COMPLEX MANIFOLDS (MTH 935)

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Abstract. These are notes for a first/second-year graduate course taught in the spring of 2020 at MSU.

Contents

References

Griffiths and Harris, *Principles of algebraic geometry*

Huybrechts, *Complex geometry: an introduction*

Kodaira and Morrow, *Complex manifolds*

Gunning and Rossi, *Analytic functions of several complex variables*

Forster, *Lectures on Riemann surfaces*

Donaldson, *Riemann surfaces.*

1. Single-variable complex analysis

We begin by reviewing complex analysis in one variable. A good reference is Rudin's *Real and Complex Analysis*, Ch. 10.

1.1. **Holomorphic functions.**

Fix a formal variable i satisfying

$$i^2 = -1.$$

The field of **complex numbers** $\mathbb{C} = \mathbb{R}[i]$ will be identified with the $x - y$ plane as follows:

$$\mathbb{C} = \{x + iy \mid (x, y) \in \mathbb{R}^2\}.$$

We endow $\mathbb{C}$ with the norm

$$|z| = \sqrt{z\bar{z}}$$

which agrees with the Euclidean norm on $\mathbb{R}^2$. A **domain** $\Omega \subset \mathbb{C}$ will refer to a *connected* open subset $\Omega \subset \mathbb{C}$.

Definition 1.1.1. A complex-valued function $f : \Omega \to \mathbb{C}$ is said to be **holomorphic** if the limit

$$f'(z) := \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z + h) - f(z)}{h}$$

exists, as a complex number, for each $z \in \Omega$. We shall denote the set of holomorphic functions on Ω by $\mathrm{Hol}(\Omega)$, and write $f(z) \in \mathrm{Hol}\left(\bar{\Omega}\right)$ if $f(z)$ is also continuous on the closure $\bar{\Omega}$.

With this definition, the ordinary rules of calculus apply to holomorphic functions. In particular, with proofs unchanged, we have:

Product rule: $(f(z) \cdot g(z))' = f'(z) \cdot g(z) + f(z) \cdot g'(z)$

Chain rule: $f(g(z))' = f'(g(z)) \cdot g'(z)$

Here complex multiplication is intended. The product rule implies that $\mathrm{Hol}(\Omega)$ is a ring under complex multiplication. The chain rule implies, more surprisingly, that the ring of entire functions $\mathrm{Hol}(\mathbb{C})$ is also closed under composition.

Example 1.1.2. (a) For $n \geq 0$, $f(z) = z^n$ is holomorphic, with $f'(z) = nz^{n-1}$.

(b) The exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^x (\cos(y) + i \sin(y))$$

is holomorphic, with $(e^z)' = e^z$. Indeed, any power series $f(z) = \sum_{n \geq 0} c_n z^n$ is holomorphic within its radius of convergence

$$R = \limsup_{n \to \infty} |c_n|^{-1/n}. \tag{1.1}$$

(c) The logarithm function can be defined as the inverse of the exponential function restricted to $\{(x,y) \mid -\infty < x < \infty, -\pi < y < \pi\}$. By the chain rule, $\log(z)$ is automatically holomorphic in its domain. In radial coordinates on $\mathbb{R}^2$, we can write out

$$\log(z) = \log r + i\theta, \quad -\pi < \theta < \pi.$$

With this convention, the domain of $\log(z)$ is $\mathbb{C} \smallsetminus \{(x,0) \mid x \leq 0\}$. So $\log(z)$ is by no means an entire function, but could be extended to a "multivalued" holomorphic function on $\mathbb{C} \smallsetminus \{0\}$.

Next, we consider the extension of the fundamental theorem of calculus to holomorphic functions. Given a piecewise C^1 path $\gamma(t) : [a,b] \to \Omega$, we define the **integral along** γ by

$$\int_\gamma F'(z)\, dz = \int_a^b F'(\gamma(t)) \cdot \gamma'(t)\, dt$$

where again, complex multiplication is intended inside the integral. As with ordinary line integrals, this is independent of the parametrization of γ (although it does depend up to $\pm$ on the direction).

Theorem 1.1.3 (FToC for holomorphic functions)**.** *Let $F(z) \in \mathrm{Hol}(\Omega)$ and let γ be a path contained in Ω. Then*

$$\int_\gamma F'(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

This can be proved exactly as in single-variable calculus, or derived as a special case of the result for line integrals.

Remark 1.1.4. We shall often be concerned with domains Ω that have **piecewise C^1 boundary**. This means that Ω is bounded, and the boundary set $\partial\Omega = \bar{\Omega} \smallsetminus \Omega$ consists of a finite collection of C^1 curves $\{\gamma(t)\}$, whose endpoints cancel (as they must). If we give $\mathbb{C}$ the standard orientation (corresponding to the ordered basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$), then each curve in $\partial\Omega$ can be oriented by the convention

$$\left\{\gamma'(t), \nu_{\gamma(t)}\right\} \sim \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$$

where ν is the inward normal to Ω. Informally, this means that we always parametrize $\partial\Omega$ such that Ω stays "on the left" of its boundary curves, as in the counterclockwise orientation of $S^1 = \partial B_1$.

1.2. **Cauchy's Theorem: two approaches.**

The foundational result of complex analysis is as follows.

Theorem 1.2.1 (Cauchy's Theorem)**.** *Let Ω be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary, and let $f(z) \in \mathrm{Hol}\left(\bar{\Omega}\right)$. Then*

$$\int_{\partial\Omega} f(z)\, dz = 0.$$

1.2.1. *Goursat's proof.* The best proof of Cauchy's theorem, due to Goursat, is based on the following simple observation.

Lemma 1.2.2. *Assume that* γ *is a* closed *curve in* Ω, *and* $F(z) \in \mathrm{Hol}(\Omega)$. *Then*

$$\int_\gamma F'(z)\,dz = 0.$$

Proof. Since γ is a closed curve, we have $\gamma(a) = \gamma(b) \in \mathbb{C}$. By the fundamental theorem of calculus for holomorphic functions, we have

$$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0$$

as claimed. □

Proposition 1.2.3. *For a closed curve* γ *in* $\mathbb{C}$, *and* $n \geq 0$, *we have*

$$\int_\gamma z^n\,dz = 0.$$

Proof. The entire function $f(z) = z^n$ has an entire, holomorphic antiderivative $F(z) = \frac{z^{n+1}}{n+1}$. Hence

$$\int_\gamma z^n\,dz = \int_\gamma F'(z)\,dz = 0$$

by the previous lemma. □

Example 1.2.4. We check the integral of $(z - z_0)^n$ over the boundary of a ball $B_r(z_0)$ directly. We have $\partial B_r(z_0) = [\gamma(\theta)]$, where $\gamma(\theta) = z_0 + re^{i\theta}, 0 \leq \theta \leq \pi$. Then $\gamma'(\theta) = ire^{i\theta}$, hence

$$\begin{aligned} \int_{\partial B_r(z_0)} (z - z_0)^n\,dz &= \int_0^{2\pi} r^n e^{in\theta} \cdot ire^{i\theta}\,d\theta \\ &= i\int_0^{2\pi} r^{n+1} e^{i(n+1)\theta} d\theta \\ &= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1. \end{cases} \end{aligned} \tag{1.2}$$

Exercise: Explain how the result of this calculation is consistent with Lemma 1.2.2.

Theorem 1.2.5 (Cauchy-Goursat for a triangle)**.** *Let* $\Delta \Subset \Omega$ *be a triangle. Then for* $f \in \mathrm{Hol}(\Omega)$, *we have*

$$\int_{\partial\Delta} f(z)\,dz = 0.$$

Proof. Let $I = |\int_{\partial\Delta} f(z)\,dz|$. Draw the lines between the midpoints, dividing Δ into four smaller triangles with half the side length. Let γ_j, $j = 1, 2, 3, 4$, denote the boundaries of these triangles, oriented counterclockwise. Then as cycles

$$\partial\Delta = \sum_{j=1}^4 \gamma_j$$

since the segments connecting the midpoints are traversed in opposite directions. Therefore

$$I = |\sum_{j=1}^{4} \int_{\gamma_j} f(z)\, dz|$$

and we may choose j_1 such that

$$|\int_{\gamma_{j_1}} f(z)\, dz| \geq I/4.$$

Let Δ_1 be the triangle with $\partial \Delta_1 = \gamma_{j_1}$.

Next, subdivide Δ_1 in the same way, and choose Δ_2 such that

$$|\int_{\partial \Delta_2} f(z)\, dz| \geq I/4^2.$$

Continuing in this fashion, we obtain nested triangles

$$\cdots \supset \Delta_{n-1} \supset \Delta_n \supset \Delta_{n+1} \supset \cdots$$

with

$$\operatorname{diam}(\Delta_n) \leq \operatorname{diam}\Delta / 2^n$$

and

$$|\int_{\partial \Delta_n} f(z)\, dz| \geq I/4^n.$$

Let

$$z_0 = \bigcap_{n \geq 0} \Delta_n.$$

Then $z_0 \in \bar{\Delta} \subset \Omega$, hence $f'(z_0)$ exists.

Let $\epsilon > 0$. Since $f'(z_0)$ exists, we may choose n such that

$$\left| f'(z_0) - \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon$$

for all $z \in \Delta_n$. Then

$$|f(z) - f(z_0) - (z - z_0) f'(z_0)| < \epsilon |z - z_0|.$$

By Proposition 1.2.3, we have

$$\int_{\partial \Delta_n} f(z)\, dz = \int_{\partial \Delta_n} \left(f(z) - f'(z_0)(z - z_0) - f(z_0) \right)\, dz.$$

But then

$$\begin{aligned} |\int_{\partial \Delta_n} f(z)\, dz| &\leq \int_{\partial \Delta_n} |f(z) - f'(z_0)(z - z_0) - f(z_0)|\, dz \\ &\leq \epsilon \int_{\partial \Delta_n} |z - z_0|\, dz \\ &\leq \epsilon (3 \operatorname{diam}\Delta_n)(\operatorname{diam}\Delta_n) \\ &\leq 3\epsilon \left(\operatorname{diam}\Delta_n \right)^2 \\ &\leq \frac{3\epsilon \left(\operatorname{diam}\Delta \right)^2}{4^n}. \end{aligned}$$

By choice of the triangles Δ_n, this yields

$$\frac{I}{4^n} \le \left| \int_{\partial\Delta_n} f(z)\, dz \right| \le \frac{3\epsilon\, (\mathrm{diam}\Delta)^2}{4^n}$$

and

$$0 \le I \le 3\epsilon\, (\mathrm{diam}\Delta)^2 .$$

Since ϵ was arbitrary, we conclude that $I = 0$, as claimed. ☐

Corollary 1.2.6 (Cauchy for a ball)**.** *Let $\gamma(t)$ be a closed curve contained in a ball $B = B_r(z_0)$, and $f(z) \in \mathrm{Hol}(B)$. Then*

$$\int_\gamma f(z)\, dz = 0. \tag{1.3}$$

Proof. We shall write $[z_0, z]$ for the straight line between z_0 and z_1 in $\mathbb{C}$. Define

$$F(z) = \int_{[z_0,z]} f(w)\, dw.$$

We claim that $F'(z) = f(z)$.

Notice that the triangle with vertices z_0, z_1, z_2, is contained in B. By Cauchy's theorem (for the triangle), we have

$$F(z_2) - F(z_1) = \int_{[z_1,z_2]} f(z)\, dz.$$

Dividing by $z_2 - z_1$ and subtracting $f(z_1)$ from both sides, we obtain

$$\begin{aligned} \frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) &= \frac{\int_{[z_1,z_2]} f(z)\, dz}{z_2 - z_1} - f(z_1) \\ &= \frac{\int_{[z_1,z_2]} (f(z) - f(z_1))\, dz}{z_2 - z_1}. \end{aligned}$$

Since $f(z) - f(z_1) \to 0$ as $z \to z_1$, and the length of $[z_1, z_2]$ is $|z_2 - z_1|$, the RHS tends to zero. Therefore $F'(z_1) = f(z_1)$, as claimed.

The result now follows from Lemma 1.2.2. ☐

Corollary 1.2.7 (Cauchy for s.c. domain)**.** *Assume Ω is simply connected. For any closed curve $\{\gamma(t)\} \subset \Omega$ and $f(z) \in \mathrm{Hol}(\Omega)$, (1.3) holds.*

Proof. Since Ω is simply connected, the path γ is nullhomotopic, *i.e.*, there exists a continuous (indeed, piecewise C^1) map $\gamma(s,t) : [0,1]^2 \to \mathbb{C}$ with $\gamma(1,t) = \gamma(t)$ and $\gamma(0,t) = \gamma(s,0) = \gamma(s,1) \equiv z_0$.

Since $[0,1]^2$ is compact, the map $\gamma(s,t)$ is uniformly continuous. We may therefore subdivide $[0,1]^2$ into n^2 rectangles, each of which has image inside a ball contained in Ω. By the previous corollary, the integral over the boundary of each rectangle vanishes. Since the sum of these is the integral over γ, this vanishes as well. ☐

Theorem 1.2.1 now follows from Corollary 1.2.7 by subdividing Ω into simply connected domains.[1]

1.2.2. *Cauchy's proof (in modern notation).* The exterior derivative d extends linearly to complex-valued functions and differential forms on $\mathbb{C} \cong \mathbb{R}^2$ in the obvious way:

$$df = d(u + iv) = du + idv.$$

In particular, we have

$$dz = dx + idy, \qquad d\bar{z} = dx - idy. \tag{1.4}$$

Definition 1.2.8. Define the two operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

The first motivation for Definition 1.2.8 is the following lemma.

Lemma 1.2.9. *For any differentiable complex-valued function f on $\mathbb{C}$, we have*

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

Proof. Check directly from (1.4) and Definition 1.2.8 (Exercise).[2] □

The second motivation for Definition 1.2.8 is the following well-known fact.

Proposition 1.2.10. *A holomorphic function $f(z)$ obeys the* Cauchy-Riemann equation[3]

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{1.5}$$

Proof. Write $f(z) = u(x,y) + iv(x,y)$. If $f(z)$ is holomorphic, then the limit along the real axis exists:

$$\begin{aligned}\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i\lim_{h\to 0}\frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.\end{aligned}$$

The limit along the imaginary axis also exists:

$$\begin{aligned}\lim_{\substack{s\to 0\\ s\in\mathbb{R}}} \frac{f(z+is) - f(is)}{is} &= \lim_{s\to 0} \frac{u(x,y+s) - u(x,y+s)}{is} + i\lim_{s\to 0}\frac{v(x,y+s) - v(x,y)}{is} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.\end{aligned}$$

[1] Strictly speaking, we need to deal with the case when $f(z)$ is not holomorphic across the boundary of Ω, but only continuous there. This can be done by moving the boundary $\partial\Omega$ slightly into Ω, and taking the limit of the line integral using continuity of $f(z)$ on $\bar{\Omega}$.

[2] The reason this Lemma works is that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, as elements of $T\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}$, form the *dual basis* to (1.4). We will adopt this perspective in §3 below.

[3] See §1.7 below for further discussion of the Cauchy-Riemann equation(s).

The two limits must be equal, giving

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0.$$

This is precisely (1.5). □

We now have the following quick proof of Theorem 1.2.1:

$$\begin{aligned}\int_{\partial\Omega} f(z)\, dz &= \int_\Omega d\,(f\, dz)\\ &= \int_\Omega df \wedge dz\\ &= \int_\Omega \frac{\partial f}{\partial \bar{z}}\, d\bar{z} \wedge dz\\ &= 0.\end{aligned}$$

Here we have used Stokes's Theorem, Lemma 1.2.9, and (1.5).

Note that applying Stokes's (*i.e.* Green's) Theorem *requires that $f(z)$ is C^1.* This is a much stronger assumption than pointwise differentiability, which is all Definition 1.1.1 gives you *a priori*. Hence Goursat's proof, although longer, is both stronger and more transparent than Cauchy's.

We will now show that this question is strictly academic.

1.3. **Cauchy Integral Formula and immediate consequences.**

Cauchy's Theorem is most potent in the following form.

Theorem 1.3.1 (Cauchy Integral Formula)**.** *Let Ω be a bounded domain with piecewise C^1 boundary. For $f \in \mathrm{Hol}(\bar{\Omega})$ and any $z \in \Omega$, there holds*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(w)}{w - z}\, dw.$$

Proof. For $0 < r < \mathrm{dist}(z, \partial\Omega)$, we let

$$\Omega_r = \Omega \smallsetminus B_r(z).$$

Then Cauchy's theorem gives

$$0 = \int_{\partial\Omega_r} \frac{f(w)}{w-z}\, dw = \int_{\partial\Omega} \frac{f(w)}{w-z}\, dw - \int_{\partial B_r(z)} \frac{f(w)}{w-z}\, dw$$

and

$$\begin{aligned}\int_{\partial\Omega} \frac{f(w)}{w-z}\, dw &= \int_{\partial B_r(z)} \frac{f(w)}{w-z}\, dw\\ &= \int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta} d\theta\\ &= i\int_0^{2\pi} f(z+re^{i\theta}) d\theta\\ &\longrightarrow 2\pi i f(z) \text{ as } r \to 0\end{aligned}$$

since f is continuous at z. This proves the formula. □

Corollary 1.3.2. *A holomorphic function is complex-analytic at each $z_0 \in \Omega$, i.e., admits a convergent power-series expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{1.6}$$

with radius of of convergence $R = R(z_0) \geq \operatorname{dist}(z_0, \partial\Omega)$. In particular, a holomorphic function is smooth on its domain.

For any $0 < r < R$, we have the formula

$$c_n = \frac{1}{2\pi i} \int_{\partial B_r} \frac{f(w)}{(w - z_0)^{n+1}}\, dw. \tag{1.7}$$

Proof. For $w \in \partial B_r(z_0)$ and $z \in B_r(z_0)$, we have $|z - z_0| < |w - z_0|$, and may write

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w - z_0 - (z - z_0)} \\ &= \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}. \end{aligned} \tag{1.8}$$

This is a uniformly convergent power series for $|z - z_0| < R' < R$. We may therefore insert (1.8) into the Cauchy integral formula and exchange limits, to obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial\Omega} f(w) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}\, dw \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{(w - z_0)^{n+1}}\, dw \right] (z - z_0)^n \end{aligned}$$

which is the desired expansion. □

Corollary 1.3.3 (Cauchy's estimates)**.** *Assume $|f(z)| \leq M$ on $B_r(z_0)$. Then*

$$|f^{(n)}(z_0)| \leq \frac{n! M}{r^n}.$$

Proof. By the above corollary, we have $f^n(z_0) = n! c_n$, for c_n given by (1.7). □

Theorem 1.3.4 (Liouville's Theorem)**.** *A bounded entire function is constant.*

Proof. Let $n = 1$ and $r \to \infty$ in the previous corollary. □

Corollary 1.3.5 (Fundamental Theorem of Algebra)**.** *A non-constant polynomial function has at least one zero on $\mathbb{C}$.*

Proof. Assuming the contrary, we may apply Liouville's theorem to the bounded entire function $1/f$. □

Definition/Lemma 1.3.6. *The* **order of vanishing** $N \in \mathbb{N} \cup \{\infty\}$ *of a holomorphic function $f(z)$ at $z_0 \in \Omega$ is the first number for which $c_N \neq 0$ in the series (1.6), or equivalently, the greatest N such that*

$$f(z) = O(z - z_0)^N \text{ as } z \to z_0.$$

Proof. Let

$$g(z) = \sum_{n=N}^{\infty} c_n (z - z_0)^{n-N}.$$

This power series has the same radius of convergence as that of $f(z)$, as one can check from (1.1). We therefore have

$$f(z) = (z - z_0)^N g(z) \tag{1.9}$$

where $g(z)$ is holomorphic in B_R with $g(z_0) = c_N \neq 0$. □

Proposition 1.3.7. *If $f(z_0) = 0$ and the order of vanishing of $f(z)$ at z_0 is less than infinity, then z_0 is an isolated zero, i.e., there exists a neighborhood $B_r(z_0)$ such that $f(z) \neq 0$ for all $z \in B_r(z_0) \setminus \{z_0\}$.*

Proof. This follows directly from (1.9). □

Theorem 1.3.8 (Identity principle)**.** *Let $f(z)$ be a holomorphic function on Ω that vanishes identically on a nonempty subdomain $\Omega' \subset \Omega$. Then $f(z)$ vanishes identically on Ω.*

Proof. Let $\tilde{\Omega}$ be the subset of Ω where the order of vanishing of $f(z)$ is ∞. Then $\tilde{\Omega}$ is open by definition, and nonempty by assumption. Moreover, $\tilde{\Omega}$ is closed, by Proposition 1.3.7. We conclude that $\tilde{\Omega}$ is both open and closed in Ω, which is connected (by definition), so $\tilde{\Omega} = \Omega$ as claimed. □

Corollary 1.3.9. *If $f(z)$ is not identically zero, then its zeroes are isolated in Ω, i.e., form a discrete subset.*

Theorem 1.3.10 (Maximum principle)**.** *If $|f(z)|$ attains a local maximum inside Ω, then $f(z)$ is constant.*

Proof. Let z_0 be such a local maximum, with $|f(z_0)| = M$. We may assume without loss of generality that $f(z_0) = M \in \mathbb{R}$.

Assume, for the sake of contradiction, that $f(z)$ is not constant in a neighborhood $B_r(z_0) \Subset \Omega$; in particular, we may choose r such that $f(z)$ is not identically equal to M on $\partial B_r(z_0)$. We now apply the Cauchy Integral Formula:

$$\begin{aligned} M = \operatorname{Re} f(z_0) &= \operatorname{Re} \int_{\partial B_r(z_0)} \frac{1}{2\pi i} \frac{f(w)}{w - z_0}\, dw \\ &= \operatorname{Re} \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w)}{re^{i\theta}}\, ire^{i\theta} \\ &= \operatorname{Re} \frac{1}{2\pi} \int_0^{2\pi} f(w)\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(w)\, d\theta. \end{aligned}$$

Since we are assuming $|f(z)| \leq M$ for all $z \in \bar{B}_r$, we have $|f(z)| = M$ if and only if $\operatorname{Re} f(z) = M$. Hence, by our assumption, $\operatorname{Re} f(z) < M$ for some points $z \in \partial B_r(z_0)$. Since $f(z)$ is continuous, we obtain a *strict* inequality

$$M = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(w)\, d\theta < \frac{1}{2\pi} \int_0^{2\pi} M\, d\theta = M$$

which is a contradiction.

We conclude that $f(z)$ is identically constant on $B_r(z_0)$. But then by the identity principle, $f(z)$ is constant on all of Ω, as claimed. $\square$

Corollary 1.3.11. *Let Ω be a bounded domain. If $f(z) \in \mathrm{Hol}(\bar{\Omega})$ with $|f(z)| \le M$ on $\partial\Omega$, then $|f(z)| \le M$ on Ω. If equality holds at any interior point, then $f(z)$ is constant.*

Proof. Since $|f(z)|$ is continuous on the compact set $\bar{\Omega}$, it attains its maximum, *i.e.*

$$|f(z_0)| = \sup_{\bar{\Omega}} f(z) = M'$$

for some $z_0 \in \bar{\Omega}$. If $z_0 \in \Omega$, then $f(z)$ is constant, and $M' \le M$. If $z_0 \in \partial\Omega$, then $M' = |f(z_0)| \le M$ by assumption, and we are again done. $\square$

Theorem 1.3.12 (Riemann's removable singularity theorem)**.** *Let $f(z) \in \mathrm{Hol}(B_r(z_0) \setminus \{z_0\})$, and assume that*

$$f(z) = o\left(\frac{1}{|z - z_0|}\right) \text{ as } z \to z_0.$$

Then $f(z)$ extends to a holomorphic function on $B_r(z_0)$.

Proof. Define the function

$$g(z) = (z - z_0)^2 f(z).$$

Then $g(z)$ is holomorphic for $z \neq z_0$ by the product rule, and

$$|g'(z_0)| = \left|\lim_{h\to 0} \frac{h^2 f(z_0 + h)}{h}\right| = |\lim_{h\to 0} h f(z_0 + h)| = 0$$

by the assumption. Therefore $g(z)$ is holomorphic on all of $B_r(z_0)$, with

$$g(z_0) = g'(z_0) = 0.$$

We therefore have

$$g(z) = \sum_{n=2}^{\infty} c_n (z - z_0)^n$$

and may define the extension of $f(z)$ to be the convergent power series

$$\sum_{n=0}^{\infty} c_{n+2}(z - z_0)^n.$$

$\square$

Lemma 1.3.13 (Schwartz Lemma)**.** *Let $f(z) \in \mathrm{Hol}\left(\bar{B}_R(0)\right)$ be a holomorphic function with a zero of order N at the origin, which satisfies $\sup_{\partial B_R} |f(z)| = M$. Then*

$$|f(z)| \le M \left(\frac{|z|}{R}\right)^N$$

for all $z \in B_R$. If equality holds at any interior point, then

$$f(z) = C\left(\frac{z}{R}\right)^N$$

for some constant with $|C| = M$.

Proof. By the proof of Lemma 1.3.6, we have $f(z)/z^N = g(z)$ holomorphic on $B_R(0)$, with

$$|g(z)| \leq \frac{|f(z)|}{R^N} \leq \frac{M}{R^N}$$

on ∂B_R. But by the maximum principle (Corollary 1.3.11), we then have

$$\frac{M}{R^N} \geq |g(z)| = |f(z)/z^N|$$

throughout B_R, which yields the claim. If equality holds at an interior point, then $g(z) = \frac{C}{R^N}$ is constant. □

1.4. **Spaces of holomorphic functions.** Recall that the L^2**-norm** of a complex-valued function is defined by

$$\|f\|_{L^2(\Omega)} = \sqrt{\int_\Omega |f(z)|^2 \, dVol}.$$

We have the following extremely strong convergence result for sequences of holomorphic functions with bounded L^2 norm.

Theorem 1.4.1 (Montel's Theorem)**.** *Let $\{f_i(z)\}$ be a sequence of holomorphic functions on $B_r(0)$ with uniformly bounded L^2 norm. Then for any $r_1 < r$, there exists a subsequence $f_{i_j}(z)$ which converges to a holomorphic function uniformly on B_{r_1}, together with all derivatives.*

Proof. Take $f = f_i$ for a single function, and let c_n be the series coefficients about $z_0 = 0$. Notice that for $n, m \geq 0$, we have

$$\int_{B_r} z^n \bar{z}^m \, dV = \begin{cases} 2\pi \dfrac{r^{2n+2}}{2n+2} & n = m \\ 0 & n \neq m. \end{cases}$$

Hence, if the L^2 norm is bounded by M, then

$$M^2 \geq \|f\|^2_{L^2(B_1)} = \sum_{n\geq 0} \frac{|c_n|^2}{2n+2}$$

and, in particular, $|c_n| \leq M\sqrt{2n+2}$. For $0 < r < 1$, we have

$$|f(z)| \leq \sum_{n\geq 0} |c_n||z|^n \leq M \sum \sqrt{2n+2} r^n \leq MC_0(r)$$

for $z \in B_r$, where $C_0(r)$ is some function of r. Similarly, we have

$$|f^{(k)}(z)| \leq MC_k(r).$$

Since the foregoing estimates apply uniformly to $f_i(z)$, we conclude from the Arzela-Ascoli Theorem that the sequence subconverges uniformly on compact subsets, together with all derivatives. This preserves the Cauchy-Riemann equations, which are equivalent to holomorphicity for C^1 functions. (Alternatively, one can appeal to the converse of Cauchy's Theorem, known as Morera's Theorem, stating that a function is holomorphic if it satisfies (1.3) for all paths). □

Corollary 1.4.2. *Let $\{f_i(z)\}$ be a sequence of holomorphic functions on a domain Ω with uniformly bounded L^2 norm. Then there exists a subsequence $f_{i_j}(z)$ which converges to a holomorphic function on Ω pointwise and uniformly on compact subdomains.*

Proof. We may cover Ω by balls and use a diagonalization argument to obtain the limit $f(z)$. We then have uniform convergence on compact subsets, since any such is covered by finitely many balls. □

Corollary 1.4.3. *Let $\{f_i(z)\}$ be a sequence of holomorphic functions as above, and let $\{z_a\}$ be a discrete subset of Ω. If each f_i vanishes to order at least N_a at z_a, then the limit $f(z)$ vanishes to order at least N_a at z_a.*

Proof. By the Theorem, we know that there exists a subsequence, again denoted $f_i(z)$, such that $f_i(z) \to f(z)$ in C^∞_{loc}.

Fix a point z_a and choose a neighborhood $B = B_r(z_a) \Subset \Omega$. By uniform convergence, we know that $|f_i(z)| \leq M$ for all $z \in \partial B$, for some constant M. But then by the Schwartz Lemma, we have

$$|f_i(z)| \leq M \left(\frac{|z - z_a|}{r} \right)^{N_a}.$$

Since $f_i \to f$, the same holds for $f(z)$. We conclude from Lemma 1.3.6 that the order of vanishing of $f(z)$ at z_a is at least N_a, as desired. □

Corollary 1.4.4. *The space of holomorphic functions on Ω with bounded L^2 norm (and vanishing to prescribed orders at a discrete set of points) is a Hilbert space, i.e., is complete with respect to the L^2 inner product.*

1.5. **Meromorphic functions.** We shall denote the *punctured ball*

$$B_r'(z_0) = B_r(z_0) \setminus \{z_0\}$$

and the *annulus*

$$U_r^R(z_0) = B_R(z_0) \setminus \bar{B}_r(z_0).$$

We have the following generalization of Corollary 1.6 to holomorphic functions on an annulus.

Theorem 1.5.1 (Laurent series)**.** *Let $f(z)$ be a holomorphic function on $\bar{U}_r^R(z_0)$. Then $f(z)$ admits a unique* **Laurent expansion**

$$f(z) = \sum_{m=-\infty}^{\infty} c_m (z - z_0)^m \tag{1.10}$$

that is uniformly convergent on compact subsets of $U_r^R(z_0)$. Indeed, for any $r \leq s \leq R$ and $m \in \mathbb{Z}$, we have

$$c_m = \frac{1}{2\pi i} \int_{\partial B_s(z_0)} \frac{f(w)}{(w - z_0)^{m+1}}\, dw. \tag{1.11}$$

Proof. We may apply the Cauchy Integral Formula on the domain $\Omega = U_r^R(z_0)$, to obtain

$$f(z) = \frac{1}{2\pi i}\left(\int_{\partial B_R(z_0)} \frac{f(w)}{w-z}\,dw - \int_{\partial B_r(z_0)} \frac{f(w)}{w-z}\,dw\right). \tag{1.12}$$

Note that we have the two expansions

$$\frac{1}{w-z} = \begin{cases} \sum_{n=0}^{\infty} \dfrac{(z-z_0)^n}{(w-z_0)^{n+1}} & |z-z_0| < |w-z_0| \\ -\sum_{n=0}^{\infty} \dfrac{(w-z_0)^n}{(z-z_0)^{n+1}} & |z-z_0| > |w-z_0|. \end{cases}$$

The first expression is convergent for $|z-z_0| \le R$, and the second for $|z-z_0| \ge r$. The Laurent series is obtained by plugging the two expressions into the two terms of (1.12), respectively, giving a series (1.10) where

$$c_m = \begin{cases} \frac{1}{2\pi i}\int_{\partial B_R} \dfrac{f(w)}{(w-z_0)^{m+1}}\,dw & m \ge 0 \\ \frac{1}{2\pi i}\int_{\partial B_r} \dfrac{f(w)}{(w-z_0)^{m+1}}\,dw & m < 0. \end{cases} \tag{1.13}$$

But by Cauchy's theorem, we may replace r or R by any $r \le s \le R$ in the integrals in (1.13), giving (1.11).

The uniqueness of the coefficients follows by plugging (1.10) into (1.11) and using uniform convergence together with Example 1.2.4 to pick out the coefficients. □

Corollary 1.5.2. *Let $f(z) \in \mathrm{Hol}\left(\bar{U}_r^R\right)$. Then we have a unique decomposition*

$$f(z) = P(z) + Q(z) \tag{1.14}$$

where $P(z) \in \mathrm{Hol}(\bar{B}_R)$ and $Q(z) \in \mathrm{Hol}(\mathbb{C} \setminus B_r)$ with $Q(z) \to 0$ as $z \to \infty$.

Furthermore, if $f(z) \in \mathrm{Hol}(B'_R)$, then (1.14) holds with $Q(z) \in \mathrm{Hol}(\mathbb{C} \setminus \{0\})$; if $f(z) \in \mathrm{Hol}(\mathbb{C} \setminus B_r)$, then $P(z)$ is an entire function.

Remark 1.5.3. As we shall see later, this corollary amounts to the statement that

$$H^1(\mathscr{O}_{\mathbb{CP}^1}) = 0.$$

Definition 1.5.4. Assume that $f(z) \in \mathrm{Hol}(B'_R(z_0))$. The **order of the pole** at z_0 is the minimal $N_\infty \in \mathbb{N} \cup \{\infty\}$ such that $c_m = 0$ for all $m < -N_\infty$ in the Laurent expansion.

The **principal part** of $f(z)$ at z_0 is

$$Q(z) = \sum_{m=1}^{N_\infty} \frac{c_{-m}}{(z-z_a)^m}.$$

The **residue** of $f(z)$ at z_0 is

$$\mathrm{Res}_{z_0}(f) = c_{-1} = \frac{1}{2\pi i}\int_{\partial B_s} f(w)\,dw$$

for any $0 < s < R$.

The **order** $N \in \mathbb{Z} \cup \{\pm\infty\}$ of a meromorphic function $f(z)$ at z_0 is defined to be the order of vanishing N, or (negative) the order of the pole $-N_\infty$, if $f(z)$ has a zero or pole at $z_0 \in \Omega$, respectively.

Theorem 1.5.5 (Residue Theorem). *Let $Z = \{z_1, \dots, z_n\} \subset \Omega$ be a finite set of points in a bounded domain with piecewise C^1 boundary. For $f(z) \in \mathrm{Hol}(\Omega \setminus \{z_1, \dots, z_n\})$ continuous up to $\partial\Omega$, we have*

$$\int_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}_{z_i}(f).$$

Proof. Choose $r > 0$ such that the balls around $B_r(z_i)$ are disjoint and compactly contained in Ω, and let $\Omega' = \Omega \setminus \cup_i B_r(z_i)$. We apply Cauchy's Theorem:

$$0 = \int_{\partial\Omega'} f(z)\,dz = \int_{\partial\Omega} f(z)\,dz - \sum_i \int_{\partial B_r(z_i)} f(z)\,dz$$

and

$$\begin{aligned}\int_{\partial\Omega} f(z)\,dz &= \sum_i \int_{\partial B_r(z_i)} f(z)\,dz \\ &= 2\pi i \sum_i \mathrm{Res}_{z_i}(f)\end{aligned}$$

by (1.11) and Definition 1.5.4, as claimed. ☐

Definition/Lemma 1.5.6. *We say that $f(z)$ is* **meromorphic** *on Ω if $f(z) \in \mathrm{Hol}(\Omega \setminus Z)$ for $Z = \{z_a\}$ a discrete subset of Ω, where*

$$\lim_{z \to z_a} f(z) = \infty \tag{1.15}$$

for each $z_a \in Z$. Equivalently, $f(z) \in \mathrm{Hol}(\Omega \setminus Z)$ has only finite-order poles at points of Z.

Proof. It is clear that a function with finite-order (nontrivial) poles satisfies (1.15).

Conversely, assume that there exists $w_0 \in \mathbb{C}$ and a neighborhood $B = B_r(z_a)$ such that $|f(z) - w_0| > \epsilon$ for all $|z - z_a| < r$, *i.e.*, the image of $B = B_r(z_a)$ is *not* dense in $\mathbb{C}$. (This is a slightly weaker assumption than (1.15)). Then the function

$$g(z) = \frac{1}{f(z) - w_0}$$

is bounded by $1/\epsilon$ on B and therefore extends to a holomorphic function at z_a, by Riemann's Theorem, with a finite-order zero at z_a. But then $f(z) = \frac{1}{g(z)} + w_0$ has a finite-order pole. ☐

Remark 1.5.7. The density statement in the proof is known as the *Casorati-Weierstrass Theorem.*

Remark 1.5.8. Notice that the quotient of two meromorphic functions is again meromorphic. The set of meromorphic functions on Ω is therefore a field, namely, the field of fractions of the set of holomorphic functions on Ω (which is an integral domain, by the identity principle).

We shall need the following two results, which show the power of the Residue Theorem when applied to meromorphic functions. The proof is direct from (1.9).

Lemma 1.5.9. *Let $f(z)$ be a meromorphic function of one variable, and assume that*

$$\mathrm{Ord}_{z_0} f(z) = N.$$

Then $f'(z)/f(z)$ has at most a simple pole at z_0, with

$$\mathrm{Res}_{z_0} \frac{f'(z)}{f(z)} = N. \tag{1.16}$$

In particular, by the Residue Theorem, we have:

Proposition 1.5.10. *Let $f(z)$ be a meromorphic function on a domain Ω, with piecewise C^1 boundary, and assume that f and f' extend continuously to $\partial\Omega$, with f nonzero there. The number of zeroes minus the number of poles inside Ω, counted with multiplicity, is given by*

$$\int_{\partial\Omega} \frac{f'(z)}{f(z)}\, dz. \tag{1.17}$$

Exercise. Use Proposition 1.17 to prove the **Open Mapping Theorem:** the image of a domain under a holomorphic function is either a point or a domain.

1.5.1. *Two classical problems.* We would be remiss not to mention the following problems from the 19th century, which are sometimes used to motivate the whole theory. We will return to each one later in the class, but there is nothing preventing you from solving them now.

Problem 1.5.11. We say that a function is **meromorphic at infinity** if $f(1/w)$ is meromorphic at $w = 0$.

(a) Identify all meromorphic functions on $\mathbb{C} \cup \{\infty\}$.

(b) Given finitely many points $\{z_1, \dots, z_n\} \subset \mathbb{C} \cup \{\infty\}$ and numbers $\{N_1, \dots, N_n\} \subset \mathbb{N}$, what is the dimension of the space of meromorphic functions with poles of order at most N_i at z_i? (Answer: $\sum_{i=1}^n N_i + 1$.)

Problem 1.5.12. Let $\tau \in \mathbb{C} \smallsetminus \mathbb{R}$ and denote the lattice

$$\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\} \subset \mathbb{C}.$$

We say that $f(z)$ is **doubly periodic** if $f(z + \lambda) = f(z)$ for every $\lambda \in \Lambda$.

(a) Prove that there does not exist a doubly periodic meromorphic function with a single, simple pole (modulo Λ).

(b) (*Tricky*) Construct a doubly periodic meromorphic function with a pole of order 2, or with two simple poles (modulo Λ). What are the residues in each case?

Remark 1.5.13. In terms of Riemann surfaces, the first problem asks for all holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$. The second asks to show that there does not exist a degree 1 holomorphic map $\mathbb{C}/\Lambda \to \mathbb{CP}^1$, and instead to construct a degree 2 map.

1.6. **The $\bar\partial$-Poincaré Lemma in dimension one.** In order to construct holomorphic functions, we shall also need to consider the inhomogeneous version of the Cauchy-Riemann equation:

$$\frac{\partial f}{\partial \bar z} = g. \tag{1.18}$$

Definition 1.6.1. We shall write $C^k(\bar\Omega)$ for functions in $C^k(\Omega)$ whose derivatives extend continuously to $\bar\Omega$.

Fix open sets $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$. We say that a function $h(x,u)$ on $\bar\Omega_1 \times \bar\Omega_2$ is **uniformly C^k in the x variable** if $h(\cdot,u)$ is in $C^k(\bar\Omega_1)$, with bounds on the C^k norm independent of $u \in \Omega_2$. We say $h(x,u)$ has **partial compact support in the u variable** if for each ball $B \Subset \Omega_1$, the restriction of h to $\bar B \times \bar\Omega_2$ has compact support.

Proposition 1.6.2 (Differentiation under the integral sign)**.** *Let $\varphi(u)$ be a function on Ω_2 that is L^1 on compact subsets. Let $h(x,u)$ be a function on $\bar\Omega_1 \times \bar\Omega_2$ that is uniformly C^k in the x variable and has partial compact support in the u variable. Define the integral*

$$I(x) = \int_{\Omega_2} h(x,u)\varphi(u)\,du.$$

Then $I(x) \in C^k(\bar\Omega_1)$, with

$$\frac{\partial I(x)}{\partial x_k} = \int_{\Omega_2} \frac{\partial h(x,u)}{\partial x_k}\varphi(u)\,du, \quad etc.$$

Proof. Fix $x \in \Omega_1$, $v \in \mathbb{R}^n$, with $|v| = 1$. Since h has partial compact support in u, we may choose a compact set $K \subset \bar\Omega_2$ such that for all $y \in \bar B_1(x) \cap \bar\Omega_1$, we have

$$I(y) = \int_K h(y,u)\varphi(u)\,du.$$

For $h \in \mathbb{R}$ with $|h| < \mathrm{dist}(x, \partial\Omega_1)$, consider the real difference quotients

$$\Delta_h = \frac{I(x+hv) - I(x)}{h} = \int_K \frac{h(x+hv,u) - g(x,u)}{h}\varphi(u)\,du.$$

Since h is uniformly C^1 in x, $\frac{h(x+hv,u)-h(x,u)}{h}$ is bounded above. The integrand of Δ_h is therefore bounded by a multiple of $\varphi(u)$, which is L^1 on K. By the Dominated Convergence Theorem, we have

$$\lim_{h\to 0}\Delta_h = \int_K \lim_{h\to 0}\frac{h(x+hv,u) - h(x,u)}{h}\varphi(u)\,du$$

and

$$D_v I(x) = \int D_v h(x,u)\varphi(u)\,du$$

which shows that I is C^1 with the desired partials. Since $D_v h(x,u)$ is uniformly bounded, $D_v I(x)$ extends continuously up to the boundary.

