RICCI FLOW AND SPHERE THEOREMS

ANDONI ROYO ABREGO

ABSTRACT. These are the notes from a talk given at the *Ricci flow reading seminar* in UW Madison on the 5th of November 2024. They are based on [5, 6, 8, 2, 4].

1. SPHERE THEOREMS

A *Sphere Theorem* seeks for a geometric condition on a (closed) Riemannian manifold that forces it to be equivalent to the the round sphere $\mathbb{S}^n$. Equivalent could mean *isometric*, *diffeomorphic* or *homeomorphic*. A classic example of the first is

Theorem 1.1 (H. Hopf [7])**.** *Let (M^n, g) be a closed, simply connected Riemannian manifold with constant sectional curvature equal to 1. Then, (M^n, g) is isometric to $\mathbb{S}^n$.*

A natural follow-up question H. Hopf posed was the following: what if all the sectional curvatures of (M^n, g) are very close to 1? This question kept Riemannian geometers occupied for a long time, until a satisfactory answer was given by M. Berger and W. Klingenberg in the early sixties, building on work of H. E. Rauch [10]:

Theorem 1.2 (M. Berger [1], W. Klingenberg [9])**.** *Let (M^n, g) be a closed, simply connected Riemannian manifold with sectional curvature $\frac{1}{4} < K \leq 1$. Then, (M^n, g) is homeomorphic to $\mathbb{S}^n$.*

Theorem 1.2 is sharp in the sense that any compact symmetric space of rank one admits a metric with $\frac{1}{4} \leq K \leq 1$. Since the work of J. Milnor we know that there exist homeomorphic Riemannian manifolds that are not diffeomorphic to each other. Then the next natural question is: can we strengthen *homeomorphic* in Theorem 1.2 to *diffeomorphic*? Hamilton's Ricci flow provided a powerful tool to answer such question and it indeed culminated with a complete answer:

Theorem 1.3 (Brendle–Schoen [3])**.** *Let (M^n, g) be a closed, simply connected Riemannian manifold with sectional curvature $\frac{1}{4} < K \leq 1$. Then, (M^n, g) is diffeomorphic to $\mathbb{S}^n$.*

It follows in particular that exotic spheres do not admit Riemannian metrics with $\frac{1}{4} < K < 1$.

The goal of this talk is to see how Ricci flow can be used to prove *diffeomorphic* sphere theorems in general and show a weaker version of Theorem 1.3 due to G. Huisken [8].

2. General Convergence Criterion

In this section we describe a general principle for which an appropriately rescaled Ricci flow will converges to a spherical space form, that is, a Rimannian manifold constant sectional curvature. The technique is due to R. Hamilton [5, 6], but it has been generalized and extended to other geometric flows by many authors. For a more powerful and abstract version, see section 5 of [6] or section 5.4 of [2].

The Riemann curvature tensor admits a well-known orthogonal decomposition

$$\mathrm{Rm} = W + V + U\,,$$

where W is traceless, V has vanishing second traces and U is contains only double traces. Denoting $\mathring{\mathrm{Rc}} = \mathrm{Rc} - \frac{1}{n}\,\mathrm{R}\,g$ the traceless Ricci tensor, these are defined by

$$\begin{aligned} U_{ijkl} &= \frac{1}{n(n-1)}\,\mathrm{R}\big(g_{ik}g_{jl} - g_{il}g_{jk}\big)\,,\\ V_{ijkl} &= \frac{1}{n-2}\big(\mathring{\mathrm{R}}_{ik}g_{jl} - \mathring{\mathrm{R}}_{il}g_{jk} - \mathring{\mathrm{R}}_{jk}g_{il} + \mathring{\mathrm{R}}_{jl}g_{ik}\big)\,,\\ W_{ijlk} &= \mathrm{R}_{ijkl} - V_{ijkl} - U_{ijkl}\,. \end{aligned}$$

We also use the notation $\mathring{\mathrm{Rm}} = \mathrm{Rm} - U = W + V$ for the non full-trace part or the Riemann tensor. We recall that a Riemannian manifold has constant sectional curvature if and only if $\mathrm{Rm} = U$.

