partial_i Y^j \partial_j + X^i Y^j \nabla_{\partial_i} \partial_j \end{aligned}$$

$\nwarrow A^k_{ij} \partial_k$ $\cancel{A^j_{ik} \partial_j}$

$$= X^i \partial_i Y^j + X^i Y^j A^k_{ij} \partial_k$$

$$= X^i \left(\partial_i Y^k + A^k_{ij} Y^j\right) \partial_k .$$

Local comp'ts determine ∇ by this rule.

Conversely, any A^k_{ij} (a matrix-val'd fcns A_i) determine a connection, locally.

⚠ A^k_{ij} is not a tensor in j and k! (not C^∞-linear in Y, by (3)).

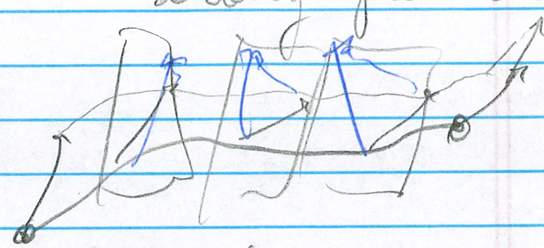
Difference of two conn's is a (1,2)-tensor:

$$(\nabla_X - \nabla'_X)(gY) = \cancel{X(g)Y} + g\nabla_X Y - \cancel{X(g)}Y - g\nabla'_X Y$$

$$= g(\nabla_X - \nabla'_X)Y \qquad \checkmark .$$

Convenient for end. eqns. (next time).

connection $\rightsquigarrow$ parallel transport rule along paths.



$$X \text{ parallel along } \gamma \iff \nabla_{\gamma'(t)}(X) \equiv 0.$$

Induced connection on $\Omega^1(M)$:

want $X(\alpha(Y)) = \nabla_X \alpha(Y) + \alpha(\nabla_X Y)$.

$$\partial_i \alpha_j Y^j = (\nabla_{\partial_i} \alpha_j) Y^j + \alpha_j (\cancel{\partial_i Y^j} + A^j_{ik} Y^k)$$

$$(\partial_i \alpha_j) Y^j + \alpha_j \cancel{\partial_i Y^j}$$

$$((\partial_i \alpha_j)\cancel{Y^j} - \alpha_k A^k_{ij})\cancel{Y^j} = \nabla_{\partial_i} \alpha_j \cancel{Y^j}$$

$$\boxed{\nabla_{\partial_i} \alpha_j = \partial_i \alpha_j - A^k_{ij} \alpha_k.}$$

Similarly: for $T \in T^{(p,q)}$,

$$(\nabla_{\partial_i} T)^{i_1 \dots i_p}_{j_1 \dots j_q} = \partial_i T^{i_1 \dots i_p}_{j_1 \dots j_q} + A^{i_1}_{ik} T^{k i_2 \dots i_p}_{j_1 \dots j_q} + \cdots - A^k_{i j_1} T^{i_1 \dots i_p}_{k j_2 \dots j_q} - \cdots.$$

$\boxed{\text{DEFINES } \nabla : T^{(p,q)} \to T^{(p,q+1)}}$ $\nabla T(X, X_1, \dots, X_q, \dots) = \nabla_X$

Now bring back the Riem. metric g.

Want to attach a canonical connection $\nabla = \nabla^g$ to g.

Add two requirements:

Metric-compatibility:

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Torsion-freeness:

$$\nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i$$

(compatibility with d.)

First fund'l Lemma of Riem. geom.

Given g, $\exists!$ tor-free compatible conn. $\nabla = \nabla^g$, called the Levi-Civita connection, with the local f'la:

$$A_{ij}^k = \Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

"Christoffel symbols"

Pf: $\partial_i g_{jk} = g(\nabla_i \partial_j, \partial_k) + g(\partial_j, \nabla_i \partial_k)$

$= \Gamma_{ij}^{l} g_{lk} + \Gamma_{ik}^{l} g_{jl}$

$= \Gamma_{ij}^{l} g_{kl} + \cancel{\Gamma_{ik}^{l}} g_{jl}$

$+$

Cyclic: $\partial_j g_{ki} = \cancel{\Gamma_{jk}^{l}} g_{il} + \Gamma_{ij}^{l} g_{kl}$

$-$

$\partial_k g_{ij} = \cancel{\Gamma_{ki}^{l} g_{jl}} + \cancel{\Gamma_{jk}^{l}} g_{il}$

$$\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} = 2\Gamma_{ij}^{l} g_{kl}$$

(which compatible)

Unique locally $\Rightarrow$ well-defined globally. □

Examples: Directional deriv., induced conn.