If h is C^k, we may iterate this argument. $\square$

Lemma 1.6.3 ($\bar\partial$-Poincaré Lemma for a compactly supported single-variable function)**.** *Given $g \in C^k_c(\mathbb{C})$, $k \geq 1$, the function*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(w)}{w-z}dw \wedge d\bar w \tag{1.19}$$

is in $C^k(\mathbb{C})$ and satisfies the inhomogeneous Cauchy-Riemann equation (1.18).

Proof. Fixing z, change variables $u = z - w$. Then $du = -dw$, and the expression (1.19) becomes

$$f(z) = -\frac{1}{2\pi i}\int \frac{g(z-u)}{u}du \wedge d\bar{u}.$$

Observe that

$$\frac{-1}{2i}dw \wedge d\bar{w} = dVol$$

and $\frac{1}{u}$ is integrable on compact sets. So we may apply the previous proposition to conclude that $f(z)$ is C^1, with

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= -\frac{1}{2\pi i}\int \frac{\partial g(z-u)}{\partial \bar{z}}\frac{du \wedge d\bar{u}}{u}\\ &= \frac{1}{2\pi i}\int \frac{\partial g(z-u)}{\partial \bar{u}}\frac{du \wedge d\bar{u}}{u}.\end{aligned}$$

But this just equals

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \lim_{r\to 0}\frac{1}{2\pi i}\int_{\mathbb{C}\setminus B_r(z)} \frac{\partial g(z-u)}{\partial \bar{u}}\frac{du \wedge d\bar{u}}{u}\\ &= -\lim_{r\to 0}\frac{1}{2\pi i}\int_{\mathbb{C}\setminus B_r(z)} d_u\left(\frac{g(z-u)du}{u}\right)\\ &= \lim_{r\to 0}\frac{1}{2\pi i}\int_{\partial B_r(z)} g(z-u)\frac{du}{u}\\ &= \lim_{r\to 0}\frac{1}{2\pi}\int_0^{2\pi} g(z-re^{i\theta})\,d\theta\\ &= g(z)\end{aligned}$$

where we have applied Stokes's Theorem and the continuity of g. $\square$

Theorem 1.6.4 ($\bar{\partial}$-Poincaré Lemma for a bounded domain in $\mathbb{C}$)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded domain. Given $g \in C^k\left(\bar{\Omega}\right)$, for $k \geq 1$, the function*

$$f(z) = \frac{1}{2\pi i}\int_\Omega \frac{g(w)}{w-z}dw \wedge d\bar{w} \tag{1.20}$$

is in $C^k_{loc}(\Omega)$ and satisfies (1.18).

Proof. Given $z_1 \in \Omega$, choose $0 < \epsilon < \dfrac{r - |z_1|}{2}$. Write

$$g(z) = g_1(z) + g_2(z)$$

where g_1 vanishes outside $B_{2\epsilon}(z_1)$ and g_2 vanishes inside $B_\epsilon(z_1)$. Define f_1 and f_2 from (1.20) corresponding to $g = g_1$ and g_2, respectively, so that

$$f = f_1 + f_2.$$

Then by the previous lemma, for $z \in B_\epsilon(z_1)$, f_1 is C^k and satisfies

$$\frac{\partial f_1}{\partial \bar{z}} = g_1.$$

Meanwhile, $\frac{g_2(w)}{w-z}$ is in $C^k(\bar{B}_\epsilon(z_1)\times\bar{\Omega})$. By Proposition 1.6.2, $f_2(z)$ is also in $C^k(B_\epsilon(z_1))$, and indeed satisfies

$$\frac{\partial f_2}{\partial \bar{z}} = \frac{1}{2\pi i}\int_\Omega \frac{\partial}{\partial \bar{z}}\left(\frac{g_2(w)}{w-z}\right) dw \wedge d\bar{w} = 0.$$

Therefore $f = f_1 + f_2$ is in $C^k(B_\epsilon(z_1))$, and satisfies

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f_1}{\partial \bar{z}} = g_1 = g.$$

But $z_1 \in \Omega$ was arbitrary, so we have in fact shown (1.18). □

1.7. **The Jacobian of a single-variable holomorphic function.** We end our discussion of single-variable complex analysis with an extremely elementary, but important, remark about the holomorphicity condition.

Consider $\mathbb{C} = \mathbb{R}^2$ with the basis $\{1, i\}$ as above. Let I be the map of $\mathbb{R}^2$ given by multiplication by i, or in this basis,

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We say that a (real)-linear map $M : \mathbb{R}^2 \to \mathbb{R}^2$ is ***I*-(anti)-linear** if

$$M(I(v)) = \pm I \cdot M(v)$$

for all $v \in \mathbb{R}^2$.

Proposition 1.7.1. *The space of linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ is a direct sum of I-linear and I-antilinear maps:*

$$\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R}\right\} \oplus \left\{\begin{pmatrix} c & d \\ d & -c \end{pmatrix} \mid c, d \in \mathbb{R}\right\}. \tag{1.21}$$

Notice that an element of the first factor corresponds to the map $z \mapsto (a+bi)\cdot z$ on $\mathbb{C}$, whereas the second corresponds to $z \mapsto (c+di)\cdot \bar{z}$.

Now, let $(u(x,y), v(x,y))^T : \mathbb{R}^2 \to \mathbb{R}^2$, be a differentiable map. The (real) Jacobian is

$$J(f) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

Notice that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are precisely the condition that the Jacobian lie in the first factor of (1.21). We conclude that *a holomorphic function is precisely one whose real Jacobian is I-linear, i.e.,* corresponds to complex multiplication on $\mathbb{C}$ by the "complex Jacobian"

$$f'(z) = \frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

which is no surprise, given Definition 1.1.1. This characterization of holomorphicity will extend to several variables.

As a corollary, we have that a single-variable holomorphic function is *conformal* and orientation-preserving at points where its derivative does not vanish (because this is true of multiplication by nonzero complex scalars). Indeed, it is easy to see that

$$\det J(f) = |f'(z)|^2 \geq 0.$$

This formula will be generalized below (see Lemma 2.5.4).

2. Rudiments of several-variable complex analysis

2.1. Holomorphicity and Hartogs' Theorem. Let $\Omega \subset \mathbb{C}^n$ be a domain, *i.e.*, a connected open set. We take real coordinates

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n$$

on $\mathbb{C}^n = \mathbb{R}^{2n}$, and complex coordinates

$$z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n.$$

We shall refer to a **polydisk**

$$D_{r_1,\ldots,r_n}(w) = B_{r_1}(w_1) \times \cdots \times B_{r_n}(w_n).$$

and write D_r if $r_i = r$ for all i.

Define the operators

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right), \qquad i = 1, \ldots, n.$$

Definition 2.1.1. A continuous function $f : \Omega \to \mathbb{C}$ is said to be **holomorphic** if it is holomorphic in each variable separately, *i.e.*,

$$\frac{\partial f}{\partial \bar{z}_i} = 0 \text{ for } i = 1, \ldots, n.$$

Lemma 2.1.2. *Let $f(z_1, \ldots, z_n)$ be a continuous function on $\Omega \subset \mathbb{C}^n$. Then TFAE:*

(a) *f is holomorphic*

(b) *The restriction of f to any complex line in $\mathbb{C}^n$ is holomorphic*

(c) *For any polydisk $D_r(w) \Subset \Omega$, f satisfies the Cauchy Integral Formula:*

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int_{|u_1 - w_1| = r} \cdots \int_{|u_n - w_n| = r} \frac{f(u_1, \ldots, u_n)}{(u_1 - z_1)(u_2 - z_2)\cdots(u_n - z_n)} du_1 \cdots du_n$$

(d) *f is complex-analytic about each $w \in \Omega$, i.e., admits a convergent power-series expansion*

$$f(z) = \sum_{i_1,\ldots,i_n=0}^{\infty} a_{i_1\cdots i_n}(z_1 - w_1)^{i_1} \cdots (z_n - w_n)^{i_n}.$$

Proof. **(a)** $\Rightarrow$ **(c)** $\Rightarrow$ **(d)** $\Rightarrow$ **(b)** $\Rightarrow$ **(a).** ☐

Many of the properties of holomorphic functions carry over to several variables: for instance, the identity principle, the maximum principle, and Liouville's theorem. Here, however, is the first big difference.

Theorem 2.1.3 (Hartogs' Theorem)**.** *Let $0 < r < R$. Any holomorphic function on $D_R \setminus \bar{D}_r$ extends to a holomorphic function on D_R.*

Proof. Let $r < r_1 < R$. Define

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|u_2| = r_1} \frac{f(z_1, u_2) du_2}{u_2 - z_2}.$$

Then F is clearly holomorphic for $z_2 \in B_{r_1}$, and also for $z_1 \in B_{r_1}$ because

$$\frac{\partial f}{\partial \bar{z}_1}(z_1, u_2) = 0$$

for

$$(z_1, u_2) \in B_{r_1} \times \partial B_{r_1} \subset D_R \setminus \bar{D}_r.$$

Therefore $F(z_1, z_2)$ is holomorphic throughout D_{r_1}.

But, by the 1-variable Cauchy integral formula, F agrees with f on the open subset

$$\left(B_{r_1} \setminus \bar{B}_r\right) \times B_{r_1} \subset D_{r_1} \setminus \bar{D}_r.$$

Since $D_{r_1} \setminus \bar{D}_r$ is connected, by the identity principle, $F(z_1, z_2) = f(z_1, z_2)$ there. Therefore, F is the desired holomorphic extension of f. □

Corollary 2.1.4. *A holomorphic function on the complement of a point in $\Omega \subset \mathbb{C}^n, n \geq 2$, extends to a holomorphic function on Ω.*

Corollary 2.1.5. *A holomorphic function on a domain in $\mathbb{C}^n, n \geq 2$, cannot vanish at an isolated point.*

Proof. Apply the previous corollary to $1/f$. □

2.2. **The Weierstrass Theorems.** We now wish to understand the zero set of a holomorphic function on $\mathbb{C}^n, n \geq 2$, which entails finding the correct generalization of the factorization property (1.15). We shall write

$$\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C} = \{(z_1, \ldots, z_{n-1}, w)\}.$$

Definition 2.2.1. A **Weierstrass polynomial** of degree d is a holomorphic function on $\mathbb{C}^n$ of the form

$$g(z, w) = w^d + a_1(z) w^{d-1} + a_2(z) w^{d-2} + \cdot + a_d(z) \tag{2.1}$$

for holomorphic functions $a_1(z), \ldots, a_d(z)$ on a domain $\Omega' \subset \mathbb{C}^{n-1}$ satisfying

$$a_1(0) = \cdots = a_d(0) = 0. \tag{2.2}$$

Theorem 2.2.2 (Weierstrass Preparation Theorem)**.** *Let $f(z, w)$ be holomorphic in a neighborhood of the origin. Assume $f(0, w) \not\equiv 0$, and $\mathrm{Ord}_0(f(0, w)) = d$. Then there exists a* unique *Weierstrass polynomial $g(z, w)$, of degree d, and a holomorphic function $h(z, w)$ on some neighborhood of the origin, with $h(0, 0) \neq 0$, such that*

$$f(z, w) = g(z, w) h(z, w). \tag{2.3}$$

2.2.1. *Digression on symmetric polynomials.* Before proving the WPT, we need to say a word about symmetric polynomials. Let $u_1, \dots, u_n$ be formal variables. We say that a polynomial $p(u_1, \dots, u_n)$ is **symmetric** if it is invariant under permuting any two coordinates. For example, we have the **elementary symmetric functions**

$$\begin{aligned} \sigma_1(u_1, \dots, u_n) &= u_1 + \cdots + u_n \\ \sigma_2(u_1, \dots, u_n) &= u_1 u_2 + u_2 u_3 + u_1 u_3 + etc. \\ &\vdots \\ \sigma_n(u_1, \dots, u_n) &= u_1 u_2 \cdots u_n \end{aligned} \tag{2.4}$$

and the **power functions**

$$p_k(u_1, \dots, u_n) = u_1^k + \cdots + u_n^k. \tag{2.5}$$

We learn in undergrad algebra that the space of all symmetric polynomials of degree $\leq k$ is generated by the elementary symmetric polynomials of degree $\leq k$ (indeed, the whole algebra is just a polynomial algebra with generators $\sigma_1, \dots, \sigma_k$). In fact, the following is even easier to prove.

Lemma 2.2.3. *The power functions of degree $\leq k$ generate all symmetric polynomials of degree $\leq k$. In particular, for each $1 \leq q \leq d$, there exists a polynomial $P_{q,n}$ in q variables such that the identity*

$$\sigma_q(u) = P_{q,n}(p_1(u), \dots, p_q(u))$$

holds as polynomials in $u = u_1, \dots, u_n$.

Proof. It is sufficient to write any symmetric function of the form

$$u_1^{k_1} u_2^{k_2} \cdots u_\ell^{k_\ell} + \text{permutations}$$

in terms of the power functions. This goes by induction: for $\ell = 1$, these are exactly the power functions. We then write

$$u_1^{k_1} u_2^{k_2} + \text{permutations} = p_{k_1} p_{k_2} - \text{const.}\, p_{k+\ell}. \tag{2.6}$$

By the same principle, we may write

$$u_1^{k_1} u_2^{k_2} u_3^{k_3} + \text{permutations} = p_{k_1} p_{k_2} p_{k_3} - (\text{terms with two factors})$$

where, by (2.6), the terms with two factors can be written using power functions. Continuing, we have

$$u_1^{k_1} \cdots u_\ell^{k_\ell} + \text{permutations} = p_{k_1} \cdots p_{k_\ell} - (\text{terms with } \ell - 1 \text{ factors}) \tag{2.7}$$

which gives us the result by induction.

The existence of the polynomials $P_{q,n}$ follows trivially, by applying the result to each symmetric polynomial σ_q. $\square$

Although it won't be necessary here, if one wants to compare the elementary symmetric functions and the power functions explicitly (*e.g.* to show using the Lemma that the former generate all symmetric polynomials), one can appeal to *Newton's identities*

$$k\sigma_k(u_1,\ldots,u_n) = (-1)^{k-1}p_k(u_1,\ldots,u_n) + \sum_{i=1}^{k-1}(-1)^{i-1}p_i(u_1,\ldots,u_n)\sigma_{k-i}(u_1,\ldots,u_n).$$

The proof of Newton's identities is based on the basic identity

$$\prod_{i=1}^{n}(z-u_i) = z^n - z^{n-1}\sigma_1(u) + z^{n-2}\sigma_2(u) - \cdots + (-1)^{n-1}\sigma_n(u) \tag{2.8}$$

which is sometimes taken as an alternative definition of the elementary symmetric functions.

We may now return to the proof of the WPT.

Proof of the Weierstrass Preparation Theorem. Let $f(z,w)$ as in the statement. Then since $f(0,w)$ is not identically zero, by (1.9), there exist $r,\delta>0$ such that

$$|f(0,w)|\geq\delta$$

for $|w|=r$. But then by continuity of f, for ϵ sufficiently small, we have

$$|f(z,w)|\geq\delta/2$$

for $|w|=r$ and all $\|z\|\leq\epsilon$. For such z, we may therefore define

$$F_q(z) = \frac{1}{2\pi i}\int_{|w|=r}\frac{w^q\frac{\partial f}{\partial w}}{f}\,dw. \tag{2.9}$$

This is holomorphic in $\|z\|\leq\epsilon$.

For a fixed z, let

$$u_1,\ldots,u_d$$

be the zeroes in w of $f(z,w)$, taken with multiplicity. (Note that by Proposition 1.5.10, there will remain d such zeroes for all $\|z\|\leq\epsilon$, since the integral (1.17) is continuous and integer-valued, hence locally constant). By Lemma 1.5.9 and the Residue Theorem, we have

$$F_q(z) = u_1^q + \cdots + u_d^q.$$

Therefore the power function in the roots of $f(z,w)$ is in fact a holomorphic function of z. By Lemma 2.2.3, there exists a polynomial $P_{q,d}$ such that

$$\sigma_q(u_1,\ldots,u_d) = P_{q,d}(F_1(z),\ldots,F_q(z)).$$

But then $\sigma_q(u_1,\ldots,u_d)=\sigma_q(z)$ is itself a holomorphic function of z, and we may define the Weierstrass polynomial

$$g(z,w) = w^d - \sigma_1(z)w^{d-1} + \cdots + (-1)^d\sigma_d(z).$$

By the identity (2.8), $g(z,w)$ vanishes on exactly the same set as $f(z,w)$ (in a neighborhood of the origin).

The quotient

$$h(z,w) = \frac{f(z,w)}{g(z,w)}$$

is, by the construction, a nonvanishing function which is holomorphic in w. But then by the 1-variable Cauchy Integral Formula, we have

$$h(z,w) = \frac{1}{2\pi i}\int_{|u|=r} \frac{h(z,u)}{u-w}du$$

which is also clearly continuous and holomorphic in z. This completes the proof of the existence of $g(z,w)$, satisfying (2.3) as desired.

The uniqueness can be seen as follows: the proof shows that the coefficients of any Weierstrass polynomial vanishing on the same set as f are determined by the power functions in the zeroes. But these power sums are determined by f, according to the formula (2.9). □

Corollary 2.2.4. *If the zero-set Z of a holomorphic function on $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$, contains the origin but does not vanish identically on $\mathbb{C}$, then Z projects surjectively onto a neighborhood of the origin in $\mathbb{C}^{n-1}$.*[4]

Corollary 2.2.5 (Holomorphic Implicit Function Theorem, first version)**.** *Let $f(z,w)$ be a holomorphic function in a neighborhood U of the origin, with $f(0,0) = 0$ but $\frac{\partial f}{\partial w}(0,0) \neq 0$. Then there exists a smaller neighborhood $U' \subset U$, and a neighborhood of the origin $V \subset \mathbb{C}^{n-1}$ such that*

$$Z \cap U' = \{(z_1,\dots,z_{n-1}, f(z_1,\dots,z_{n-1})) \mid (z_1,\dots z_{n-1}) \in V\}. \tag{2.10}$$

Proof. By assumption, $f(0,w)$ has a zero of order exactly one at the origin. From the WPT, we have that the zero set of f agrees with that of a degree one Weierstrass polynomial $g(z,w) = w + a_1(z)$ near the origin. The result follows by taking $f(z) = -a_1(z)$. □

Theorem 2.2.6 (Riemann Extension Theorem)**.** *Let $Z = \{f = 0\}$ be the vanishing set of a holomorphic function. Suppose $g(z,w)$ is holomorphic on $\Omega \setminus Z$ and bounded. Then g extends to a holomorphic function on Ω.*

Proof. Assume wlog that $\Omega = B_R(0), 0 \in Z$, and $f(0,w) \not\equiv 0$. Choose r,δ,ϵ as in the previous proof, so that $|f(z,w)| \geq \delta$ for $|w| = r, \|z\| \leq \epsilon$. Then $f(z,w) = 0$ only if $|w| < r$, and so Z does not meet $|w| = r$ and $g(z,w)$ is well-defined and holomorphic there.

By Theorem 1.3.12, $g(z,w)$ extends to $\tilde{g}(z,w)$. But then by the 1-variable Cauchy integral formula, the function

$$\tilde{g}(z,w) = \frac{1}{2\pi i}\int_{|u|=r} \frac{g(z,u)}{u-w}du \tag{2.11}$$

is continuous, and holomorphic in z also, so we are done. □

Theorem 2.2.7 (Weierstrass Division Theorem)**.** *Let $g(z,w)$ be a Weierstrass Polynomial of degree d. For any f holomorphic in a neighborhood of the origin, we can write*

$$f = gh + r \tag{2.12}$$

on a smaller neighborhood, where $r(z,w)$ is a polynomial in w of degree less than d.

[4]See Proposition 2.4.2 below for a more precise statement.

Proof. Choose r, δ, ϵ as usual, so that $g(z,w)$ does not vanish for $|w| = r, \|z\| \le \epsilon$. Define

$$h(z,w) = \frac{1}{2\pi i} \int_{|u|=r} \frac{f(z,u)}{g(z,u)} \frac{du}{u-w}.$$

Then $h(z,w)$ is holomorphic, and so is

$$r := f - gh.$$

We then write

$$\begin{aligned} f(z,w) = f - gh &= \frac{1}{2\pi i} \int \left(f(z,u) - g(z,w) \frac{f(z,u)}{g(z,u)} \right) \frac{du}{u-w} \\ &= \frac{1}{2\pi i} \int \frac{f(z,u)}{g(z,u)} \left(\frac{g(z,u) - g(z,w)}{u-w} \right) du \end{aligned} \tag{2.13}$$

But inspection shows that

$$p(z,u,w) = \frac{g(z,u) - g(z,w)}{u-w}$$

is a polynomial of degree less than d in u and w. Therefore

$$r(z,w) = \frac{1}{2\pi i} \int \frac{f(z,u)}{g(z,u)} p(z,u,w) du$$

is also a polynomial of degree $< d$ in w, since w appears only in $p(z,u,w)$ on the RHS. □

2.3. **The local ring $\mathscr{O}_n$.** We define the **ring of germs of holomorphic functions at** z_0 (also sometimes called just the **local ring** at z_0) to be the ring $\mathscr{O}_{n,z_0}$ of equivalence classes

$$\{[(U,f)] \mid U \ni z_0 \text{ open}, f \in \mathrm{Hol}(U)\}$$

where

$$(U,f) \sim (V,g) \Leftrightarrow f = g \text{ on } U \cap V.$$

The multiplication operation

$$[(U,f) \cdot (V,g)] = [(U \cap V, f \cdot g)]$$

is clearly well-defined.

Henceforth, we shall suppress the open set U from our notation, and will simply refer to a local function $f \in \mathscr{O}_{n,z_0}$. We will also abbreviate $\mathscr{O}_n = \mathscr{O}_{n,0}$ for the ring of germs of holomorphic functions at origin.

Theorem 2.3.1. *The ring $\mathscr{O}_n$ is a local, Noetherian, UFD.*

We already know that $\mathscr{O}_n$ is an integral domain, by the identity principle (any holomorphic function is nonvanishing on an open, dense subset of its domain). It is also clearly a local ring, meaning that it has a unique maximal ideal, namely:

$$\mathfrak{m} = \{f \in \mathscr{O}_n \mid f(0) = 0\}.$$

This is clearly an ideal (*i.e.* closed under addition and scalar multiplication by elements of $\mathscr{O}_n$), and is maximal because for any f with $f(0) \neq 0$, $1/f$ is holomorphic in a neighborhood of the origin, and therefore belongs to $\mathscr{O}_n$.

2.3.1. *Digression on factorization in polynomial rings.* To show that $\mathscr{O}_n$ is Noetherian, and a UFD, we have to recall some facts from ring theory. For more detailed proofs, one may consult Artin's *Algebra* or Atiyah and MacDonald's *Commutative Algebra* (or Wikipedia, for that matter).

Let R be an integral domain, *i.e.* a ring without zero-divisors. Recall that an element $f \in R$ is said to be **irreducible** if for any u and v such that $uv = f$, either u or v is necessarily a unit (*i.e.* an invertible element). A **unique factorization domain** (UFD) is an integral domain in which every nonzero element $f \in R$ admits a decomposition

$$f = \prod_{i=1}^{k} g_i \tag{2.14}$$

where $g_i \in R$ are irreducible, which is *unique* up to permuting the g_i and multiplying by units.

Assuming that R is a domain in which factorizations into irreducibles (2.14) exist, it is easy to convince yourself that R is a UFD if and only if *every irreducible element is* **prime**, *i.e.*

$$f \mid uv \Rightarrow f \mid u \text{ or } f \mid v.$$

Lemma 2.3.2 (Gauss's Lemma)**.** *If R is a UFD, then the polynomial ring $R[t]$ is a UFD.*

Proof sketch. Let K be the fraction field of R. It follows from the division algorithm that the polynomial ring $K[t]$ is a principal ideal domain (choose the element of lowest degree in a given ideal). But in a PID, any irreducible element is prime, as one shows by the following famous trick. Assume that f is irreducible and divides uv, so there exists $g \in R$ such that

$$fg = uv. \tag{2.15}$$

Assume that f and u are relatively prime, i.e. have no common factors other than units. Then the ideal $(f, u) = R$ is the whole ring, so there exist a and b such that

$$af + bu = 1.$$

Multiplying (2.15) by b, we obtain

$$fgb = buv = (1 - af)v$$

and

$$f(gb + av) = v.$$

Therefore f divides v, if f and u are relatively prime. But otherwise f and u are not relatively prime, so have a non-unit factor, which is equal to f up to a unit. Therefore f divides u, which is equally good.

To finish the proof, you just have to lift the question back from $K[t]$ to $R[t]$ by cancelling denominators (appropriately). This amounts to showing that if f and g are *primitive* in $R[t]$, *i.e.* each has relatively prime coeffients, then fg is again primitive. To this end, one can observe that a polynomial is primitive iff it is nonzero in the integral domain $R/(u)$ for each irreducible element $u \in R$. □

Lemma 2.3.3 (The resultant of two polynomials)**.** *Let R be a UFD. Then two polynomials $u, v \in R[t]$ are relatively prime in $K[t]$ if and only if there exists $\gamma \neq 0 \in R$, called the* **resultant** *of u and v, as well as $\alpha, \beta \in R[t]$, with $\deg \alpha < \deg v$ and $\deg \beta < \deg u$, such that*

$$\alpha u + \beta v = \gamma. \tag{2.16}$$

Assuming that α and β are relatively prime in $R[t]$, the resultant is unique up to multiplication by units. Moreover, the resultant is a polynomial function of the coefficients of u and v.

Proof. If α and β are relatively prime in $K[t]$, which is a PID, then the existence of a solution of (2.16) is clear. One can show using the division algorithm that it is possible to reduce the degree of α (and so too of β) as stated. The converse is also clear.

Since we can bound the degrees of α and β, solving the equation (2.16) can be reduced to a matrix equation on the coefficients of α and β whose entries are coefficients of u and v, as I wrote in class. The determinant of this matrix determines the solvability of (2.16), and for a matrix A with nonvanishing determinant, we can always solve (2.16) with $detA$ as the RHS, over the ring R (using the adjugate matrix). □

Definition 2.3.4. Given a polynomial u, define the **discriminant** $D(u)$ to be the resultant of u and u'. Then $D(u) = 0$ if and only if u is coprime to u'. But this is true exactly when u has multiple roots (in the algebraic closure of K). Hence the discriminant, which is a polynomial in the coefficients of u, vanishes if and only if the polynomial u has multiple roots.

Example 2.3.5. For a quadratic polynomial $at^2 + bt + c$, the discriminant is $b^2 - 4ac$. For a cubic of the form $t^3 + pt + q$ (which any cubic is equivalent to under a change of coordinates), the discriminant is given by

$$-4p^3 - 27q^2.$$

We now end our digression and return to the proof that $\mathscr{O}_n$ is a Noetherian UFD.

Proof that $\mathscr{O}_n$ is a UFD. We proceed by induction on n. We have $\mathscr{O}_n = \mathbb{C}$, which is a field, hence a UFD.

Assume for induction that $\mathscr{O}_{n-1}$ is a UFD. Let $f \in \mathscr{O}_n$, which we may assume (by changing coordinates, if necessary) is nonvanishing along the w-axis. By the WPT, $f = gu$, for $g \in \mathscr{O}_{n-1}[w]$ and $u \in \mathscr{O}_n$ a unit. By Gauss's Lemma, we have a unique factorization $g = \prod g_i$, giving

$$f = \prod g_i u. \tag{2.17}$$

Suppose now that

$$f = \prod f_i \tag{2.18}$$

is another decomposition into irreducibles. We then have $f_i(0, w) \not\equiv 0$, since otherwise f would vanish identically. Again by the WPT, we may write $f_i = \tilde{g}_i \tilde{u}_i$. Notice that $\tilde{g} = \prod \tilde{g}_i$ is again a Weierstrass polynomial, and $\tilde{u} = \prod \tilde{u}_i$ is a unit. We now have

$$\prod g_i u = f = \prod \tilde{g}_i \tilde{u}_i$$

and

$$gu = \tilde{g}\tilde{u}.$$

But by uniqueness in the WPT, we must have $g = \tilde{g}$. Therefore

$$g = \prod g_i = \prod \tilde{g}_i$$

and by Gauss's Lemma, the two factorizations must be the same up to permutations. But this shows that the factorization (2.18) is equivalent to (2.17), which shows the uniqueness. □

Proof that $\mathscr{O}_n$ is Noetherian. We again proceed by induction on n, the case $n = 0$ being trivial. Assume that $\mathscr{O}_{n-1}$ is Noetherian. Then by the Hilbert Basis Theorem, $\mathscr{O}_{n-1}[w]$ is again Noetherian.

Let $I \subset \mathscr{O}_n$ be a nontrivial ideal. Choose a nonzero $f \in I$ with $f(0,w) \not\equiv 0$, by changing coordinates if necessary. By the WPT, we have $f = gu$ for a Weierstrass polynomial g. But then since u is a unit, we have $g \in I$ as well, from which we conclude

$$\tilde{I} = I \cap \mathscr{O}_{n-1}[w]$$

is nonempty.

Now, choose a finite generating set $\{g_i\}_{i=1}^k$, consisting of Weierstrass polynomials, for the ideal

$$I \cap \mathscr{O}_{n-1}[w]$$

over the ring $\mathscr{O}_{n-1}[w]$. We claim that this is also a generating set for I over $\mathscr{O}_n$. Let $f \in I$ be arbitrary. We now apply the Weierstrass Division Theorem to divide f by g_1, giving

$$f = g_1 h + r$$

for a polynomial $r \in \mathscr{O}_{n-1}[w]$. But then $r = f - gh$ also belongs to I, and so to $\tilde{I}$, and we have

$$r = \sum a_i g_i.$$

This gives

$$f = g_1(h + a_1) + \sum_{i=2}^{k} a_i g_i.$$

Since $f \in I$ was arbitrary, we conclude that $I = (g_1, \cdots, g_k)$ is finitely generated, completing the induction. □

We end with a last fact, which tells us that factorization and divisibility in the ring $\mathscr{O}_n$, although "local" by definition, is not entirely so.

Proposition 2.3.6. *If f and g are relatively prime in $\mathscr{O}_{n,0}$ they remain relatively prime in $\mathscr{O}_{n,x}$ for all x sufficiently close to zero.*

Proof. Assume wlog that f and g are both Weierstrass polynomials of nonzero degree. Then for z in an open neighborhood, $f(z,w)$ and $g(z,w)$ do not vanish identically in w.

Let γ be the resultant of f and g, so that there exist $\alpha, \beta \in \mathscr{O}_{n-1}[w]$ such that

$$\alpha f + \beta g = \gamma. \tag{2.19}$$

Assume for contradiction that $x = (z_0, w_0)$ is such that $f(z_0, w_0) = g(z_0, w_0) = 0$, and there exists a nontrivial common factor $h(z,w)$ in $\mathscr{O}_{n,(z_0,w_0)}$, which we may assume is a Weierstrass polynomial in $(w - w_0)$, with $h(z_0, w_0) = 0$. This means that $h \mid f$ and $h \mid g$ in $\mathscr{O}_{n,(z_0,w_0)}$. By (2.19), which holds in a neighborhood of 0, we have $h \mid \gamma$.

But then $h(z,w)$ must have degree zero in w. (This can be seen by looking at points z near z_0 where $\gamma(z) \neq 0$, but $h(z,w)$ would have nontrivial zeroes if it had positive degree.) Therefore

$$h(z_0, w) = h(z_0, w_0) \equiv 0$$

and so $f(z_0, w) = 0 = h(z_0, w)$. But this contradicts our observation that $f(z_0, w)$ does not vanish identically in w for z_0 near the origin. □

2.4. **Analytic germs and ideals in $\mathscr{O}_n$.**

We will now give the main payoffs of our study of $\mathscr{O}_n$. The first is the following refinement of Corollary 2.2.4 above, which uses the following slightly informal definition. The meaning will always be clear in context.

Definition 2.4.1. We say that a certain property holds **generically** (or for a generic point) if it is true on an open dense subset that is the complement of the vanishing set of a holomorphic function (or functions).

Proposition 2.4.2. *If $f \in \mathscr{O}_n$ is irreducible, with $\mathrm{Ord}_0 f(0,w) = d < \infty$, then the fiber over a generic point z near the origin of C^{n-1} consists of d distinct points. In other words, the zero set of f is a "branched cover" of a neighborhood of $\mathbb{C}^{n-1}$.*

Proof. Assume without loss that $f = g$ is a Weierstrass polynomial of degree d. If g is irreducible, then the discriminant $D(g)(z) \in \mathscr{O}_{n-1}$ is not identically zero. By the above discussion, for z near the origin such that $D(g)(z) \neq 0$, there are d distinct solutions of $g(z,w) = 0$, as claimed. □

Theorem 2.4.3 (Weak nullstellensatz)**.** *If $g(z,w) \in \mathscr{O}_n$ is irreducible, and $f \in \mathscr{O}_n$ vanishes on the zero set $\{z \mid g(z) = 0\}$, then $g \mid f$ in $\mathscr{O}_n$.*

Proof. We apply the Weierstrass Division Theorem:

$$f = gh + r$$

where $r \in \mathscr{O}_{n-1}[w]$ has $\deg(r) < d = \deg(g)$.

By the previous proposition, we may choose $z \in \mathbb{C}^{n-1}$ arbitrarily close to zero, such that

$$\#\{w \mid g(z,w) = 0\} = d.$$

By assumption, for any such w, we have

$$0 = f(z,w) = g(z,w)h(z,w) + r(z,w) = r(z,w).$$

But then $r(z,w)$ has at least d distinct roots, which implies that $r(z,w)$ is identically zero in w. Since z was a generic point close to the origin, we conclude that r vanishes identically. Therefore $f = gh$, as desired. □

Recall that our motivation for defining the local ring $\mathscr{O}_n$ was that we had to keep shrinking the domain in which we could adequately describe the zero set of a holomorphic function. Since we are shrinking the domain of the functions, it is also convenient to be allowed to shrink the domain of the zero sets.

Definition 2.4.4 (Analytic germs)**.** The **germ** of a set X at the origin is the equivalence class of X under the relation that two sets $X \equiv Y$ if and only if there exists an open set $U \ni 0$ such that

$$X \cap U = Y \cap U.$$

Given a finite collection $f_1, \ldots, f_k \in \mathscr{O}_n$, we define the **germ of the zero set**

$$\mathbf{Z}_0(f_1, \ldots, f_k) = [\{z \mid 0 = f_1(z) = \cdots = f_k(z)\}]$$

to be the germ of the common vanishing set of the f_i. A germ X is said to be **analytic** if it is of the form

$$X = \mathbf{Z}_0(f_1, \cdots, f_k)$$

for such a finite collection. In the case that

$$X = \mathbf{Z}_0(f)$$

for a single function f, we say that X is **the germ of a hypersurface** at the origin. The **ideal** $I(X) \subset \mathscr{O}_n$ of an analytic germ is equal to

$$I(X) = \{f \in \mathscr{O}_n \mid f(z) = 0 \; \forall \, z \in X\}.$$

Conversely, given an ideal $I \subset \mathscr{O}_n$, the germ of the zero-set of I is given by

$$\mathbf{Z}_0(I) := \cap_{f \in I} \mathbf{Z}_0(f).$$

Lemma 2.4.5. *For any ideal $I \subset \mathscr{O}_n, \mathbf{Z}_0(I)$ is an analytic germ. In particular, we have*

$$\begin{aligned} \mathbf{Z}_0(I(X)) &= X \\ \mathbf{Z}_0(I(X) + I(Y)) &= X \cap Y \\ \mathbf{Z}_0(I(X) \cap I(y)) &= X \cup Y \\ X \subset Y &\Rightarrow I(Y) \subset I(X). \end{aligned}$$

Proof. Since $\mathscr{O}_n$ is Noetherian, the ideal I is finitely generated by $f_1, \ldots, f_k$, and so

$$\mathbf{Z}_0(I) = \mathbf{Z}_0(f_1, \ldots, f_k)$$

is indeed analytic. The remaining claims are tautological. □

It remains to determine $I(\mathbf{Z}_0(I))$, for an ideal $I \subset \mathscr{O}_n$. This is the content of a deep theorem due to Hilbert, which we have already proved in a special case: assume that $I = (g)$ is a principal ideal, with g irreducible. Then $\mathbf{Z}_0((g)) = \mathbf{Z}_0(g)$ is the germ of a hypersurface. The Weak Nullstellensatz above exactly states that if $f \in I(\mathbf{Z}_0(g)) = I(\mathbf{Z}_0(I))$ then $f \in (g) = I$. The version for general analytic germs is as follows; we will not give the proof of this result, although it is in the same spirit as the weak version (with more algebra).

Theorem 2.4.6 (General Nullstellensatz)**.** *Let $I \in \mathscr{O}_n$ be an ideal. If $f \in I(\mathbf{Z}_0(I))$ (i.e. f vanishes on the zero set of I), then $f^n \in I$ for some $n \geq 0$. Restated, f belongs to the* radical *$\sqrt{I}$ of I.*

Lastly, to finish the correspondence between germs of functions and germs of sets, recall that an ideal $\mathfrak{p}$ is said to be **prime** if $uv \in \mathfrak{p} \Rightarrow u$ or $v \in \mathfrak{p}$.

Definition/Lemma 2.4.7. *We say that an analytic germ X is* **irreducible** *if it cannot be written as $X = X_1 \cup X_2$ for two proper subsets $X_1, X_2 \subset X$ that are themselves analytic germs. An analytic germ $\mathbf{Z}_0(I)$ is irreducible if and only if the ideal I is prime.*

Proof. The proof is straightforward, and no different from the case of hypersurfaces (where $I = (f)$ is prime iff f is irreducible). □

Proposition 2.4.8. *Let X be the germ of a hypersurface. Then X has a unique decomposition*

$$X = X_1 \cup \cdots \cup X_k$$

into irreducible germs of hypersurfaces.

Proof. Let $f = f_1^{k_1} \cdots f_\ell^{k_\ell}$ be a decomposition of $f \in \mathscr{O}_n$ into irreducibles, which exists since $\mathscr{O}_n$ is a UFD. Then clearly

$$X = \mathbf{Z}_0(f) = \mathbf{Z}_0(f_1) \cup \cdots \cup \mathbf{Z}_0(f_\ell)$$

is the desired irreducible decomposition.

We can show the uniqueness of the decomposition as follows: let

$$X = Y_1 \cup \cdots \cup Y_p$$

be another such decomposition into irreducible analytic germs. Then we must have $Y_1 \supset X_i$ for some i, because

$$Y_1 = Y_1 \cap X = \cup_{i=1}^{\ell} (Y_1 \cap X_i).$$

If $Y_1 \cap X_i$ is a proper analytic subset of Y_1 for each i, then their union is again a proper subset, because Y_1 is assumed irreducible. Hence we must have $Y_1 \supset X_i$ for some i. By the weak nullstellensatz, we have $I(Y_1) \subset (f_i) \subset \mathscr{O}_n$, so $I(Y_1) = (f_i) \cdot I'$ for some ideal $I' \notin (f_i)$. If I' is not the whole $\mathscr{O}_n$, then $\mathbf{Z}_0(I')$ is a proper analytic germ. But then $Y_1 = X_i \cup \mathbf{Z}_0(I')$ is a proper decomposition into analytic germs, which is a contradiction. Therefore $I(Y_1) = (f_i)$, and $Y_1 = \mathbf{Z}_0(f_i) = X_i$. Continuing, one gets the uniqueness of the decomposition. □

Remark 2.4.9. The last result holds for general analytic germs as well, as one can show using factorization by prime ideals and the general Nullstellensatz.

We note that the last several facts form the basis of a subject called "algebraic geometry," which I strongly recommend you to pursue (at a later time).

2.5. **The holomorphic Implicit Function Theorem.** We now turn to the question of when an analytic germ is in fact the germ of a smooth (indeed, a complex) manifold. We begin with the following Lemma, which could have been proved much sooner.