Theorem 2.1 (General Convergence Criterion). *Let $\big(M^n, g(t)\big)$ be a Ricci flow on a maximal time interval $t \in [0, T)$ with $n \geq 3$ and $T < \infty$. Suppose there exist some $0 < \sigma < 1$ and $C_0 < \infty$ such that*

$$|\mathring{\mathrm{Rm}}|^2 \leq C_0\,\mathrm{R}^{2-\sigma} \tag{1}$$

holds on $M \times [0, T)$. Then, the rescaled metrics $\frac{1}{2(n-1)(T-t)}g(t)$ converge smoothly as $t \to T$ to a metric of constant sectional curvature equal to 1 on M.

Observe that (1) breaks the scaling invariance: if $\mathrm{R} \to \infty$ as $t \to T$, then

$$\frac{|\mathring{\mathrm{Rm}}|^2}{\mathrm{R}^2} \leq C_0 R^{-\sigma} \to 0\,.$$

This means that in regions where the scalar curvature is becoming unbounded, it dominates over W and V. Later we will see that the pinching (1) further implies that the scalar curvature must blow-up as $t \to T$.

Lemma 2.2 (**Algebraic estimate**).

$$|\nabla \mathrm{Rc}\,|^2 \geq \frac{3n-2}{2(n-1)(n-2)}|\nabla \mathrm{R}\,|^2$$

Proof. Algebraic trace decomposition of $\nabla \mathrm{Rc}$. □

Proposition 2.3 (**Gradient estimate**). *There exists some $\eta_0 > 0$ depending only on n and C_0 with the following property. For every $0 < \eta \leq \eta_0$ there exists some C_η depending on n, σ, η, C_0 and the initial data such that*

$$|\nabla \mathrm{R}\,|^2 \leq \eta\,\mathrm{R}^3 + C_\eta\,.$$

Proof. We begin by computing

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla \mathrm{R}\,|^2 &= 2\,\mathrm{Rc}(\nabla \mathrm{R}, \nabla \mathrm{R}) + 2\big\langle \nabla \mathrm{R}, \nabla\big(\Delta \mathrm{R} + 2|\,\mathrm{Rc}\,|^2\big)\big\rangle \\ &= \Delta|\nabla \mathrm{R}\,|^2 - 2|\nabla^2 \mathrm{R}\,|^2 + 6\,\mathrm{Rc}(\nabla \mathrm{R}, \nabla \mathrm{R}) + 4\big\langle \nabla \mathrm{R}, \nabla|\,\mathrm{Rc}\,|^2\big\rangle \\ &\leq \Delta|\nabla \mathrm{R}\,|^2 - 2|\nabla^2 \mathrm{R}\,|^2 + C\big(n, C_0\big)\,\mathrm{R}\,|\nabla \mathrm{Rc}\,|^2\,.\end{aligned}$$

In the last inequality we have used the basic inequality $|\nabla \mathrm{R}\,| \leq C(n)|\nabla \mathrm{Rc}\,|$ as well as $|\,\mathrm{Rc}\,| \leq C\big(n, C_0\big)\,\mathrm{R}$ due to the pinching (1). Since the scalar curvature will remain positive, we may compute

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla \mathrm{R}\,|^2}{\mathrm{R}}\right) \leq \frac{1}{\mathrm{R}}&\Big(\Delta|\nabla \mathrm{R}\,|^2 - 2|\nabla^2 \mathrm{R}\,|^2 + C\big(n, C_0\big)\,\mathrm{R}\,|\nabla \mathrm{Rc}\,|^2\Big) \\ &- \frac{1}{\mathrm{R}^2}|\nabla \mathrm{R}\,|^2\Big(\Delta \mathrm{R} + 2|\,\mathrm{Rc}\,|^2\Big)\end{aligned}$$