4. Geometric properties of a connection?

Example:

Curvature!

<u>Defn</u>. $X, Y, Z \in \mathcal{X}(M)$

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

↳ Riemann curvature operator

$$R(X,Y) : \mathcal{X}(M) \to \mathcal{X}(M).$$

<u>Second fund'l lemma of Riem. geom</u>:

$R(X,Y)Z$ is tensorial in X, Y, and Z!

<u>Pf</u>: $\nabla_X \nabla_Y (f \cdot Z) - \nabla_Y \nabla_X (f \cdot Z) - \nabla_{[X,Y]}(f \cdot Z)$

$$= \nabla_X \big(Y(f) Z + f \nabla_Y Z\big) - \nabla_Y \big(X(f) \cdot Z + f \nabla_X Z\big)$$

$$- [X,Y](f) Z + f \nabla_{[X,Y]} Z$$

$$= f\, R(X,Y) Z$$

✓. Can also check X & Y.

$\Rightarrow$ $R(\cdot,\cdot)\cdot$ is $(3,1)$-tensor.

<u>Defn</u>. The <u>Riemann curvature tensor</u> is the $(0,4)$-tensor defined by:

$$R(X,Y,Z,W) = g\big(R(X,Y)W, Z\big).$$

Example: $S^n \subset \mathbb{R}^{n+1}$

$$\nabla = \pi \circ \nabla^0 \quad \text{, induced conn.}$$

componentwise.

X, Y tangent to S^n. $\Rightarrow$ $\nabla_X Y = \pi(X(Y))$

$$= X(Y) + \langle X, Y\rangle \vec{r} .$$

$\langle Y, \vec{r}\rangle = 0$

$0 = X\langle Y, \vec{r}\rangle = \langle X(Y), \vec{r}\rangle + \langle Y, X(\vec{r})\rangle$

$\qquad X(\vec{r}) = X$.

$$\nabla_X \nabla_Y Z = X\left(Y(Z) + \langle Y, Z\rangle \vec{r}\right) + \langle X, Y(Z) + \cancel{\langle Y, Z\rangle \vec{r}}\rangle \vec{r}$$

$$= X(\cancel{Y(Z)}) + \langle Y, Z\rangle X + \left(\langle X(Y), Z\rangle + \cancel{\langle Y, X(Z)\rangle} + \langle X, Y(Z)\rangle\right)\vec{r}$$

$$\nabla_Y \nabla_X Z = Y(\cancel{X(Z)}) + \langle X, Z\rangle Y + \left(\langle Y(X), Z\rangle + \langle X, Y(Z)\rangle + \cancel{\langle Y, X(Z)\rangle}\right)\vec{r}$$

$$\nabla_{[X,Y]} Z = [\cancel{X, Y}] Z + \langle [X, \cancel{Y}], Z\rangle \vec{r}$$

$$R(X, Y) Z = \langle Y, Z_w\rangle X - \langle X, Z_w\rangle Y$$

$$R(X, Y, Z, W) = \langle Y, W\rangle\langle X, Z\rangle - \langle X, W\rangle\langle Y, Z\rangle .$$

$R_{ijkl} = g_{ik} g_{jl} - g_{il} g_{jk}$. Can see here!

$\boxed{\text{Constant positive curvature.}}$

Bianchi identities: $R(Y, X, Z, W) = -R(X, Y, Z, W) = R(X, Y, W, Z)$

$$R(X, Y, Z, W) = R(Z, W, X, Y)$$

(1st BIANCHI)

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$$

$\nabla R(U, X, Y, Z, W)$
$+ \nabla R(X, Y, U, Z, W)$
$+ \nabla R(Y, U, X, Z, W) = 0.$

(2ND BIANCHI)