Lemma 2.5.1 (Chain rule for complex-valued functions)**.** *Let $h(w) = h(w^1, \dots, w^n) : \mathbb{C}^n \to \mathbb{C}$ be a complex-valued function, and let $w(z) = (w^1(z), \dots, w^n(z)) : \mathbb{C} \to \mathbb{C}^n$ be an n-tuple of complex-valued functions. Then*

$$\begin{aligned} \frac{\partial h(w(z))}{\partial z} &= \frac{\partial h}{\partial w^i}\frac{\partial w^i}{\partial z} + \frac{\partial h}{\partial \bar{w}^i}\frac{\partial \bar{w}^i}{\partial z} \\ \frac{\partial h(w(z))}{\partial \bar{z}} &= \frac{\partial h}{\partial w^i}\frac{\partial w^i}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{w}^i}\frac{\partial \bar{w}^i}{\partial \bar{z}}. \end{aligned} \tag{2.20}$$

Proof. Let $z = x + iy$ and $w^j(z) = u^j(z) + iv^j(z)$. Then by the real chain rule, we have

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial h}{\partial u^j}\frac{\partial u^j}{\partial x} + \frac{\partial h}{\partial v^j}\frac{\partial v^j}{\partial x} \\ \frac{\partial h}{\partial y} &= \frac{\partial h}{\partial u^j}\frac{\partial u^j}{\partial y} + \frac{\partial h}{\partial v^j}\frac{\partial v^j}{\partial y}. \end{aligned} \tag{2.21}$$

This gives

$$\frac{\partial h}{\partial z} = \frac{1}{2}\left(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial h}{\partial u^j}\frac{\partial u^j}{\partial x} + \frac{\partial h}{\partial v^j}\frac{\partial v^j}{\partial x} - i\left(\frac{\partial h}{\partial u^j}\frac{\partial u^j}{\partial y} + \frac{\partial h}{\partial v^j}\frac{\partial v^j}{\partial y}\right)\right). \tag{2.22}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial h}{\partial w^j}\frac{\partial w^j}{\partial z} + \frac{\partial h}{\partial \bar{w}^j}\frac{\partial \bar{w}^j}{\partial z} &= \frac{1}{2}\left(\frac{\partial h}{\partial u^j} - i\frac{\partial h}{\partial v^j}\right)\left(\frac{1}{2}\left(\frac{\partial u^j}{\partial x} + i\frac{\partial v^j}{\partial x} - i\left(\frac{\partial u^j}{\partial y} + i\frac{\partial v^j}{\partial y}\right)\right)\right) \\ &\quad + \frac{1}{2}\left(\frac{\partial h}{\partial u^j} + i\frac{\partial h}{\partial v^j}\right)\left(\frac{1}{2}\left(\frac{\partial u^j}{\partial x} - i\frac{\partial v^j}{\partial x} - i\left(\frac{\partial u^j}{\partial y} - i\frac{\partial v^j}{\partial y}\right)\right)\right) \\ &= \frac{1}{2}\frac{\partial h}{\partial u^j}\left(\frac{\partial u^j}{\partial x} - i\frac{\partial u^j}{\partial y}\right) + \frac{1}{2}\frac{\partial h}{\partial v^j}\left(\frac{\partial v^j}{\partial x} - i\frac{\partial v^j}{\partial y}\right). \end{aligned} \tag{2.23}$$

Observe that (2.22) agrees with (2.23), giving the first equation of (2.20). The second is proved similarly. ☐

Definition 2.5.2. We say that a continuous map $f : \mathbb{C}^n \to \mathbb{C}^m$ is **holomorphic** if $f^i(z^1, \dots, z^n)$ is holomorphic, for $i = 1, \dots, m$. Define the **complex Jacobian** to be the $m \times n$ matrix of complex numbers

$$\mathscr{J}(f)(z) = \left(\frac{\partial f^i}{\partial z^j}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}.$$

Lemma 2.5.3. *Given two holomorphic functions*

$$\mathbb{C}^p \xrightarrow{g} \mathbb{C}^n \xrightarrow{f} \mathbb{C}^m$$

we have the holomorphic chain rule

$$\mathscr{J}(f \circ g)(z) = \mathscr{J}(f)(g(z)) \cdot \mathscr{J}(g)(z)$$

where $\cdot$ denotes matrix multiplication.

Proof. Fix i and j. The holomorphic chain rule as stated is equivalent to the claim that

$$\frac{\partial f^i(g(z))}{\partial z^j} = \frac{\partial f^i}{\partial w^k}(g(z))\frac{\partial g^k}{\partial z^j}(z). \tag{2.24}$$

Setting $h(w) = f^i(w)$, and $w^k(z) = g^k(z^1, \dots, z, \dots, z^m)$, with z in the j'th place, and applying the previous lemma, we see that all but the first term on the RHS of (2.20) drop out. This gives (2.24). □

We should now compare the real and the complex Jacobians. Let $J(f)(z)$ be the real Jacobian, considered as a map from $T_z\mathbb{R}^{2n} \to T_{f(z)}\mathbb{R}^{2m}$. With respect to the bases

$$B_1 = \left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\}$$

for $T_z\mathbb{R}^{2n}$, and

$$B_2 = \left\{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m}, \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^m}\right\}$$

for $T_{f(z)}\mathbb{R}^{2m}$, the matrix of $J(f)$ is given by

$$(J(f))_{B_1,B_2} = \begin{pmatrix} \frac{\partial u^i}{\partial x^j} & \frac{\partial u^i}{\partial y^j} \\ \frac{\partial v^i}{\partial x^j} & \frac{\partial v^i}{\partial y^j} \end{pmatrix}. \tag{2.25}$$

If we complexify the tangent spaces, then the map $J(f)$ extends canonically to a map

$$T_z\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \to T_{f(z)}\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

which we shall continue to denote by $J(f)$. Letting $z^j = x^j + iy^j$ and $w^k = u^k + iv^k$, we may choose the *complex* bases

$$B_1' = \left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n}\right\}$$

for $T_z\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$, and

$$B_2' = \left\{\frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^m}, \frac{\partial}{\partial \bar{w}^1}, \dots, \frac{\partial}{\partial \bar{w}^m}\right\}$$

for $T_z\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$. With respect to this basis, Lemma 2.5.1 (or an equivalent calculation) implies that the matrix of $J(f)$ is given by

$$\begin{aligned}(J(f))_{B_1',B_2'} &= \begin{pmatrix} \frac{\partial f^i}{\partial z^j} & \frac{\partial f^i}{\partial \bar{z}^j} \\ \frac{\partial \bar{f}^i}{\partial z^j} & \frac{\partial \bar{f}^i}{\partial \bar{z}^j} \end{pmatrix} \\ &= \begin{pmatrix} \mathscr{J}(f) & 0 \\ 0 & \overline{\mathscr{J}(f)} \end{pmatrix}\end{aligned} \tag{2.26}$$

if f is holomorphic. This gives the following:

Lemma 2.5.4. *If $f : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic, then*

$$\det J(f)(z) = |\det \mathscr{J}(f)(z)|^2. \tag{2.27}$$

Proof. The determinant of $J(f)$ is unchanged under complexification and change of basis, and the relation (2.27) holds for any matrix that is block-diagonal of the form (2.26). ☐

Theorem 2.5.5 (Holomorphic Inverse Function Theorem)**.** *Let $f : \Omega \to \Omega' \subset \mathbb{C}^n$ be a holomorphic map between two domains of $\mathbb{C}^n$, and let $z_0 \in \Omega$ be such that*

$$\det \mathscr{J}(f)(z_0) \neq 0. \tag{2.28}$$

Then there exists a neighborhood $U' \ni f(z_0) \subset \Omega'$ and an inverse map $f^{-1} : U' \to \Omega$ that is holomorphic.

Proof. According to Lemma 2.5.4 and (2.28), the determinant of the real Jacobian is nonzero at z_0. By the real inverse function theorem, there exists a C^∞ inverse map f^{-1} as stated. It remains to check that f^{-1} is holomorphic. We have

$$\begin{aligned} z &= f^{-1}(f(z)) \\ 0 &= \frac{\partial}{\partial \bar{z}^j} f^{-1}(f(z)) \\ &= \frac{\partial f^{-1}}{\partial w^k}\frac{\partial f^k}{\partial \bar{z}^j} + \frac{\partial f^{-1}}{\partial \bar{w}^k}\frac{\partial \bar{f}^k}{\partial \bar{z}^j} \end{aligned}$$

where we have applied Lemma 2.5.1. But $\frac{\partial f^k}{\partial \bar{z}^j} = 0$, and we are left with

$$0 = \frac{\partial f^{-1}}{\partial \bar{w}^k}\overline{\frac{\partial f^k}{\partial z^j}}.$$

Since $\mathscr{J}(f)$ is nonsingular, the complex conjugate $\overline{\frac{\partial f^k}{\partial z^j}}$ is as well. We conclude that $\frac{\partial f^{-1}}{\partial \bar{w}^k} = 0$ for all k, as desired. ☐

Theorem 2.5.6 (Holomorphic Implicit Function Theorem)**.** *Given $f^1, \ldots, f^k \in \mathscr{O}_{n,z_0}$ with*

$$\det\left(\frac{\partial f^i}{\partial z^j}(z_0)\right)_{1 \le i,j \le k} \neq 0 \tag{2.29}$$

there exist open sets $U \subset \mathbb{C}^{n-k}, V \subset \mathbb{C}^k$, with $z_0 \in U \times V$, and $g : U \to V$ holomorphic such that

$$f^i(z) = f^i(z_0) \ \text{for}\ i = 1, \ldots, k \quad \Leftrightarrow \quad z = \left(g(z^{k+1}, \ldots, z^n), z^{k+1}, \ldots, z^n\right) \tag{2.30}$$

for $z \in U \times V$.

Proof. This follows from the Inverse Function Theorem in the usual way. Define a map $\tilde{f} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\tilde{f}^i(z) = \left(f^1(z), \ldots, f^k(z), z^{k+1}, \ldots, z^n\right).$$

Then

$$\mathscr{J}(\tilde{f}) = \begin{pmatrix} \mathscr{J}(f) & \\ 0 & \mathrm{Id} \end{pmatrix}$$

and $\det \mathscr{J}(\tilde{f})(z_0) = \det\left(\frac{\partial f^i}{\partial z^j}(z_0)\right)_{1 \le i,j \le k} \neq 0$. By the Inverse Function Theorem, there exists a holomorphic inverse $\tilde{f}^{-1}(z)$ in a neighborhood. We let

$$g(z^{k+1}, \ldots, z^n) = \tilde{f}^{-1}\left(f^1(z_0), \ldots, f^k(z_0), z^{k+1}, \ldots, z^n\right).$$

Then (2.30) holds by definition, as one can check, and g is clearly holomorphic since $\tilde{f}^{-1}$ is. $\square$

2.6. **Generic smoothness and biholomorphisms.** We now turn briefly back to analytic germs, to make another point using the Implicit Function Theorem. Recall the informal definition of the term "generic" made in Definition 2.4.1, whose meaning will be clear in each statement. (Or if not in the statement, then definitely in the proof.)

Lemma 2.6.1 (Generic smoothness)**.** *Let $X = \mathbf{Z}_0(f)$ be the germ of a hypersurface. Then the generic point of X is smooth.*

Proof. Assume first that $X = \mathbf{Z}_0(f)$ is irreducible, *i.e.*, $f \in \mathscr{O}_n$ is irreducible.

We may assume without loss that $f = f(z,w)$ is an irreducible Weierstrass polynomial. Then f is relatively prime to $\frac{\partial f}{\partial w}$, hence the discriminant $D(f)(z) \in \mathscr{O}_{n-1}$ (*i.e.* the resultant of f and $\frac{\partial f}{\partial w}$) does not vanish identically.

The vanishing locus $\mathbf{Z}_0(D(f)(z))$ is an analytic germ at the origin in $\mathbb{C}^{n-1}$. For any point z_0 outside $\mathbf{Z}_0(D(f)(z))$, the polynomials $f(z_0,w)$ and $\frac{\partial f}{\partial w}(z_0,w)$ have distinct roots in w. This means that for any w_0 such that $f(z_0,w_0) = 0$, we have $\frac{\partial f}{\partial w} \neq 0$. By the implicit function theorem, the vanishing set $\{f(z,w) = 0\}$ near (z_0,w_0) is a smooth manifold, as claimed.

For the case that X is not irreducible, by Proposition 2.4.8, it is a finite union of irreducible analytic hypersurfaces. But these are generically smooth, and the finite union of generically smooth things is again generically smooth (because a finite intersection of analytic germs is again an analytic germ, and a finite union of dense open sets is again open and dense). $\square$

Remark 2.6.2. Generic smoothness is also true of general analytic germs.

Lastly, we turn to the following converse of the inverse function theorem. Note that the result fails over the real numbers, as seen from the map $x \mapsto x^3$ (which is bijective over $\mathbb{R}$, but not over $\mathbb{C}$).

Theorem 2.6.3. *Let $f : \Omega \to \Omega' \subset \mathbb{C}^n$ be a bijective holomorphic map between two domains in $\mathbb{C}^n$. Then the Jacobian determinant* $\det J(f)(z) \neq 0$ *for all $z \in \Omega$. In particular, f is a biholomorphism, i.e., there exists a holomorphic inverse map $f^{-1} : \Omega' \to \Omega$.*

Proof. We proceed by induction on the dimension n. The base case $n = 1$ goes as follows. Given $z_0 \in \Omega$, we know that there exists $N \geq 0$ such that $f(z) = f(z_0) + (z - z_0)^N \tilde{f}(z)$, where $\tilde{f}(z_0) \neq 1$. Since $\tilde{f}(z_0) \neq 0$, we may choose an N'th root $g(z)$ of $\tilde{f}(z)$ in a neighborhood of z_0. Then

$$f(z) = f(z_0) + ((z - z_0)g(z))^N$$

and $((z - z_0)g(z))'(z_0) \neq 0$, so by the Inverse Function Theorem, $(z - z_0)g(z)$ is a bijection in a neighborhood. But the composition of a bijection with an N-to-1 map is N-to-1. So if $f(z)$ is bijective, we must have $N = 1$. Then indeed the Jacobian $f'(z_0)$ does not vanish, as claimed.

Next, we assume that the result has been established for $1 \le k < n$. We first claim that for $z \in \Omega$, the implication

$$\det \mathscr{J}(f)(z) = 0 \quad \Rightarrow \quad \mathscr{J}(f)(z) = 0 \tag{2.31}$$

holds. We can show this using the induction hypothesis and the implicit function theorem, as follows.

Assume for contradiction that $\det \mathscr{J}(f)(z_0) = 0$, but $1 \le k = \mathrm{rk}\, \mathscr{J}(f)(z_0) < n$. We may choose coordinates so that

$$\det \left(\frac{\partial f^i}{\partial z^j}(z_0) \right)_{1 \le i,j \le k} \neq 0. \tag{2.32}$$

Then the implicit function theorem gives the existence of $g(z_{k+1}, \ldots, z_n)$ such that

$$f^i(g(z^{k+1}, \ldots, z^n), z^{k+1}, \ldots, z^n) = f^i(z_0)$$

for $1 \le i \le k$. Define the holomorphic function

$$\begin{gathered} h : \mathbb{C}^{n-k} \to \mathbb{C}^{n-k} \\ h(z^{k+1}, \ldots, z^n) = \left(f^{k+1}(g(z^{k+1}, \ldots, z^n), z^{k+1}, \ldots, z^n), \ldots, f^n(g(z^{k+1}, \ldots, z^n), z^{k+1}, \ldots, z^n) \right). \end{gathered} \tag{2.33}$$

Since f is bijective, $f(g(z^{k+1}, \ldots, z^n), z^{k+1}, \ldots, z^n)$ must be bijective from $\{(z_0^1, \ldots, z_0^k)\} \times U'$ to $\{(f^1(z_0), \ldots, f^k(z_0))\} \times V'$. Therefore h is also a bijection between these neighborhoods. But some thought using the chain rule shows that the Jacobian of h must vanish, by our assumptions on f. This contradicts our induction hypothesis, establishing the implication (2.31).

Now, let z_0 be a point where $\det J(f)(z_0) = 0$. Then $X = \mathbf{Z}_0(\det J(f)(z))$ is the germ of a nontrivial analytic hypersurface at z_0. By Lemma 2.6.1, we can choose a point z_1 close to z_0 with $\det J(f)(z_1) = 0$ but such that X is smooth near z_1, *i.e.*, there exists a bijective map $g : U \to X$ for a neighborhood of the origin $U \subset \mathbb{C}^{n-1}$. But then we have

$$\mathscr{J}(f \circ g) = \mathscr{J}(f) \cdot \mathscr{J}(g) = 0 \tag{2.34}$$

because $\mathscr{J}(f) \equiv 0$ on X, by (2.31). A holomorphic map with vanishing Jacobian is constant; therefore $f(z)$ is constant along X, which is a contradiction to the bijectivity (since $n > 1$ and therefore X is not an isolated point). This completes the induction. $\square$

3. Complex structures and differential forms

In this section, we will add a layer of abstraction to what we have already (essentially) done, before moving onward to geometry.

3.1. **Complex structures and complexification.** Let V be a $2n$-dimensional real vector space. A **complex structure** is an endomorphism $I : V \to V$ with

$$I^2 = -1.$$

Notice that if V carries a complex structure I, then it also carries the structure of an n-dimensional $\mathbb{C}$-vector space, by the rule

$$(a + bi) \cdot v = av + bI(v) \tag{3.1}$$

which one can check gives a valid scalar multiplication. However, for reasons which will become apparent, we shall not use the complex multiplication (3.1) but will continue to refer to the action of I by name.

We now let

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}. \tag{3.2}$$

Then V is naturally contained in $V_{\mathbb{C}}$ by the map $v \mapsto v \otimes 1$. Also, $V_{\mathbb{C}}$ has a complex conjugation map

$$\overline{v \otimes \lambda} = v \otimes \bar{\lambda}.$$

The real subspace $V \subset V_{\mathbb{C}}$ is precisely the fixed set of the conjugation map.

We now canonically extend the complex structure I to the vector space $V_{\mathbb{C}}$, by the rule

$$I(v \otimes \lambda) = I(v) \otimes \lambda. \tag{3.3}$$

Then $V_{\mathbb{C}}$ has *two* complex structures, I and i, the first given by (3.3), and the second given by complex multiplication using the attached scalars:

$$i \cdot (v \otimes \lambda) = v \otimes (i\lambda). \tag{3.4}$$

When multiplying elements of $V_{\mathbb{C}}$ by i, we shall always mean in the sense of (3.4).

Because $I^2 = -1$, its eigenvalues must be $\pm i$. The eigenspaces are therefore subspaces of $V_{\mathbb{C}}$, given by

$$V^{1,0} = \{v \in V_{\mathbb{C}} \mid I(v) = i \cdot v\}, \qquad V^{0,1} = \{v \in V_{\mathbb{C}} \mid I(v) = -i \cdot v\}.$$

Lemma 3.1.1. *We have*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

and $\overline{V^{1,0}} = V^{0,1}$. *In particular,* $V^{1,0}$ *and* $V^{0,1}$ *are both complex subspaces of dimension* n.

Proof. Since $V^{1,0} \cap V^{0,1} = 0$, the canonical map

$$\kappa : V^{1,0} \oplus V^{0,1} \to V_{\mathbb{C}}$$

is injective. But we also have projection maps

$$\begin{aligned} \pi_{1,0} : V_{\mathbb{C}} &\to V^{1,0} \\ v &\mapsto \frac{1}{2}\left(v - iI(v)\right) \end{aligned}$$

and

$$\pi_{0,1} : V_{\mathbb{C}} \to V^{0,1}$$
$$v \mapsto \frac{1}{2}(v + iI(v))$$

which one checks yield elements of the claimed eigenspaces. Then the map

$$\pi_{1,0} \oplus \pi_{0,1} : V_{\mathbb{C}} \to V^{1,0} \oplus V^{0,1}$$

is a right-inverse of κ, since

$$\kappa(\pi_{1,0} \oplus \pi_{0,1}(v)) = \frac{1}{2}(v - iI(v)) + \frac{1}{2}(v + iI(v)) = v.$$

Therefore κ is also surjective, hence an isomorphism.

To see that conjugation exchanges the factors, note that by definition, we have $\overline{I(v)} = I(\bar{v})$. Letting $v \in V^{1,0}$, we have

$$v = \pi_{1,0}(v) = \frac{1}{2}(v - iI(v))$$

and

$$\bar{v} = \frac{1}{2}(\bar{v} + iI(\bar{v})) = \pi_{0,1}(\bar{v}).$$

Hence $\bar{v} \in V^{0,1}$, as claimed. □

Definition 3.1.2. Let (V, I) and (W, J) be vector spaces with complex structures. We say that a real-linear map $\alpha : V \to W$ is *complex-linear* if

$$\alpha(I(v)) = J(\alpha(v)) \tag{3.5}$$

for all $v \in V$.

Remark 3.1.3. Notice that by definition, the natural map

$$V \to V_{\mathbb{C}} \to V^{1,0}$$

gives a complex-linear map between (V, I) and $(V^{1,0}, i)$. The two are therefore canonically isomorphic as complex vector spaces.

Proposition 3.1.4. *Fix two vector spaces (V, I) and (W, J) with complex structures, and a complex-linear map $\alpha : V \to W$. Then the canonical extension $\alpha : V_{\mathbb{C}} \to W_{\mathbb{C}}$ satisfies*

$$\overline{\alpha(v)} = \alpha(\bar{v})$$
$$\alpha(V^{1,0}) \subset W^{1,0}$$
$$\alpha(V^{0,1}) \subset W^{0,1}.$$

Proof. The first identity is by definition, and the next two are also easy to check. □

Theorem 3.1.5. *Any complex-linear map $\alpha : (V, I) \to (V, I)$ is orientation-preserving.*

Proof. Denote the canonical extension of α to $V_{\mathbb{C}}$ again by α. Choose any basis $v_1, \ldots, v_n$ for $V^{1,0}$ and pick the basis

$$B = \{v_1, \ldots, v_n, \bar{v}_1, \ldots, \bar{v}_n\} \tag{3.6}$$

for $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Then by the previous Proposition, in the basis (3.6), the matrix of α is of the form

$$(\alpha)_B = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}. \tag{3.7}$$

Since the determinant of α is unchanged under complexification, we have

$$\det \alpha = \det A \det \bar{A} = \det A \overline{\det A} = |\det A|^2 > 0. \tag{3.8}$$

Therefore the map α is orientation-preserving, *i.e.*, the two orientations are equivalent. $\square$

Remark 3.1.6. Notice that the proof of Theorem 3.1.5 gives a streamlined proof of Lemma 2.5.4.

Corollary 3.1.7. *A complex structure I induces a canonical orientation on V.*

Proof. Let $\{e_1, \dots, e_n\} \subset V$ be nonzero vectors such that

$$\{e_1, \dots, e_n, I(e_1), \dots, I(e_n)\} \tag{3.9}$$

form a basis for V; such a choice is clearly possible. Define the orientation on V to be given by the ordered basis (3.9).

We claim that any two bases chosen in this way induce the same orientation on V. Given $\{e_i\}$ and $\{e_i'\}$ as above, we may define an endomorphism of V by

$$\begin{aligned} \alpha : e_i &\mapsto e_i' \\ I(e_i) &\mapsto I(e_i'). \end{aligned}$$

One checks that this is I-linear. By the Theorem, it is orientation-preserving; so the two choices induce the same orientation on V. $\square$

3.2. **Dual spaces and exterior powers.** Let $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ be the (real) dual space of V. We give this a complex structure by the rule

$$I(\alpha)(v) = \alpha(I(v)) \tag{3.10}$$

for $\alpha \in V^*, v \in V$. Then

$$(V^*)_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = (V_{\mathbb{C}})^*$$

where the last * is in the complex sense. Through this identification $(V^*)_{\mathbb{C}} = (V_{\mathbb{C}})^*$, we have

$$\begin{aligned} (V^*)^{1,0} &= \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = i f(v)\} \\ &= \mathrm{Hom}_{\mathbb{C}}((V, I), \mathbb{C}) \\ &\cong \left(V^{1,0}\right)^*. \end{aligned}$$

In fact, the natural pairing

$$(V^*)_{\mathbb{C}} \otimes_{\mathbb{C}} V_{\mathbb{C}} \to \mathbb{C} \tag{3.11}$$

induces isomorphisms

$$(V^*)^{1,0} \cong \left(V^{1,0}\right)^* \tag{3.12}$$
$$(V^*)^{0,1} \cong \left(V^{0,1}\right)^*.$$

This follows because for $\alpha \in (V^*)^{1,0}$ and $\beta \in V^{0,1}$, we have

$$\alpha(\beta) = -iI(\alpha)(\beta) = -i\alpha(I\beta) = (-i)^2\alpha(\beta = -\alpha(\beta)$$

which implies that $\alpha(\beta) = 0$.

Next, we have the real and complex exterior algebras on V and $V_{\mathbb{C}}$, respectively, given by

$$\begin{aligned} \Lambda^* V &= \bigoplus_{k=0}^{2n} \Lambda^k_{\mathbb{R}} V \subset \otimes^*_{\mathbb{R}} V \\ \Lambda^* V_{\mathbb{C}} &= \bigoplus_{k=0}^{2n} \Lambda^k_{\mathbb{C}} V_{\mathbb{C}} \subset \otimes^*_{\mathbb{C}} V_{\mathbb{C}} \\ &= \Lambda^* V \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned} \tag{3.13}$$

Definition/Lemma 3.2.1. *Define the subspace of alternating elements of type (p,q):*

$$\Lambda^{p,q}V := \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1} \subset \Lambda^{p+q} V_{\mathbb{C}}.$$

We then have

$$\begin{aligned} \Lambda^k V_{\mathbb{C}} &= \bigoplus_{p+q=k} \Lambda^{p,q} V \\ \overline{\Lambda^{p,q} V} &= \Lambda^{q,p} V \\ \wedge : \Lambda^{p,q} V \otimes_{\mathbb{C}} \Lambda^{r,s} V &\to \Lambda^{p+r,q+s} V. \end{aligned} \tag{3.14}$$

Proof. These all follow formally from the direct sum decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. □

Define the operator

$$\mathbb{I} = \otimes^k I : \Lambda^k V \to \Lambda^k V. \tag{3.15}$$

Then for $\omega = \alpha \otimes \beta \in \Lambda^{p,q}V$, we have

$$\mathbb{I}(\omega) = \mathbb{I}(\alpha \otimes \beta) = i^p \alpha \otimes i^{-q} \beta = i^{p-q} \omega. \tag{3.16}$$

Therefore $\Lambda^{p,q}V \subset \Lambda^{p+q}V$ lies inside the eigenspace with eigenvalue i^{p-q} inside the $(p+q)$-forms $\Lambda^{p+q}V_{\mathbb{C}}$.

We may perform the exterior power operations for V^* in an identical manner. Then the natural pairing

$$\Lambda^* V^*_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda^* V_{\mathbb{C}} \to \mathbb{C}$$

induces isomorphisms

$$\Lambda^{p,q}(V^*) \cong (\Lambda^{p,q}V)^* \tag{3.17}$$

for each p and q. This just means that for $\omega \in \Lambda^{p,q}(V^*)$ and $\eta \in \Lambda^{r,s}V$, we have $\omega(\eta) \neq 0$ only if $p = r$ and $q = s$, and the pairing is nondegenerate. The isomorphism (3.17) follows formally from the $k = 1$ case, given by (3.12).

3.3. **Holomorphic (co)tangent spaces.** We now come to the key example of the constructions of the previous subsection: given a point $z \in \mathbb{C}^n$, let

$$V = T_z\mathbb{C}^n.$$

Then we have the standard basis for V as a real vector space:

$$\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\right\}.$$

The standard complex structure is given by the rule

$$I\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}, \quad I\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}.$$

The action of the complex structure on the dual space $V^* = T_z^*\mathbb{C}^n$, *i.e.* the cotangent space, is determined by

$$I(dx^j)\left(\frac{\partial}{\partial y^j}\right) = dx^j\left(I\left(\frac{\partial}{\partial y^j}\right)\right) = dx^j\left(-\frac{\partial}{\partial x^j}\right) = -1 \tag{3.18}$$

and so is

$$I(dx^j) = -dy^j, \quad I(dy^j) = dx^j. \tag{3.19}$$

According to Lemma 3.1.1, we have a splitting

$$T_{\mathbb{C},z}\mathbb{C}^n := (T_z\mathbb{C}^n)\otimes_{\mathbb{R}}\mathbb{C} = T_z^{1,0}\mathbb{C}^n \oplus T_z^{0,1}\mathbb{C}^n \tag{3.20}$$

where $T_z^{1,0}\mathbb{C}^n$ is called the **holomorphic tangent space** of $\mathbb{C}^n$ at z. A basis for $T_z^{1,0}\mathbb{C}^n$ may be given by the projections

$$\left\{\pi_{1,0}\left(\frac{\partial}{\partial x^j}\right)\right\}_{j=1}^n = \left\{\frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)\right\}_{j=1}^n = \left\{\frac{\partial}{\partial z^j}\right\}_{j=1}^n.$$

Similarly, a basis for the anti-holomorphic tangent space $T_z^{0,1}$ is given by $\left\{\frac{\partial}{\partial \bar{z}^j}\right\}_{j=1}^n$.

We also have the **holomorphic cotangent space** $(T_z^*)^{1,0}\mathbb{C}^n$ as well as $(T_z^*)^{0,1}\mathbb{C}^n$. These are spanned by the dual bases to the above, $dz^j = dx^j + idy^j$ and $d\bar{z}^j = dx^j - idy^j$, $j = 1, \dots, n$, respectively.

Proposition 3.3.1. *Let $f : U \to V$ be a holomorphic map between open subsets $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$. Then the $\mathbb{C}$-linear extension of the differential $J(f)(z) : T_z\mathbb{C}^n \to T_{f(z)}\mathbb{C}^m$ respects the above decomposition, i.e.*

$$J(f)(z)\left(T_z^{1,0}\mathbb{C}^n\right) \subset T_{f(z)}^{1,0}\mathbb{C}^n, \quad J(f)(z)\left(T_z^{0,1}\mathbb{C}^n\right) \subset T_{f(z)}^{0,1}\mathbb{C}^n.$$

Proof. The real Jacobian of a holomorphic map is I-linear, for the standard complex structures on $\mathbb{C}^n$ and $\mathbb{C}^m$ (exercise). Then the claims follow from Proposition 3.1.4. $\square$

3.4. **Differential forms on $\mathbb{C}^n$.** We now come back to complex analysis. Given a domain $\Omega \subset \mathbb{C}^n$, write $A^k_{\mathbb{R}}(\Omega)$ for the space of C^∞ real-valued differential k-forms on Ω. We shall write

$$A^k(\Omega) = A^k_{\mathbb{R}}(\Omega) \otimes_{\mathbb{R}} \mathbb{C}$$

for the space of C^∞ complex-valued differential forms on Ω.

Definition 3.4.1. Define the space of (p,q)**-differential forms**

$$A^{p,q}(\Omega) = \left\{ \omega \in A^{p+q}(\Omega) \mid \omega \in \Lambda^{p,q} T^*_z \mathbb{C}^n \; \forall \, z \in \Omega \right\}. \tag{3.21}$$

Note that this is a module over the space of complex-valued smooth functions $A^0(\Omega)$.

As above, we canonically extend the exterior derivative operator d to complex-valued forms $A^{p,q}$. We now define two new operators

$$\partial = \pi^{p+1,q} \circ d, \qquad \bar{\partial} = \pi^{p,q+1} \circ d. \tag{3.22}$$

Explicitly, these operators are given as follows. We use the following notation for a (p,q)-form:

$$\alpha = \alpha_{IJ}(z) dz^I \wedge d\bar{z}^J = \sum_{\substack{\{i_1 < \cdots < i_p\} \subset \{1,\ldots,n\} \\ \{j_1 < \cdots < j_q\} \subset \{1,\ldots,n\}}} \alpha_{i_1 \cdots i_q j_1 \cdots j_p}(z) dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}. \tag{3.23}$$

Then we have

$$\begin{aligned} \partial \alpha &= \frac{\partial \alpha_{IJ}}{\partial z^k} dz^k \wedge dz^I \wedge d\bar{z}^J \\ \bar{\partial} \alpha &= \frac{\partial \alpha_{IJ}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J. \end{aligned} \tag{3.24}$$

Notice that we now have a more manifestly coordinate-invariant definition of holomorphicity, namely:

$$f(z) \text{ is holomorphic } \Leftrightarrow \bar{\partial} f(z) = 0. \tag{3.25}$$

The algebraic properties of the operators ∂ and $\bar{\partial}$ can be summarized as follows.

Proposition 3.4.2. *We have*

$$\begin{gathered} d = \partial + \bar{\partial} \\ \partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial \bar{\partial} = -\bar{\partial} \partial. \end{gathered}$$

We have also the following commutation rules, for $\alpha \in A^{p,q}$ and $\beta \in A^{r,s}$:

$$\begin{aligned} \bar{\partial}(\alpha \wedge \beta) &= \bar{\partial} \alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \bar{\partial} \beta \\ \partial(\alpha \wedge \beta) &= \partial \alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \partial \beta. \end{aligned}$$

Proof. We can either use the explicit formulae above, or argue as follows. Letting $f \in A^0$, from Lemma 1.2.9 (or now simply by definition), we have

$$df = \partial f + \bar{\partial} f.$$

The case of $\alpha \in A^{p,q}$ follows from the formula

$$d(\alpha_{IJ} dz^I \wedge d\bar{z}^J) = d(\alpha_{IJ}) \wedge dz^I \wedge d\bar{z}^J. \tag{3.26}$$

The next claim follows by writing

$$0 = d^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial$$

and observing that the forms of each type must vanish individually. The commutation rules are proved similarly. □

The point of defining these spaces and operators is the following invariance property:

Proposition 3.4.3. *Let $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$ be domains, and $f : \Omega \to \Omega'$ a holomorphic map. Then for $\alpha \in A^{p,q}(\Omega')$, we have*

$$f^*\alpha \in A^{p,q}(\Omega)$$
$$\bar{\partial}\left(f^*\alpha\right) = f^*\left(\bar{\partial}\alpha\right).$$

Proof. The first claim follows from the fact that the Jacobian of a holomorphic map is complex-linear: by Proposition 3.1.4, it preserves the (anti)-holomorphic tangent spaces, and by duality, so too the cotangent spaces and all exterior powers. More explicitly, one can simply pull back the formula (3.23) using the fact that

$$df^i = \partial f^i, \qquad d\bar{f}^i = \bar{\partial}\bar{f}^i.$$

This gives

$$\begin{aligned} f^*\alpha(z) &= \alpha_{i_1\cdots i_p j_1\cdots j_q}(f(z))d\left(f^{i_1}\right)\wedge\cdots\wedge d\left(f^{i_p}\right)\wedge d\left(\bar{f}^{j_1}\right)\wedge\cdots\wedge d\left(\bar{f}^{j_q}\right) \\ &= \alpha_{i_1\cdots i_p j_1\cdots j_q}(f(z))\frac{\partial f^{i_1}}{\partial z^{k_1}}dz^{k_1}\wedge\cdots\wedge\frac{\partial f^{i_p}}{\partial z^{k_p}}dz^{k_p}\wedge\frac{\partial \bar{f}^{j_1}}{\partial \bar{z}^{k_1}}d\bar{z}^{k_1}\wedge\cdots\wedge\frac{\partial \bar{f}^{j_q}}{\partial \bar{z}^{k_q}}d\bar{z}^{k_q} \\ &= \alpha_{i_1\cdots i_p j_1\cdots j_q}(f(z))\frac{\partial f^{i_1}}{\partial z^{k_1}}\cdots\frac{\partial f^{i_p}}{\partial z^{k_p}}\cdot\frac{\partial \bar{f}^{j_1}}{\partial \bar{z}^{k_1}}\cdots\frac{\partial \bar{f}^{j_q}}{\partial \bar{z}^{k_q}}dz^{k_1}\wedge\cdots\wedge dz^{k_p}\wedge d\bar{z}^{k_1}\wedge\cdots\wedge d\bar{z}^{k_q} \end{aligned}$$

which is again a (p,q)-form.

The second identity follows from the first identity and the property $d \circ f^* = f^* \circ d$ of the ordinary exterior derivative operator. □

3.5. The $\bar{\partial}$-Poincaré Lemma in several variables.

We now prove the general version of Theorem 1.6.4. Recall that we denote a polydisk of radius r by

$$D_r = B_r \times \cdots \times B_r \subset \mathbb{C}^n.$$

We shall write $\beta \in A^{p,q}(\bar{D}_r)$ for the subspace of (smooth) forms in $A^{p,q}(D_r)$ that extend continuously to the boundary.

Lemma 3.5.1 ($\bar{\partial}$-Poincaré Lemma in a closed polydisk)**.** *Let $\beta \in A^{p,q}(\bar{D}_r)$, with $q \geq 1$, satisfy $\bar{\partial}\beta = 0$. Then there exists $\alpha \in A^{p,q-1}(D_r)$ such that*

$$\bar{\partial}\alpha = \beta.$$

Proof. Notice that if $\beta = g_{IJ}dz^I \wedge d\bar{z}^J$, then $\bar{\partial}\beta = (-1)^p dz^I \wedge \bar{\partial}\left(g_{IJ}d\bar{z}^J\right)$ vanishes if and only if $\bar{\partial}\left(g_{IJ}d\bar{z}^J\right) = 0$ for each I. Moreover, if $\alpha = f_{IJ'}dz^I d\bar{z}^{J'}$, then $\bar{\partial}\alpha = (-1)^p dz^I \bar{\partial}\left(f_{IJ'}d\bar{z}^{J'}\right)$. Hence $\bar{\partial}\alpha = \beta$ if and only if $\bar{\partial}\left(f_{IJ'}d\bar{z}^{J'}\right) = \left(g_{IJ}d\bar{z}^J\right)$ for each I. It therefore suffices to consider the case $p = 0$. We shall also prove only the case $n = 2$, since the general case is only notationally more complex.

First, let $q = 2$. Then $\beta = g(z^1, z^2)d\bar{z}^1 \wedge d\bar{z}^2$. By Theorem 1.6.4, we may solve

$$\frac{\partial}{\partial \bar{z}^1} f(z^1, z^2) = g(z^1, z^2)$$

by the formula

$$f(z^1, z^2) = \frac{1}{2\pi i}\int_{B_r} \frac{g(w, z^2)}{w^1 - z^1} dw \wedge d\bar{w}. \tag{3.27}$$

Letting $\alpha = f(z^1, z^2)d\bar{z}^2$, we have

$$\bar{\partial}\alpha = \frac{\partial f}{\partial \bar{z}^1} d\bar{z}^1 \wedge d\bar{z}^2 = g d\bar{z}^1 \wedge d\bar{z}^2 = \beta$$

as desired.

Next, let $q = 1$. Then $\beta = g_1(z^1, z^2)d\bar{z}^1 + g_2(z^1, z^2)d\bar{z}^2$. By (3.27), we may let f_2 solve

$$\frac{\partial}{\partial \bar{z}^1} f_2(z^1, z^2) = g_2(z^1, z^2).$$

Then $\bar{\partial} f_2 = \frac{\partial}{\partial \bar{z}^1} f_2 d\bar{z}^1 + g_2 d\bar{z}^2$, and we have

$$\begin{aligned} \tilde{\beta} := \beta - \bar{\partial} f_2 &= \left(g_1 - \frac{\partial f_2}{\partial \bar{z}^1}\right) d\bar{z}^1 + (g_2 - g_2)d\bar{z}^2 \\ &=: \tilde{g}_1(z_1, z_2)d\bar{z}^1. \end{aligned}$$

This still solves

$$\begin{aligned} \bar{\partial}\tilde{\beta} = \bar{\partial}\beta - \bar{\partial}^2 f_2 = 0 \\ = \frac{\partial \tilde{g}_1}{\partial \bar{z}^2} d\bar{z}^2 \wedge d\bar{z}^1. \end{aligned}$$

Therefore

$$\frac{\partial \tilde{g}_1}{\partial \bar{z}^2} = 0. \tag{3.28}$$

Now let f_1 solve

$$\frac{\partial f_1}{\partial \bar{z}^1} = \tilde{g}_1$$

by (3.27). Then

$$\frac{\partial f_1}{\partial \bar{z}^2} = \frac{1}{2\pi i}\int_{B_r} \frac{\frac{\partial}{\partial \bar{z}^2} g(w, z^2)}{w - z^1} dw \wedge d\bar{w} = 0$$

by (3.28). Hence

$$\bar{\partial} f_1 = \frac{\partial f_1}{\partial \bar{z}^1} d\bar{z}^1 + \frac{\partial f_1}{\partial \bar{z}^2} d\bar{z}^2 = \tilde{g}_1 d\bar{z}^1 = \tilde{\beta} = \beta - \bar{\partial} f_2.$$

Now let $\alpha = f_1 + f_2$. Then

$$\bar{\partial}\alpha = \bar{\partial} f_1 + \bar{\partial} f_2 = \left(\beta - \bar{\partial} f_2\right) + \bar{\partial} f_2 = \beta$$

as desired. This completes the case $n = 2, q = 1$.

The case $n > 2$ follows by a similar strategy of knocking off the factors $d\bar{z}^k$ from β one-by-one. □

Theorem 3.5.2 ($\bar{\partial}$-Poincaré Lemma in an open polydisk)**.** *Let $\beta \in A^{p,q}(D_r)$, with $q \geq 1$, satisfy $\bar{\partial}\beta = 0$. Then there exists $\alpha \in A^{p,q-1}(D_r)$ such that*

$$\bar{\partial}\alpha = \beta.$$

Proof. As before, it suffices to prove the theorem for $p = 0$.

Choose an increasing sequence $r_m \nearrow r$, and write $D_m = D_{r_m} \Subset D_r = D$.

Claim 1. For each m, there exists $\alpha_m \in A^{p,q-1}(D)$ with $\bar{\partial}\alpha_m = \beta$ on D_m.

Since β is continuous on D_m, by the previous lemma, there exists $\alpha'_m \in A^{p,q-1}(D_{m+1})$ with $\bar{\partial}\alpha'_m = \beta$ on D_{m+1}. Choose a smooth cutoff ψ with $\mathrm{supp}\psi \subset D_{m+1}$ and $\psi \equiv 1$ on D_m, and let

$$\alpha_m = \psi\alpha'_m. \tag{3.29}$$

This proves Claim 1.

We now proceed by induction on q. We will do the induction step first; so fix $q > 1$ and assume that the Theorem has been proven for $1, \ldots, q-1$.

Claim 2. For $q > 1$, it is possible to choose $\{\alpha_m\}$ such that

$$\alpha_{m+1} = \alpha_m$$

on D_{m-1}.