which combined with

$$\begin{aligned}\Delta\left(\frac{|\nabla \mathrm{R}\,|^2}{\mathrm{R}}\right) &= \frac{\Delta|\nabla \mathrm{R}\,|^2}{\mathrm{R}} - 2\frac{\langle \nabla \mathrm{R}, \nabla|\nabla \mathrm{R}\,|^2\rangle}{\mathrm{R}^2} - \frac{|\nabla \mathrm{R}\,|^2\Delta \mathrm{R}}{\mathrm{R}^2} + 2\frac{|\nabla \mathrm{R}\,|^4}{\mathrm{R}^3} \\ &= \frac{\Delta|\nabla \mathrm{R}\,|^2}{\mathrm{R}} - \frac{2}{\mathrm{R}}|\nabla^2 \mathrm{R}\,|^2 - \frac{|\nabla \mathrm{R}\,|^2\Delta \mathrm{R}}{\mathrm{R}^2} + \frac{2}{\mathrm{R}^3}\Big|\,\mathrm{R}\,\nabla^2 \mathrm{R} - \nabla \mathrm{R}\,\nabla \mathrm{R}\,\Big|^2\end{aligned}$$

it yields

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}\right) \leq -\frac{2}{\mathrm{R}^3}\Big|\mathrm{R}\,\nabla^2 \mathrm{R} - \nabla \mathrm{R}\,\nabla \mathrm{R}\Big|^2 + C_1(n,C_0)|\nabla \mathrm{Rc}|^2$$

In order to control the bad term on the right-hand-side, we compute the evolution of $\big|\mathring{\mathrm{Rc}}\big|^2 = |\mathrm{Rc}|^2 - \frac{1}{n}\mathrm{R}^2$. We begin with

$$\begin{aligned}\frac{\partial}{\partial t}|\mathrm{Rc}|^2 &= 4\,\mathrm{R}^{ij}\,\mathrm{R}_{ip}\,\mathrm{R}_j{}^p + 2\big\langle \mathrm{Rc}, \Delta\,\mathrm{Rc} + 2\,\mathrm{R}_{ipjq}\,\mathrm{R}^{pq} - 2\,\mathrm{R}_{ip}\,\mathrm{R}_j{}^p\big\rangle \\ &= \Delta|\mathrm{Rc}|^2 - 2|\nabla \mathrm{Rc}|^2 + 4\,\mathrm{R}_{ipjq}\,\mathrm{R}^{ij}\,\mathrm{R}^{pq}\end{aligned}$$

and

$$\frac{\partial}{\partial t}\mathrm{R}^2 = 2\,\mathrm{R}\,\Delta\,\mathrm{R} + 4\,\mathrm{R}\,|\mathrm{Rc}|^2 = \Delta\,\mathrm{R}^2 - 2|\nabla \mathrm{R}|^2 + 4\,\mathrm{R}\,|\mathrm{Rc}|^2$$

separately and put them together with the Lemma 2.2 to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\big|\mathring{\mathrm{Rc}}\big|^2 &= \Delta\big|\mathring{\mathrm{Rc}}\big|^2 - 2\left(|\nabla \mathrm{Rc}|^2 - \frac{1}{n}|\nabla \mathrm{R}|^2\right) + 4\left(\mathrm{R}_{ipjq}\,\mathrm{R}^{ij} - \frac{1}{n}\,\mathrm{R}\,\mathrm{R}_{pq}\right)\mathrm{R}^{pq} \\ &\leq \Delta\big|\mathring{\mathrm{Rc}}\big|^2 - \frac{(n-2)^2}{(3n-2)n}|\nabla \mathrm{Rc}|^2 + 4\,\mathrm{R}_{ipjq}\,\mathrm{R}^{ij}\,\mathring{\mathrm{R}}^{pq}\,.\end{aligned}$$

Note that $C_2(n) := \frac{(n-2)^2}{(3n-2)n} \geq 0$ for $n \geq 3$. Using the pinching assumption (1) we may estimate the last term too as

$$\begin{aligned}\mathrm{R}_{ipjq}\,\mathrm{R}^{ij}\,\mathring{\mathrm{R}}^{pq} &= \mathrm{R}_{ipjq}\,\mathring{\mathrm{R}}^{ij}\mathring{\mathrm{R}}^{pq} + \frac{1}{n}\,\mathrm{R}\,\mathring{\mathrm{R}}_{pq}\mathring{\mathrm{R}}^{pq} \\ &\leq C(n)\Big(|W||V|^2 + |V|^3 + \mathrm{R}\,|V|^2\Big) \leq C_3\big(n,C_0\big)\,\mathrm{R}^{3-\sigma}\,,\end{aligned}$$

leading to

$$\left(\frac{\partial}{\partial t}-\Delta\right)\big|\mathring{\mathrm{Rc}}\big|^2 \leq -C_2|\nabla \mathrm{Rc}|^2 + C_3\,\mathrm{R}^{3-\sigma}\;.$$

Finally, we can combine all these estimates to obtain that the function

$$f = \frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} + \frac{2C_1}{C_2}\big|\mathring{\mathrm{Rc}}\big|^2 - \eta\,\mathrm{R}^2$$

satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)f &\leq -C_1|\nabla \mathrm{Rc}|^2 + C_3\,\mathrm{R}^{3-\sigma} + 2\eta|\nabla \mathrm{R}|^2 - 4\eta\,\mathrm{R}\,|\mathrm{Rc}|^2 \\ &\leq \big(-C_1 + C(n)\eta\big)|\nabla \mathrm{Rc}|^2 + C_3\,\mathrm{R}^{3-\sigma} - \frac{4\eta}{n}\,\mathrm{R}^3\,,\end{aligned}$$

simply by $|\nabla \mathrm{R}|^2 \leq C(n)|\nabla \mathrm{Rc}|^2$ from Lemma 2.2 and $|\mathrm{Rc}|^2 \geq \frac{1}{n}\mathrm{R}^2$. For each $0 < \eta \leq C_1/C(n)$ it follows, using Young's inequality, that

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leq C_3 \mathrm{R}^{3-\sigma} - \frac{4\eta}{n}\mathrm{R}^3 \leq C_4(n,\sigma,\eta,C_0)$$

and the maximum principle implies that

$$f \leq \sup_M f(0) + C_4 T\,.$$

In particular,

$$\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} \leq \eta\,\mathrm{R}^2 + C_5 T$$

with C_5 depending on C_4 and the initial data. Multiplying by R and using Young's inequality we end up with

$$|\nabla \mathrm{R}|^2 \leq \eta\,\mathrm{R}^3 + C_5\,\mathrm{R}\,T \leq 2\eta\,\mathrm{R}^3 + \eta^{-1}\big(C_5 T\big)^{3/2}$$

and the claim follows after redefining η and bounding T with the initial lower bound of the scalar curvature. □

Proposition 2.4. *As $t \to T$ it holds that*

$$\mathrm{R}_{max} \to \infty \quad \textit{and} \quad \frac{\mathrm{R}_{max}}{\mathrm{R}_{min}} \to 1\,.$$

Proof. By the pinching assumption (1) we know that $\mathrm{R}_{max} \to \infty$ as $t \to T$. Thus, for any $\eta > 0$ there is some $0 \leq \bar{t} < T$ such that the constant on the right-hand-side of Proposition 2.3 satisfies $C_\eta \leq \eta\,\mathrm{R}^3_{max}$ for all $\bar{t} \leq t < T$. Therefore, we may safely assume (after redefining η) that for any small $\eta > 0$

$$|\nabla \mathrm{R}| \leq \eta^2\,\mathrm{R}^{3/2}_{max}$$

holds for all times close enough to T. Fix a point $p \in M$ where R assumes its maximum and let γ be any geodesic with $\gamma(0) = p$. Then,

$$\begin{aligned}\mathrm{R}\big(\gamma(s)\big) &= \mathrm{R}\big(\gamma(0)\big) - \int_0^s \frac{d}{d\theta}\big|\mathrm{R}\big(\gamma(\theta)\big)\big|\,d\theta \\ &\geq \mathrm{R}_{max} - \eta^2\,\mathrm{R}^{3/2}_{max}\,L_g(\gamma)\end{aligned}$$