Assume that $\alpha_1, \ldots, \alpha_m$ have already been chosen. As in Claim 1, may choose $\tilde{\alpha}_m \in A^{0,q-1}(D)$ such that $\bar{\partial}\tilde{\alpha}_{m+1} = \beta$ on D_{m+1}. Then

$$\bar{\partial}\left(\alpha_m - \tilde{\alpha}_{m+1}\right) = \beta - \beta = 0$$

on D_m. By the induction hypothesis, there exists $\gamma \in A^{0,q-2}$ such that

$$\bar{\partial}\gamma = \alpha_m - \tilde{\alpha}_{m+1}.$$

Let

$$\alpha_{m+1} = \tilde{\alpha}_{m+1} + \bar{\partial}(\psi\gamma).$$

We then have

$$\bar{\partial}\alpha_{m+1} = \bar{\partial}\tilde{\alpha}_{m+1} + 0 = \beta$$

on D_{m+1}, and

$$\alpha_{m+1} = \tilde{\alpha}_{m+1} + \bar{\partial}\gamma = \alpha_m$$

on D_{m-1}, as claimed.

We now have a sequence α_m which agree on the open sets D_m, hence converge trivially to $\alpha \in A^{0,q-1}(D)$ satisfying $\bar{\partial}\alpha = \beta$. This proves the theorem for $q > 1$, assuming it also holds for $q = 1$.

Claim 3. For $q = 1$, we can choose a sequence $\alpha_m \in A^0(D)$ with $\bar{\partial}\alpha_m = \beta$ on D_m, and

$$|\alpha_{m+1} - \alpha_m|_{C^m(D_m)} < 2^{-m}. \tag{3.30}$$

Assume $\alpha_1, \dots, \alpha_m$ have been chosen. Let $\tilde{\alpha}_{m+1} \in A^0(D)$ such that $\bar{\partial}\tilde{\alpha}_{m+1} = \beta$ on D_{m+1} as before. Then

$$\bar{\partial}\left(\alpha_m - \tilde{\alpha}_{m+1}\right) = 0$$

on D_{m+1}. But α_m are now functions, hence we conclude that $\alpha_m - \tilde{\alpha}_{m+1}$ is holomorphic on D_{m+1}. It therefore has a uniformly convergent Taylor series on D_m. We can truncate the series to obtain a polynomial $P = P(z^1, \dots, z^n)$ such that

$$|\alpha_m - \tilde{\alpha}_{m+1} - P|_{C^m(D_m)} < 2^{-m}. \tag{3.31}$$

We now let $\alpha_{m+1} = \tilde{\alpha}_{m+1} + P$, which is well-defined on D, and satisfies

$$\bar{\partial}\alpha_{m+1} = \bar{\partial}\tilde{\alpha}_{m+1} = \beta \tag{3.32}$$

on D_{m+1}. Moreover, by (3.31), α_{m+1} satisfies (3.30), which proves Claim 3.

By (3.30), the sequence $\{\alpha_m\}$ is uniformly convergent in $C^k(D_n)$ for each $n, k > 0$. We therefore have $\alpha_m \to \alpha \in A^0(D)$, satisfying $\bar{\partial}\alpha = \beta$ on D, as desired. □

Remark 3.5.3. Notice that the proof also works with $r = \infty$, so with $\mathbb{C}^n$ replacing D_r. We will also need the following generalization. We write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. (The theorem also works with $\mathbb{C}$ replaced by a ball and $\mathbb{C}^*$ replaced by an annulus.)

Theorem 3.5.4. *Let $r, s \in \mathbb{N}$ with $r + s = n$, and put*

$$\Omega = \mathbb{C}^r \times (\mathbb{C}^*)^s \subset \mathbb{C}^n.$$

Let $\beta \in A^{p,q}(\Omega)$, with $q \geq 1$, satisfy $\bar{\partial}\beta = 0$. Then there exists $\alpha \in A^{p,q-1}(\Omega)$ such that

$$\bar{\partial}\alpha = \beta.$$

Proof. The proof is the same as that of Theorem 3.5.2, except that one uses a truncation of the Laurent series instead of the Taylor series in (3.31). □

4. Complex manifolds

This section finally begins the main business of the class, which is the study of complex manifolds. We shall be particularly interested in *compact* complex manifolds. The first clear differences between the categories of compact smooth (*i.e.* real) manifolds and compact complex manifolds are as follows.

(1) Any holomorphic function on a compact complex manifold is locally constant.

(2) It is impossible to holomorphically embed a compact complex manifold of positive dimension in $\mathbb{C}^N$, for any N.

(3) The coordinate charts of a complex manifold cannot always be taken to be $\mathbb{C}^n$.

(4) There exist holomorphic families of compact complex manifolds that are not isotrivial, *i.e.*, in which nearby members are not isomorphic.

The corresponding (false) statements in the smooth category are obtained by replacing "holomorphic" by "smooth" and $\mathbb{C}$ by $\mathbb{R}$.

We will establish (1-4) over the course of this section.

4.1. Definitions and first properties.

Let M be a smooth manifold of real dimension $2n$, with an **atlas** of coordinate charts $\underline{U} = \{U_\alpha, \varphi_\alpha\}$. Recall that an atlas is an open cover of M, together with maps $\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subset \mathbb{R}^{2n}$ that are homeomorphisms onto their images, for which the transition functions

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

are smooth maps. We say that an atlas is **holomorphic** if, identifying $\mathbb{R}^{2n} = \mathbb{C}^n$ in the standard way (as above), the transition functions are holomorphic maps in the sense of Definition 2.5.2. Two holomorphic atlases $\underline{U}$ and $\underline{V}$ are *equivalent* if the union $\underline{U} \cup \underline{V}$ is again a holomorphic atlas.

Definition 4.1.1. A **complex manifold** M of (complex) dimension n is a smooth manifold of real dimension $2n$, equipped with an equivalence class of holomorphic atlases. A complex manifold of dimension $n = 1$ is called a **Riemann surface**.

Definition 4.1.2. A continuous map $f : M \to N$ between two complex manifolds is said to be **holomorphic** if it restricts to a holomorphic map between coordinate charts. In other words, for charts $U \subset M, V \subset N$ and $\varphi : U \to \mathbb{C}^n, \psi : V \to \mathbb{C}^p$, the map

$$\psi \circ f \circ \varphi : \varphi(f^{-1}(V) \cap U) \to \psi(V) \tag{4.1}$$

is holomorphic.

We say that f is **biholomorphic** if it is also bijective. By Theorem 2.6.3, such an f has a holomorphic inverse function f^{-1}, hence is an isomorphism in the category of complex manifolds and holomorphic maps.

Lastly, a **holomorphic function** on M is a holomorphic map $f : M \to \mathbb{C}$. In particular, on any coordinate chart, f restricts to a holomorphic function on a domain in $\mathbb{C}^n$.

Theorem 4.1.3. *Any holomorphic function f on a connected, compact, complex manifold M is constant.*

Proof. Since M is compact, $|f|$ attains its maximum at a point p lying inside some coordinate chart U. But U is a domain in $\mathbb{C}^n$, so by the maximum principle, f is constant on U. Since M is connected, the usual argument shows that f must be constant throughout M. □

Corollary 4.1.4. *If M is connected, the image of any holomorphic map $M \to \mathbb{C}^N$ is a point.*

Proof. Given a holomorphic map $f : M \to \mathbb{C}^N$, each coordinate function on $\mathbb{C}^N$ pulls back to a holomorphic function on M, which must be constant by the Theorem. □

Definition 4.1.5. Let $\pi : M \to N$ be a holomorphic submersion, *i.e.,* a holomorphic map whose differential is surjective at all points—equivalently, by (2.26), whose complex Jacobian $\mathscr{J}(\pi)$ is surjective. Theorem 2.5.6 implies that for each $t \in N$, the fiber $M_t = \pi^{-1}(t)$, is a complex manifold, where the coordinate charts are obtained from those of M by restricting to coordinate hyperplanes appropriately. A **holomorphic family** of complex manifolds, parametrized by N, is simply the collection of fibers $\{M_t\}_{t\in N}$ of a holomorphic submersion. The dimension of the fibers is

$$\dim M_t = \dim M - \dim N.$$

Definition 4.1.6. Recall that for any point p in a smooth manifold M, the tangent space T_pM is simply the tangent space to $T_{\varphi_i(p)}\mathbb{C}^n$ in any coordinate chart, where tangent vectors are identified under pushforward by the transition functions. Since these functions are holomorphic, they preserve the $(1,0)$ and $(0,1)$ parts of the complexification $T_{\mathbb{C},p}M = T_pM \otimes_{\mathbb{R}} \mathbb{C}$, by Proposition 3.1.4. We may therefore define the **holomorphic tangent space**

$$T_p^{1,0}M = (T_pM)^{1,0}.$$

As discussed in Remark 3.1.3, the space T_pM is canonically isomorphic to $T_p^{1,0}M$. We can therefore expect all the geometry of a complex manifold to be reflected in the holomorphic tangent spaces.

Similarly, we may define the (anti)-holomorphic cotangent spaces $(T_p^*)^{1,0}M$ and $(T_p^*)^{0,1}M$ at each point, and the spaces of (p,q)-differential forms

$$A^{p,q}(U) = \left\{\omega \in A^{p+q}(U) \mid \omega \in \Lambda^{p,q}T_p^*\mathbb{C}^n \ \forall\, p \in U\right\}$$

for any open set $U \subset M$. Moreover, we may define the operators ∂ and $\bar\partial$ exactly as in (3.22), which again satisfy the conclusions of Proposition 3.4.2.

Definition 4.1.7. Given an open set $U \subset M$, define the space

$$Z_{\bar\partial}^{p,q}(U) = \ker\bar\partial \subset A^{p,q}(U).$$

Since $\overline{\partial}^2 = 0$ (Proposition 3.4.2), we know that the image $\bar\partial A^{p,q-1}(M)$ is contained in $Z_{\bar\partial}^{p,q}(M)$. We may therefore define the **Dolbeault cohomology groups** of M by

$$H_{\bar\partial}^{p,q}(M) = \frac{Z_{\bar\partial}^{p,q}(M)}{\bar\partial A^{p,q-1}(M)}. \tag{4.2}$$

By the $\bar\partial$-Poincaré Lemma (Theorem 3.5.2), for an open polydisk $D_r \subset M$ and any $p \geq 0, q \geq 1$, the kernel $Z^{p,q}_{\bar\partial}(D_r)$ of the $\bar\partial$ operator is identical to its image, $\bar\partial A^{p,q-1}(D_r)$, and therefore

$$H^{p,q}_{\bar\partial}(D_r) = 0.$$

In this sense, the Dolbeault cohomology groups with $q \geq 1$ are "locally trivial," and can be expected to detect the global holomorphic "shape" of M. As we shall see (in part), the Dolbeault groups are a refinement of the DeRham cohomology groups (where d is in place of $\bar\partial$ in (4.2)) in the holomorphic category. In particular, they are functorial with respect to holomorphic maps:

Proposition 4.1.8. *Dolbeault cohomology defines a contravariant functor from the category of complex manifolds to the category of complex vector spaces. In other words, given a holomorphic map $f : M \to N$ between complex manifolds, the pullback map on differential forms induces a linear map*

$$H^{p,q}_{\bar\partial}(N) \xrightarrow{f^*} H^{p,q}_{\bar\partial}(M) \tag{4.3}$$

which is functorial. In particular, f^* is an isomorphism if f is a biholomorphism.

Proof. By Proposition 3.4.3, we have $f^* Z^{p,q}(N) \subset Z^{p,q}(M)$ and $\bar\partial f^* A^{p,q-1}(N) = f^* \bar\partial A^{p,q-1}(N)$. Therefore f^* descends to a well-defined map $H^{p,q}_{\bar\partial}(N) \to H^{p,q}_{\bar\partial}(M)$, which retains its functoriality properties. □

Definition 4.1.9. Define the **Hodge numbers**

$$h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(M).$$

These are the most basic invariants (*i.e.*, biholomorphism invariants) of a complex manifold.

4.2. **Examples.** This section describes the first few examples in the subject.

Example 4.2.1. The **Riemann sphere** $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ has two coordinate charts

$$U = \mathbb{C}$$

with coordinate z, and

$$V = \mathbb{C}^* \cup \{\infty\}$$

with coordinate $w = 1/z$, and $w(\infty) = 0$. The transition function is holomorphic on the overlap

$$U \cap V = \mathbb{C}^*.$$

Example 4.2.2. The n-dimensional **projective space** is given by

$$\begin{aligned} \mathbb{CP}^n &= \left\{\mathbb{C}\text{-lines through the origin in } \mathbb{C}^{n+1}\right\} \\ &= \{[Z] \mid Z \neq 0 \in \mathbb{C}^{n+1}\} / \left([Z] \sim [\lambda Z] \text{ for } \lambda \in \mathbb{C}^*\right). \end{aligned} \tag{4.4}$$

We shall denote points of $\mathbb{CP}^n$ by equivalence classes $[Z] = [Z^0, \dots, Z^n]$, where

$$\left[Z^0, \dots, Z^n\right] \sim \left[\lambda Z^0, \dots, \lambda Z^n\right] \tag{4.5}$$

for any $\lambda \neq 0 \in \mathbb{C}$.

We have $n+1$ standard coordinate charts on $\mathbb{CP}^n$, given by

$$U_i = \left\{ \left[Z^0, \ldots, Z^n \right] \mid Z^i \neq 0 \right\}$$

for $i = 0, \ldots, n$, with coordinate maps

$$\begin{aligned} \varphi_i : U_i \xrightarrow{\sim} \mathbb{C}^n &= \left\{ \left(z^0, \ldots, \hat{z}^i, \ldots, z^n \right) \right\} \\ \left[Z^0, \ldots, Z^n \right] &\mapsto \left(\frac{Z^0}{Z^i}, \ldots, \frac{\hat{Z}^i}{Z^i}, \ldots, \frac{Z^n}{Z^i} \right). \end{aligned}$$

The notation "$\hat{z}^i$" means that we omit that element of the sequence. Notice that φ_i is well-defined under the equivalence relation (4.5). The inverse map is given by

$$\varphi_i^{-1} : \left(z^0, \ldots, \hat{z}^i, \ldots, z^n \right) \mapsto \left[z^0, \ldots, 1, \ldots, z^n \right]$$

where 1 is in the i'th coordinate entry. The transition function on

$$U_i \cap U_j = \left\{ [Z] \mid Z^i \neq 0, Z^j \neq 0 \right\}$$

is given by

$$\varphi_j \circ \varphi_i^{-1} : (z^0, \ldots, \hat{z}^i, \ldots, z^n) \mapsto \left(\frac{z^0}{z^j}, \ldots, \frac{\hat{z}^j}{z^j}, \ldots, \frac{1}{z^j}, \ldots, \frac{z^n}{z^j} \right)$$

which is holomorphic, as required.

Notice that we may write

$$\begin{aligned} \mathbb{CP}^n &= \mathbb{C}^n \sqcup \mathbb{CP}^{n-1} \\ &= U_0 \sqcup \left\{ \left[0, Z^1, \ldots, Z^n \right] \right\}. \end{aligned} \tag{4.6}$$

In this way, $\mathbb{CP}^n$ can be seen as a compactification of $\mathbb{C}^n$ by adding a "plane at infinity," whose points in turn correspond to complex lines through the origin in $\mathbb{C}^n$.

Given an injective linear map

$$\left(a^i{}_j \right)_{\substack{1 \leq i \leq n+1 \\ 1 \leq j \leq k+1}} : \mathbb{C}^{k+1} \to \mathbb{C}^{n+1}$$

we obtain a holomorphic inclusion

$$\begin{aligned} \mathbb{CP}^k &\hookrightarrow \mathbb{CP}^n \\ \left[w^0, \ldots, w^k \right] &\mapsto \left[a^0{}_j w^j, \ldots, a^n{}_j w^j \right]. \end{aligned}$$

which we refer to as a **projective k-plane**. Notice that any $k+1$ linearly independent points in $\mathbb{C}^{n+1}$ determine a projective k-plane. A projective $(n-1)$-plane in $\mathbb{CP}^n$ is called a **hyperplane**. The space of hyperplanes is parametrized by the **dual projective space**

$$\mathbb{CP}^{n*} = \left(\mathbb{C}^{n+1} \right)^* \setminus \{0\} / \sim$$

which is of course biholomorphic to $\mathbb{CP}^n$, but not canonically.

Example 4.2.3. Given a complex manifold M and a group Γ acting on M properly discontinuously by biholomorphisms, the quotient space M/Γ is again a complex manifold, with coordinate charts inherited from M. For example, let $m \in \mathbb{Z}$ act on $\mathbb{C}^n$ by

$$(z^1, \ldots, z^n) \mapsto (2^m z^1, \ldots, 2^m z^n).$$

The quotient $\mathbb{C}^n/\mathbb{Z}$ is called a **Hopf manifold**, and is easily seen to be diffeomorphic to $S^1 \times S^{2n-1}$.

Example 4.2.4. Let $\Lambda \subset \mathbb{C}$ be a lattice, *i.e.*

$$\Lambda = \{m\tau_1 + n\tau_2 \mid m, n \in \mathbb{Z}\} \subset \mathbb{C}$$

for $\tau_1/\tau_2 \notin \mathbb{R}$. Then the quotient $\mathbb{C}/\Lambda$ is a one-dimensional **complex torus**. In particular, for $\tau \notin \mathbb{R}$, we let

$$\Lambda_\tau = \langle 1, \tau \rangle$$

and

$$X_\tau = \mathbb{C}/\Lambda_\tau.$$

Exercise: Show that there can be no holomorphic, injective map from $\mathbb{C}$ to X_τ.

Lemma 4.2.5. *For a complex torus $X = \mathbb{C}/\Lambda$, we have $h^{1,0}(X) = 1$.*

Proof. Observe that the space of holomorphic $(1,0)$-forms on X contains the element dz. Since dz spans $T_z^{(1,0)}X$ at each point $z \in X$, any holomorphic 1-form on X is of the form $\alpha(z)dz$ for a doubly periodic holomorphic function $\alpha(z)$ (see Problem 1.5.12 above). By Liouville's Theorem, $\alpha(z)$ must be constant. Therefore

$$H^{1,0}(X) = \{cdz \mid c \in \mathbb{C}\}$$

which has rank one, as claimed. □

Proposition 4.2.6. *Given two lattices Λ and $\Lambda' \subset \mathbb{C}$, the complex tori $X = \mathbb{C}/\Lambda$ and $Y = \mathbb{C}/\Lambda'$ are biholomorphic if and only if there exists $c \in \mathbb{C}^*$ such that $c \cdot \Lambda = \Lambda'$.*

Proof. Let $f : X \to Y$ be a biholomorphism. Then f lifts to a holomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{C}$, which we may choose with $\tilde{f}(0) = 0$, so $\tilde{f}(\Lambda) = \Lambda'$. But then f^*dz is a holomorphic differential on X, which is equal to cdz by the Lemma. On $\mathbb{C}$, we have

$$f^*dz = cdz = \tilde{f}'(z)dz.$$

Therefore $\tilde{f}(z) = cz$, and $\tilde{f}(\Lambda) = c\Lambda = \Lambda'$, as desired. □

According to the proposition, the set of isomorphism classes of complex tori is identical to the set of lattices in $\mathbb{C}$ modulo complex scalars. Given any such Λ, we may assume, after multiplication by a scalar, that $1 \in \Lambda$ is an element of shortest length. With this choice, Λ will have nontrivial intersection with the strip

$$S = \{z \in \mathbb{C} \mid -1/2 \leq \operatorname{Re} z \leq 1/2, \operatorname{Im} z > 0, |z| \geq 1\}.$$

Letting $\tau \in \Lambda \cap S$ be the element with minimal imaginary part, we have $\Lambda = \Lambda_\tau$. It is easy to convince yourself that for $\tau \in S$, Λ_τ is unique up to isomorphism, except for the identifications

$$\tau \sim \tau + 1$$

and

$$\tau = e^{i\theta} \sim e^{i(\pi - \theta)}.$$

These identifications only affect the boundary of S. We have shown the following:

Theorem 4.2.7. *The set of all isomorphism classes of one-dimensional complex tori is given by*

$$\{X_\tau \mid \tau \in S/\sim\}.$$

Exercise: Show that

$$\left(S \setminus \left\{\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}, i\right\}\right)/\sim$$

parametrizes a holomorphic family of complex tori, in the sense of Definition 4.1.5.

4.3. **Subvarieties and submanifolds.** A closed subset $V \subset M$ is called an **analytic subvariety** if, for each $p \in V$, the germ of V at p is analytic in the sense of Definition 2.4.4.

Equivalently, for each $p \in V$, there exists a coordinate neighborhood $U \ni p$ and holomorphic functions $f^1, \ldots, f^{k_p}$ such that

$$S \cap U = \left\{f^1(z) = \cdots = f^{k_p}(z) = 0\right\} =: \mathbf{Z}(f^1, \cdots, f^{k_p}). \tag{4.7}$$

We say that V is an **analytic hypersurface** if $k_p = 1$ for all $p \in V$.

Definition 4.3.1. A **complex submanifold** $S \subset M$ of codimension k is an analytic subvariety such that for all $p \in S$, we have $k_p = k$, and for $f^1, \ldots, f^k$ as in (4.7), the matrix

$$\left(\frac{\partial f^i}{\partial z^j}(p)\right) \tag{4.8}$$

has full rank (equal to k). By Theorem 2.5.6, S is a complex manifold in its own right, of dimension $n - k$.

Notice that the fibers of a holomorphic family, per Definition 4.1.5, are complex submanifolds of the total space. Indeed, this is the special case where the functions $f^1, \ldots, f^k$ are each defined on an open neighborhood of S inside M.

A point on an analytic subvariety where (4.8) holds is called a *smooth point*, as opposed to a *singular point*. According to Lemma 2.6.1, at least in the hypersurface case, the generic point of an analytic variety is smooth. In this sense, an analytic variety is a submanifold with "analytic singularities" along a proper subset.

Definition 4.3.2. Given an entire holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, the vanishing locus $X = \mathbf{Z}(f) \subset \mathbb{C}^n$ is called an **affine analytic hypersurface**. If f is a polynomial, then X is called an **affine algebraic hypersurface**.

Notice that if zero is a regular value, *i.e.*, for each $p \in \mathbb{C}^n$ with $f(p) = 0$, there exists z^i such that

$$\frac{\partial f}{\partial z^i}(p) \neq 0, \tag{4.9}$$

then $\mathbf{Z}(f)$ is a smooth hypersurface in $\mathbb{C}^n$.

Example 4.3.3. Consider the affine algebraic hypersurface $X_\lambda = \mathbf{Z}(P_\lambda) \subset \mathbb{C}^2$, where

$$P_\lambda(x, y) = y^2 - x(x-1)(x-\lambda).$$

We have

$$\frac{\partial P_\lambda}{\partial y} = 2y$$
$$\frac{\partial P_\lambda}{\partial x} = -\left((x-1)(x-\lambda) + x(x-\lambda) + x(x-1)\right).$$

Then $\frac{\partial P_\lambda}{\partial y}$ is nonvanishing except at $(0,0),(1,0)$, and $(\lambda,0)$. But if $\lambda \neq 0$ or 1, then $\frac{\partial P_\lambda}{\partial x}$ is nonvanishing at these points and we conclude that X_λ is a complex submanifold. As λ varies, these form a holomorphic family of smooth affine hypersurfaces $\{X_\lambda\}$ called **elliptic curves**, which is intimately related to the family $\{X_\tau\}$ of Example 4.2.4.

For $\lambda = 1$, $X_1 = \{y^2 = x(x-1)^2\}$ is no longer a submanifold, but an affine analytic hypersurface singular at $(1,0)$. In fact, X_1 can be parametrized by

$$x = t^2, \quad y = t(t^2-1), \quad t \in \mathbb{C}. \tag{4.10}$$

This map is 1-to-1 except for $\pm 1 \mapsto (1,0)$. Hence, X_1 is isomorphic to $\mathbb{C}$ with two points identified (this is called a *rational nodal curve*).

Definition 4.3.4. Given k *homogeneous* polynomials

$$P^1(Z^0,\ldots,Z^n),\ldots,P^k(Z^0,\ldots,Z^n)$$

the common vanishing set

$$X = \mathbf{Z}(P^1,\ldots,P^k) \subset \mathbb{CP}^n$$

is well-defined under the equivalence relation (4.5). This is called a **projective algebraic variety**. In particular, X is an analytic subvariety of $\mathbb{CP}^n$: for example, on the coordinate chart $U_0 \cong \mathbb{C}^n$, we clearly have

$$X \cap U_0 = \mathbf{Z}\left(P^1(1,u^1,\ldots,u^n),\ldots,P^k(1,u^1,\ldots,u^n)\right).$$

Example 4.3.5. Let

$$\hat{P}_\lambda(X,Y,Z) = ZY^2 - X(X-Z)(X-\lambda Z)$$

and put

$$\hat{X}_\lambda = \mathbf{Z}(\hat{P}_\lambda(X,Y,Z)) \subset \mathbb{CP}^2.$$

Then clearly $\hat{X}_\lambda \cap U_2 = X_\lambda$, per Example 4.3.3. Notice that $\hat{X}_\lambda \setminus U_2 = \{[0,1,0]\}$, and on $U_1 = \{[x,1,z]\}$, we have

$$\hat{X}_\lambda \cap U_1 = \mathbf{Z}\left(z - x(x-z)(x-\lambda z)\right).$$

Then $\frac{\partial P_\lambda(x,1,z)}{\partial z}(0,0) = 1$, hence $[0,1,0]$ is a smooth point as well. For $\lambda \neq 0,1$, $\hat{X}_\lambda$ is therefore a smooth projective algebraic variety in $\mathbb{CP}^2$ (also called a *smooth projective curve*). Since $\mathbb{CP}^2$ is compact and $\hat{X}_\lambda$ is a closed subset, it is also compact; hence, $\hat{X}_\lambda$ is a "compactification" of X_λ.

Every affine *algebraic* variety can be compactified to a projective *algebraic* variety in a similar way. It is easy to check whether a projective variety is smooth, using the following criterion:

Proposition 4.3.6. *Let $X = \mathbf{Z}(P^1, \ldots, P^k)$ be a projective algebraic variety. Then X is smooth if and only if*

$$\left(\frac{\partial P^i}{\partial Z^j}\right)_{\substack{1 \le i \le k \\ 0 \le j \le n}} \tag{4.11}$$

has rank k at each point of X.

Proof. On U_ℓ, let $p^i(z^0, \ldots, \hat{z}^\ell, \ldots, z^n) = P^i(z^0, \ldots, 1, \ldots, z^n)$. Then $X \cap U_\ell = \mathbf{Z}(p^1, \ldots, p^k)$, and

$$\frac{\partial p^i}{\partial z^j}(z^0, \ldots, \hat{z}^\ell, \ldots, z^n) = \frac{\partial P^i}{\partial Z^j}(z^0, \ldots, 1, \ldots, z^n), \quad 1 \le i \le k, 0 \le j \ne \ell \le n. \tag{4.12}$$

But for any homogeneous polynomial of degree d, we have the identity

$$d \cdot P = Z^k \frac{\partial P}{\partial Z^k}.$$

For a point $Z \in U_\ell$ where $P^i(Z) = 0$, we have

$$\frac{\partial P^i}{\partial Z^\ell}(Z) = \frac{-Z^j}{Z^\ell} \frac{\partial P^i}{\partial Z^j}(Z).$$

The vector $\left(\frac{\partial P^i}{\partial Z^\ell}(Z)\right)_{i=1}^k$ is therefore a linear combination of the vectors $\left(\frac{\partial P^i}{\partial Z^j}(Z)\right)_{i=1}^k$, for $j \ne \ell$. It follows that the matrix (4.12) has the same rank as (4.11), which gives the result. □

Notice that there is a difference between the definition of an analytic hypersurface in $\mathbb{C}^n$, which may be defined by a different function near each point, and an *affine* analytic hypersurface, which is defined by a single entire function. This brings up the following question:

Problem 4.3.7 (Cousin problem)**.** Is every analytic hypersurface in $\mathbb{C}^n$ an affine analytic hypersurface?

Note that an analytic hypersurface in a compact complex manifold can never be the zero locus of an entire function, since this would necessarily be constant. In the case of projective space, we have the following analogue of the above question:

Problem 4.3.8. Is every analytic subvariety of $\mathbb{CP}^n$ a projective algebraic variety?

The first question will be answered (affirmatively) below in §6.4.1, and the second will be partially answered by Theorem 9.2.4.

5. Sheaves

5.1. **Motivation.** At the end of the last section, we saw two problems that have to do with passing from local to global data. We begin this section by describing another, more central, motivating problem, which (in some form or other) will occupy us for most of the rest of the semester.

Let Σ be a Riemann surface and $\{p_\beta\}$ a discrete collection of points. Choose coordinate charts $\{U_\alpha\}$ for Σ such that each p_β is contained in exactly one U_β, and corresponds to the origin in this chart. For each β, fix a "principal part"

$$Q_\beta(z) = \sum_{i=1}^{N_\beta} a_i^\beta z^{-i}. \tag{5.1}$$

Problem 5.1.1 (Mittag-Leffler)**.** Does there exist a global meromorphic function[5] on Σ which is holomorphic on $\Sigma \setminus \{p_\beta\}$ and has principal part $Q_\beta(z)$ at p_β, for each β?

Notice that the problem is locally trivial, since on the patch U_β, we may take the function $Q_\beta(z)$ as our solution. In fact, it is also trivial in the cases $\Sigma = \mathbb{C}$ and $\Sigma = \mathbb{CP}^1$. However, we have seen in Problem 1.5.12 that for $\Sigma = \mathbb{C}/\Lambda$, there does not exist a meromorphic function with a single, simple pole; so the answer to the global question is sometimes negative. The goal is to describe a general approach—in fact, two approaches that will turn out to be equivalent—which will allow us in principle either to solve the problem or to identify the "obstruction" to the existence of a solution in a given case.

5.1.1. *Approach via Čech cohomology.* Let $\underline{U} = \{U_\alpha\}$ be the open cover of Σ described above. Choose Q_β per (5.1), and let

$$Q_{\alpha\beta} = Q_\alpha - Q_\beta,$$

which is a holomorphic function on $U_\alpha \cap U_\beta$, for each α and β. Notice that on $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$Q_{\alpha\beta} + Q_{\beta\gamma} + Q_{\gamma\alpha} = 0. \tag{5.2}$$

To make a global solution, we need to find a *holomorphic* function g_α on U_α, for each α, such that

$$Q_\alpha + g_\alpha = Q_\beta + g_\beta \tag{5.3}$$

on $U_\alpha \cap U_\beta$, for each α and β. This would give a well-defined meromorphic function f with

$$f = Q_\alpha + g_\alpha$$

on each chart U_α.

Notice that (5.3) is equivalent to

$$g_\beta - g_\alpha = Q_{\alpha\beta}. \tag{5.4}$$

Define

$$Z^1(\underline{U}, \mathscr{O}) = \{\{f_{\alpha\beta}\} \mid f_{\alpha\beta} \in \mathrm{Hol}(U_\alpha \cap U_\beta), f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0\}$$

[5] See 5.3.9 below for the formal definition.

and

$$B^1(\underline{U}, \mathscr{O}) = \{\{f_{\alpha\beta}\} \mid f_{\alpha\beta} = g_\beta - g_\alpha, \text{ for some } \{g_\alpha\}\}.$$

In this formulation, the obstruction to choosing $\{g_\alpha\}$ as required lies in the quotient group

$$\check{H}^1(\underline{U}, \mathscr{O}) = \frac{Z^1(\underline{U}, \mathscr{O})}{B^1(\underline{U}, \mathscr{O})}. \tag{5.5}$$

5.1.2. *Approach via Dolbeault cohomology.* Assume that the cover U_α is locally finite, and choose smooth cutoff functions $\rho_\alpha \in C_c^\infty(U_\alpha)$ such that $\rho_\alpha \equiv 1$ in a neighborhood of p_α. Let

$$\beta = \sum_\alpha \bar\partial(\rho_\alpha Q_\alpha).$$

Observe that this is a global, smooth $(0,1)$-form (vanishing identically near p_α), and satisfies

$$\bar\partial\beta = 0.$$

If we can solve

$$\bar\partial\alpha = \beta$$

for $\alpha \in C^\infty(M)$, then

$$f = \sum_\alpha \rho_\alpha f_\alpha - \gamma$$

will satisfy $\bar\partial f = 0$, with the required principal parts at p_α. In this setup, the obstruction to solving the problem therefore lies in the Dolbeault cohomology group (4.2) already defined above:

$$H^{0,1}_{\bar\partial}(\Sigma) = \frac{Z^{0,1}_{\bar\partial}(\Sigma)}{\bar\partial A^0(\Sigma)}. \tag{5.6}$$

5.2. **Presheaves and sheaves.** Let M be a topological space. A **presheaf** $\mathscr{F}$ on M assigns to each open set $U \subset M$ a set $\mathscr{F}(U)$, together with *restriction maps*

$$r_{VU} : \mathscr{F}(V) \to \mathscr{F}(U)$$

for each $U \subset V$, satisfying

i) $r_{UU} = \mathrm{Id}$

ii) $r_{VU} \circ r_{WV} = r_{WU}$ for each triple $U \subset V \subset W$.

For an element $s \in \mathscr{F}(W)$, we shall often write $s|_U = r_{WU}(s)$. When the sets $\mathscr{F}(U)$ are endowed with an algebraic structure and the restriction maps are morphisms in the relevant category, we say that $\mathscr{F}$ is a *presheaf of abelian groups, rings, modules, etc.*

Let $\underline{U} = \{U_\alpha\}$ be an open cover of $U \subset M$. We say that $\mathscr{F}$ is a **sheaf** if it satisfies two further axioms, for any such open cover:

iii) If $f, g \in \mathscr{F}(U)$ satisfy $f|_{U_\alpha} = g|_{U_\alpha}$ for each α, then $f = g$.

iv) If $f_\alpha \in \mathscr{F}(U_\alpha)$ satisfy

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$$

for each α, β, then there exists $f \in \mathscr{F}(U)$ such that

$$f|_{U_\alpha} = f_\alpha$$

for all α.

Example 5.2.1. The **sheaf of continuous functions** $\mathscr{C}^0_M$ assigns to each open set $U \subset M$ the set of continuous functions on U:

$$\mathscr{C}^0_M(U) = C^0(U).$$

This clearly satisfies the axioms, and is a sheaf of *rings*.

Example 5.2.2. More generally, given any continuous map $\pi : X \to M$ between topological spaces, define the **sheaf of sections** $\mathscr{X}$ of M by

$$\mathscr{X}(U) = \{\sigma : U \to X \text{ continuous} \mid \pi \circ \sigma = Id_U\}.$$

It is trivial to check that this is a sheaf of sets. If the fibers of X are endowed with additional structure (such as if X is a vector bundle over M), then $\mathscr{X}$ is naturally a sheaf of abelian groups (or modules over $C^0(M)$).

The previous example can be recovered by taking $X = M \times \mathbb{R}$, and π the projection to the first factor. Indeed, you will show on the homework that *any* sheaf is isomorphic to the sheaf of sections of some map of topological spaces $X \to M$.

Example 5.2.3. Consider the constant presheaf $\mathbb{R}$, which assigns $\mathbb{R}(U) = \mathbb{R}$ for each open subset. If the space $M = U \sqcup V$ is disconnected, then this is a presheaf, but not a sheaf. For, we have sections $0 \in \mathbb{R}(U)$ and $1 \in \mathbb{R}(V)$, but no section exists in $\mathbb{R}(M)$ that restricts to each.

To remedy the situation, define the **sheaf of locally constant functions**, $\underline{\mathbb{R}}$, by the prescription

$$\underline{\mathbb{R}}(U) = C^0(U, \mathbb{R}^{discrete}).$$

By the previous example(s), $\underline{\mathbb{R}}$ is clearly a sheaf. This is an example of *sheafification*—see Definition 5.3.7 below.

We may similarly define locally constant sheaves for any abelian group, such as $\underline{\mathbb{Z}}$ or $\underline{\mathbb{C}}$.

Example 5.2.4. On any smooth manifold, we have the following sheaves:

$\mathscr{C}^\infty$, the sheaf of complex-valued smooth functions

$\mathscr{C}^*$, the sheaf of nonvanishing smooth functions, viewed as a sheaf of groups under multiplication

$\mathscr{A}^k$, the sheaf of smooth k-forms

$\mathscr{Z}^p_d$, the sheaf of closed k-forms.

The first and third are sheaves of modules over the $\mathscr{C}^\infty$.

Example 5.2.5. On any complex manifold, we have also the following sheaves:

$\mathscr{O}$, the sheaf of holomorphic functions

$\mathscr{O}^*$, the sheaf of nonvanishing holomorphic functions, viewed as a sheaf of groups under multiplication

Ω^p, the sheaf of holomorphic p-forms, *i.e.*, $\bar{\partial}$-closed $(p,0)$-forms

$\mathscr{A}^{p,q}$, the sheaf of *smooth* (p,q)-forms

$\mathscr{Z}^{p,q}_{\bar\partial}$, the sheaf of $\bar\partial$-closed (p,q)-forms

$\mathscr{I}_V$, the sheaf of holomorphic functions vanishing along an analytic subvariety $V \subset M$.

All but the second are sheaves of modules over $\mathscr{O}$.

5.3. **Basic constructions in the sheaf category.**

Definition 5.3.1. Given $x \in M$, define the **stalk** of $\mathscr{F}$ at x by

$$\mathscr{F}_x = \{(U,s) \mid x \in U, s \in \mathscr{F}(U)\} / \sim$$

where $(U_1,s_1) \sim (U_2,s_2)$ if and only if there exists $x \in U \subset U_1 \cap U_2$ such that

$$s_1|_U = s_2|_U .$$

Notice that any section $s \in \mathscr{F}(U)$ defines a canonical element $s_x \in \mathscr{F}_x$, for each $x \in U$. Moreover, according to the sheaf axiom iii) above, these images uniquely determine the section s.

Example 5.3.2. The stalk at $x \in M$ of the sheaf of holomorphic functions $\mathscr{O}_x$ is isomorphic to the ring of germs of holomorphic functions $\mathscr{O}_n$ of §2.3.

Definition 5.3.3. Let $\mathscr{F},\mathscr{G}$ be presheaves. A **morphism** (or *map of sheaves*

$$\alpha : \mathscr{F} \to \mathscr{G}$$

is given by a map $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for each $U \subset M$, satisfying

$$\alpha_U \circ r^{\mathscr{F}}_{VU} = r^{\mathscr{G}}_{VU} \circ \alpha \tag{5.7}$$

for any $U \subset V$. We say that α is **injective** (resp. surjective) if the induced maps on stalks

$$\alpha_x : \mathscr{F}_x \to \mathscr{G}_x$$

are injective (resp. surjective), for each $x \in M$.

Proposition 5.3.4. *If $\mathscr{F}$ is a sheaf and $\alpha : \mathscr{F} \to \mathscr{G}$ is injective, then for any $U \subset M$, the induced maps*

$$\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$$

are injective.

Proof. Let $s \in \mathscr{F}(U)$ be an element with $\alpha(s) = 0$. We have $\alpha_x(s_x) = \alpha(s)_x = 0$ for all $x \in U$. But α_x is injective by assumption, so we conclude that $s_x = 0$ for all $x \in U$. By sheaf axiom iii), this implies that $s \equiv 0$. □

Definition 5.3.5. We define the **Kernel** $\operatorname{Ker}\alpha$ of a sheaf morphism by the prescription:

$$(\operatorname{Ker}\alpha)(U) = \operatorname{Ker}\left(\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)\right).$$

One can check using axiom (iv) that $\operatorname{Ker}\alpha$ is a sheaf, and indeed there exists a canonical injective map of sheaves

$$\operatorname{Ker}\alpha \to \mathscr{F}.$$

In other words, $\operatorname{Ker}\alpha$ is a **subsheaf** of $\mathscr{F}$.

Example 5.3.6. Define the **exponential map**

$$\exp : \mathscr{O} \to \mathscr{O}^* \\ f \mapsto e^{2\pi i f}$$

This is a surjective map of sheaves, per Definition 5.3.3. However, notice that the element

$$z \in \mathscr{O}^*(\mathbb{C} \smallsetminus \{0\})$$

is *not* the image of an element $f \in \mathscr{O}(\mathbb{C} \smallsetminus \{0\})$. For, if $z = e^{2\pi i f}$, then $dz/z = 2\pi i df$. But then

$$2\pi i = \int_{S^1} \frac{dz}{z} = 2\pi i \int_{S^1} df = 0$$

by the fundamental theorem, which is a contradiction.

We conclude that a surjective map of sheaves is not necessarily surjective on sections over each open set. One consequence is that the *image presheaf* $\mathrm{im}(\alpha)$ is not necessarily a sheaf: in other words, there can exist elements $t \in \mathscr{G}$ such that $\mathrm{im}\mathscr{F}_x \ni t_x$ for all $x \in U$, but for which there exists no $s \in \mathscr{F}(U)$ with $\alpha(s) = t$, violating axiom iv). This is a fundamental problem in sheaf theory.

A first step toward remedying the problem is to make the following definition.

Definition 5.3.7. Given a presheaf $\mathscr{F}$ on M, we define the **sheafification** $\mathscr{F}^+$ by declaring that $\mathscr{F}^+(U)$ consists of the set of all maps $s \to \sqcup_{x \in U} \mathscr{F}_x$, with $s(x) \in \mathscr{F}_x$, such that for all $x \in U$ there exists an open subset $x \in V \subset U$ and a section $t \in \mathscr{F}(V)$ such that $t_y = s(y)$ for all $y \in V$.