and it follows that $\mathrm{R} \geq (1-\eta)\,\mathrm{R}_{max}$ at all points at distance at most $L = \eta^{-1}\,\mathrm{R}^{-1/2}_{max}$. Now, since the hypothesis (1) implies (exercise) the Ricci pinching $\mathrm{Rc} \geq C_1^2(n,\sigma,C_0)\,\mathrm{R}\,g$, the inequality

$$\mathrm{Rc} \geq C_1^2(1-\eta)\,\mathrm{R}_{max}\,g \geq \frac{C_1^2}{4}\,\mathrm{R}_{max}\,g$$

holds along the geodesics γ (w.l.o.g. $\eta < \frac{3}{4}$). Myers Theorem then implies that geodesics of length greater than $L_0 = 2C_1^{-1}\,\mathrm{R}_{max}^{-1/2}$ contain conjugate points. It follows that we can cover the whole manifold by geodesics of length L by choosing $\eta > 0$ small enough and establish

$$\mathrm{R}_{min} \geq (1-\eta)\,\mathrm{R}_{max}$$

for all times close to T. Since $\eta > 0$ was arbitrary, the claim follows. □

Proposition 2.5. *For any $\varepsilon > 0$ there exists some $\delta > 0$ such that*

$$(1-\varepsilon)\left(\frac{n}{2}+\varepsilon\right) \leq (T-t)\,\mathrm{R} \leq \frac{n}{2}$$

holds for all $T-\delta < t < T$.

Proof. From the evolution equation,

$$\frac{\partial}{\partial t}\mathrm{R} = \Delta\mathrm{R} + 2|\mathrm{Rc}|^2 \geq \Delta\mathrm{R} + \frac{2}{n}\mathrm{R}^2$$

we get by ODE comparison that

$$\mathrm{R}^{-1}(s) - \mathrm{R}^{-1}(t) \leq -\frac{2}{n}(s-t)$$

and the upper bound follows by letting $s \to T$ and using Proposition 2.4 to say that $\mathrm{R}(s) \to \infty$ everywhere.

For the lower bound, we use Proposition 2.4 to say that for each fixed $\varepsilon > 0$ there is some time $\delta > 0$ such that

$$\mathrm{R}_{min}^{\sigma} \geq 2C_0\,\varepsilon \quad \text{and} \quad \mathrm{R}_{min} \geq (1-\varepsilon)\,\mathrm{R}_{max}$$

for all $T-\delta \leq t < T$ and therefore

$$\frac{|\mathrm{Rc}|^2}{\mathrm{R}^2} = \frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} + \frac{1}{n} \leq C_0\,\mathrm{R}^{-\sigma} \leq \frac{\varepsilon}{2} + \frac{1}{n}\,.$$

It follows that

$$\frac{\partial}{\partial t}\mathrm{R} = \Delta\mathrm{R} + 2|\mathrm{Rc}|^2 \leq \Delta\mathrm{R} + \left(\varepsilon + \frac{2}{n}\right)\mathrm{R}_{max}^2$$

and thus the maximum of the scalar curvature satisfies (justify rigorously)

$$\frac{\partial}{\partial t}\mathrm{R}_{max} \leq \left(\varepsilon + \frac{2}{n}\right)\mathrm{R}_{max}^2$$

and consequently

$$\mathrm{R}_{max} \geq \left(\varepsilon + \frac{2}{n}\right)\frac{1}{T-t}\,.$$

Then, for all times $T-\delta<t<T$

$$\mathrm{R}_{min} \geq (1-\varepsilon)\,\mathrm{R}_{max} \geq (1-\varepsilon)\left(\frac{n}{2}+\varepsilon\right)$$

as clamed. □

Finally we are ready to prove the General Convergence Critirion.

Proof Theorem 2.1. Combining the pinching condition (1) and Proposition 2.5, we obtain the bound

$$|\operatorname{Rm}|^2 = |W|^2+|V|^2+|U|^2 \leq C\big(n,C_0\big)\,\mathrm{R}^2 \leq \frac{C(n,C_0)}{(T-t)^2}\,.$$

In other words, the singularity is Type I. This implies that the rescaled metrics

$$\bar{g}(t) = \frac{1}{2(n-1)(T-t)}g(t)$$

satisfy uniform curvature bounds

$$\sup_{M\times[0,T)} |\operatorname{Rm}_{\bar{g}(t)}|_{\bar{g}(t)} = 2(n-1)\sup_{M\times[0,T)}(T-t)|\operatorname{Rm}_{g(t)}|_{g(t)} \leq C(n,C_0)$$

and the usual long time existence obstruction theorem implies that the metrics $\bar{g}(t)$ converge smoothly as $t\to T$ to a smooth limit metric $\bar{g}(T)$. Since the ratio $\mathrm{R}_{max}/\mathrm{R}_{min}$ is scaling invariant, Proposition 2.4 and Proposition 2.5 imply that $\mathrm{R}_{\bar{g}(T)} = n(n-1)$ in the whole M. Moreover, by Proposition 2.5

$$\begin{aligned}|\mathring{\operatorname{Rm}}_{\bar{g}(t)}|^2_{\bar{g}(t)} &= 4(n-1)^2(T-t)^2|\mathring{\operatorname{Rm}}_{g(t)}|^2_{g(t)}\\ &\leq 4(n-1)^2(T-t)^2C_0\,\mathrm{R}^{2-\sigma} \leq 4(n-1)^2C_0(T-t)^\sigma\end{aligned}$$

and it follows that $\operatorname{Rm}_{g(T)} = U_{g(T)}$. We conclude that $\big(M,g(T)\big)$ has constant sectional curvature equal to 1. □

3. HUISKEN'S PINCHING

With Theorem 2.1 at hand, one aims to find conditions on the initial data such that (1) holds on $M\times[0,T)$. In three dimensions, $\mathrm{Rc}>0$ is one such condition [5]. In higher dimensions, one has

Theorem 3.1 (G. Huisken [8]). *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ with $\mathrm{R} > 0$ satisfying*

$$|\mathring{\mathrm{Rm}}|^2 < \frac{2\delta_n}{n(n-1)}\, \mathrm{R}^2 \tag{2}$$

with

$$\delta_4 = \frac{1}{5}\,, \quad \delta_5 = \frac{1}{10}\,, \quad \delta_n = \frac{2}{(n-2)(n+1)}\,.$$

Then, there exist constants $0 < \sigma < 1$ and $C_0 < \infty$ depending only on n and the initial data such that the unique solution to Ricci flow with initial data g satisfies

$$|\mathring{\mathrm{Rm}}|^2 \leq C_0\, \mathrm{R}^{2-\sigma}$$

on $M \times [0, T)$, where $T < \infty$ is the maximal time of existence.

Combined with the general convergence criterion it follows

Corollary 3.2. *Every closed Riemannian manifold of positive scalar curvature satisfying* (2) *is diffeomorphic to a spherical space form.*

In the remaining, let us discuss the proof of Theorem 3.1. By the maximum principle, we know that the scalar curvature will remain positive for all future times, so we may consider the function

$$f_\sigma = \frac{|\mathring{\mathrm{Rm}}|^2}{\mathrm{R}^{2-\sigma}}$$

for some $0 \leq \sigma < \frac{1}{2}$. This way, our task reduces to showing that f_σ is bounded above for some $\sigma > 0$. Since M is compact and the inequality (2) is strict, there exists some $\varepsilon > 0$ depending on the initial data such that

$$f_0 \leq (1-\varepsilon)^2 \frac{2\delta_n}{n(n-1)} \tag{3}$$

holds at $t = 0$. A lengthy but straightforward computation gives

Lemma 3.3.