Equivalently, a section $s \in \mathscr{F}^+(U)$ is given by an open cover $\underline{U} = \{U_\alpha\}$ of U, together with sections $s_\alpha \in \mathscr{F}(U_\alpha)$ for which

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta} \tag{5.8}$$

modulo equivalence under refinements. From this perspective, it is clear that $\mathscr{F}^+$ is a sheaf. Moreover, one checks that the obvious map

$$\mathscr{F} \to \mathscr{F}^+$$

induces isomorphisms on all stalks:

$$\mathscr{F}_x \to \mathscr{F}_x^+.$$

Lastly, the sheafification has the property that if $\mathscr{G}$ is a sheaf, then any morphism $\alpha : \mathscr{F} \to \mathscr{G}$ factors as $\mathscr{F} \to \mathscr{F}^+ \to \mathscr{G}$, agreeing with α on stalks.

Definition 5.3.8. Define the **image sheaf** $\mathrm{Im}\,\alpha = \mathrm{im}\,\alpha^+$. This is naturally a subsheaf of $\mathscr{G}$. Define the **cokernel** $\mathrm{Coker}\,\alpha$ to be the sheaf associated to the presheaf

$$U \mapsto \mathscr{G}(U)/\alpha\left(\mathscr{F}(U)\right).$$

Definition 5.3.9. Define the sheaf of **meromorphic functions** $\mathscr{M}$ on M to be the sheaf associated to the presheaf

$$U \mapsto \mathrm{Frac}(\mathscr{O}_U)$$

where $\mathrm{Frac}(\mathscr{O}_U)$ is the fraction field of the ring of holomorphic functions on U. We denote by $\mathscr{M}^*$ the multiplicative sheaf of meromorphic functions that are not identically zero on any connected component of M.

5.4. **The sheaf of divisors.** This subsection gives another important example of a sheaf of abelian groups.

Definition 5.4.1. A **divisor** D on an open set $U \subset M$ is, by definition, a formal linear combination

$$D = \sum_\beta n_\beta Z_\beta,$$

where $n_\beta \in \mathbb{Z}$, Z_β is an irreducible analytic subvariety of U, and the sum is *locally* finite. We say that a divisor D is **effective** if $n_\beta \geq 0$ for all β. The **sheaf of divisors** $\mathscr{D}iv$ on M assigns to each open set $U \subset M$ the abelian group $\mathscr{D}iv(U)$ of divisors on U. In particular, the stalk $\mathscr{D}iv_x$ is the group of *finite* formal linear combinations of irreducible analytic germs at x.

Given any meromorphic function f on U, we may define an element

$$\mathrm{div}(f) \in \mathscr{D}iv(U)$$

as follows. By Theorem 2.3.1, for any open set $x \in U$, there exists $V \ni x$ such that

$$f = \frac{h_1^{n_1} \cdots h_k^{n_k}}{g_1^{m_1} \cdots g_\ell^{m_\ell}}$$

on V, with h_i and g_i all relatively prime in $\mathscr{O}_x$. We define the stalk of the map div by

$$\mathrm{div}(f)_x = \sum_{\alpha=1}^{k} n_\alpha \mathbf{Z}_0(h_\alpha) - \sum_{\beta=1}^{\ell} n_\beta \mathbf{Z}_0(g_\beta).$$

One can prove using Proposition 2.3.6 that this gives a well-defined, surjective map of sheaves

$$\mathrm{div} : \mathscr{M}^* \to \mathscr{D}iv.$$

Definition 5.4.2. A (global) divisor D on M is said to be a **principal divisor** if there exists a global meromorphic function f on M such that

$$D = \mathrm{div}(f).$$

5.5. **Complexes and the global sections functor.** We say that a sequence of maps of sheaves

$$\cdots \to \mathscr{F}_i \overset{\alpha_i}{\to} \mathscr{F}_{i+1} \overset{\alpha_{i+1}}{\to} \mathscr{F}_{i+2} \to \cdots$$

is a **complex** if $\alpha_{i+1} \circ \alpha_i = 0$ for all i. The complex is said to be **exact** if $\mathrm{Ker}\, \alpha_{i+1} = \mathrm{Coker}\, \alpha_i$ for each i. A **short exact sequence** is an exact sequence of the form

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0.$$

Example 5.5.1. Given any map of sheaves $\alpha : \mathscr{F} \to \mathscr{G}$, the sequence

$$0 \to \operatorname{Ker}\alpha \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \to \operatorname{Coker}\alpha \to 0$$

is exact.

Example 5.5.2. The **exponential exact sequence** is given on any complex manifold by

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \to 0.$$

Example 5.5.3. Given a complex submanifold S, the ideal sheaf $\mathscr{I}_S$ fits into a short exact sequence:

$$0 \to \mathscr{I}_S \to \mathscr{O}_M \to \mathscr{O}_S \to 0.$$

Here $\mathscr{O}_S$ is the "extension by zero" of the sheaf of holomorphic functions on S, whose sections are given by

$$\mathscr{I}_S(U) = \mathscr{O}_S(U \cap S).$$

Example 5.5.4. The ordinary Poincaré Lemma implies that the complex of sheaves

$$0 \to \mathbb{R} \to \mathscr{C}^\infty \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \to \cdots \to \mathscr{A}^n \to 0$$

is a long exact sequence.

Example 5.5.5. The $\bar{\partial}$-Poincaré Lemma implies that the complex of sheaves

$$0 \to \Omega^p \to \mathscr{A}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,1} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,2} \to \cdots \to \mathscr{A}^{p,n} \to 0$$

is a long exact sequence.

Definition 5.5.6. Define the **global sections functor** $\Gamma : \{\text{sheaves on M}\} \to \text{Ab}$ by

$$\Gamma(\mathscr{F}) = \mathscr{F}(M).$$

As we have seen in Example 5.3.6 above, the global sections functor is not always right-exact, *i.e.*, from a short exact sequence of sheaves only gives an exact sequence of abelian groups:

$$0 \to \Gamma(\mathscr{E}) \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G}). \tag{5.9}$$

Example 5.5.7. We may define the **sheaf of principal parts** $\mathscr{P}\mathscr{P}$ as the cokernel of the inclusion map

$$0 \to \mathscr{O} \to \mathscr{M} \to \mathscr{P}\mathscr{P} \to 0. \tag{5.10}$$

In the Riemann surface case, this is exactly the set of "Mittag-Leffler data." We may therefore reformulate the Mittag-Leffler Problem 5.1.1 as one of determining the image

$$\Gamma(\mathscr{M}) \to \Gamma(\mathscr{P}\mathscr{P}).$$

Example 5.5.8. The sheaf of divisors $\mathscr{D}iv$ on M fits into a short exact sequence

$$0 \to \mathscr{O}^* \to \mathscr{M}^* \xrightarrow{\operatorname{div}} \mathscr{D}iv \to 0. \tag{5.11}$$

We may therefore formulate the Cousin Problem 4.3.7 as one of determining the part of the image

$$\Gamma(\mathscr{M}^*) \to \Gamma(\mathscr{D}iv)$$

that is also effective; *i.e.*, the space of effective principal divisors on M.

6. Sheaf cohomology

Sheaf cohomology gives a way to remedy the failure of right-exactness of the global sections functor. There are many equivalent definitions. Our approach will be to list a set of axioms guaranteeing that the groups have the desired properties, and also defining them uniquely up to isomorphism (if they exist). We will then give a construction of the groups and check that they satisfy the axioms.

We shall assume henceforth that all sheaves are sheaves of abelian groups (possibly with additional structure). We shall also assume always that M is paracompact, *i.e.*, every open cover has a locally finite subcover.

6.1. **Axioms.** A **sheaf cohomology theory** is an assignment of abelian groups $H^i(\mathscr{F})$, for $i = 0, \dots, \infty$, to any given sheaf $\mathscr{F}$, together with certain maps between the groups, satisfying several axioms.

Axiom 1. For each $i = 0, \dots, \infty$, the sheaf cohomology group $H^i(-)$ is a (covariant) functor from the category of sheaves to the category of abelian groups.

Axiom 2. $H^0(-) = \Gamma(-)$.

Axiom 3. Given any short exact sequence of sheaves

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$$

on M, there exist maps $\delta^{i*} : H^i(\mathscr{G}) \to H^{i+1}(\mathscr{E})$, so that the sheaf cohomology groups form a long exact sequence

$$0 \to \Gamma(\mathscr{E}) \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G}) \xrightarrow{\delta^{0*}} H^1(\mathscr{E}) \to H^1(\mathscr{F}) \to H^1(\mathscr{G}) \xrightarrow{\delta^{1*}} H^2(\mathscr{E}) \to \cdots. \tag{6.1}$$

This assignment gives a functor from the category of short exact sequences of sheaves to long exact sequences of abelian groups.

Before stating the remaining axioms, we need to make the following definitions.

Definition 6.1.1. A sheaf $\mathscr{I}$ on M is said to be **faithful** if, given any exact sequence of sheaves

$$0 \to \mathscr{I} \to \mathscr{F} \to \mathscr{G} \to 0$$

on M, and any open set $U \subset M$, the induced sequence of abelian groups

$$0 \to \mathscr{I}(U) \to \mathscr{F}(U) \to \mathscr{G}(U) \to 0$$

is exact. We say that a sheaf $\mathscr{I}$ is **acyclic** (for the given theory) if $H^i(\mathscr{I}) = 0$ for all $i \geq 1$.

Axiom 4. Any faithful sheaf $\mathscr{I}$ satisfies $H^1(\mathscr{I}) = 0$.

Axiom 5. On a compact space M, any faithful sheaf is acyclic.

For a more detailed set of axioms, determining the sheaf cohomology groups up to *canonical* isomorphism, see Warner, *Foundations of differentiable manifolds and Lie groups*, Bredon, *Sheaf Theory*, or any of the many other books covering the subject.

6.2. **The Čech cohomology groups.** We now give a concrete construction of sheaf cohomology, which we will take as our definition of the groups.

Let $\underline{U}$ be a locally finite open cover of M. Given a sheaf $\mathscr{F}$ on M, define the abelian groups

$$\begin{aligned} C^0(\underline{U},\mathscr{F}) &= \Pi_\alpha \mathscr{F}(U_\alpha) \\ C^1(\underline{U},\mathscr{F}) &= \Pi_{\alpha\neq\beta} \mathscr{F}(U_\alpha \cap U_\beta) \\ &\vdots \\ C^p(\underline{U},\mathscr{F}) &= \Pi_{\alpha_0\neq\cdots\neq\alpha_p} \mathscr{F}(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}) \\ &\vdots \end{aligned}$$

We call $C^p(\underline{U},\mathscr{F})$ the group of p**-cochains**, an element of which is denoted

$$\sigma = \{\sigma_I \in \mathscr{F}(\cap_{k=0}^p U_{i_k})\}_{\#I=p+1}.$$

Define the **coboundary operator**

$$\delta : C^p(\underline{U},\mathscr{F}) \to C^{p+1}(\underline{U},\mathscr{F})$$

$$(\delta\sigma)_{i_0\cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \, \sigma_{i_0\cdots\hat{\imath}_j\cdots i_{p+1}}\big|_{U_{i_0}\cap\cdots\cap U_{i_p}}. \tag{6.2}$$

For example, if $\tau = \{\tau_U\}$ is a 0-cycle, we have

$$(\delta\tau)_{UV} = \tau_V - \tau_U.$$

If $\sigma = \{\sigma_{UV}\}$ is a 1-cycle, we have

$$(\delta\sigma)_{UVW} = \sigma_{VW} - \sigma_{UW} + \sigma_{UV}.$$

Notice that

$$\left(\delta^2\tau\right)_{UVW} = \tau_W - \tau_V - (\tau_W - \tau_U) + \tau_V - \tau_U = 0.$$

In general, we have

Proposition 6.2.1. $\delta^2 = 0$.

Proof. We calculate

$$\begin{aligned} \left(\delta^2\sigma\right)_{i_0\cdots i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k \, (\delta\sigma)_{i_0\cdots\hat{\imath}_k\cdots i_{p+2}} \\ &= \sum_{k=0}^{p+2} (-1)^k \left(\sum_{j=0}^{k-1} (-1)^j \sigma_{i_0\cdots\hat{\imath}_j\cdots\hat{\imath}_k\cdots i_{p+2}} + \sum_{j=k}^{p+1} (-1)^j \sigma_{i_0\cdots\hat{\imath}_k\cdots\hat{\imath}_{j+1}\cdots i_{p+2}} \right) \\ &= \sum_{j<k} (-1)^{k+j} \sigma_{i_0\cdots\hat{\imath}_j\cdots\hat{\imath}_k\cdots i_{p+2}} + \sum_{k<j'} (-1)^{k+j'-1} \sigma_{i_0\cdots\hat{\imath}_k\cdots\hat{\imath}_{j'}\cdots i_{p+2}} \\ &= 0 \end{aligned}$$

as claimed. □

The proposition shows that the *Čech complex*

$$C^\bullet(\underline{U},\mathscr{F}) : 0 \to C^0(\underline{U},\mathscr{F}) \xrightarrow{\delta} C^1(\underline{U},\mathscr{F}) \xrightarrow{\delta} C^2(\underline{U},\mathscr{F}) \to \cdots$$

is a complex of abelian groups. We define the **Čech cohomology groups of $\mathscr{F}$ with respect to $\underline{U}$** as

$$\check{H}^i(\underline{U}, \mathscr{F}) = h^i\left(C^\bullet(\underline{U}, \mathscr{F})\right) = \frac{\mathrm{Ker}\left(\delta : C^i(\underline{U}, \mathscr{F}) \to C^{i+1}(\underline{U}, \mathscr{F})\right)}{\mathrm{Im}\left(\delta : C^{i-1}(\underline{U}, \mathscr{F}) \to C^i(\underline{U}, \mathscr{F})\right)}. \tag{6.3}$$

The elements of the numerator of the RHS of (6.3) are called *Čech cocycles*, and the elements of the denominator are called *Čech coboundaries.*

To satisfy all of the required axioms, we must remove the dependence on the open cover $\underline{U}$, which can be done as follows. We say that $\underline{U}' = \{U'_\beta\}$ is a **refinement** of the open cover $\underline{U}$ if, for each β, we have

$$U'_\beta \subset U_\alpha \tag{6.4}$$

for some some α. For each β, we may choose $\varphi(\beta) = \alpha$ per (6.4), and define a map

$$\rho_\varphi : C^p(\underline{U}, \mathscr{F}) \to C^p(\underline{U}', \mathscr{F})$$
$$\left(\rho_\varphi(\sigma)\right)_{\beta_0\cdots\beta_p} = \sigma_{\varphi(\beta_0)\cdots\varphi(\beta_p)}\Big|_{U_{\beta_0}\cap\cdots\cap U_{\beta_p}}.$$

This is clearly a chain map $C^\bullet(\underline{U}, \mathscr{F}) \to C^\bullet(\underline{U}', \mathscr{F})$, hence gives a map on cohomology

$$\rho : H^p(\underline{U}, \mathscr{F}) \to H^p(\underline{U}', \mathscr{F}). \tag{6.5}$$

One can check that ρ is independent of the choice of map φ on the indexing set of the refinement.

Definition 6.2.2. The p'th sheaf cohomology group of $\mathscr{F}$ is defined to be the direct limit

$$H^p(\mathscr{F}) = \lim_{\overrightarrow{\underline{U}}} \check{H}^p(\underline{U}, \mathscr{F})$$

over refinements of open covers of M. This simply means that we consider Čech classes for all open covers, where two elements are equivalent if they agree under a common refinement.

Lemma 6.2.3. *Let $\underline{U}$ be a locally finite open cover of M. Given an open set $V \subset M$, let $\underline{V} = \underline{U} \cup \{V\}$.*

Suppose that $\mathscr{F}$ is a sheaf on M such that the restriction $\mathscr{F}|_V$ satisfies

$$\check{H}^p(\underline{U}, \mathscr{F}|_V) = 0$$

for all $p \geq 1$. Then

$$\check{H}^p(\underline{U}, \mathscr{F}) \cong \check{H}^p(\underline{V}, \mathscr{F})$$

for all $p \geq 1$.

Proof. By the definition of the Čech complex above, we have an exact sequence of complexes:

$$0 \to C^{\bullet-1}\left(\underline{U}, \mathscr{F}|_V\right) \to C^\bullet\left(\underline{V}, \mathscr{F}\right) \to C^\bullet\left(\underline{U}, \mathscr{F}\right) \to 0.$$

Since

$$C^{\bullet-1}\left(\underline{U}, \mathscr{F}|_V\right) : 0 \to \mathscr{F}(V) \to C^0\left(\underline{U}|_V, \mathscr{F}|_V\right) \to \cdots$$

is exact by assumption, the Snake Lemma yields the stated identity of Čech groups. □

Theorem 6.2.4. *The groups $H^p(-)$ of Definition 6.2.2 satisfy Axioms 1-5 of the previous section.*

Proof. Axiom 1 is clear from the definition.

For Axiom 2, given any open cover $\underline{U}$, we in fact have

$$\begin{aligned}\check{H}^0(\underline{U},\mathscr{F}) &= \{\{g_\alpha\} \mid g_\alpha - g_\beta = 0 \text{ on } U_\alpha \cap U_\beta\} \\ &= \mathscr{F}(M) = \Gamma(\mathscr{F})\end{aligned}$$

by the sheaf axioms iii) and iv). This persists in the direct limit.

For Axiom 3, given a short exact sequence of sheaves

$$0 \to \mathscr{E} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{G} \to 0$$

we obtain an exact sequence between the Čech complexes:

$$0 \to C^\bullet(\underline{U},\mathscr{E}) \xrightarrow{\alpha^\bullet} C^\bullet(\underline{U},\mathscr{F}) \xrightarrow{\beta^\bullet} C^\bullet(\underline{U},\mathscr{G}). \tag{6.6}$$

It is often possible to choose the cover $\underline{U}$ so that the maps $\beta^\bullet$ are also surjective, *i.e.*, (6.6) forms an exact sequence with $\to 0$ on the right. In this case, the existence of the connecting maps δ^* forming the long exact sequence follows directly from the Snake Lemma.

In general, the connecting maps δ^* can be defined by passing to a refinement, as follows. Because the sheaf map β is surjective, given a class $[\sigma] \in H^i(\underline{U},\mathscr{G})$, it is possible to choose a refinement $\underline{U}'$ such that there exists $\tau \in C^i(\underline{U}',\mathscr{F})$ with $\beta(\tau) = \sigma|_{C^i(\underline{U}',\mathscr{G})}$. Since σ is a Čech cocycle, we have

$$\beta(\delta(\tau)) = \delta(\beta(\tau)) = \delta(\sigma) = 0.$$

Therefore $\delta(\tau) = \alpha(\mu)$, for some $\mu \in C^{i+1}(\underline{U}',\mathscr{E})$. We then have

$$\alpha(\delta(\mu)) = \delta(\alpha(\mu)) = \delta^2(\tau) = 0.$$

But since α is injective, we conclude that $\delta(\mu) = 0$, and $[\mu]$ represents a class in $h^i\left(C^\bullet(\underline{U}',\mathscr{E})\right) = \check{H}^i(\underline{U}',\mathscr{E})$. We then define

$$\delta^*([\sigma]) = [\mu].$$

Imitating the proof of the Snake Lemma, one can check that this class is independent of the various choices of cocycle representatives, and gives the required long exact sequence of cohomology groups.

For Axiom 4, we argue as follows; let $\mathscr{I}$ be a faithful sheaf. Given an open cover $\underline{U}$, we define a "sheafy" Čech complex

$$0 \to \mathscr{C}^0(\underline{U},\mathscr{I}) \to \mathscr{C}^1(\underline{U},\mathscr{I}) \to \mathscr{C}^2(\underline{U},\mathscr{I}) \to \cdots$$

where

$$\begin{aligned}\mathscr{C}^0(\underline{U},\mathscr{I}) &= \Pi_\alpha\ \mathscr{I}|_{U_\alpha} \\ \mathscr{C}^1(\underline{U},\mathscr{I}) &= \Pi_{\alpha\neq\beta}\ \mathscr{I}|_{U_\alpha\cap U_\beta} \\ &\vdots\end{aligned}$$

and the differentials are defined as in (6.2).

We claim that

$$0 \to \mathscr{I} \to \mathscr{C}^0(\underline{U},\mathscr{I}) \to \mathscr{C}^1(\underline{U},\mathscr{I}) \to \mathscr{C}^2(\underline{U},\mathscr{I}) \to \cdots \tag{6.7}$$

is an exact sequence of sheaves. To see this, let $x \in M$ and consider the induced sequence on stalks:

$$0 \to \mathscr{I}_x \to \mathscr{C}^0(\underline{U}, \mathscr{I})_x \to \mathscr{C}^1(\underline{U}, \mathscr{I})_x \to \cdots$$

Let $\sigma = (\sigma_{\alpha_0 \cdots \alpha_p}) \in \operatorname{Ker} \delta_x$. Choosing an index β such that $U_\beta \ni x$, define the stalk of a cochain at x by

$$\tau_{\alpha_0 \cdots \alpha_{p-1}} = \sigma_{\beta \alpha_0 \cdots \alpha_{p-1}}.$$

Then τ is a well-defined element of $\mathscr{C}^{p-1}(\underline{U}, \mathscr{I})_x$, which one can check satisfies $\delta_x \tau = \sigma$.

Because $\mathscr{I}$ is a faithful sheaf and $\Gamma(\mathscr{C}^p(\underline{U}, \mathscr{I})) = C^p(\underline{U}, \mathscr{I})$, we get from (6.7) a short exact sequence of groups:

$$0 \to \Gamma(\mathscr{I}) \to C^0(\underline{U}, \mathscr{I}) \to C^1(\underline{U}, \mathscr{I}) \to \operatorname{Ker}\left(C^1(\underline{U}, \mathscr{I}) \to C^2(\underline{U}, \mathscr{I})\right) \to 0.$$

This shows that $\check{H}^1(\underline{U}, \mathscr{I}) = 0$ for all open covers $\underline{U}$, and therefore $H^1(\mathscr{I}) = 0$.

To verify Axiom 5, assume that the base space M is compact. This allows us to restrict to finite open covers (after taking refinements) and argue by induction. The result is trivial for a cover with one element. Assume that the result has been established for all covers with n elements, and let $\underline{V}$ be a cover of M with $n+1$ elements. Choosing any open set $W \in \underline{V}$, write $\underline{V} = \underline{U} \cup \{W\}$ and $U = \cup U_\alpha$. We have an exact sequence of sheaves

$$0 \to \mathscr{I} \to \mathscr{I}|_U \oplus \mathscr{I}|_W \to \mathscr{I}|_{U \cap W} \to 0.$$

Since $\mathscr{I}$ is faithful, we obtain an exact sequence of Čech complexes:

$$0 \to C^\bullet(\underline{V}, \mathscr{I}) \to C^\bullet(\underline{V}, \mathscr{I}|_U) \oplus C^\bullet(\underline{V}, \mathscr{I}|_W) \to C^\bullet(\underline{V}, \mathscr{I}|_{U \cap W}) \to 0. \tag{6.8}$$

But for the faithful sheaves $\mathscr{I}|_U$, $\mathscr{I}|_W$, and $\mathscr{I}|_{U \cap W}$, the cover $\underline{V}$ has at least one "redundant" open set. By Lemma 6.2.3 and the induction hypothesis, we get that the cohomology groups $\check{H}^p(\underline{V}, \mathscr{I}|_U)$, $\check{H}^p(\underline{V}, \mathscr{I}|_W)$, and $\check{H}^p(\underline{V}, \mathscr{I}|_{U \cap W})$ are all zero, for $p \geq 1$. Applying the Snake Lemma to (6.8), we conclude that $\check{H}^p(\underline{V}, \mathscr{I}) = 0$ for all $p \geq 2$, as required. □

6.3. **Resolutions.** Underlying the Čech definition of sheaf cohomology is the complex of sheaves (6.7), which is an example of the following. We say that a complex

$$\mathscr{A}^\bullet : 0 \to \mathscr{A}^0 \to \mathscr{A}^1 \to \mathscr{A}^2 \to \cdots \tag{6.9}$$

is a **resolution** of a sheaf $\mathscr{F}$ if there exists an injective map $\mathscr{F} \to \mathscr{A}^0$ such that the complex

$$0 \to \mathscr{F} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \cdots \tag{6.10}$$

is exact. If the sheaves in the resolution have a certain property, for instance, are acyclic, then (6.9) is said to be an **acyclic resolution**.

Theorem 6.3.1. *Given an acyclic resolution $\mathscr{A}^\bullet$ of a sheaf $\mathscr{F}$, we have*

$$H^p(\mathscr{F}) = h^p(\Gamma(\mathscr{A}^\bullet)) \tag{6.11}$$

for $p \geq 0$. In other words, the sheaf cohomology groups are equal to the cohomology groups of the complex of global sections of an acyclic resolution.

Proof. For $i \geq 1$, let $\mathscr{K}^i = \text{Ker}\left(\mathscr{A}^i \to \mathscr{A}^{i+1}\right)$. Then $\mathscr{K}^0 = \mathscr{F}$, and we have exact sequences of sheaves

$$0 \to \mathscr{K}^i \to \mathscr{A}^i \to \mathscr{K}^{i+1} \to 0 \tag{6.12}$$

for each $i \geq 0$. Applying the long exact sequence in cohomology to (6.12), for $i = 0$, gives

$$\begin{aligned} 0 &\to H^0(\mathscr{F}) \to H^0(\mathscr{A}^0) \to H^0(\mathscr{K}^1) \\ &\to H^1(\mathscr{F}) \to 0 \to H^1(\mathscr{K}^1) \\ &\to H^2(\mathscr{F}) \to 0 \to H^2(\mathscr{K}^1) \\ &\to H^3(\mathscr{F}) \to 0 \to \cdots. \end{aligned}$$

This gives

$$H^0(\mathscr{F}) = \text{Ker}\left(\Gamma(\mathscr{A}^0) \to \Gamma(\mathscr{A}^1)\right) = h^0(\Gamma(\mathscr{A}^\bullet))$$

and

$$H^1(\mathscr{F}) = h^1\left(\Gamma(\mathscr{A}^\bullet)\right)$$

which is the desired result, for $p = 0, 1$.

The long exact sequence corresponding to the i'th short exact sequence in (6.12) reads

$$\begin{aligned} 0 &\to H^0(\mathscr{K}^i) \to H^0(\mathscr{A}^i) \to H^0(\mathscr{K}^{i+1}) \\ &\to H^1(\mathscr{K}^i) \to 0 \to H^1(\mathscr{K}^{i+1}) \\ &\to H^2(\mathscr{K}^i) \to 0 \to \cdots \end{aligned}$$

This gives

$$H^1(\mathscr{K}^i) \cong h^{i+1}(\Gamma(\mathscr{A}^\bullet))$$

and

$$H^p(\mathscr{K}^i) \cong H^{p-1}(\mathscr{K}^{i+1}).$$

for each $p \geq 2$. Therefore, we have

$$\begin{aligned} H^p(\mathscr{F}) \cong H^p(\mathscr{K}^0) &\cong H^{p-1}(\mathscr{K}^1) \\ &\cong H^{p-2}(\mathscr{K}^2) \\ &\ \ \vdots \\ &\cong H^1(\mathscr{K}^{p-1}) \\ &\cong h^p(\Gamma(\mathscr{A}^\bullet)) \end{aligned} \tag{6.13}$$

as desired. □

Corollary 6.3.2. *Given a faithful resolution $\mathscr{I}^\bullet$ of a sheaf $\mathscr{F}$ over a compact base space M, we have*

$$H^p(\mathscr{F}) = h^p(\Gamma(\mathscr{I}^\bullet))$$

for $p \geq 0$.

Proof. By Axiom 5, a faithful resolution over a compact base is acyclic. □

Remark 6.3.3. It follows from the previous corollary that, at least for any sheaf that admits a faithful resolution, the sheaf cohomology groups over a compact space are determined up to isomorphism by the axioms. An alternative approach (originating in the famous "Tohoku paper" of Grothendieck) is to define the sheaf cohomology groups directly using a resolution by sufficiently flexible sheaves. This puts sheaf cohomology in the more general framework of "right-derived functors;" the reader may consult Hartshorne, Ch. 3.

As another application of Theorem 6.3.1, we can do away with the need for taking direct limits over the open covers $\underline{U}$ in the definition of Čech cohomology.

Corollary 6.3.4 (Leray Theorem)**.** *Let $\mathscr{F}$ be a sheaf, and let $\underline{U}$ be an open cover such that for any choice of indices $\alpha_0, \ldots, \alpha_p$, the restriction*

$$\mathscr{F}|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}}$$

is acyclic. Then

$$H^p(\mathscr{F}) = \check{H}^p(\underline{U}, \mathscr{F}).$$

Proof. Under this assumption, the sheafy Čech complex (6.7) is an acyclic resolution of $\mathscr{F}$. The complex of abelian groups obtained by taking global sections therefore computes the sheaf cohomology; but this is just the Čech complex with respect to the open cover $\underline{U}$. □

6.4. **Fine resolutions and the DeRham and Dolbeault Theorems.** We will now describe the class of faithful/acyclic sheaves that is most relevant to our situation. A sheaf of abelian groups $\mathscr{F}$ is said to be **fine** if, for any open set $U \subset M$ and locally finite open cover $\underline{U}$ of U, there exist maps $\eta_\alpha : \mathscr{F}(U_\alpha) \to \mathscr{F}(U)$ such that:

For any $\tau \in \mathscr{F}(U_\alpha)$, the support of $\eta_\alpha(\tau)$ is contained in U_α,[6] and;

For any $\sigma \in \mathscr{F}(U)$, we have

$$\sum_\alpha \eta_\alpha(\sigma|_{U_\alpha}) = \sigma.$$

Proposition 6.4.1. *The sheaf $\mathscr{A}^p$ (resp. $\mathscr{A}^{p,q}$) on a smooth (resp. complex) manifold is fine.*

Proof. Let $U \subset M$ and $\underline{U}$ be a locally finite cover of U. Choose a partition of unity $\{\rho_\alpha\} \in C^0(\underline{U}, \mathscr{C}^\infty)$ subordinate to $\underline{U}$, and define

$$\eta_\alpha(\tau) = \rho_\alpha \tau$$

for $\tau \in \mathscr{A}^{p(,q)}(U_\alpha)$. Then $\eta_\alpha(\tau)$ is a global section of $\mathscr{A}^{p(,q)}$ supported on U_α, and we have

$$\begin{aligned}\sum_\alpha \eta_\alpha\big(\sigma|_{U_\alpha}\big) &= \sum_\alpha \rho_\alpha\left(\sigma|_{U_\alpha}\right)\\ &= \sum_\alpha (\rho_\alpha \sigma)\\ &= \left(\sum_\alpha \rho_\alpha\right)\sigma = \sigma\end{aligned}$$

[6]The support of a section σ is the set of all x such that $\sigma_x \neq 0$.

as desired. □

Theorem 6.4.2. *A fine sheaf $\mathscr{F}$ is faithful and acyclic.*

Proof. Given an exact sequence $0 \to \mathscr{F} \to \mathscr{H} \to \mathscr{G} \to 0$ of sheaves, we claim that the induced sequence

$$0 \to \mathscr{F}(U) \to \mathscr{H}(U) \to \mathscr{G}(U) \to 0$$

is exact. For $\mu \in \mathscr{G}(U)$, we have an open cover $\underline{U}$ and elements $\sigma_\alpha \in \mathscr{H}(U_\alpha)$ such that $\sigma_\alpha \mapsto \mu|_{U_\alpha}$. Let

$$\sigma_{\alpha\beta} = \sigma_\beta - \sigma_\alpha \in \mathscr{F}(U_\alpha \cap U_\beta)$$

and let

$$\tau_\alpha = \sum_\beta \rho_\beta \sigma_{\beta\alpha},$$

which is a well-defined element of $\mathscr{F}(U_\alpha)$. Then

$$\begin{aligned}\tau_\gamma - \tau_\alpha &= \sum_\beta \rho_\beta \left(\sigma_{\beta\gamma} - \sigma_{\beta\alpha}\right)\\ &= \sum_\beta \rho_\beta \sigma_{\gamma\alpha}\\ &= \sum_\beta \rho_\beta \sigma_{\gamma\alpha}\\ &= \sigma_\alpha - \sigma_\gamma\end{aligned}$$

on $U_\alpha \cap U_\gamma$. Therefore $\{\sigma_\alpha + \tau_\alpha\}$ glues to form a well-defined section of $\mathscr{F}(U)$ that maps to $\mu \in \mathscr{H}(U)$. This shows that $\mathscr{F}$ is faithful, which implies that it is acyclic if the base M is compact.

To show that $\mathscr{F}$ is acylic in general, one can use a similar trick: given a cocycle σ, one defines

$$\tau_{\alpha_0 \cdots \alpha_{p-1}} = \sum_\beta \rho_\beta \sigma_{\beta\alpha_0\cdots\alpha_{p-1}} \tag{6.14}$$

to obtain an element with $\delta\tau = \sigma$. □

Corollary 6.4.3 (DeRham Theorem)**.** *For a smooth manifold M, we have*

$$H^p(\underline{\mathbb{R}}_M) \cong H^p_{dR}(M) \tag{6.15}$$

where $\underline{\mathbb{R}}_M$ is the locally constant sheaf on M.

Proof. Let $\mathscr{A}^p_{\mathbb{R}}$ be the sheaf of smooth real-valued p-forms as above. By the Poincaré Lemma, the complex

$$0 \to \mathscr{A}^0_{\mathbb{R}} \xrightarrow{d} \mathscr{A}^1_{\mathbb{R}} \xrightarrow{d} \mathscr{A}^2_{\mathbb{R}} \to \cdots \to \mathscr{A}^n_{\mathbb{R}} \to 0$$

is a resolution of the locally constant sheaf $\underline{\mathbb{R}}$. By Proposition 6.4.1, this is a fine resolution, which is acyclic by Theorem 6.4.2. By Theorem 6.3.1, the sheaf cohomology of $\underline{\mathbb{R}}$ is isomorphic to the cohomology of the global sections of this complex, which are simply the DeRham cohomology groups of M. □

Remark 6.4.4. Strictly speaking, the DeRham Theorem asserts the equivalence between singular cohomology and DeRham cohomology of a manifold. See p. 42 of Griffiths and Harris for a simple proof that $H^p(\underline{\mathbb{Z}})$ computes CW cohomology, which is equal to singular cohomology (as you know from Hatcher, presumably).

Corollary 6.4.5 (Dolbeault Theorem)**.** *For a complex manifold M, we have*

$$H^q(\Omega^p_M) \cong H^{p,q}_{\bar\partial}(M). \tag{6.16}$$

Proof. By the $\bar\partial$-Poincaré Lemma, the complex

$$0 \to \mathscr{A}^{p,0} \xrightarrow{\bar\partial} \mathscr{A}^{p,1} \xrightarrow{\bar\partial} \mathscr{A}^{p,2} \to \cdots \to \mathscr{A}^{p,n} \to 0$$

is a fine resolution of the sheaf of holomorphic p-forms on M. As in the proof of Corollary 6.4.3, the global sections compute sheaf cohomology. □

Corollary 6.4.6. *If M is a complex manifold of dimension n, then*

$$H^q(\Omega^p_M) = 0, \quad q > n.$$

6.4.1. **Solution of the Cousin problem**. Consider the long exact sequence in cohomology associated to the exponential sheaf sequence (Example 5.5.2) on $\mathbb{C}^n$:

$$\cdots \to H^q(\mathscr{O}_{\mathbb{C}^n}) \to H^q(\mathscr{O}^*_{\mathbb{C}^n}) \to H^{q+1}(\underline{\mathbb{Z}}) \to \cdots. \tag{6.17}$$

By the Dolbeault Theorem and the Poincaré Lemma, we have

$$H^q(\mathscr{O}_{\mathbb{C}^n}) = H^{0,q}_{\bar\partial}(\mathbb{C}^n) = 0$$

for $q \geq 1$. By the DeRham Theorem, we have $H^{q+1}(\underline{\mathbb{Z}}) = 0$ on $\mathbb{C}^n$. Therefore, for $q \geq 1$, (6.17) implies

$$H^q(\mathscr{O}^*_{\mathbb{C}^n}) = 0.$$

Now, consider the long exact cohomology sequence associated to (5.11):

$$\cdots \to \Gamma(\mathscr{M}^*) \xrightarrow{\text{div}} \Gamma(\mathscr{D}iv) \to H^1(\mathscr{O}^*) = 0.$$

This shows that the map div is surjective on global sections. Hence, every divisor on $\mathbb{C}^n$ is equal to div of a global meromorphic function; in particular, an effective divisor is the vanishing set of an entire holomorphic function.

6.5. **Calculations using Čech cohomology.** We can now use the Leray Theorem in conjunction with the Dolbeault Theorem to make a few cohomology calculations.

Theorem 6.5.1.

$$H^q\left(\Omega^p_{\mathbb{CP}^1}\right) = \begin{cases} \mathbb{C} & p = q = 0 \ \textit{or}\ 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. As for any compact complex manifold, we have $H^0(\mathscr{O}_{\mathbb{CP}^1}) = \mathbb{C}$.

Choose the standard cover $\{U_0, U_1\}$ for $\mathbb{CP}^1$, with coordinate z on $U_0 \subset \mathbb{CP}^1$ and $w = 1/z$ on $U_1 \subset \mathbb{CP}^1$. We have $H^0(\mathscr{O}_{\mathbb{CP}^1}) = H^{1,0}_{\bar\partial}(\mathbb{CP}^1) = 0$ by Problem 4 of HW # 3, or as follows: let

$$\alpha|_{U_0} = \sum_{i=0}^{\infty} a_i z^i dz$$

be a holomorphic 1-form on U_0. Then on U_0, we have $dz = -dw/w^2$, which gives

$$\alpha|_{U_1} = \sum_{i=0}^{\infty} a_i w^{-i-2} dw.$$

This is holomorphic if and only if $a_i \equiv 0$ for all i.

Now, notice that $U_0 \cap U_1 = \mathbb{C}^*$, so by the Dolbeault Theorem[7] and the general version of the Poincaré Lemma (Theorem 3.5.4), we have

$$H^q(\Omega^p|_{U_0 \cap U_1}) \cong H^{p,q}_{\bar\partial}(\mathbb{C}^*) = 0.$$

Therefore $\{U_0, U_1\}$ is an acyclic cover of $\mathbb{CP}^1$. By the Leray Theorem, we can use it to compute the remaining cohomology groups.

A Čech 1-cochain for $\mathscr{O}_{\mathbb{CP}^1}$ is represented by a holomorphic function $h \in \mathrm{Hol}(\mathbb{C}^*)$. By Corollary 1.5.2, there exist $P(z) \in \mathrm{Hol}(U_0)$ and $Q(w) \in \mathrm{Hol}(U_1)$ such that

$$h = P + Q$$

Therefore, $h = \delta\{P, Q\}$ is a Čech coboundary. We conclude that

$$H^1(\mathscr{O}_{\mathbb{CP}^1}) = 0.$$

Given a holomorphic 1-form $h(z)dz$, with $h(z) \in \mathrm{Hol}(\mathbb{C}^*)$, representing a 1-cochain for $\Omega^1_{\mathbb{CP}^1}$, we may write

$$h(z) = \sum_{i=-\infty}^{\infty} a_i z^i$$

uniquely by Laurent expansion. The image under δ of a 0-cochain

$$\left\{ \sum_{i=0}^{\infty} b_i z^i dz, \sum_{i=0}^{\infty} c_i w^i dw \right\}$$

is

$$\sum_{i=0}^{\infty} b_i z^i dz - \sum_{i=0}^{\infty} c_i z^{-2-i} dz.$$

Therefore $h(z)$ is a coboundary if and only if $a_{-1} = 0$, which gives

$$H^1(\Omega^1_{\mathbb{CP}^1}) = \mathbb{C}$$

as claimed. □

Proposition 6.5.2. $H^q(\mathscr{O}_{\mathbb{CP}^n}) = 0, \quad q \geq 1.$

[7] Strictly speaking, we are not using the Dolbeault Theorem as stated, but rather the fact that $\mathscr{A}^{p,\bullet}|_{U_0 \cap U_1}$ is a fine resolution of $\Omega^p|_{U_0 \cap U_1}$ (over M).

Proof. We will only prove that $H^1(\mathscr{O}_{\mathbb{CP}^2}) = 0$; the general case requires more bookkeeping.

Write
$$\mathbb{CP}^2 = \{[X_0, X_1, X_2]\}.$$
Choose the standard cover $\underline{U} = \{U_0, U_1, U_2\}$, which is acyclic as above. We then have $U_0 \cong \mathbb{C}^* \times \mathbb{C}$ and $U_0 \cap U_1 \cong (\mathbb{C}^*)^2$, *etc.* Any holomorphic function on these sets has a convergent Laurent series in the local coordinates, which are ratios of the homogeneous coordinates X_i. So for example, any holomorphic function f on U_0 has a unique Laurent expansion of the form
$$f = \sum_{i=-\infty}^{\infty} \sum_{\substack{i+j+k=0 \\ j,k\geq 0}} a_{ijk} X_0^i X_1^j X_2^k.$$
A 1-cocycle σ is therefore given by
$$\begin{aligned} \sigma_{01} &= \sum_{\substack{i+j+k=0 \\ k\geq 0}} a_{ijk} X_0^i X_1^j X_2^k \\ \sigma_{12} &= \sum_{\substack{i+j+k=0 \\ i\geq 0}} b_{ijk} X_0^i X_1^j X_2^k \\ \sigma_{20} &= \sum_{\substack{i+j+k=0 \\ j\geq 0}} c_{ijk} X_0^i X_1^j X_2^k. \end{aligned} \tag{6.18}$$
We may assume without loss of generality that $a_{000} = b_{000} = c_{000} = 0$. We then have
$$0 = (\delta\sigma)_{012} = \sum_{i,j,k} (a_{ijk} + b_{ijk} + c_{ijk})\, X_0^i X_1^j X_2^k.$$
Since $b_{ijk} = c_{ijk} = 0$ if $i, j < 0$, we conclude that $a_{ijk} = 0$; so $a_{ijk} = 0$ unless either $j, k \geq 0$ or $i, k \geq 0$. Similar conclusions hold for b and c. The expression reduces to
$$0 = \sum_{\substack{i<0,j,k\geq 0 \\ i+j+k=0}} (a_{ijk} + c_{ijk})\, X_0^i X_1^j X_2^k + \sum_{\substack{j<0,i,k\geq 0 \\ i+j+k=0}} (a_{ijk} + b_{ijk})\, X_0^i X_1^j X_2^k + \sum_{\substack{k<0,i,j\geq 0 \\ i+j+k=0}} (b_{ijk} + c_{ijk})\, X_0^i X_1^j X_2^k.$$
We conclude
$$\begin{aligned} a_{ijk} &= -c_{ijk}, \quad i < 0 \\ b_{ijk} &= -a_{ijk}, \quad j < 0 \\ c_{ijk} &= -b_{ijk}, \quad k < 0. \end{aligned} \tag{6.19}$$
We can let
$$\begin{aligned} \tau_0 &= \sum_{i<0} a_{ijk} X_0^i X_1^j X_2^k \\ \tau_1 &= \sum_{j<0} b_{ijk} X_0^i X_1^j X_2^k \\ \tau_2 &= \sum_{k<0} c_{ijk} X_0^i X_1^j X_2^k \end{aligned}$$
which define holomorphic functions on U_0, U_1, U_2, respectively. One checks from (6.19) that the 0-cochain $\tau = (\tau_i)_{i=0,1,2}$ satisfies
$$\delta\tau = \sigma.$$
We have shown that $\check{H}^1(\underline{U}, \mathscr{O}_{\mathbb{CP}^2}) = 0$, and the result follows from the Leray Theorem. ∎

Remark 6.5.3. With more work (which we may carry out later), one can show

$$H^q\left(\Omega^p_{\mathbb{CP}^n}\right) = \begin{cases} \mathbb{C} & p = q \le n \\ 0 & \text{otherwise.} \end{cases}$$

This result also follows from the Hodge Theorem, which we would prove next semester.

Example 6.5.4. Let $M = \mathbb{C}^2 \smallsetminus \{0\}$. Take the cover $U_1 = \{z^1 \neq 0\}, U_2 = \{z^2 \neq 0\}$, which is acyclic as before. Then $\mathscr{O}(U_1 \cap U_2)$ consists of Laurent series

$$f(z^1, z^2) = \sum_{m,n=-\infty}^{\infty} a_{mn}(z^1)^m (z^2)^n \tag{6.20}$$

but, for example, $\mathscr{O}(U_1)$ consists of series

$$f(z^1, z^2) = \sum_{m \ge 0} b_{mn}(z^1)^m (z^2)^n. \tag{6.21}$$

Therefore all the terms with $m, n < 0$ in (6.20) represent cohomology classes, and

$$\dim H^1(\mathscr{O}_M) = \infty. \tag{6.22}$$

By contrast, according to Hartogs' Theorem, every class in $H^0(\mathscr{O}_M)$ extends to $H^0(\mathscr{O}_{\mathbb{C}^2})$.

6.6. **The Euler characteristic.** Recall the *Rank-Nullity Theorem*, which states that for an exact sequence

$$0 \to A \to B \to C \to 0$$

of finite-dimensional vector spaces, we have

$$\dim B = \dim A + \dim C. \tag{6.23}$$

Given a complex of finite-dimensional vector spaces

$$C^\bullet : 0 \to C^n \overset{\alpha^n}{\to} C^{n+1} \overset{\alpha^{n+1}}{\to} \cdots \to \cdots \overset{\alpha^{m-1}}{\to} C^m \to 0 \tag{6.24}$$

we obtain

$$\begin{aligned} \sum_{i=n}^{m} (-1)^i h^i(C^\bullet) &= \sum (-1)^i \left(\dim\ker\alpha^i - \dim\operatorname{im}\alpha^{i-1}\right) \\ &= \sum (-1)^i \left(\dim\ker\alpha^i + \dim\operatorname{im}\alpha^i\right) \\ &= \sum (-1)^i \dim C^i. \end{aligned} \tag{6.25}$$

The best-known instance of this formula is when (6.24) is the CW chain complex of a finite CW complex M. Then (6.25) is the **topological Euler characteristic**

$$\chi_{top}(M).$$

The right-hand side of (6.25) is given in terms of the concrete data of the CW cochain complex $C^\bullet$, but the left-hand side is given in terms of cohomology groups, which depend only on the topology of M. The conclusion is that $\chi_{top}(M)$, which is trivial to compute from a particular CW structure, is a homotopy invariant.

We now make the following generalization to sheaves.

Definition 6.6.1. Given a sheaf $\mathscr{F}$ of vector spaces on a topological space M such that $\dim H^i(\mathscr{F}) < \infty$ for all i and $H^i(\mathscr{F}) = 0$ for i sufficiently large, define the **(sheaf) Euler characteristic**

$$\chi(\mathscr{F}) = \sum(-1)^i \dim H^i(\mathscr{F}). \tag{6.26}$$

According to the DeRham Theorem, for a manifold M of finite topological type (and indeed for any finite CW complex), we have

$$\chi_{top}(M) = \chi\left(\underline{\mathbb{R}}_M\right)$$

where $\underline{\mathbb{R}}_M$ is the locally constant sheaf on M; so this is a strict generalization of the topological Euler characteristic.

We have the following generalization of the Rank-Nullity Theorem.

Lemma 6.6.2. *Given an exact sequence of sheaves*

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$$

each with finite Euler characteristic, we have

$$\chi(\mathscr{F}) = \chi(\mathscr{E}) + \chi(\mathscr{G}).$$

Proof. The long exact sequence in sheaf cohomology reads:

$$0 \to H^0(\mathscr{E}) \to H^0(\mathscr{F}) \to H^0(\mathscr{G}) \overset{\delta^{0*}}{\to} H^1(\mathscr{E}) \to H^1(\mathscr{F}) \to H^1(\mathscr{G}) \overset{\delta^{1*}}{\to} H^2(\mathscr{E}) \to \cdots. \tag{6.27}$$

Applying (6.24), since the sequence is exact, we obtain

$$\begin{aligned} 0 &= H^0(\mathscr{E}) - H^0(\mathscr{F}) + H^0(\mathscr{G}) - H^1(\mathscr{E}) + H^1(\mathscr{F}) - H^1(\mathscr{G}) + \cdots \\ &= \left(H^0(\mathscr{E}) - H^0(\mathscr{F}) + H^0(\mathscr{G})\right) - \left(H^1(\mathscr{E}) - H^1(\mathscr{F}) + H^1(\mathscr{G})\right) + \cdots \\ &= \chi(\mathscr{E}) - \chi(\mathscr{F}) + \chi(\mathscr{G}) \end{aligned}$$

as desired. □

As in the topological case, the Euler characteristic of a sheaf is often easier to compute than the individual cohomology groups. It is also sufficient for many applications.

7. Holomorphic vector bundles

The constructions of the last two sections are most commonly applied to sheaves of sections of holomorphic vector bundles, which we now introduce. We focus on theory in this section, and will defer any nontrivial concrete examples to §8.

7.1. **Definitions.** Recall that a (topological) **vector bundle** of rank r is given by a surjective map $\pi : E \to M$ of topological spaces, where each fiber $E_x = \pi^{-1}(x)$ has the structure of an r-dimensional complex vector space. For each $x \in M$, we require the existence of a neighborhood $U \ni x$ and a *local frame* of sections $\{e_i\}_{i=1}^r$ of π over U—meaning that $\{e_i(x)\}_{i=1}^r$ forms a basis for each fiber E_x, and for any $V \subset U$, the space of sections of π over V is given by

$$\left\{ \sum_{i=1}^r s^i e_i\big|_V \;\middle|\; \{s^i\}_{i=1}^r \in C^0_{\mathbb{C}}(V)^r \right\}. \tag{7.1}$$

A vector bundle of rank $r = 1$ is called a **line bundle**.

If E and M are smooth (resp. complex) manifolds, and all the objects mentioned are smooth (resp. holomorphic), then we say that E is a **smooth (resp. holomorphic) vector bundle**.

Example 7.1.1. The *trivial bundle* of rank r over M is given by the Cartesian product $M \times \mathbb{C}^r$. If M is a complex manifold, this is naturally a holomorphic vector bundle.

Given two smooth (resp. holomorphic) vector bundles E and F over M, a **bundle map** $\varphi : E \to F$ is a smooth (resp. holomorphic) map such that $\varphi(E_x) \subset F_x$, for all x, and φ induces a complex-linear map $\varphi(x) : E_x \to F_x$ *with* $\mathrm{rk}\varphi(x)$ *independent of* x. Two bundles are *isomorphic* if this map has an inverse, which is equivalent to being a linear isomorphism on each fiber (exercise).

Now, let $\underline{U} = \{U_\alpha\}$ be an open cover of M by coordinate charts, over which E has local frames $\{e_i^\alpha\}_{i=1}^r$. Notice that for each α, the map

$$\begin{aligned} U_\alpha \times \mathbb{C}^r &\to E|_{U_\alpha} \\ (x, (z^1, \ldots, z^r)) &\mapsto \sum_{i=1}^r z^i e_i(x) \in E_x \end{aligned} \tag{7.2}$$

is a bundle isomorphism over U_α, also known as a *local trivialization* of E. For this reason, the condition in Definition 7.1 above is known as *local triviality.*

Applying (7.1) with $U = U_\beta, V = U_\alpha$, and $s = e_j^\alpha$, we may write

$$e_j^\alpha\big|_{U_\alpha \cap U_\beta} = \sum_i (g_{\alpha\beta})^i{}_j \, e_i^\beta \tag{7.3}$$

in order to define the *transition functions* $(g_{\alpha\beta})^i{}_j$ on $U_\alpha \cap U_\beta$. For any section

$$s \overset{loc}{=} \sum_j (s^\alpha)^j \, e_j^\alpha$$

we get from (7.3) the transformation law

$$\left(s^\beta\right)^i = \sum_j (g_{\alpha\beta})^i{}_j \, (s^\alpha)^j \, . \tag{7.4}$$

By definition, the transition functions satisfy the *cocycle conditions*[8]

$$g_{\beta\gamma} \cdot g_{\alpha\beta} = g_{\alpha\gamma} \tag{7.5}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$, where $\cdot$ denotes matrix multiplication.

Conversely, given any collection of invertible matrix-valued functions satisfying (7.5), one can define a vector bundle $E \to M$ by gluing together the trivial bundles $U_\alpha \times \mathbb{C}^r$ according to (7.4), which is an equivalence relation. Moreover, if the transition functions $(g_{\alpha\beta})^i{}_j$ are holomorphic on each coordinate chart, then E will be a holomorphic vector bundle.

We now come to the following **meta-theorem:** any canonical construction that can be made with vector spaces carries over naturally to (holomorphic) vector bundles. In each case, the naturality of the construction will imply that the transition functions satisfy the cocycle condition (7.5), making the bundle well-defined.

Examples 7.1.2. Let E and F be vector bundles over M. Let $U, V \subset M$ be open subsets over which both are trivial, and denote the respective transition functions by $g(x)$ and $h(x)$, for $x \in U \cap V$.

1. The direct sum $E \oplus F$ has fiber $E_x \oplus F_x$, and transition function

$$\begin{pmatrix} g(x) & 0 \\ 0 & h(x) \end{pmatrix}.$$

2. The dual bundle E^* has fiber $E^*_x = \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$, and transition function $(g^T)^{-1}$. A bundle map $E \to F$ induces a bundle map $F^* \to E^*$ of the same (constant) rank.

3. The tensor product $E \otimes F$ has fiber $E_x \otimes_{\mathbb{C}} F_x$, with transition function $g(x) \otimes h(x)$.

4. The alternating product $\wedge^k E \subset \otimes^k E$ has transition function $\wedge^k g(x)$. In particular, we define the **determinant line bundle**

$$\det E = \wedge^{\mathrm{rk} E} E$$

with transition function given by $\det g(x)$. In this case, (7.5) follows from multiplicativity of the determinant of a matrix.

5. Given a subbundle $E \subset F$ (*i.e.* the image of an injective bundle map), we may define the quotient bundle F/E, with fiber F_x/E_x, as follows. After possibly shrinking our coordinate neighborhoods, we may choose a frame for F of the form

$$\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_s\}$$

where the first r sections are a frame for E. Then the transition function is necessarily of the form

$$h(x) = \begin{pmatrix} g(x) & \ell(x) \\ 0 & k(x) \end{pmatrix}. \tag{7.6}$$

We may take the $(s - r) \times (s - r)$ matrix $k(x)$ as the transition function for F/E, since the cocycle condition on h implies it for k. Moreover, if both E and F are holomorphic,

[8] Notice the similarity with (5.2); see also Theorem 8.2.1 below.

then the quotient is holomorphic by this construction, independently of the choices involved (exercise).

There is another construction of central importance in the study of vector bundles: given a map $f : N \to M$ and a bundle $E \to M$, define the **pullback bundle**

$$f^*E = E \times_M N \to N. \tag{7.7}$$

The fibers are

$$(f^*E)_x = E_{f(x)}$$

for $x \in N$. Given local frames $(U_\alpha, \{e_i^\alpha\})$ for E, local frames for f^*E may be given by

$$\left(f^{-1}(U_\alpha), \{e_i^\alpha(f(x))\}_{i=1}^r\right).$$

Lastly, given a holomorphic vector bundle $\pi : E \to M$, we denote its **sheaf of holomorphic sections**, in the sense of Example 5.2.2, by $\mathscr{E}$. We shall often abuse notation and denote the holomorphic bundle itself by $\mathscr{E}$, since this sheaf carries the same data. When we refer to a holomorphic bundle $\mathscr{E}$, we will write E for the underlying smooth vector bundle. We shall also sometimes use the notation

$$\Gamma(U, E) = \mathscr{E}(U)$$

to denote the space of holomorphic sections of E over an open set U.

Remark 7.1.3. One has to be slightly careful with sheaves of sections when discussing canonical operations. For instance, the sheaf of sections of $E \otimes_{\mathbb{C}} F$ is the tensor product

$$\mathscr{E} \otimes_{\mathscr{O}_M} \mathscr{F}$$

in the category of sheaves of $\mathscr{O}_M$-modules, *i.e.*, the sheaf associated to the presheaf $U \mapsto \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$. This means that $E \otimes F$ may have more global sections than just $\Gamma(\mathscr{E}) \otimes_{\Gamma(\mathscr{O})} \Gamma(\mathscr{F})$; in fact, this is a key feature of the subject.

There is also a potential for confusion between the *fiber* E_x of a vector bundle at x, which is an r-dimensional complex vector space, and the *stalk* $\mathscr{E}_x$ of the corresponding sheaf of sections, which is a free module of rank r over the local ring $\mathscr{O}_{n,x}$. At least it turns out that any sheaf with the latter property is the sheaf of sections of a vector bundle—see Huybrechts, Proposition 2.2.19.

7.2. **The (co)tangent bundle and the (co)normal sequence.** Given a complex manifold M, the **(holomorphic) tangent bundle** TM consists of the collection of all tangent spaces $\{T_xM \mid x \in M\}$, with local trivializations inherited from the coordinate charts. Recall that T_xM is canonically isomorphic to the holomorphic tangent space $T_x^{(1,0)}M$; in a coordinate chart $U = \{z^1, \ldots, z^n\}$, we therefore have the local frame

$$\left\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\right\}$$

for TM. Given a different coordinate chart $V = \{w^1, \ldots, w^n\}$, the transition function for TM on $U \cap V$ is given by the holomorphic Jacobian matrix

$$\left(\frac{\partial w^i}{\partial z^j} \right)_{i,j=1,\ldots,n}.$$

In this way, the tangent bundle TM canonically inherits the structure of a holomorphic vector bundle from the complex manifold M.

Similarly, the cotangent bundle $T^*M = TM^*$ is holomorphic, as are the exterior powers $\Lambda^k T^*M$. The sheaf of sections of the latter is of course Ω^k_M, the sheaf of holomorphic k-forms studied above. The **canonical line bundle** of M is defined to be

$$K_M = \Omega^n_M = \det T^*M.$$

The canonical bundle is of central importance in complex geometry.

Now, given a complex submanifold $S \subset M$, the tangent bundle TS has a natural inclusion into TM over points of S. We define the **normal bundle** via the following exact sequence of holomorphic vector bundles on S:

$$0 \to TS \to TM|_S \to N_S \to 0. \tag{7.8}$$

(See Example 7.1.2.5 for the definition of the quotient as a holomorphic bundle.) This is known as the **normal sequence**. Taking duals, we obtain the **conormal sequence**

$$0 \to N^*_S \to T^*M|_S \to T^*S \to 0 \tag{7.9}$$

which is again an exact sequence of holomorphic vector bundles on S, by Example 7.1.2.2 above.

Notice that, given any holomorphic bundle $\mathscr{E}$ and a global section $s \in \Gamma(\mathscr{E})$, the vanishing set $\mathbf{Z}(s)$ is an analytic subvariety of M. For, in any coordinate chart U_α over which E is trivial, we have

$$\mathbf{Z}(s) \cap U_\alpha = \mathbf{Z}\left((s^\alpha)^1, \ldots, (s^\alpha)^r \right).$$

Since the local components are holomorphic (by definition), and $\{U_\alpha\}$ cover M, this is indeed analytic, according to the definition in §4.3. We say that a global section $s \in \Gamma(\mathscr{E})$ **vanishes transversely** if $S = \mathbf{Z}(s)$ is a complex submanifold defined by the local components of s, per Definition 4.3.1.

The relationship between holomorphic bundles and subvarieties will be explored further in §8.

7.3. **Finiteness theorem.**

We now come to the following fundamental theorem.

Theorem 7.3.1. *For a holomorphic vector bundle $\mathscr{E}$ over a* compact *complex manifold, we have*

$$\dim H^i(\mathscr{E}) < \infty$$

for all $i \geq 0$.

This result can be approached either via Čech or via Dolbeault cohomology. Although the Dolbeault approach is more powerful, it requires PDE techniques that we would not develop until next semester. The Čech approach requires only complex variables and a bit of functional analysis, and has the virtue of generalizing to coherent analytic sheaves (see Gunning and Rossi).

For starters, we can prove the $i = 0$ case of the Theorem.

7.3.1. *Proof of finiteness of $H^0(\mathscr{E})$.* We make $H^0(\mathscr{E})$ into a Hilbert space as follows. Choose a finite open cover $\underline{U} = \{U_\alpha\}$ of M over which E is locally (holomorphically) trivialized, as well as a refinement $\underline{V} = \{V_\alpha\}$ with the property that

$$V_\alpha \Subset U_\alpha$$

for each α. Denote by $z_\alpha : U_\alpha \to \mathbb{C}^n$ the local coordinate on U_α, whose image we assume to be a bounded domain; we may also assume that all the transition functions of E are bounded.

Given two 0-cocycles $f = \{f^i_\alpha\}, g = \{g^i_\alpha\} \in H^0(\underline{V}, \mathscr{E}) \cong H^0(\mathscr{E})$ (*i.e.* global sections of $\mathscr{E}$), define the Hermitian inner product:

$$(f, g) = \sum_{\alpha, i} \int_{V_\alpha} f^i_\alpha(z_\alpha)\overline{g^i_\alpha(z_\alpha)} dVol_{z_\alpha}.$$

This inner product is finite: because $H^0(\underline{V}, \mathscr{E}) \cong H^0(\underline{U}, \mathscr{E})$, each f_α is the restriction of a holomorphic function on U_α, and therefore has finite L^2 norm on V_α. By Corollary 1.4.4, $H^0(\mathscr{E})$ is a Hilbert space.

We claim that the unit ball in $H^0(\mathscr{E})$ is relatively compact, by the following argument. Given a sequence f^k of cocycles in $H^0(\mathscr{E})$ for which $\|f^k\| = 1$ in the above norm, each f^k is represented by a cocycle $\{(f^k)^i_\alpha\} \in C^0(\underline{U}, \mathscr{E})$ for which $(f^k)^i_\alpha$ also has uniformly bounded L^2 norm on the chart U_α (because, modulo transition functions, f^k_α is equal to f^k_β in any overlapping chart). By Montel's Theorem 1.4.1, there exists a subsequence of f^k that converges uniformly on V_α, for each α, and this subsequence converges with respect to the above inner product. This establishes the claim.

It is clear (by choosing an orthonormal basis) that a Hilbert space whose unit ball is relatively compact is finite-dimensional. □

7.3.2. *Digression on Fréchet spaces.* The proof for higher cohomology groups is similar in spirit, but requires some deeper functional analysis. It is necessary to make the space of Čech cochains into a topological vector space, which is most naturally done within the following framework. We follow Appendix B of Gunning and Rossi.

Let $\{p_n\}$ be a sequence of **pseudonorms** on a vector space X, *i.e.*, norms that are allowed to have $p_n(x) = 0$ for some nonzero $x \in X$. We define a basis of open neighborhoods of X from the open balls with respect to the pseudonorms p_n. So, a subset $U \subset X$ is open if and only if for each $x \in U$, there exist $n, \delta > 0$ such that the δ-neighborhood of x with respect to p_n is contained in U. In particular, a subsequence converges in X if and only if it converges with respect to every pseudonorm p_n.

Definition 7.3.2. We say that the vector space X with the topology defined by the pseudonorms $\{p_n\}$ is a **Fréchet space** if $p_n(x) = 0\,\forall\, n \Rightarrow x = 0$ (*i.e.* X is Hausdorff), and X is complete in the topology defined above.

Examples 7.3.3. 1. The space of C^∞ functions is often considered as a Fréchet space, defined by the C^k norms. Completeness follows from the Arzela-Ascoli theorem.

2. Given an open set $U \subset M$ in a complex manifold, we make the space of holomorphic functions $\mathscr{O}(U)$ into a Fréchet space based on the collection of pseudonorms defined by

$$\|f\|_K = \sup_{x \in K} |f(x)|$$

where K is any compact subset of U. Completeness follows from the Montel Theorem, as in §7.3.1 above.

Lemma 7.3.4 (Open mapping theorem)**.** *Let X and Y be Fréchet spaces and $\varphi : X \to Y$ a continuous, surjective map. Then φ is an open map.*

Proof. See G & R Appendix B, Lemma 6. ☐

Theorem 7.3.5. *Every locally compact topological vector space has finite dimension.*

Proof. See Rudin, Functional Analysis, Theorem 1.22 on p. 17. ☐

Definition 7.3.6. A continuous map $\psi : X \to Y$ of Fréchet spaces is said to be **compact** if there exists a neighborhood V of the origin in X such that the image $\psi(V)$ in Y has compact closure.

Theorem 7.3.7 (L. Schwartz)**.** *Let $\varphi, \psi : X \to Y$ be continuous linear transformations between Fréchet spaces, with φ surjective and ψ compact. Then $\varphi + \psi$ has closed range, and the cokernel*

$$Y / (\varphi + \psi)(X)$$

is finite-dimensional.

Proof. For the closed range statement, see the proof in G & R, Appendix B, Theorem 12, which follows from Lemma 7.3.4 and the corresponding statement for Banach spaces.

Since the range is closed, $Y' = Y / (\varphi + \psi)(X)$ is again a Fréchet space. By Theorem 7.3.5, it suffices to show that Y' is locally compact, *i.e.*, for any $N, \epsilon > 0$ and any sequence $\{y_n\} \in Y$ with $p_N(y_n) < \epsilon$, the sequence $\{[y_n]\} \subset Y'$ has a convergent subsequence. Let $V \subset X$ be the neighborhood of the origin in the statement of Definition 7.3.6 for the map ψ. By Lemma 7.3.4, we may choose ϵ sufficiently small and N sufficiently large that for each y_n, there exists $x_n \in V$ such that

$$\varphi(x_n) = y_n.$$

Then, by choice of V, $\{\psi(-x_n)\}$ has a convergent subsequence (again labelled x_n). But in Y', we have

$$[y_n] = [\varphi(x_n)] = -[\psi(x_n)].$$

Therefore $[y_n]$ has a convergent subsequence, as claimed. ☐

We can now return to Theorem 7.3.1. The proof uses Leray coverings, *i.e.*, open covers satisfying the condition of Corollary 6.3.4. These in fact exist generally for complex manifolds, as shown in G & R, Ch. VI. For Riemann surfaces, any open covering by simply connected open sets with simply connected overlaps is sufficient: by the Riemann mapping theorem and the $\bar{\partial}$-Poincaré Lemma, these are acyclic.[9]

7.3.3. *Proof of Theorem 7.3.1.* Let $\underline{U} = \{U_\alpha\}$ and $\underline{V} = \{V_\alpha\}$ each be finite Leray covers of M, chosen such that

$$V_\alpha \Subset U_\alpha$$

for each α.

According to Example 7.3.3.2, spaces of holomorphic sections are Fréchet, as are the spaces of p-cochains $C^p(\underline{U}, \mathscr{E})$ and $C^p(\underline{V}, \mathscr{E})$, with the product topology. (This requires the covers to be finite.) Moreover, since the Čech differential δ is continuous, the space of cocycles

$$Z^p(\underline{U}, \mathscr{E}) = \ker \delta \subset C^p(\underline{U}, \mathscr{E})$$

is a closed subspace, and so too a Fréchet space. Denote the restriction map

$$\psi : Z^p(\underline{U}, \mathscr{E}) \to Z^p(\underline{V}, \mathscr{E}).$$

By Montel's Theorem, this is a compact map.

By the Leray Theorem, the restriction map

$$H^p(\underline{U}, \mathscr{E}) \to H^p(\underline{V}, \mathscr{E})$$

is an isomorphism. This implies that the sum

$$\varphi = (\psi \oplus \delta) : X = Z^p(\underline{U}, \mathscr{E}) \oplus C^{p-1}(\underline{V}, \mathscr{E}) \to Z^p(\underline{V}, \mathscr{E})$$

is surjective. But we then have

$$0 \oplus \delta = \varphi - (\psi \oplus 0).$$

By Theorem 7.3.7, the cokernel

$$Z^p(\underline{V}, \mathscr{E}) / (0 \oplus \delta)(X) = Z^p(\underline{V}, \mathscr{E}) / \delta\left(C^{p-1}(\underline{V}, \mathscr{E})\right)$$

is finite-dimensional. Since $\underline{V}$ is a Leray cover, this implies the claim.

[9]In fact, for the Leray Theorem on a Riemann surface, no assumption on the cohomology of the overlaps is necessary—see Forster, Theorem 12.8. We also note that in the case of Riemann surfaces, the finiteness theorem can be proved without introducing Fréchet spaces—see Forster, Theorem 14.9, or (simpler yet) Gunning, *Lectures on Riemann surfaces*, Theorem 7 on p. 64.

7.4. **The $\bar\partial$-operator and integrability.** As mentioned above, it is also extremely fruitful to study holomorphic bundles via Dolbeault cohomology, *i.e.*, using real analysis. Let $\mathscr{A}^{p,q}(E)$ denote the sheaf of E-valued (p,q)-forms, meaning the sheaf of *smooth* sections of the vector bundle $\Lambda^{p,q}T^*M \otimes E$.

The basic observation is as follows: every holomorphic bundle, $\mathscr{E}$, comes with a canonical differential operator

$$\bar\partial_{\mathscr{E}} : \mathscr{A}^{p,q}(E) \to \mathscr{A}^{p,q+1}(E). \tag{7.10}$$

This is simply defined to agree with the ordinary $\bar\partial$-operator in any holomorphic trivialization. So, given a local *holomorphic* frame e_i^α for E, we write

$$\begin{aligned} \bar\partial_{\mathscr{E}} &: \mathscr{A}^{0,0}(E) \to \mathscr{A}^{0,1}(E) \\ \bar\partial_{\mathscr{E}}(s^i e_i^\alpha) &= \left(\bar\partial s^i\right) e_i^\alpha \end{aligned} \tag{7.11}$$

where we are using the Einstein summation convention. One has to check that (7.11) is well-defined: given an equivalent frame $e_j^\beta = g^i{}_j e_i^\alpha$, we get

$$\begin{aligned} \bar\partial_{\mathscr{E}}(s^j e_j^\beta) &= \bar\partial_{\mathscr{E}}(s^j g^i{}_j e_i^\alpha) \\ &= \left(\bar\partial s^j\right) g^i{}_j e_i^\alpha + s^j \left(\bar\partial g^i{}_j\right) e_i^\alpha \\ &= \left(\bar\partial s^j\right) g^i{}_j e_i^\alpha \\ &= \left(\bar\partial s^j\right) e_j^\beta \end{aligned}$$

since $g^i{}_j$ is holomorphic. This shows that the definition did not depend on the choice of frame. The maps (7.10) on (p,q)-forms are defined via the Leibniz rule:

$$\bar\partial_{\mathscr{E}}(s\alpha) = \left(\bar\partial_{\mathscr{E}} s\right) \wedge \alpha + s\,\bar\partial\alpha. \tag{7.12}$$

Here s is a smooth section of E and $\alpha \in \mathscr{A}^{p,q}$.

Now, given the expression (7.11), we clearly have $\bar\partial_{\mathscr{E}}^2 = 0$. Moreover, since the operator is identical to (several copies of) $\bar\partial$ in local coordinates, the proof of the $\bar\partial$-Poincaré Lemma carries over without change. We conclude that complex

$$0 \to \mathscr{A}^{p,0}(E) \xrightarrow{\bar\partial_{\mathscr{E}}} \mathscr{A}^{p,1}(E) \xrightarrow{\bar\partial_{\mathscr{E}}} \mathscr{A}^{p,2}(E) \to \cdots \to \mathscr{A}^{p,n}(E) \to 0 \tag{7.13}$$

is an acyclic resolution of the sheaf $\Omega^p \otimes \mathscr{E}$. By Theorem 6.3.1, we obtain the following generalization of Theorem 6.4.5:

Theorem 7.4.1. $H^q(\Omega^p \otimes \mathscr{E}) \cong H^{p,q}_{\bar\partial_{\mathscr{E}}}$.

Corollary 7.4.2. *For a holomorphic vector bundle $\mathscr{E}$ over an n-dimensional complex manifold, we have*

$$H^i(\mathscr{E}) = 0, \quad i > n.$$

Proof. This follows from the $p = 0$ case of the previous theorem, and the fact that the acyclic resolution (7.13) terminates after n steps. □

Corollary 7.4.3. *For a holomorphic vector bundle $\mathscr{E}$ over an n-dimensional* compact *complex manifold, the Euler characteristic $\chi(\mathscr{E})$ is finite.*

Proof. We have seen in Theorem 7.3.1 that each cohomology group is finite-dimensional, and Corollary 7.4.2 implies that only finitely many are nonzero. □

Remark 7.4.4. The analytic approach receives a major bonus from the following result, which shows that holomorphic vector bundles live and die by the $\bar{\partial}$-operator.

Theorem 7.4.5. *Let E be a smooth vector bundle together with a differential operator $\bar{\partial}_E$ of the form (7.11), satisfying the Leibniz rule (7.12) and*

$$\bar{\partial}_E^2 = 0.$$

Then there exists a holomorphic structure $\mathscr{E}$ on E such that $\bar{\partial}_E = \bar{\partial}_{\mathscr{E}}$.

For a beautiful proof, see Donaldson and Kronheimer, *The geometry of four-manifolds*, pp.50-53.

Remark 7.4.6. There is a deeper integrability question in the subject, which asks when a smooth manifold M that admits an "almost-complex structure," *i.e.* a bundle map $I : TM \to TM$ with $I^2 = -1$, is a complex manifold, *i.e.* possesses a holomorphic atlas. For instance, although the 6-sphere possesses an almost-complex structure (coming from the octonionic cross product on $\mathbb{R}^7$), we still do not know if it is a complex manifold.

8. Line bundles, divisors, and linear systems

Recall that a (holomorphic) *line bundle* is simply a (holomorphic) vector bundle of rank one. The equivalent formulations of this concept are very ample, as we now describe.

8.1. Main examples.

We have already seen that any complex manifold M carries a *canonical line bundle*, $K_M = \Omega_M^n$, whose sections are holomorphic forms of top degree. In general, the canonical bundle and the trivial bundle are the only line bundles that are guaranteed to exist on a complex manifold. However, in the most important cases—compact Riemann surfaces and projective varieties—the following constructions guarantee that nontrivial line bundles will always be floating around.

Example 8.1.1. The **tautological bundle** $\mathscr{O}(-1)$ on $\mathbb{CP}^n$ is the subbundle of the trivial bundle $\underline{\mathbb{C}}^{n+1} = \mathbb{CP}^n \times \mathbb{C}^{n+1}$ given by

$$\{(\ell, t) \mid t \in \ell\} \subset \underline{\mathbb{C}}^{n+1}.$$

The fiber over a point $\ell \in \mathbb{CP}^n$ is exactly the corresponding line. A holomorphic frame for $\mathscr{O}(-1)$ over the standard chart $U_\alpha = \{[z_0, \ldots, \hat{z}_\alpha, \ldots, z_n]\}$ may be given by

$$e^\alpha ([z_0, \ldots, \hat{z}_\alpha, \ldots, z_n]) = (z_0, \ldots, 1, \ldots, z_n).$$

The transition function (per 7.3) is then given on $U_\alpha \cap U_\beta$ by

$$g_{\alpha\beta} = z_\beta = \frac{Z_\beta}{Z_\alpha}$$

in homogeneous coordinates.

The dual of the tautological bundle is denoted by

$$\mathscr{O}(1) = \mathscr{O}(-1)^*.$$

This is sometimes called the **hyperplane bundle** (see Example 8.3.1 below). Given $k \in \mathbb{Z}$, we will denote the tensor power by

$$\mathscr{O}(k) = \mathscr{O}(1)^{\otimes k}$$

where $\mathscr{O}(0) = \mathscr{O}$, and for $k < 0$, we mean $\mathscr{O}(k) = \mathscr{O}(-1)^{\otimes |k|}$. Given any holomorphic vector bundle $\mathscr{E}$ on $\mathbb{CP}^n$, it is standard to write

$$\mathscr{E}(k) = \mathscr{E} \otimes \mathscr{O}(k).$$

Also, given any projective variety $X \subset \mathbb{CP}^n$, we shall denote the restriction to X by

$$\mathscr{O}_X(k) = \mathscr{O}(k)|_X = \iota_X^* \mathscr{O}(k)$$

where $\iota_X : X \hookrightarrow \mathbb{CP}^n$ is the inclusion. By the pullback construction (7.7), this is a holomorphic line bundle on X. We shall see later that $\mathscr{O}_X(k)$ is a nontrivial bundle for all $k \neq 0$, if $\dim X > 0$.

Example 8.1.2. Let Σ be a Riemann surface. Given a point $p \in \Sigma$, we define the **point bundle** $\mathscr{O}(p)$ as follows. Let U be a coordinate chart in which $p = z_0$, and let $V = \Sigma \setminus \{p\}$. Then $\mathscr{O}(p)$ is defined to be trivial on the two open sets U and V, with transition function

$$g_{UV} = (z - z_0)^{-1}$$

where z is the coordinate on U. This is holomorphic and nonvanishing on $U \cap V = U \smallsetminus \{p\}$, and satisfies the cocycle condition (7.5) trivially. Hence, $\mathscr{O}(p)$ is a holomorphic line bundle.

We claim that if Σ is compact, then $\mathscr{O}(p)$ is a nontrivial bundle. For, we may construct a global holomorphic section:

$$\begin{aligned} s_U &= z - z_0 \\ s_V &= 1. \end{aligned} \tag{8.1}$$

This clearly satisfies (7.4). However, s has an isolated zero at p, so is not identically constant, as would necessarily be the case if $\mathscr{O}(p)$ were the trivial bundle (whose sections are global holomorphic functions on Σ).

In fact, given any meromorphic function $f(x)$ on Σ that is holomorphic on V and has a pole of order at most one at p, we may define a *holomorphic* section of $\mathscr{O}(p)$ by

$$\begin{aligned} s_U &= (z - z_0) f \\ s_V &= f. \end{aligned}$$

By Lemma 1.5.6, s_U and s_V are holomorphic on their respective domains.

Conversely, given any holomorphic section s of $\mathscr{O}(p)$, we may define a meromorphic function by

$$\begin{aligned} f|_U &= (z - z_0)^{-1} s_U \\ f|_V &= s_V. \end{aligned}$$

We conclude that this sheaf has the following equivalent description:

$$\mathscr{O}(p) \cong \begin{pmatrix} \text{sheaf of meromorphic functions on } \Sigma \\ \text{with a pole of order at most one at } p \end{pmatrix} \subset \mathscr{M}. \tag{8.2}$$

We also define

$$\mathscr{O}(-p) = \mathscr{O}(p)^*. \tag{8.3}$$

Either by a similar argument or directly from (8.3), we can obtain that $\mathscr{O}(-p)$ is isomorphic to the sheaf of holomorphic functions *vanishing at* $p \in \Sigma$.

Example 8.1.3. Now fix a divisor

$$D = \sum_\alpha n_\alpha p_\alpha - \sum_\beta m_\beta q_\beta \tag{8.4}$$

where $n_\alpha, m_\beta \in \mathbb{N}$, and $\{p_\alpha\}, \{q_\beta\}$ are discrete sets of points on the Riemann surface Σ. We may define the **line bundle associated to** D:

$$\mathscr{O}(D) = \bigotimes_\alpha \mathscr{O}(p_\alpha)^{\otimes n_\alpha} \otimes \bigotimes_\beta \mathscr{O}(-q_\beta)^{\otimes m_\beta}. \tag{8.5}$$

As in Example 8.1.2, we then have:

$$\mathscr{O}(D) \cong \begin{pmatrix} \text{sheaf of meromorphic functions with} \\ \text{poles of order at most } n_\alpha \text{ at } p_\alpha \text{ and} \\ \text{zeroes of order at least } m_\beta \text{ at } q_\beta \end{pmatrix} \subset \mathscr{M}. \tag{8.6}$$

It is very convenient that this subsheaf of $\mathscr{M}$ is isomorphic to the sheaf of sections of a line bundle.

This construction will be carried out in §8.3 below for divisors in complex manifolds of general dimension, with almost no essential changes.

Remark 8.1.4. Note that in the previous examples, $\mathscr{O}$(number) and $\mathscr{O}$(divisor) have different (although closely related) meanings.

8.2. **The Picard group and the first Chern class.** It is a special feature of line bundles, as opposed to general holomorphic bundles, that the tensor product of two line bundles is again a line bundle. The tensor product is of course associative and commutative, up to isomorphism. Moreover, for any line bundle L, there is a natural isomorphism

$$L^* \otimes L \cong \mathscr{O} \tag{8.7}$$

coming from the fact that $L^* \otimes L \cong \mathrm{Hom}(L, L)$, which has an obvious nonvanishing global section. The set of *isomorphism classes* of holomorphic line bundles on M is thus an abelian group, with multiplication given by the tensor product and inversion given by taking duals. This is called the **Picard group** $\mathrm{Pic}(M)$.

The Picard group has the following very handy cohomological interpretation. Given an isomorphism class $[L]$ represented by a line bundle L, choose a cover $\underline{U} = \{U_\alpha\}$ of M over which L is trivialized. The transition functions

$$\{g_{\alpha\beta}\}$$

determine a Čech 1-cochain with values in $\mathscr{O}^*$, and the compatibility condition (7.5) states precisely that this cochain is a multiplicative cocycle, lying in $\check{Z}^1(\underline{U}, \mathscr{O}^*)$.

Theorem 8.2.1. *The above correspondence determines a natural isomorphism*

$$\mathrm{Pic}(M) \cong H^1(\mathscr{O}^*_M).$$

Proof. We claim that the cocycle $g = \{g_{\alpha\beta}\}$ gives a well-defined element in $H^1(\mathscr{O}^*)$. Let $L' \in [L]$, with an isomorphism $\varphi : L \to L'$. Then there exists a refinement $\underline{V} = \{V_\alpha\}$ of $\underline{U} = \{U_\alpha\}$ over which both L and L' are trivialized, with frames e^α and f^α, respectively, and φ is given by

$$\varphi(e^\alpha) = q_\alpha(z) f^\alpha$$

for nonvanishing holomorphic functions $q_\alpha(z)$, for each α. Define a 0-cochain $q = \{q_\alpha\}$. Let $h_{\alpha\beta}$ be the transition functions of L', defined by

$$f^\alpha = h_{\alpha\beta} f^\beta.$$

Then we have

$$f^\alpha = q_\alpha^{-1}\varphi\left(e^\alpha\right) = q_\alpha^{-1}\varphi\left(g_{\alpha\beta}e^\beta\right) = q_\alpha^{-1}g_{\alpha\beta}\varphi\left(e^\beta\right) = q_\alpha^{-1}g_{\alpha\beta}q_\beta f^\beta \quad \text{(no summation)}$$

which gives

$$h_{\alpha\beta} = q_\alpha^{-1}g_{\alpha\beta}q_\beta.$$

Therefore the cocycles g and h differ by $\delta(q)$, which implies that the class in $H^1(\mathscr{O}^*)$ determined above is well defined.

Conversely, given a class in $H^1(\mathscr{O}^*)$ represented by a cocycle $\{g_{\alpha\beta}\}$, these satisfy (7.5) by definition, and can be used to construct a holomorphic line bundle. Changing by a coboundary only changes the resulting bundle by an isomorphism, and this construction is clearly inverse to the one above. Hence we have the claimed isomorphism. □

Recall the exponential sequence of Example 5.5.2 above. The long exact cohomology sequence reads:

$$\cdots \to H^1(\underline{\mathbb{Z}}) \to H^1(\mathscr{O}_M) \xrightarrow{\exp} H^1(\mathscr{O}_M^*) \cong \mathrm{Pic}(M) \to H^2(\underline{\mathbb{Z}}) \to \cdots. \tag{8.8}$$

Notice that $H^1(\mathscr{O}_M)$ is a complex vector space, whereas $H^2(\underline{\mathbb{Z}}) \cong H^2(M,\mathbb{Z})$ is a $\mathbb{Z}$-module. In this sense, $\mathrm{Pic}(M)$ has both a "continuous" and a "discrete" part. The latter is encoded by:

Definition 8.2.2. The **first Chern class** $c_1(L)$ is defined to be the image of $[L] \in \mathrm{Pic}(M)$ in $H^2(M,\mathbb{Z})$, via (8.8).

Proposition 8.2.3. *We have*

$$\begin{aligned} c_1(L\otimes L') &= c_1(L) + c_1(L') \\ c_1(L^*) &= -c_1(L) \\ c_1(f^*L) &= f^*c_1(L) \end{aligned} \tag{8.9}$$

for a map $f : N \to M$.

Proof. The first two items follow because $c_1(\cdot)$ is a group homomorphism, by Theorem 8.2.1 and the definition. The third item follows from the fact that the transition functions of f^*L (and the corresponding cocycle in $H^1(N)$) are given by the pullbacks of the transition functions of L. □

Remark 8.2.4. Notice that none of the previous discussion required L to be holomorphic: indeed, smooth line bundles are classified by $H^1(\mathscr{A}^*)$, where $\mathscr{A}^*$ is the space of nonvanishing, complex-valued smooth functions. We have the exponential sequence

$$0 \to \underline{\mathbb{Z}} \to \mathscr{A} \to \mathscr{A}^* \to 0 \tag{8.10}$$

which gives the first Chern class of a smooth bundle as above. Since the holomorphic exponential sequence is a subsequence of (8.10), this agrees with the above definition for a holomorphic bundle. However, by Proposition 6.4.1, $H^1(\mathscr{A}) = 0$, so the sequence (8.8) becomes an injective map:

$$0 \to H^1(\mathscr{A}^*) \to H^2(\underline{\mathbb{Z}}) \to \cdots.$$

We conclude that *a smooth line bundle is determined up to isomorphism by its first Chern class.* In particular, any holomorphic line bundle with vanishing first Chern class is trivial as a smooth bundle, even while it might not be nontrivial as a holomorphic bundle.

8.3. **Div, Pic, and linear equivalence.** Recall from §5.4 and Example 5.5.8 that the *sheaf of divisors* $\mathscr{D}iv_M$ on a complex manifold, M, assigns to each open subset U the abelian group of divisors on U, per Definition 5.4.1. We shall denote its group of global sections by

$$\Gamma(\mathscr{D}iv_M) = \mathrm{Div}(M)$$

which consists of the set of locally finite linear combinations of global irreducible hypersurfaces in M.

According to Example 5.5.8, the sheaf $\mathscr{D}iv$ is precisely the quotient sheaf $\mathscr{M}^*/\mathscr{O}^*$ (as argued above, because any divisor is locally defined by a meromorphic function, uniquely up to multiplication by nonvanishing holomorphic functions). The corresponding long exact sequence in cohomology reads:

$$0 \to \Gamma(\mathscr{O}^*) \to \Gamma(\mathscr{M}^*) \xrightarrow{\mathrm{div}} \mathrm{Div}(M) \xrightarrow{\mathscr{O}(\cdot)} H^1(\mathscr{O}^*) \cong \mathrm{Pic}(M) \to H^1(\mathscr{M}^*) \to \cdots. \tag{8.11}$$

In particular, we have a natural map

$$\begin{aligned} \mathrm{Div}(M) &\to \mathrm{Pic}(M) \\ D &\mapsto [\mathscr{O}(D)]. \end{aligned}$$

The map defined by this cohomological procedure directly generalizes the construction given by Example 8.1.3. Explicitly, let $\{U_\alpha\}$ be an open cover of M such that D has a local defining function f_α on each U_α, *i.e.*, f_α generates $\mathscr{I}_{D,x} \subset \mathscr{O}_x$ for each $x \in U_\alpha$. Define the transition functions of $\mathscr{O}(D)$ by

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

By the Nullstellensatz, these extend to nonvanishing holomorphic functions on $U_\alpha \cap U_\beta$. This construction makes $\mathscr{O}(D)$ well defined up to isomorphism.

The bundle $\mathscr{O}(-D) = \mathscr{O}(D)^*$, for D an irreducible effective divisor, is easier to describe: it is isomorphic to the ideal sheaf $\mathscr{I}_D \subset \mathscr{O}$ of holomorphic functions vanishing along D.

Example 8.3.1. Let $H \cong \mathbb{CP}^{n-1}$ be any hyperplane in $\mathbb{CP}^n$. Then

$$\mathscr{O}(H) \cong \mathscr{O}(1).$$

For, assume that H is defined by the the linear equation

$$\ell(Z) = a^0 Z_0 + \cdots + a^n Z_n = 0 \tag{8.12}$$

and choose $f_\alpha = \frac{\ell(Z)}{Z_\alpha}$ for its defining function on U_α. Then the transition functions are clearly identical to those of $\mathscr{O}(1)$, per Example 8.1.1 above. More generally, we have

$$\mathscr{O}(kH) \cong \mathscr{O}(k). \tag{8.13}$$

Definition 8.3.2. Given a global meromorphic *function* f on M, it is standard to write

$$(f) = \mathrm{div}(f).$$

We say that two divisors D and D' are **linearly equivalent** if there exists a global meromorphic function $f \in \Gamma(\mathscr{M})$ such that

$$D' = D + (f). \tag{8.14}$$

In this case, according to the above cohomological construction, the bundles $\mathscr{O}(D)$ and $\mathscr{O}(D')$ should be isomorphic. Indeed, if $\{f_\alpha\}$ is a system of defining functions for D, then according to (8.14), $\{ff_\alpha\}$ will be a system of defining functions for D', which gives the bundles the same transition functions.

Definition 8.3.3. A **meromorphic section** of a line bundle L is a section of the sheaf $\mathscr{L} \otimes_{\mathscr{O}_M} \mathscr{M}$. Explicitly, such a section is given by meromorphic functions $\{(s^\alpha)^i\}$ in each coordinate chart, obeying the compatibility conditions (7.4). We shall sometimes write $\operatorname{div}(s)$ for the divisor of zeroes and poles of a meromorphic section of a line bundle.

Caution. In the case $L \cong \mathscr{O}(D)$, a meromorphic *section* s of L corresponds via (8.6) to a meromorphic *function* on M. In this case, $\operatorname{div}(s)$ and (s) have different meanings; in particular

$$(s) = \operatorname{div}(s) + D.$$

We can summarize the essential points of this construction as follows.

Theorem 8.3.4. **(a)** *For each divisor $D \in \operatorname{Div}(M)$, there exists a line bundle $\mathscr{O}(D)$, unique up to isomorphism, such that $\mathscr{O}(D)$ carries a global meromorphic section s with $\operatorname{div}(s) = D$. If $D \sim D'$, then $\mathscr{O}(D) \cong \mathscr{O}(D')$; in other words, the map $D \mapsto \mathscr{O}(D)$ descends to an injective homomorphism*

$$\operatorname{Div}(M)/\sim \quad \hookrightarrow \quad \operatorname{Pic}(M). \tag{8.15}$$

(b) *Conversely, if a line bundle L has a nontrivial*[10] *global meromorphic section s, then*

$$L \cong \mathscr{O}(D)$$

where $D = \operatorname{div}(s)$.

If the divisors concerned are effective then the sections are holomorphic, and vice-versa.

Corollary 8.3.5. *A divisor D is principal if and only if $\mathscr{O}(D) \cong \mathscr{O}$.*

Remark 8.3.6. We shall see in §10 that for compact Riemann surfaces, the map (8.15) is an isomorphism. The same is true for projective varieties, although we would not develop the tools to prove this until next semester.

8.4. **Linear systems and maps to projective space.** There is yet another viewpoint on line bundles and divisors that is important for applications.

Definition/Lemma 8.4.1. *Given a holomorphic line bundle $L \to M$ and a nonzero subspace $V \subset \Gamma(M, L)$, the* **linear system** *associated to V is the family of effective divisors*

$$\operatorname{Div}_V = \{\operatorname{div}(s) \mid s \in V\}.$$

The **base locus** *of V is defined to be*

$$\operatorname{Bs}(V) = \cap_{D \in \operatorname{Div}_V} D = \{x \in M \mid s(x) = 0 \;\forall\, s \in V\}.$$

[10] By a *nontrivial section*, we mean one that does not vanish identically on any connected component of M.

A linear system is said to be **complete** *if* $V = \Gamma(M, L)$. *Given a divisor* D *on* M, *write* $|D|$ *for the complete linear system associated to* $H^0(\mathscr{O}(D))$; *equivalently,*

$$|D| = \{D + (f) \geq 0 \mid f \in \mathscr{M}(M)\}$$

is the set of all effective divisors linearly equivalent to D.

Proof. The equivalence follows by interpreting $\mathscr{O}(D) \subset \mathscr{M}$ according to (8.6), which is of course not limited to Riemann surfaces. $\square$

Notice that a subspace V of dimension $d+1$ on M defines a holomorphic map

$$M \setminus \mathrm{Bs}(V) \to \mathbb{CP}^d.$$

For, we can choose a basis $s_0, \ldots, s_d$, and send

$$x \mapsto [s_0(x), \ldots, s_d(x)]. \tag{8.16}$$

Since the transition functions are scalars, this gives a well-defined map away from the base locus. One can avoid choosing a basis for V by instead associating to $x \in M \setminus \mathrm{Bs}(x)$ the linear functional

$$s \mapsto s(x) \in L_x \cong \mathbb{C}. \tag{8.17}$$

This gives an element in the dual space V^* of V that is well-defined up to the choice of isomorphism in (8.17). Hence, we actually have a canonical map

$$\varphi_V : M \setminus \mathrm{Bs}(V) \to \mathbb{P}(V^*).$$

If once does choose a basis for V, hence an isomorphism of V with V^*, then this agrees with (8.16).

Lastly, one can observe that the pullback of the hyperplane bundle on $\mathbb{CP}^n$ is the line bundle L:

$$\varphi_V^* \mathscr{O}(1) \cong L.$$

So, in fact, the divisors in the linear system are just "hyperplane sections" of the image $\varphi_V(M)$.

Definition 8.4.2. We say that a holomorphic line bundle L is **ample** if, for some $k > 0$, the complete linear system associated to L^k is an embedding. The bundle L is said to be **very ample** if L itself gives an embedding.

By definition, a compact complex manifold is projective if and only if it admits an ample line bundle. This point of view leads to the *Kodaira embedding theorem*, which we would prove next semester.

8.5. **The degree of a line bundle on a Riemann surface.** Before moving on, we should actually calculate the first Chern class of the bundle $\mathscr{O}(D)$ constructed in Examples 8.1.2-8.1.3.

Let Σ be a Riemann surface, and fix $p \in \Sigma$. Let (U, z) be a coordinate neighborhood identified with $B_1(0) \subset \mathbb{C}$, with p corresponding to the origin in the z coordinate. Let

$$\begin{aligned} V &= \Sigma \setminus [0,1] \\ W &= \Sigma \setminus [-1,0]. \end{aligned}$$

Then $\{U, V, W\}$ is an open cover of Σ. Notice that

$$U \cap V \cap W = U \setminus \mathbb{R} = U_+ \cup U_-$$

where $U_\pm$ corresponds to the part of B_1 within the upper (resp. lower) half-plane.

To compute $c_1(\mathscr{O}(p))$ using this cover, we must calculate the connecting homomorphism in the exponential sequence, applied to the cocycle

$$g = \{g_{UV} = z^{-1}, g_{UW} = z^{-1}, g_{VW} = 1\} \in H^1(\{U,V,W\}, \mathscr{O}^*).$$

Choose $\sigma \in C^1(\{U,V,W\}, \mathscr{O})$ such that $e^{2\pi i \sigma_{UV}} = g_{UV}$, *etc.*, as follows:

$$\begin{aligned} \sigma_{UV} &= \frac{-\log r - \theta_V i}{2\pi i}, \quad -\pi < \theta_V < \pi \\ \sigma_{UW} &= \frac{-\log r - \theta_W i}{2\pi i}, \quad 0 < \theta_W < 2\pi \\ \sigma_{VW} &= 0. \end{aligned}$$

We then have

$$\begin{aligned} \omega_{UVW} := (\delta\sigma)_{UVW} &= \sigma_{VW} - \sigma_{UW} + \sigma_{UV} \\ &= \frac{1}{2\pi i}(\log r + \theta_W i - \log r - \theta_V i) \\ &= \frac{\theta_W - \theta_V}{2\pi} \\ &= \begin{cases} 0 & z \in U_+ \\ 1 & z \in U_- \end{cases} \end{aligned}$$

and $\omega_{UWV} = -\omega_{UVW}$, *etc.* Then $\omega \in H^2(\{U,V,W\}, \underline{\mathbb{Z}})$ represents $c_1(\mathscr{O}(p))$.

Now, assuming that Σ is compact, we have

$$H^2(\underline{\mathbb{Z}}) \cong H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}.$$

We need to evaluate the class ω against the fundamental class $[\Sigma] \in H_2(\Sigma, \underline{\mathbb{Z}})$, which requires unpacking the DeRham isomorphism of Corollary 6.4.3. To this end, we choose a special partition of unity subordinate to $\{U, V, W\}$. Let $\varphi(r)$ be a smooth, compactly-supported function on the unit ball U, with

$$\varphi(r) = \begin{cases} 1 & 0 \le r \le 1/4 \\ 0 & 3/4 \le r. \end{cases}$$

Let $\psi(x)$ be a smooth function on Σ with $0 \le \psi(x) \le 1$ as follows: assume that for $1/4 \le r \le 1$, $\psi(r,\theta) = \psi(\theta)$ is independent of r, and satisfies

$$\psi(\theta) = \begin{cases} 1 & -\pi/4 \le \theta \le \pi/4 \\ 0 & 3\pi/4 \le \theta \le 5\pi/4. \end{cases}$$

Let

$$\rho_U = \varphi, \quad \rho_V = (1-\varphi)\,\psi, \quad \rho_W = (1-\varphi)\,(1-\psi)\,.$$

Then ρ_U, ρ_V, and ρ_W have compact support in U, V, and W, respectively, and clearly satisfy

$$\rho_U + \rho_V + \rho_W = 1$$

i.e., they form a partition of unity subordinate to $\{U, V, W\}$.

Now, per the proof of Theorem 6.3.1, we need to trace through the isomorphisms coming from the exact sequences:

$$0 \to \underline{\mathbb{R}} \to \mathscr{A}^0 \to \mathscr{Z}^1_d \to 0 \tag{8.18}$$

and

$$0 \to \mathscr{Z}^1_d \to \mathscr{A}^1 \to \mathscr{Z}^2_d \to 0. \tag{8.19}$$

By the proof of Theorem 6.4.2, we know that the 1-cocycle $\eta \in H^1(\mathscr{Z}^1_d)$ defined by

$$\eta_{\beta\gamma} = d\left(\sum_\alpha \rho_\alpha \omega_{\alpha\beta\gamma}\right) = \sum_\alpha d\rho_\alpha \omega_{\alpha\beta\gamma}$$

satisfies $\delta(\eta) = \omega$, in the long exact sequence associated to (8.18). Then, the 0-cocycle $\tau \in H^0\left(\mathscr{Z}^2_d\right)$ defined by

$$\tau_\gamma = d\left(\sum_\beta \rho_\beta \eta_{\beta\gamma}\right) = \sum_{\alpha,\beta} d\rho_\beta \wedge d\rho_\alpha \omega_{\alpha\beta\gamma} \tag{8.20}$$

satisfies $\delta(\tau) = \eta$, in the long exact sequence associated to (8.19). Therefore $\tau \in H^2_{DR}(\Sigma)$ represents the class of ω under the DeRham isomorphism.

From (8.20), we see that the support of τ is contained in the support of ω, so lies within $U_- \subset U$, where $\tau = \tau_U$. We have

$$\begin{aligned} \tau = \tau_U &= d\rho_W \wedge d\rho_V \omega_{VWU} + d\rho_V \wedge d\rho_W \omega_{WVU} \\ &= -2 d\rho_V \wedge d\rho_W \omega_{UVW}. \end{aligned}$$

We calculate

$$\begin{aligned} d\rho_V &= -d\varphi\psi + (1-\varphi)d\psi \\ d\rho_W &= -d\varphi(1-\psi) - (1-\varphi)d\psi \\ d\rho_V \wedge d\rho_W &= \psi(1-\varphi)d\varphi \wedge d\psi - (1-\varphi)(1-\psi)d\psi \wedge d\varphi \\ &= (1-\varphi)\,(\psi + 1 - \psi)\,d\varphi \wedge d\psi \\ &= (1-\varphi)d\varphi \wedge d\psi \\ &= -\frac{1}{2} d\,(1-\varphi)^2 \wedge d\psi. \end{aligned}$$

This yields

$$\begin{aligned}\tau &= d\,(1-\varphi)^2 \wedge d\psi\,\omega_{UVW}\\ &= d\,(1-\varphi)^2 \wedge d\psi\big|_{U_-}.\end{aligned}$$

Since the support of $d\,(1-\varphi)^2$ is contained in $1/4 \le r \le 3/4$, we have $\psi = \psi(\theta)$ by assumption. Integrating over Σ, we obtain

$$\begin{aligned}\int_\Sigma \tau &= \int_{U_-} d(1-\varphi(r))^2 \wedge d\psi(\theta)\\ &= \int_0^1 \frac{d}{dr}\,(1-\varphi(r))^2\,dr \cdot \int_{-\pi}^0 \frac{d}{d\theta}\psi(\theta)\,d\theta\\ &= 1.\end{aligned}$$

If Σ is compact then, because it is orientable, it has a fundamental homology class $[\Sigma]$ generating $H_2(\Sigma,\mathbb{Z}) \cong \mathbb{Z}$. Let $\langle , \rangle$ denote the natural pairing between $H_2(\Sigma,\mathbb{R})$ and $H^2(\Sigma,\mathbb{R})$, which on DeRham classes is just given by integration. Then the result of the above calculation is:

$$\langle c_1(\mathscr{O}(p)), [\Sigma]\rangle = 1.$$

This will generalize to line bundles associated to a divisors, per Example 8.1.3, as soon as we make the following definition.

Definition 8.5.1. The **degree** of a divisor D on a Riemann surface is defined by

$$\deg(D) = \sum_\alpha n_\alpha - \sum_\beta m_\beta$$

where D is of the form (8.4).

Theorem 8.5.2. *For a divisor D on a compact Riemann surface Σ, we have*

$$\langle c_1(\mathscr{O}(D)), [\Sigma]\rangle = \deg D.$$

Proof. This follows immediately from the definition (8.5), Proposition 8.2.3, and the above calculation of $c_1(\mathscr{O}(p))$. ☐

Corollary 8.5.3. *Let L be a holomorphic bundle on a compact Riemann surface, Σ. For any nontrivial meromorphic section s of L, we have*

$$\deg \operatorname{div}(s) = \langle c_1(L), [\Sigma]\rangle.$$

In other words, the number of zeroes minus the number of poles of any meromorphic section of L, counted with multiplicity, is given by evaluating the first Chern class.

Proof. According to Theorem 8.3.4*b*, a line bundle with a nontrivial meromorphic section, s, is isomorphic to $\mathscr{O}(\operatorname{div}(s))$. The claim then follows from the previous theorem. ☐

Corollary 8.5.4. *On a compact Riemann surface, Σ, we have a commutative diagram*

$$\begin{tikzcd} \operatorname{Div}(\Sigma)/\sim \arrow[r] \arrow[dr, "\deg"'] & \operatorname{Pic}(\Sigma) \arrow[d, "{\langle c_1(\cdot),[\Sigma]\rangle}"] \\ & \mathbb{Z} \end{tikzcd}$$

where the maps to $\mathbb{Z}$ *are surjective. In particular, for a meromorphic function* $f \in \mathscr{M}(\Sigma)$, *we have* $\deg((f)) = 0$.

9. Bundles and divisors on $\mathbb{CP}^n$

This section discusses the basics of line bundles and divisors on projective space (and some projective varieties).

9.1. **The topology of projective space.** Recall that $\mathbb{CP}^1 \simeq S^2$ and therefore has (co)homology $\mathbb{Z}$ in degrees 0 and 2. More generally, according to (4.6), $\mathbb{CP}^n$ can be decomposed as:

$$\begin{aligned} \mathbb{CP}^n &= \mathbb{C}^n \sqcup \mathbb{CP}^{n-1} \\ &= \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \mathbb{CP}^{n-2} \\ &\vdots \\ &= \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \{pt\}. \end{aligned}$$

The disjoint unions can be extended continuously to give a CW decomposition of $\mathbb{CP}^n$. Since each cell has even dimension, we get

$$H_i(\mathbb{CP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd}. \end{cases} \tag{9.1}$$

Notice that according to this description, a k-plane ($\mathbb{CP}^k$) generates $H_{2k}(\mathbb{CP}^n)$.

As there is no torsion, the cohomology also takes the form (9.1). It is instructive to write down generators for the DeRham cohomology groups. To this end, define the **Fubini-Study form**

$$\omega = \frac{i}{2\pi} \partial \bar{\partial} \log |Z|^2 \tag{9.2}$$

where $|Z|^2 = Z_0^2 + \cdots + Z_n^2$. This requires some explanation. *A priori*, ω is only a well-defined differential form on $\mathbb{C}^{n+1} \smallsetminus \{0\}$, but we claim that it descends to a differential form on $\mathbb{CP}^n$ (in the following way).

Let $U \subset \mathbb{CP}^n$ be any open set such that there exists a holomorphic section of the projection $\mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$, which we denote again by $Z(x)$, for $x \in U$. (For instance, $U = U_\alpha$ can be taken to be a standard coordinate chart). Then ω descends to $U \subset \mathbb{CP}^n$ via the formula (9.2). One must then check that the resulting form on $\mathbb{CP}^n$ does not depend on the choice of section: let $f \cdot Z$ an another such section, where $f = f(x)$ is a nonvanishing holomorphic function on U. By shrinking U, we may choose a branch of log that is well-defined on $f(U)$. Then we have

$$\begin{aligned} \frac{i}{2\pi} \partial \bar{\partial} \log |fZ|^2 &= \frac{i}{2\pi} \partial \bar{\partial} \log f \bar{f} |Z|^2 \\ &= \frac{i}{2\pi} \partial \bar{\partial} \left(\log f + \log \bar{f} + \log |Z|^2 \right) \\ &= \frac{i}{2\pi} \left(\partial \bar{\partial} \log f - \bar{\partial} \partial \log \bar{f} + \partial \bar{\partial} \log |Z|^2 \right) \\ &= \frac{i}{2\pi} \partial \bar{\partial} \log |Z|^2 = \omega. \end{aligned}$$

This shows that ω indeed descends to a well-defined, closed form of type $(1,1)$ on $\mathbb{CP}^n$.

To show that $[\omega]$ is the integral generator of the second cohomology, we just need to integrate against $\mathbb{CP}^1$; in other words, we may assume without loss that $n = 1$. In a standard coordinate chart $\mathbb{C} \subset \mathbb{CP}^1$, we have

$$\begin{aligned}
\omega &= \frac{i}{2\pi}\partial\bar\partial \log\left(1 + |z|^2\right) \\
&= \frac{i}{2\pi}\partial\left(z\frac{d\bar z}{1+|z|^2}\right) \\
&= \frac{i}{2\pi}\frac{dz\wedge d\bar z}{1+|z|^2}\left(1 - \frac{z\bar z}{1+|z|^2}\right) \\
&= \frac{i}{2\pi}\frac{dz\wedge d\bar z}{\left(1+|z|^2\right)^2} \\
&= \frac{1}{\pi}\frac{r\,dr}{\left(1+r^2\right)^2}.
\end{aligned}$$

This clearly integrates to 1 over $\mathbb{C} = \mathbb{R}^2$.

On a standard coordinate chart of $\mathbb{CP}^n$, a similar calculation gives the coordinate expression

$$\omega(z) = \frac{i}{2\pi}\left[\frac{dz^i\wedge d\bar z^i}{1+|z|^2} - \frac{\bar z^j dz^j \wedge z^\ell d\bar z^\ell}{(1+|z|^2)^2}\right]$$

where i, j, and ℓ are summed over. Notice that at the point $0 \in \mathbb{C}^n$, and for the coordinate plane $\mathbb{C}^k = \{(z^1,\ldots,z^k,0,\ldots,0)\}$, we have

$$\omega^k(0)\big|_{\mathbb{C}^k} = k!\left(\frac{i}{2\pi}\right)^k dz^1\wedge d\bar z^1\wedge\cdots\wedge dz^k\wedge d\bar z^k$$

which is a positive multiple of the volume form on $\mathbb{C}^k$. Since ω is invariant under the biholomorphism group $SU(n+1)$ of $\mathbb{CP}^n$, which acts transitively, we conclude that the same is true at any point. This implies that

$$\int_{\mathbb{CP}^k}\omega^k > 0.$$

Hence ω^k represents a nonzero cohomology class; in fact, one can check directly that the value of this integral is 1. The ring structure on the cohomology of $\mathbb{CP}^n$ is therefore given by

$$H^*(\mathbb{CP}^n) = \mathbb{Z}[\omega]/[\omega]^{n+1}.$$

The Fubini-Study form (metric) plays a starring role in complex geometry, as we would see next semester.

9.2. **Cohomology of the line bundles on $\mathbb{CP}^n$.** One consequence of the above discussion is the following:

Lemma 9.2.1. $c_1(\mathscr{O}(k)) = k[\omega]$.

Proof. First assume $n = 1$. Then $\mathscr{O}_{\mathbb{CP}^1}(1) = \mathscr{O}_{\mathbb{CP}^1}(pt)$, which by Theorem 8.5.2, has first Chern class $1 \in H^2(\mathbb{CP}^1, \mathbb{Z})$, corresponding to the DeRham class $[\omega]$. Therefore $c_1(\mathscr{O}_{\mathbb{CP}^1}(k)) = k[\omega]$.

For $n \geq 1$, let $\iota : \mathbb{CP}^1 \to \mathbb{CP}^n$ be the inclusion of a line. Then by Proposition 8.2.3, we have

$$\iota^* c_1(\mathscr{O}_{\mathbb{CP}^n}(k)) = c_1(\iota^* \mathscr{O}_{\mathbb{CP}^n}(k)) = c_1(\mathscr{O}_{\mathbb{CP}^1}(k)) = k[\omega_{\mathbb{CP}^1}].$$

But $\iota^* : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, with $\iota^* \omega_{\mathbb{CP}^n} = \omega_{\mathbb{CP}^1}$, so we conclude that

$$c_1(\mathscr{O}_{\mathbb{CP}^n}(k)) = k[\omega]$$

as claimed. $\square$

Remark 9.2.2. We would see a direct proof of this fact using Chern-Weil theory in the next semester.

Theorem 9.2.3. $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z} \cong \{[\mathscr{O}(k)] \mid k \in \mathbb{Z}\}$.

Proof. Recall from Proposition 6.5.2 that $H^1(\mathscr{O}_{\mathbb{CP}^n}) = 0$. From the long exact sequence (8.8) defining the first Chern class, we see that $c_1 : \mathrm{Pic}(\mathbb{CP}^n) \to H^2(\mathbb{CP}^n, \mathbb{Z})$ is injective. But, by the previous subsection, we have $H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$. Lemma 9.2.1 then implies that $\mathscr{O}(1)$ generates the group, as claimed. $\square$

We now calculate the global sections of $\mathscr{O}(k)$. Given a nonzero linear functional $\ell(Z) : \mathbb{C}^{n+1} \to \mathbb{C}$ of the form (8.12), we obtain a nonzero linear function on $\mathscr{O}(-1) \subset \underline{\mathbb{C}}^{n+1}$ by restriction. Hence $\ell(Z)$ defines a global section of $\mathscr{O}(1)$. More generally, for each $k \geq 0$, any homogeneous polynomial $P(Z)$ of degree k defines a linear functional on $\left(\underline{\mathbb{C}}^{n+1}\right)^{\otimes k}$, and on $\mathscr{O}(-k) = \mathscr{O}(-1)^{\otimes k} \subset \left(\underline{\mathbb{C}}^{n+1}\right)^{\otimes k}$ by restriction. This gives an injective map from homogeneous polynomials of degree k to global sections of $\mathscr{O}(k)$:

$$\mathbb{C}\left[Z^0, \ldots, Z^n\right]_k \to H^0(\mathscr{O}(k)). \tag{9.3}$$

Theorem 9.2.4. *The above map is an isomorphism. We therefore have*

$$\dim H^0(\mathscr{O}_{\mathbb{CP}^n}(k)) = \binom{n+k}{k}.$$

Proof. It remains to show that (9.3) is surjective.

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the projection. Notice that the pullback $\pi^* \mathscr{O}(-1) \subset (\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}^{n+1}$ is trivial, with a nonvanishing section given by $e = (x, x)$. The pullback $\pi^* \mathscr{O}(-k)$ also has a nonvanishing section $e^{\otimes k}$.

Given a section $s \in H^0(\mathscr{O}(k))$, the pullback $\pi^* s$ is a global section of $\pi^* \mathscr{O}(k) \cong (\pi^* \mathscr{O}(-k))^*$. We may therefore evaluate

$$f = \pi^* s(e^{\otimes k})$$

to obtain a nontrivial holomorphic function $f(Z)$ on $\mathbb{C}^{n+1} \setminus 0$. The construction implies that f is k-homogeneous, *i.e.*

$$f(\lambda Z) = \lambda^k f(Z).$$

But, by Hartogs's Theorem, $f(Z)$ extends to a holomorphic function on $\mathbb{C}^{n+1}$. Any k-homogeneous smooth function on $\mathbb{C}^{n+1}$ must be a k-homogeneous polynomial (since its k+1-st partials are -1-homogeneous, and must therefore vanish). We conclude that $f(Z)$ is a homogeneous polynomial, which implies that s is precisely the image of $f(Z)$ under the map (9.3). $\square$

Corollary 9.2.5. *Any analytic hypersurface $X \subset \mathbb{CP}^n$ is algebraic, i.e., is the vanishing locus of a homogeneous polynomial $P(Z)$. If $P(Z)$ has no repeated factors, then it is unique up to multiplication by a constant, and vanishes transversely at a generic point of X.*

Proof. By Theorem 8.3.4, any hypersurface X defines a line bundle $\mathscr{O}(X)$, which carries a section s that vanishes to order one along X. But by Theorem 9.2.4, we have $\mathscr{O}(X) \simeq \mathscr{O}(k)$ for some $k \geq 0$, and $s = P(Z)$ is given by a homogeneous polynomial of degree k. The ratio of any two such polynomials defining X is holomorphic on $\mathbb{CP}^n$, hence constant.

Lastly, since $P(Z)$ has no repeated factors, $p(z) = P(1, z^1, \ldots, z^n)$ will also not have repeated factors as long as $\deg p = \deg P$, which can be arranged by changing coordinates. Then p and $\frac{\partial p}{\partial z^1}$ (say) are relatively prime as polynomials, and their discriminant (see Definition 2.3.4) $D(z^2, \ldots, z^n)$ does not vanish identically. So p vanishes transversely (and X is smooth) at all points not lying over the vanishing locus of the discriminant. $\square$

Remark 9.2.6. This is the first instance of Serre's GAGA[11] principle: see Griffiths and Harris, pp. 164-171.

Next, we calculate the higher cohomology groups of these line bundles. Let $H \cong \mathbb{CP}^{n-1}$ be a hyperplane in $\mathbb{CP}^n$. Recall that we have $\mathscr{O}(-1) \cong \mathscr{O}(-H) \cong \mathscr{I}_H$, the ideal sheaf of H. The ideal sheaf sequence of Example 5.5.3 therefore takes the form

$$0 \to \mathscr{O}(-1) \to \mathscr{O} \to \mathscr{O}_H \to 0.$$

Here we are abusing notation and writing $\mathscr{O}_H = \iota_* \mathscr{O}_H$ for the pushforward of the structure sheaf of $\mathscr{O}_H$. (By an exercise on your homework, this does not change the cohomology.) Hence, the above sequence is *not* an exact sequence of vector bundles over $\mathbb{CP}^n$, but is still an exact sequence of $\mathscr{O}$-modules, to which we can apply cohomology. Tensoring by $\mathscr{O}(k)$, we obtain a more general exact sequence (homework exercise):

$$0 \to \mathscr{O}_{\mathbb{CP}^n}(k-1) \to \mathscr{O}_{\mathbb{CP}^n}(k) \to \mathscr{O}_{\mathbb{CP}^{n-1}}(k) \to 0. \tag{9.4}$$

Theorem 9.2.7. *We have*

$$\dim H^q\left(\mathscr{O}_{\mathbb{CP}^n}(k)\right) = \begin{cases} \binom{n+k}{k} & q = 0,\ k \geq 0 \\ \binom{-k-1}{-k-1-n} & q = n,\ k \leq -n-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The case $q = 0$ has already been established in Theorem 9.2.4, and the case $k = 0$ in Proposition 6.5.2. We can prove the remaining items of the formula by induction, using the formula (9.4).

For the base case $n = 0$, we have $\mathbb{CP}^0 = pt$, and $H^0(\mathscr{O}_{pt}(k)) = \mathbb{C}$ for all k, which agrees with the formula.

[11] *Géométrie Algébrique et Géométrie Analytique.*

We first prove the result for $k \geq 1$ by a double induction. Assume that the result is established for up to $n-1$ and $k-1$. The long exact sequence associated to (9.4) reads:

$$\begin{aligned} 0 \to & H^0\left(\mathscr{O}(k-1)\right) \to H^0\left(\mathscr{O}(k)\right) \to H^0\left(\mathscr{O}_{\mathbb{CP}^{n-1}}(k)\right) \\ \to & 0 \to H^1\left(\mathscr{O}(k)\right) \to 0 \\ \to & \cdots \end{aligned} \tag{9.5}$$

which gives $H^q(\mathscr{O}(k)) = 0$ for $q \geq 1$.

Next, let $n = 1$ and $k < 0$. The long exact sequence is

$$\begin{aligned} & 0 \to 0 \to H^0\left(\mathscr{O}_{\mathbb{CP}^1}(k+1)\right) \to \mathbb{C} \\ & \to H^1\left(\mathscr{O}_{\mathbb{CP}^1}(k)\right) \to H^1\left(\mathscr{O}_{\mathbb{CP}^1}(k+1)\right) \to 0. \end{aligned} \tag{9.6}$$

This gives $H^1\left(\mathscr{O}_{\mathbb{CP}^1}(-1)\right) = 0$, and $\dim H^1(\mathscr{O}_{\mathbb{CP}^1}(k)) = \dim H^1(\mathscr{O}_{\mathbb{CP}^1}(k+1)) + 1$, for $k \leq -2$, which gives

$$\dim H^1(\mathscr{O}_{\mathbb{CP}^1}(k)) = -k-1$$

agreeing with the formula.

Finally, let $n \geq 2$ and $k \leq -1$, and assume that the result is known up to $n-1$ and down to $k+1$. The long exact sequence reads:

$$\begin{aligned} 0 \to & H^0\left(\mathscr{O}(k)\right) \to H^0\left(\mathscr{O}(k+1)\right) \to H^0\left(\mathscr{O}_{\mathbb{CP}^{n-1}}(k+1)\right) \\ \to & H^1\left(\mathscr{O}(k)\right) \to 0 \to 0 \\ \to & \cdots \to \\ \to & H^{n-1}\left(\mathscr{O}(k)\right) \to 0 \to H^{n-1}\left(\mathscr{O}_{\mathbb{CP}^{n-1}}(k+1)\right) \\ \to & H^n\left(\mathscr{O}(k)\right) \to H^n\left(\mathscr{O}(k+1)\right) \to 0 \end{aligned} \tag{9.7}$$

This gives $H^q(\mathscr{O}(k)) = 0$ for all $1 \leq q \leq n-1$, and

$$\begin{aligned} \dim H^n(\mathscr{O}(k)) &= \binom{-k-2}{-k-n-1} + \binom{-k-2}{k-n-2} \\ &= \binom{-k-1}{-k-n-1} \end{aligned}$$

by "inclusion-exclusion," establishing the formula by induction. ☐

Remark 9.2.8. The dimension formula of Theorem 9.2.7 can also be established using the Serre duality theorem, which we will prove below in the special case of Riemann surfaces, and the Kodaira vanishing theorem, which we would prove next semester.

9.3. **The Euler sequence and the adjunction formula.** Now that we know all the line bundles on $\mathbb{CP}^n$, it is worth determining which one is the canonical bundle. For this, we need a global description of the tangent bundle $T\mathbb{CP}^n$.

Recall that we have a holomorphic submersion

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n.$$

The tangent bundle $T\mathbb{C}^{n+1}$ has the global frame $\{\frac{\partial}{\partial Z^i}\}_{i=0}^n$. Given a point $Z = (Z^0, \ldots, Z^n) \in \mathbb{C}^{n+1}$, the images $\pi_{*,Z}\frac{\partial}{\partial Z^i}$ span $T_{[Z]}\mathbb{CP}^n$. Moreover, the kernel is given by the 1-dimensional subspace

$$\ker \pi_{*,Z} = \left(Z^i \frac{\partial}{\partial Z^i} \right) \tag{9.8}$$

where the summation convention is used. For, this vector is clearly annihilated by the projection map, which has rank n, so its kernel must have dimension one.

Observe that for any linear functional $\ell(Z)$ on a fiber $W \cong \mathbb{C} \subset \mathbb{C}^{n+1}$ over $[Z] \in \mathbb{CP}^n$, the vector field

$$X(Z) = \ell(Z)\frac{\partial}{\partial Z^i}$$

on W descends to a well-defined tangent vector on $\mathbb{CP}^n$. This is because for $\lambda \neq 0 \in \mathbb{C}$, $(Z, X(Z))$ and $(\lambda Z, X(\lambda Z)) = \lambda(Z, X(Z))$ correspond to the same tangent vector to $\mathbb{CP}^n$. At the bundle level, this means that for any holomorphic section s of $\mathscr{O}(1)$ over $\mathbb{CP}^n$, the expression

$$s(x)\frac{\partial}{\partial Z^i}$$

gives a well-defined, holomorphic (as one could check) section of $T\mathbb{CP}^n$.

This discussion can be summed up by the existence of an exact sequence of holomorphic vector bundles

$$0 \to \mathscr{O} \to \mathscr{O}(1)^{\oplus(n+1)} \to T\mathbb{CP}^n \to 0 \tag{9.9}$$

called the **Euler sequence**. The first map sends $1 \mapsto Z^i\frac{\partial}{\partial Z^i}$, and the second sends an $(n+1)$-tuple of sections $(s^0(x), \ldots, s^n(x))$ to $s^i(x)\frac{\partial}{\partial Z^i}$. Dualizing (9.9), we obtain

$$0 \to \Omega^1_{\mathbb{CP}^n} \to \mathscr{O}(-1)^{\oplus(n+1)} \to \mathscr{O} \to 0. \tag{9.10}$$

Lemma 9.3.1. *Given an exact sequence of holomorphic vector bundles*

$$0 \to E \to F \xrightarrow{f} G \to 0$$

we have a canonical isomorphism:

$$\det F \cong \det E \otimes \det G. \tag{9.11}$$

Proof. (*Cf.* HW 2 # 8.) Write $s = \mathrm{rk}E$, $r = \mathrm{rk}F$, and let U be a sufficiently small open set containing a given point. Over U, one defines a map

$$\det E \otimes \det G \to \det F$$

$$\alpha_1 \wedge \cdots \wedge \alpha_s \otimes \gamma_1 \wedge \cdots \wedge \gamma_{r-s} \mapsto \alpha_1 \wedge \cdots \wedge \alpha_s \wedge \tilde{\gamma}_1 \wedge \cdots \wedge \tilde{\gamma}_{r-s}$$

where $\tilde{\gamma}_i \in \Gamma(U, F)$ are chosen such that $f(\tilde{\gamma}_i) = \gamma_i$. This map is manifestly holomorphic on U, and well-defined: changing $\tilde{\gamma}_i$ to $\tilde{\gamma}_i + \alpha$, for $\alpha \in \Gamma(U, E)$, we have

$$\begin{aligned}
\alpha_1 \wedge \cdots \wedge \alpha_s \wedge & \tilde{\gamma}_1 \wedge \cdots \wedge (\tilde{\gamma}_i + \alpha) \wedge \cdots \wedge \tilde{\gamma}_{r-s} \\
&= \alpha_1 \wedge \cdots \wedge \alpha_s \wedge \tilde{\gamma}_1 \wedge \cdots \wedge \tilde{\gamma}_{r-s} + \alpha_1 \wedge \cdots \wedge \alpha_r \wedge \tilde{\gamma}_1 \wedge \cdots \wedge \alpha \wedge \cdots \wedge \tilde{\gamma}_{r-s} \\
&= \alpha_1 \wedge \cdots \wedge \alpha_s \wedge \tilde{\gamma}_1 \wedge \cdots \wedge \tilde{\gamma}_{r-s}
\end{aligned}$$

since $\alpha_1 \wedge \cdots \wedge \alpha_r \wedge \alpha = 0$. Since the map is well-defined, it gives the global isomorphism (9.11).

Alternatively, one can examine the transition functions directly from (7.6). ☐

Proposition 9.3.2. $K_{\mathbb{CP}^n} \cong \mathscr{O}(-n-1)$.

Proof. This follows by applying the Lemma to the dual Euler sequence (9.10). □

Next, we return for a moment to the general situation: let $\mathscr{E}$ be a holomorphic vector bundle over a complex manifold M.

Lemma 9.3.3. *Given a submanifold $S \subset M$ along which a global section $s \in \Gamma(\mathscr{E})$ vanishes transversely, we have*

$$E|_S \cong N_S.$$

Proof. Choose coordinates and a local frame for E. Then

$$ds = \left(\frac{\partial s^\alpha}{\partial z^j}\right)$$

gives a well-defined map $TM|_S \to E|_S$, as one checks from the fact that $s(x) = 0$ for $x \in S$. By assumption, this matrix has full rank, so gives a surjective, holomorphic bundle map whose kernel is precisely TS. We therefore have a holomorphic bundle isomorphism

$$E|_S \cong TM|_S /TS \cong N_S$$

as claimed. □

Theorem 9.3.4 (Adjunction formula)**.** *The canonical bundle of a complex submanifold $S \subset M$ is given by*

$$K_S \cong K_M|_S \otimes \det N_S. \tag{9.12}$$

In particular, if S is the zero-set of a transverse section of a vector bundle E, we have

$$K_S \cong K_M \otimes \det E|_S\,. \tag{9.13}$$

Proof. Applying Lemma 9.3.1a to the conormal sequence (7.9) yields

$$K_M|_S \cong K_S \otimes \det N_S^*.$$

Tensoring with $\det N_S$ and using the fact that $\det N_S \otimes \det N_S^* \cong \mathscr{O}$ gives the adjunction formula (9.12). Applying Lemma 9.3.3 gives (9.13). □

Corollary 9.3.5. *Let $S \subset \mathbb{CP}^n$ be a smooth projective hypersurface (i.e., the transverse vanishing locus of a homogeneous polynomial) of degree d. Then*

$$K_S \cong \mathscr{O}_S(d-n-1).$$

Proof. Since $S \subset M$ is a hypersurface, N_S is a line bundle, and we have $\det N_S = N_S \cong \mathscr{O}_S(d)$. Then (9.13) reads

$$\begin{aligned} K_S &\cong K_{\mathbb{CP}^n} \otimes \mathscr{O}(d)|_S \\ &= \mathscr{O}_S(-n-1+d) \end{aligned}$$

as claimed. □

9.4. **Plane curves.** A hypersurface $X \subset \mathbb{CP}^2$ is referred to as a **plane curve**. By Theorem 9.2.4, any such Y is the vanishing locus of a homogeneous polynomial P of degree d without repeated factors, unique up to a constant multiple. We will always assume that defining polynomials have no repeated factors, and will sometimes refer equally to d as the *degree* of X (not to be confused with the *degree* of a divisor on a Riemann surface, per Definition 8.5.1).

It is convenient at this point to make the following definition.

Definition 9.4.1. Let X and Y be plane curves defined by polynomials P and Q of degree d and e, respectively. Given $p \in C \cap X$, define the **intersection multiplicity**

$$\iota_p(X,Y) = \dim_{\mathbb{C}} \mathscr{O}_{\mathbb{CP}^2,p}/\left(P,Q\right)_p.$$

This notation requires some explanation. Since P is a section of $\mathscr{O}(d)$ and Q is a section of $\mathscr{O}(e)$, after choosing frames near the point p, they give elements of the local ring $\mathscr{O}_p$, well-defined up to multiplication by nonvanishing functions. So the ideal $(P,Q)_p$ is well-defined in $\mathscr{O}_p$. (Note that the intersection multiplicity may be infinity.)

Lemma 9.4.2. *For X and Y as above, if $p \in X$ is a smooth point (where P vanishes transversely), then*

$$\iota_p(X,Y) = \operatorname{Ord}_p Q|_X. \tag{9.14}$$

Here, $Q|_X$ denotes the restriction to X of Q, which is a section of $\mathscr{O}_X(e)$.

Proof. We claim that $R = \mathscr{O}_{\mathbb{CP}^2,p}/(P)_p \cong \mathscr{O}_1$, the ring of germs of holomorphic functions at the origin in $\mathbb{C}^1$. For, by the implicit function theorem, there exists a local chart $\{(z,w)\}$ near p in which $P(z,w) = w$. Hence R is isomorphic to the ring of convergent power series in the z variable, as claimed. Then the restriction of Q is given by $Q(z) = z^m g(z)$, with $g(0) \neq 0$, where m computes both sides of (9.14). □

Lemma 9.4.3. *Given a smooth plane curve X of degree d, we have*

$$\langle c_1(\mathscr{O}_X(k)), [X]\rangle = dk. \tag{9.15}$$

In particular, for any divisor $D \in |\mathscr{O}_X(k)|$, we have

$$\deg(D) = dk. \tag{9.16}$$

Proof. Notice that by Theorem 8.5.2 and (8.13), (9.15) and (9.16) are equivalent; it suffices to prove (9.16). In fact, by additivity of the Chern class, we may assume $k = 1$ without loss of generality.

Let P be the defining polynomial of X. Choose a hyperplane $H \cong \mathbb{CP}^1$ not contained in X. By the Fundamental Theorem of Algebra, the restriction of P to H vanishes in $d = \deg P$ points, counted with multiplicity. By Lemma 9.4.2, this gives:

$$\sum_{p \in H \cap X} \iota_p(H,X) = d.$$

But, by definition, we have

$$\iota_p(H,X) = \iota_p(X,H). \tag{9.17}$$

Let Q be the linear functional (section of $\mathscr{O}(1)$) defining H, and let D be the divisor of zeroes of the restriction of Q to X. Again by Lemma 9.4.2, and (9.17), we have

$$\deg(D) = \sum_{p \in X \cap H} \iota_p(X, H) = \sum_{p \in H \cap X} \iota_p(H, X) = d$$

which is the desired statement, for $k = 1$. The statement for general k follows by additivity of the degree (Chern class) under tensor products. □

We have the following direct application of Lemma 9.4.3.

Theorem 9.4.4 (Bézout's Theorem). *Let $X \neq Y$ be plane curves in $\mathbb{CP}^2$ of degree d and e, respectively, defined by homogeneous polynomials f and g, and assume that X is smooth. Then*

$$\sum_{p \in X \cap Y} \iota_p(X, Y) = de.$$

Proof. Let Q be a defining equation for Y. Since X is connected (exercise), and $X \neq Y$, the restriction of Q to X does not vanish identically, and its divisor of zeros D on X is well-defined. But by Lemmas 9.4.2 and 9.4.3, we have

$$\sum_{p \in X \cap Y} \iota_p(X, Y) = \deg(D) = de$$

as claimed. □

Remark 9.4.5. By developing intersection theory topologically (see Griffiths and Harris, pp. 49-65), this result becomes obvious, and can be vastly generalized.

Next, we have another very classical result:

Theorem 9.4.6 ("Degree-genus formula"). *Let X be a smooth plane curve of degree d. The first Chern class of the canonical bundle K_X is given by*

$$\langle c_1(K_X), [X] \rangle = d(d-3). \tag{9.18}$$

In particular, the divisor of zeroes and poles of any meromorphic 1-form on X has degree $d(d-3)$.

We also have

$$h^{1,0}(X) = h^{0,1}(X) = \binom{d-1}{2}. \tag{9.19}$$

Proof. By Corollary 9.3.5, we have $K_X \cong \mathscr{O}_X(d-3)$. Then (9.18) follows from Lemma 9.4.3.

To prove (9.19), consider the ideal sheaf sequence of X, which takes the form

$$0 \to \mathscr{O}_{\mathbb{CP}^2}(-d) \to \mathscr{O}_{\mathbb{CP}^2} \to \mathscr{O}_X \to 0.$$

The exact sequence in cohomology is:

$$H^1(\mathscr{O}_{\mathbb{CP}^2}) = 0 \to H^1(\mathscr{O}_X) \xrightarrow{\sim} H^2(\mathscr{O}_{\mathbb{CP}^2}(-d)) \to 0. \tag{9.20}$$

By Theorem 9.2.7, this gives

$$\dim H^1(\mathscr{O}_X) = h^{0,1}(X) = \binom{d-1}{d-3} = \binom{d-1}{2}$$

as claimed. By Theorem 9.2.4 and a similar exact sequence argument, we also have

$$h^{1,0}(X) = \dim H^0(K_X) = \dim H^0\left(\mathscr{O}_X(d-3)\right) = \dim H^0(\mathscr{O}_{\mathbb{CP}^2}(d-3)) = \binom{d-3+2}{d-3} = \binom{d-1}{2}$$

as claimed. ☐

Remark 9.4.7. Since $TX = K_X^*$, (9.18) implies that

$$\langle c_1(TX), [X]\rangle = d(3-d). \tag{9.21}$$

Because X is a compact Riemann surface, (9.21) determines its topology entirely, as can be seen in many different ways. The main point is that the Chern class of the tangent bundle is equal to the *Euler class*, which gives the topological Euler characteristic when evaluated against $[X]$. So (9.21) is equivalent to

$$\chi_{top}(X) = d(3-d).$$

Now, recall that the Euler characteristic of a smooth, compact, orientable surface of topological genus g is equal to $2-2g$. We therefore have

$$\begin{aligned} d(3-d) &= 2-2g \\ 2g &= d^2-3d+2 = (d-1)(d-2) \\ g &= \binom{d-1}{2}. \end{aligned}$$

By Theorem 9.4.6, we obtain the fundamental identity

$$h^{1,0}(X) = h^{0,1}(X) = g \tag{9.22}$$

in the case that X is a smooth plane curve. By the end of the next section, we will have proved (9.22) for general compact Riemann surfaces.

10. The Riemann-Roch Theorem

We will now apply our modern technology to prove *the* classical theorem about meromorphic functions on a Riemann surface. The goal is to effectively solve the following version of the Mittag-Leffler problem (5.1.1):

Given a finite collection of points $\{p_\alpha\}$ on a compact Riemann surface Σ, and integers n_α, what is the dimension of the space of meromorphic functions with poles of order at most n_α at p_α?

Notice that this problem has two aspects. The first is to determine the number of "constraints" on the principal parts of such a meromorphic function (*i.e.*, to bound the dimension from above); the second is to actually "construct" meromorphic functions with controlled poles (*i.e.*, to bound the dimension from below). We shall see that our cohomological tools can handle both aspects in a remarkable way.

10.1. **Motivation for the formula.** Let Σ be a compact Riemann surface. For a divisor

$$D = \sum_\alpha n_\alpha p_\alpha - \sum_\beta m_\beta q_\beta$$

we shall write

$$\ell(D) = \dim H^0\left(\mathscr{O}(D)\right) = \dim\left\{f \in \mathscr{M}(M) \mid (f) \geq D\right\}.$$

for the dimension of the space of meromorphic functions with poles of order at most n_α at p_α, and zeroes of order at least m_β at q_β. Recall that if $D \sim D'$ are linearly equivalent, then $\mathscr{O}(D) \cong \mathscr{O}(D')$, and consequently $\ell(D) = \ell(D')$.

Assume for the moment that D is effective, of degree $d = \sum n_\alpha$. We want to make an estimate of $\ell(D)$. First of all, notice that

$$\ell(D) \leq d + 1 \tag{10.1}$$

for obvious reasons: the principal part of a meromorphic function at p_α has dimension n_α, and any two functions with the same principal parts differ by a holomorphic function, which must be constant.

We have seen above (Problem 1.5.11) that for the case of $\mathbb{CP}^1$, there are no further constraints, and (10.1) is sharp. But a general Riemann surface Σ may carry holomorphic differential forms, which impose constraints in the following way. Given a meromorphic function f with $(f) \geq D$, and a holomorphic differential form ω, we obtain a global meromorphic differential form by taking the product:

$$\eta = f\omega. \tag{10.2}$$

Lemma 10.1.1. *Given a meromorphic differential form η on a compact Riemann surface Σ, with poles at $\{p_\alpha\}$, we have*

$$\sum_\alpha \mathrm{Res}_{p_\alpha}\eta = 0. \tag{10.3}$$

Here $\mathrm{Res}_{p_\alpha}\eta$ *is defined to be the coefficient* a_{-1} *in the Laurent expansion*

$$\eta = \sum_{i=-n_\alpha}^{\infty} a_i z^i\, dz$$

with respect to any local coordinate z in which $p_\alpha = 0$.

Proof. Choose a triangulation of Σ such that each face Δ_γ is contained in a coordinate chart, and each p_α is in the interior of a face. Then, since Σ is closed, we have

$$0 = \partial\Sigma = \partial\sum_\gamma \Delta_\gamma = \sum_\gamma \partial\Delta_\gamma.$$

Integrating η over this 1-chain, we get

$$0 = \sum_\gamma \int_{\partial\Delta_\gamma} \eta = \sum_\alpha 2\pi i \mathrm{Res}_{p_\alpha}\eta$$

by the Residue Theorem, applied in each face Δ_γ. □

As discussed above, since a meromorphic function is determined by its principal parts, up to a constant, we have:

$$0 \to \mathbb{C} \to H^0(\mathscr{O}(D)) \to \mathbb{C}^d.$$

Putting this together with the assignment (10.2), by Lemma 10.1.1, gives a complex:

$$0 \to \mathbb{C} \to H^0(\mathscr{O}(D)) \to \mathbb{C}^d \to H^0(\Omega^1_\Sigma)^*.$$

We can also determine the cokernel of the right-hand map: a holomorphic form has residue zero at p_α, when multiplied by any possible principal part, if and only if it *vanishes* to order n_α at p_α. We therefore obtain a complex

$$0 \to \mathbb{C} \to H^0(\mathscr{O}(D)) \to \mathbb{C}^d \to H^0(\Omega)^* \to H^0(\Omega(-D))^* \to 0 \tag{10.4}$$

which is exact, except possibly at $\mathbb{C}^d$. By (6.25), we obtain

$$\ell(D) - 1 \le d - h^{1,0} + \dim H^0(\Omega(-D)). \tag{10.5}$$

The **Riemann-Roch Theorem** states that (10.5) is an equality (and so the complex (10.4) was indeed exact).

Having provided this motivation, we will now attempt to give a maximally efficient proof of the theorem.[12]

10.2. **First version.**

Recall that for a holomorphic line bundle $\mathscr{L}$, we write

$$\mathscr{L}(D) = \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}(D)$$

which is again isomorphic to a line bundle. If $D = \sum_\alpha n_\alpha p_\alpha$ is effective, there is (by definition) an exact sequence of sheaves of $\mathscr{O}$-modules

$$0 \to \mathscr{L} \to \mathscr{L}(D) \to \oplus_\alpha (\mathbb{C}^{n_\alpha})_{p_\alpha} \to 0. \tag{10.6}$$

Here $(\mathbb{C}^{n_\alpha})_{p_\alpha}$ is the "skyscraper sheaf" whose sections are $\mathbb{C}^{n_\alpha}$ for open sets containing p_α and zero otherwise. We also write

$$h^{0,1} = h^{0,1}_\Sigma = H^1(\mathscr{O}_\Sigma)$$

[12] This approach has the disadvantage of making it slightly difficult to see the connection between the proof and the motivation. For a proof more directly connected to the above residue argument, see Griffiths and Harris, Ch. 2.

as above. This number, sometimes called the “arithmetic genus” of Σ, is finite by Theorem 7.3.1.

Theorem 10.2.1 (Riemann-Roch, first version)**.** *For a holomorphic line bundle $\mathscr{L}$ over a compact Riemann surface Σ, we have*

$$\dim H^0(\mathscr{L}) - \dim H^1(\mathscr{L}) = \langle c_1(\mathscr{L}), [\Sigma]\rangle + 1 - h^{0,1}_\Sigma. \tag{10.7}$$

Proof. Recalling the definition (6.6.1) of the Euler characteristic, notice that the LHS of (10.7) is simply $\chi(\mathscr{L})$. We have seen in Corollary 7.4.3 that this is a finite quantity.

Let D_0 be any effective divisor of degree d, and consider the exact sheaf sequence (10.6). Applying the additivity of the Euler characteristic, Lemma 6.6.2, we obtain

$$\chi(\mathscr{L}(D_0)) = \chi(\mathscr{L}) + d. \tag{10.8}$$

Taking $d > -\chi(\mathscr{L})$, we conclude that

$$\chi(\mathscr{L}(D_0)) = H^0(\mathscr{L}(D_0)) - H^1(\mathscr{L}(D_0)) > 0$$

and therefore

$$H^0(\mathscr{L}(D_0)) > 0.$$

Consequently, the bundle $\mathscr{L}(D_0)$ has nontrivial global holomorphic sections, which correspond to nontrivial *meromorphic* sections of $\mathscr{L}$. Letting s be any such section of $\mathscr{L}$ and $D = \operatorname{div}(s)$, we conclude from Theorem 8.3.4*b* that

$$\mathscr{L} \cong \mathscr{O}(D).$$

Therefore, all line bundles on Σ are isomorphic to $\mathscr{O}(D)$ for some divisor; it suffices to prove the theorem for bundles of this form.

We can now prove the formula by induction. The base case $\mathscr{L} = \mathscr{O}$ reads:

$$\dim H^0(\mathscr{O}) - \dim H^1(\mathscr{O}) = 0 + 1 - h^{0,1}$$

which is true by definition. Now, let

$$D = D_0 - D_1$$

be an arbitrary divisor, where D_0 and D_1 are both effective, and write $d = \deg D = d_0 - d_1$. By (10.8), since $\mathscr{O} = (\mathscr{O}(-D_1))\,(D_1)$, we have

$$\chi(\mathscr{O}) = \chi(\mathscr{O}(-D_1)) + d_1$$

and

$$\chi(\mathscr{O}(-D_1)) = -d_1 + 1 - h^{0,1}.$$

Applying (10.8) again, we obtain

$$\begin{aligned}\chi(\mathscr{O}(D)) &= \chi(\mathscr{O}(-D_1)) + d_0 = d_0 - d_1 + 1 - h^{0,1}\\ &= d + 1 - h^{0,1}\end{aligned}$$

as desired. ∎

Corollary 10.2.2. *Every holomorphic line bundle $\mathscr{L}$ over Σ is isomorphic to $\mathscr{O}(D)$ for some divisor D.*

Proof. This was shown during the proof of the last theorem. □

Remark 10.2.3. In view of the previous corollary, we will refer to the "degree" of any holomorphic line bundle $\mathscr{L}$ as the degree of any divisor such that $\mathscr{L} \cong \mathscr{O}(D)$. By Theorem 8.5.2, this is just the first Chern class of $\mathscr{L}$ evaluated against $[\Sigma]$.

Corollary 10.2.4. *Given any divisor D with $\deg(D) > h^{0,1}$, there exists a nonconstant meromorphic function f with $(f) \geq -D$.*

Proof. This follows because $H^1(\mathscr{O}(D))$ (fortunately) appears with a negative sign on the right-hand side of (10.7), so the assumption implies that $\ell(D) \geq 2$. □

Corollary 10.2.5. *Any Riemann surface with $h^{0,1} = 0$ is biholomorphic to $\mathbb{CP}^1$.*

Proof. For any point $p \in \Sigma$, we have $\dim H^0(\mathscr{O}(p)) \geq 2$, so there exists a nonconstant meromorphic function f with a simple pole at p. Then

$$[1, f]$$

defines a holomorphic map from Σ to $\mathbb{CP}^1$, and the degree of this map is clearly one (since f has a simple pole). Hence, by Theorem 2.6.3 (or much more elementary arguments), this is a biholomorphism. □

10.3. **Serre vanishing and projective embeddings.** Theorem 10.2.1 gives us a powerful method for manufacturing meromorphic functions on a Riemann surface. To gain more precise control over the output, we need to better understand the groups $H^1(\mathscr{L})$, in particular the group $H^1(\mathscr{O})$ whose dimension appears negatively on the RHS of (10.7).

We will first give a direct proof of a vanishing theorem due to Serre. The following lemmas will prove convenient.

Lemma 10.3.1. *For any holomorphic map*[13] *between line bundles*

$$\psi : \mathscr{L} \to \mathscr{N} \tag{10.9}$$

over Σ, there exists an effective divisor D such that ψ factorizes as

$$\mathscr{L} \to \mathscr{L}(D) \xrightarrow{\sim} \mathscr{N} \tag{10.10}$$

where the first map is the canonical inclusion and the second is an isomorphism.

Proof. This is a simple exercise in the definitions, using the local description (1.9) of a single-variable holomorphic function. □

Lemma 10.3.2. *Any nonzero holomorphic bundle map ψ as in (10.9) induces an epimorphism*

$$H^1(\mathscr{L}) \twoheadrightarrow H^1(\mathscr{N}).$$

[13]This means a map of $\mathscr{O}_\Sigma$-modules that is not necessarily a "bundle map," per §7.1. Locally, it is just given by a holomorphic function that may have zeroes.

Proof. Let D be the effective divisor produced by the previous lemma. Consider the long exact sequence in cohomology associated to (10.6), which was implicitly used in the proof of Theorem 10.2.1:

$$0 \to H^0(\mathscr{L}) \to H^0(\mathscr{L}(D)) \to \oplus_\alpha \mathbb{C}^{n_\alpha} \to H^1(\mathscr{L}) \to H^1(\mathscr{L}(D)) \to 0. \tag{10.11}$$

This shows that $H^1(\mathscr{L}) \to H^1(\mathscr{L}(D))$ is an epimorphism. But ψ factorizes as (10.10), and so the induced map on cohomology also factorizes as

$$H^1(\mathscr{L}) \twoheadrightarrow H^1(\mathscr{L}(D)) \xrightarrow{\sim} H^1(\mathscr{N}).$$

Since the first map is surjective and the second is an isomorphism, the composition is surjective. ☐

Theorem 10.3.3 (Serre vanishing)**.** *Let $\mathscr{L}$ be a line bundle on Σ and D any divisor with*

$$\deg D = d \geq H^1(\mathscr{L}) + h^{0,1}. \tag{10.12}$$

Then

$$H^1(\mathscr{L}(D)) = 0.$$

In particular, we have

$$H^1(\mathscr{O}(D)) = 0$$

for all divisors with $\deg D \geq 2h^{0,1}$.

Proof. Let $\mathscr{N} = \mathscr{L}(D)$. By HW 5 # 8, we have

$$\mathrm{Hom}_{\mathscr{O}}(\mathscr{L}, \mathscr{N}) \cong H^0(\mathscr{L}^* \otimes \mathscr{N}) = H^0(\mathscr{O}(D)).$$

Here, Hom is the space of all holomorphic maps $\mathscr{L} \to \mathscr{N}$. By Lemma 10.3.2, any nonzero map $\psi : \mathscr{L} \to \mathscr{N}$ induces a surjective map on cohomology, and so an *injective* map on dual spaces:

$$\psi^* : H^1(\mathscr{N})^* \hookrightarrow H^1(\mathscr{L})^*.$$

Assume, for the sake of contradiction, that $H^1(\mathscr{N}) \neq 0$, and let $\lambda \neq 0 \in H^1(\mathscr{N})^*$ be a nonzero element. Since, for any nonzero ψ as above, the induced map ψ^* is an injection, the element $\psi^*(\lambda)$ is nonzero. We therefore obtain an injective map

$$\begin{aligned} H^0(\mathscr{O}(D)) &\hookrightarrow H^1(\mathscr{L})^* \\ \psi &\mapsto \psi^*(\lambda). \end{aligned} \tag{10.13}$$

But Theorem 10.2.1 gives

$$\dim H^0(\mathscr{O}(D)) \geq d + 1 - h^{0,1}. \tag{10.14}$$

Hence, if

$$d - h^{0,1} \geq \dim H^1(\mathscr{L})$$

then (10.13) cannot possibly be injective. We conclude that $\lambda = 0$ for d as in (10.12). Since λ was arbitrary, we are done. ☐

Corollary 10.3.4. *For* $\deg D = d \geq 2h^{0,1}$, *we have*

$$\ell(D) = d + 1 - h^{0,1}.$$

Recall from §8.4 that spaces of holomorphic sections of line bundles (or equivalently, *linear systems* of divisors on Σ) define maps to projective space in an obvious way. We can easily use Corollary 10.3.4 to obtain the following result.

Theorem 10.3.5. *For any divisor D with $\deg D \geq 2h^{0,1}+2$, the map to $\mathbb{CP}^{h^{0,1}+2}$ associated to the complete linear system $|D|$ is an embedding.*

Proof. There are three things to check. First, we must verify that the base locus $\mathrm{Bs}(|D|) = \cap_{D' \in |D|} D'$ is empty, so that the map is defined on all of Σ. This is equivalent to showing that for each point $p \in \Sigma$, there exists a section $s \in H^0(\mathscr{O}(D))$ with $s(p) \neq 0$. Supposing the contrary, we would have

$$H^0(\mathscr{O}(D-p)) = H^0(\mathscr{O}(D)).$$

But this is impossible, by Corollary 10.3.4, since $\deg(D-p) = \deg D - 1$ for $\deg D \geq 2h^{0,1}+1$, as we have assumed. Therefore $\mathrm{Bs}(|D|) = \varnothing$.

Next, we must check that the map associated to $|D|$ is injective. This is equivalent to showing that for any two points $p \neq q \in \Sigma$, there is a section s with $s(p) = 0$ but $s(q) \neq 0$. (This is called "separating points.") But again by our degree assumption, we have

$$\dim H^0(\mathscr{O}(D-p)) > \dim H^0(\mathscr{O}(D-p-q))$$

so not all sections that vanish at p also vanish at q.

Lastly, we must check that the derivative of the map associated to $|D|$ is nonvanishing at every point $p \in \Sigma$. With a tiny bit of thought, this is equivalent to showing that some section s vanishes at p to order exactly one (*i.e.* the derivative of this coordinate is nonzero at p). Again, there must be such a section, because

$$\dim H^0(\mathscr{O}(D-p)) > \dim H^0(\mathscr{O}(D-2p))$$

by Corollary 10.3.4. We have shown that the map is an embedding, as required. □

Remark 10.3.6. We will see below that the degree thresholds in these theorems can be improved by one.

Corollary 10.3.7. *Every line bundle $\mathscr{L}$ of positive degree on a Riemann surface is ample (per Definition 8.4.2).*

Corollary 10.3.8. *Every compact Riemann surface is biholomorphic to a smooth projective variety.*

10.4. **Second version (with Serre duality).** Although we know that it is finite and vanishes for effective divisors of high-enough degree, the cohomology group $H^1(\mathscr{O}(D))$ appearing in (10.7) is still a bit mysterious. We can reinterpret it geometrically as follows.

Recall from §7.4 that the Dolbeault isomorphism $H^1(\mathscr{L}) \cong H^{0,1}_{\bar{\partial}_{\mathscr{L}}}$ holds for any line bundle; therefore any class $H^1(\mathscr{L})$ is represented by an equivalence class of $(0,1)$-forms on Σ with values in the underlying smooth bundle, L. Notice that since Σ is a Riemann surface, any such form is $\bar{\partial}$-closed. We now define the following pairing:

$$\begin{aligned} &\mathscr{A}^{1,0}(L^*) \otimes \mathscr{A}^{0,1}(L) \to \mathbb{C} \\ &(\alpha \otimes \lambda, \beta \otimes s) \mapsto \int_\Sigma \lambda(s)\alpha \wedge \beta. \end{aligned} \tag{10.15}$$

Notice that the integral makes sense, because $\lambda(s)$ is a scalar and $\alpha \wedge \beta$ is a 2-form.

Lemma 10.4.1. *The pairing (10.15) descends to a pairing*

$$\langle , \rangle : H^0(\Omega \otimes \mathscr{L}^*) \otimes H^1(\mathscr{L}) \to \mathbb{C}.$$

Moreover, the induced map

$$\iota : H^0(\Omega \otimes \mathscr{L}^*) \to H^1(\mathscr{L})^* \tag{10.16}$$

is injective.

Proof. The first claim amounts to the statement that given any holomorphic section $\omega \in \Gamma(\Omega \otimes \mathscr{L}^*)$, the function $\langle \omega, - \rangle$ vanishes identically on the image $\bar{\partial}(\mathscr{A}^0(L))$. To see this, write

$$\begin{aligned} \langle \omega, \bar{\partial} s \rangle &= \int_\Sigma \omega \wedge \bar{\partial}_{\mathscr{L}} s \\ &= \int_\Sigma \bar{\partial}(\omega(s)) \\ &= \int_\Sigma d(\omega(s)) \\ &= 0. \end{aligned}$$

Here we have used the Leibniz rule and the assumption that ω is holomorphic:

$$\bar{\partial}(\omega(s)) = \left(\bar{\partial}_{\Omega \otimes \mathscr{L}^*} \omega\right)(s) + \omega \otimes \bar{\partial}_{\mathscr{L}} \omega = \omega \otimes \bar{\partial}_{\mathscr{L}} \omega.$$

This is sufficient to show that the pairing $\langle , \rangle$ induced by (10.15) is well defined.

To show the injectivity, let $\omega \neq 0 \in H^0(\Omega \otimes \mathscr{L}^*)$, and let $p \in \Sigma$ be a point where $\omega(p) \neq 0$. Choose a coordinate chart U and local frame near $p = 0$, in which we have

$$\omega = \lambda(z)\, dz$$

with $\lambda(0) \neq 0$. We may choose a smooth section s of $\mathscr{L}$ over U, such that $\lambda(s)(z) \equiv 1$ for z in a neighborhood $U' \Subset U$ of p. Let $\chi \geq 0$ be a smooth cutoff supported on U', with $\chi(0) = 1$, and let

$$\eta = \chi \lambda d\bar{z}.$$

This is an element of $\mathscr{A}^{0,1}(L)$ with $\bar{\partial}_{\mathscr{L}} \eta = 0$ (trivially). We then have

$$\begin{aligned} \langle \omega, \eta \rangle &= \int_{U'} \chi \lambda(s)(z) dz \wedge d\bar{z} \\ &= \int_{U'} \chi dz \wedge d\bar{z} \neq 0. \end{aligned} \tag{10.17}$$

Therefore the functional $\langle \omega, - \rangle$ is not identically zero, as claimed. ∎

This simple argument establishes that nonzero global holomorphic 1-forms pair nontrivially with cohomology classes. The question is then whether any cohomology class pairs nontrivially with a holomorphic 1-form. This is a famous theorem of Serre, which can be seen as a complex version of the Poincaré Duality Theorem for an orientable smooth manifold.

Theorem 10.4.2 (Serre duality)**.** *The pairing $\langle , \rangle$ above is a perfect pairing, i.e. the map ι of (10.16) is an isomorphism. In particular, we have*

$$\dim H^1(\mathscr{L}) = \dim H^0(\Omega \otimes \mathscr{L}^*).$$

We will prove a special case of this result in §10.4.1, and the general case in §10.5.

Corollary 10.4.3. *We have*

$$h^{1,0}(\Sigma) = h^{0,1}(\Sigma).$$

Proof. This is the case $\mathscr{L} = \mathscr{O}$ in the previous Theorem. □

Definition 10.4.4. Define the **genus** of Σ by

$$g = g_\Sigma = h^{0,1}(\Sigma) = h^{0,1}(\Sigma).$$

We shall write

$$K = K_\Sigma$$

for the divisor of any meromorphic section of Ω_Σ, the canonical bundle of Σ, called a **canonical divisor** (really a linear equivalence class of divisors).

We can now rephrase the Riemann-Roch Theorem using Serre duality.

Theorem 10.4.5 (Riemann-Roch, second version)**.** *For any divisor D on Σ of degree d, we have*

$$\ell(D) - \ell(K - D) = d + 1 - g$$

Corollary 10.4.6. *The degree of the canonical bundle of Σ is given by*

$$\deg K = 2g - 2.$$

10.4.1. *Proof of Serre duality for plane curves.* For the special case of plane curves, there is a quick-and-dirty proof of Theorem 10.4.2 that goes as follows. According to Remark 9.4.7, we have already seen by direct calculation that the degree of the canonical bundle is

$$\deg K = d(d-3) = 2\binom{d-1}{2} - 2 = 2g - 2$$

agreeing with Corollary 10.4.6, where the "genus" g is given by Definition 10.4.4. (From the analytic perspective, this is actually the central point of the whole theory.)

Notice that by Lemma 10.4.1, it is sufficient to establish that the dimensions of the two spaces are equal. Also, applying Lemma 10.4.1 with $\mathscr{L} = \mathscr{O}(D)$, we have

$$\dim H^0(\Omega(-D)) = \ell(K - D) \leq \dim H^1(\mathscr{O}(D)). \tag{10.18}$$

Putting this together with Theorem 10.2.1, we get[14]

$$\ell(D) - \ell(K-D) \geq \ell(D) - \dim H^1(\mathscr{O}(D)) = d + 1 - g. \tag{10.19}$$

Applying (10.19) with $K - D$ in place of D, we obtain

$$\ell(K-D) - \ell(D) \geq (2g-2-d) + 1 - g = -d - 1 + g. \tag{10.20}$$

Adding (10.19) and (10.20), we obtain

$$0 \geq 0.$$

But this implies that the inequality (10.18) must have been an equality. □

10.5. **Proof of Serre duality.** We will now give a clever proof of Theorem 10.4.2 for a general Riemann surface, relying on a similar trick to the proof of Serre vanishing above (Theorem 10.3.3). I learned the proof from Forster's book (§17).

Recall that given a holomorphic map $\psi : \mathscr{L}_0 \to \mathscr{L}$ between line bundles, not identically zero, the induced map $H^1(\mathscr{L}_0) \to H^1(\mathscr{L})$ is a surjection, by Lemma 10.3.2. The corresponding map on dual spaces

$$\psi^* : H^1(\mathscr{L})^* \hookrightarrow H^1(\mathscr{L}_0)^* \tag{10.21}$$

is therefore an injection. We also have a natural inclusion

$$H^0(\Omega \otimes \mathscr{L}^*) \hookrightarrow H^0(\Omega \otimes \mathscr{L}_0^*) \tag{10.22}$$

induced by the dual holomorphic map $\mathscr{L}^* \to \mathscr{L}_0^*$. The following Lemma is crucial.

Lemma 10.5.1. *Let ψ as above and $\omega \in H^0(\Omega \otimes \mathscr{L}_0^*)$. If the element $\iota(\omega) = \langle \omega, - \rangle \in H^1(\mathscr{L}_0)^*$ lies in the image of $H^1(\mathscr{L})^*$ under ψ^* (per (10.21)), then ω lies in the image of $H^0(\Omega \otimes \mathscr{L}^*)$ under (10.22).*

Proof. By Lemma 10.3.1, we may assume without loss of generality that $\mathscr{L}_0 = \mathscr{L}(-D)$, for D an effective divisor, and $\mathscr{L}_0 \to \mathscr{L}$ is the natural map. In fact, we can assume that $D = p$ is an effective divisor of degree one, and the general result will follow by induction. Then to show that $\omega \in H^0(\mathscr{L}_0^*)$ belongs to $H^0(\mathscr{L}^*) = H^0(\mathscr{L}_0^*(-p))$, we must simply show that $\omega(p) = 0$.

As in the proof of Lemma 10.4.1, let (U, z) be a coordinate neighborhood of p on which $\mathscr{L}_0$ is trivialized. Let χ be a cutoff supported in U with $\chi \equiv 1$ on a ball $B \ni p$. Consider the element

$$\eta = \bar{\partial}\left(\frac{\chi}{z}\right) \in Z^{0,1}_{\bar{\partial}_{\mathscr{L}_0}}.$$

Notice that since $\mathscr{L} = \mathscr{L}_0(p)$, η belongs to the image $\bar{\partial}\mathscr{A}^0(L)$, and is therefore equivalent to zero in $H^{0,1}_{\bar{\partial}_{\mathscr{L}}} \cong H^1(\mathscr{L})$. So the assumption on $\iota(\omega)$ implies that

$$\langle \omega, \eta \rangle = 0.$$

[14]For an effective divisor, we could also finish the proof here by combining this inequality with (10.5), which goes in the opposite direction.

But, writing $\omega = f(z)\,dz$ on U, we have

$$\begin{aligned} 0 &= \int_U f(z)\,dz \wedge \bar\partial\left(\frac{\chi}{z}\right) \\ &= -\int \bar\partial\left(\frac{\chi f(z)\,dz}{z}\right) \\ &= \int_{\partial B} \frac{f(z)}{z}\,dz \\ &= 2\pi i f(0) \end{aligned}$$

by the Cauchy Integral Formula (!). Therefore $f(0) = 0$ and ω vanishes at p, as claimed. ☐

Proof of Theorem 10.4.2. Let $\lambda \neq 0 \in H^1(\mathscr{L})^*$. We must produce a section $\omega \in H^0(\Omega \otimes \mathscr{L}^*)$ such that $\iota(\omega) = \lambda$, where ι is the injective map defined by (10.16).

Let

$$n = \max\left[\deg \mathscr{L} + 1, 3h^{0,1} - \deg \Omega\right]. \tag{10.23}$$

Choose any line bundle $\mathscr{L}_0$ with

$$\deg \mathscr{L}_0 + n = \deg \mathscr{L}.$$

(For instance, we may take any effective divisor D_0 of degree n, and let $\mathscr{L}_0 = \mathscr{L}(-D_0)$.) We shall write $\iota_0 = \iota_{\mathscr{L}_0}$.

The space of holomorphic maps $\mathscr{L}_0 \to \mathscr{L}$ is isomorphic to the space of global sections of the tensor product:

$$\mathrm{Hom}_{\mathscr{O}}(\mathscr{L}_0, \mathscr{L}) \cong \Gamma(\mathscr{L}_0^* \otimes \mathscr{L}).$$

Since

$$\deg \mathscr{L}_0^* \otimes \mathscr{L} = -\deg \mathscr{L}_0 + \deg \mathscr{L}_0 + n = n$$

we have

$$\dim \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}) \geq n + 1 - h^{0,1} \tag{10.24}$$

by Theorem 10.7.

Given $\psi : \mathscr{L}_0 \to \mathscr{L}$, let $\psi^* : H^1(\mathscr{L})^* \to H^1(\mathscr{L}_0)^*$ be the induced map (10.21). Consider the subspace

$$\Lambda = \{\psi^*\lambda \mid \psi \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L})\} \subset H^1(\mathscr{L}_0)^* \tag{10.25}$$

By Lemma 10.3.1, ψ^* is is an injection for $\psi \neq 0$, so $\psi^*\lambda \neq 0$. Therefore we have an isomorphism

$$\begin{aligned} \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}) &\xrightarrow{\sim} \Lambda \\ \psi &\mapsto \psi^*\lambda. \end{aligned}$$

From (10.24) and Theorem 10.2.1, we obtain

$$\dim \Lambda \geq n + 1 - h^{0,1}. \tag{10.26}$$

Now, by Theorem 10.2.1 the space of holomorphic sections of $\Omega \otimes \mathscr{L}_0^*$ has dimension

$$\dim H^0\left(\Omega \otimes \mathscr{L}_0^*\right) \geq \left(\deg \Omega - \deg \mathscr{L} + n\right) + 1 - h^{0,1}. \tag{10.27}$$

Define the subspace

$$I = \iota_0\left(H^0\left(\Omega \otimes \mathscr{L}_0^*\right)\right) \subset H^1\left(\mathscr{L}_0\right)^*.$$

Recall from Lemma 10.4.1 that the map ι_0 is an injection, so (10.27) gives

$$\dim I \geq \deg \Omega - \deg \mathscr{L} + n + 1 - h^{0,1}. \tag{10.28}$$

On the other hand, since $n > \deg \mathscr{L}$ by (10.23), we have $\deg \mathscr{L}_0 < 0$, and Theorem 10.2.1 gives

$$-\dim H^1\left(\mathscr{L}_0\right) = \deg \mathscr{L} - n + 1 - h^{0,1}$$

and

$$\dim H^1\left(\mathscr{L}_0\right) = n + h^{0,1} - 1 - \deg \mathscr{L}. \tag{10.29}$$

Combining (10.26) and (10.28), we now have

$$\dim I + \dim \Lambda \geq 2n + 2 + \deg \Omega - \deg \mathscr{L} - 2h^{0,1}. \tag{10.30}$$

Comparing (10.30) and (10.29), the assumption (10.23) guarantees that the sum of the dimensions of the two subspaces I and Λ is greater than that of $H^1\left(\mathscr{L}_0\right)^*$. Consequently, there exists a nonzero element

$$\lambda_0 \in I \cap \Lambda \subset H^1\left(\mathscr{L}_0\right)^*$$

and, by definition, a section $\omega \in H^0\left(\Omega \otimes \mathscr{L}_0^*\right)$ and a holomorphic map $\psi : \mathscr{L}_0 \to \mathscr{L}$, such that

$$\iota_0\left(\omega\right) = \lambda_0 = \psi^* \lambda. \tag{10.31}$$

But, using ψ to identify $\mathscr{L}_0$ as a subsheaf of $\mathscr{L}$, Lemma 10.5.1 implies that in fact $\omega \in H^0\left(\Omega \otimes \mathscr{L}^*\right)$, where $\mathscr{L}^* \to \mathscr{L}_0^*$ is induced by ψ. Then ω satisfies

$$\iota\left(\omega\right) = \lambda$$

as desired. $\square$

11. Applications to Riemann surfaces

11.1. **The Riemann-Hurwitz formula.**

11.2. **Genus one.**

11.3. **Genus two and three.**

11.4. **The Hodge Theorem.** Michigan State University, East Lansing, MI 48823
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