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) f_\sigma = {}& \frac{2(1-\sigma)}{\mathrm{R}} \langle \nabla \mathrm{R}, \nabla f_\sigma \rangle - \frac{2}{\mathrm{R}^{4-\sigma}} \big|\mathrm{R}\, \nabla \mathrm{Rm} - \mathrm{Rm}\, \nabla \mathrm{R}\big|^2 \\ & - \frac{\sigma(1-\sigma)}{\mathrm{R}^{4-\sigma}} |\mathring{\mathrm{Rm}}|^2 + \frac{4}{\mathrm{R}^{3-\sigma}} \left(P + \frac{\sigma}{2} |\mathring{\mathrm{Rm}}|^2 |\,\mathrm{Rc}\,|^2\right)\end{aligned}$$

where

$$P = 2\, \mathrm{R}\, \mathrm{R}_{ijkl}\, \mathrm{R}^{ipkq}\, \mathrm{R}_{p\ q}^{\ j\ l} + \frac{1}{2}\, \mathrm{R}\, \mathrm{R}_{ijkl}\, \mathrm{R}^{klpq}\, \mathrm{R}_{pq}^{\ \ ij} - |\,\mathrm{Rm}\,|^2 |\,\mathrm{Rc}\,|^2\,.$$

In order to apply the maximum principle, we would like to estimate the absolute term P. This can be successfully done by inserting the $\mathrm{R}_{ijkl} = W_{ijkl} + V_{ijkl} + U_{ijkl}$ in the definition and using the pinching condition Eq. (2). The following purely algebraic Lemma is indeed where the hard work of the result is:

Lemma 3.4. *Suppose there exists some* $\varepsilon > 0$ *such that*

$$|\mathring{\mathrm{Rm}}|^2 \leq \delta_n (1-\varepsilon)^2 |U|^2 \,.$$

Then,

$$P \leq -\frac{\varepsilon}{n}\, \mathrm{R}^2\, |\mathring{\mathrm{Rm}}|^2 \,.$$

In other words, we have the estimate

$$P \leq -\frac{\varepsilon}{n}\, \mathrm{R}^{4-\sigma}\, f_\sigma$$

provided that $f_0 \leq (1-\varepsilon)^2 \frac{2\delta_n}{n(n-1)}$.

CLAIM 1. The estimate (3) holds for all $t > 0$.

Proof. Since (3) holds at $t = 0$, in particular the strict inequality

$$f_0(0) < (1-\varepsilon+\eta)^2\, \frac{2\delta_n}{n(n-1)}$$

holds for all $0 < \eta \leq \frac{\varepsilon}{2}$. Suppose now there is some first event $(x_0, t_0) \in M \times (0, T)$ such that

$$f(t_0, x_0) = (1-\varepsilon+\eta)^2\, \frac{2\delta_n}{n(n-1)}$$

for some $\eta > 0$. Then, by Lemma 3.3 and Lemma 3.4

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta\right) f_0(x_0, t_0) \leq 4\, \frac{P(x_0, t_0)}{\mathrm{R}^3(x_0, t_0)} \leq -\frac{(\varepsilon-\eta)}{n}\, \mathrm{R}(x_0, t_0) f_0^2(x_0, t_0) < 0 \,,$$

which is a contradiction. Therefore, $f_0 < (1-\varepsilon+\eta)^2 \frac{2\delta_n}{n(n-1)}$ for all $0 < t < T$ and all $0 < \eta \leq \frac{\varepsilon}{2}$ and the claim follows by letting $\eta \to 0$. □

At this point, Lemma 3.4 allows us to deduce that $P \leq -\frac{\varepsilon}{n}\, \mathrm{R}^2\, |\mathring{\mathrm{Rm}}|^2$ for all future times and hence we can show

CLAIM 2. There exists $\sigma > 0$ small enough such that $f_\sigma(t) \leq \sup_M f_\sigma(0)$ holds for all $0 \leq t < T$.

Proof. Using that $P \leq -\frac{\varepsilon}{n}\, \mathrm{R}^2\, |\mathring{\mathrm{Rm}}|^2$ and $f_0 < \frac{2\delta_n}{n(n-1)}$ we estimate the last term in the evolution equation of f_{σ_0} as

$$\frac{4}{\mathrm{R}^{3-\sigma}} \left(P + \frac{\sigma}{2} |\mathring{\mathrm{Rm}}|^2 |\,\mathrm{Rc}\,|^2\right) \leq \frac{4}{\mathrm{R}^{1-\sigma}} \left(-\frac{\varepsilon}{n} + \frac{\sigma}{2} \frac{|\,\mathrm{Rc}\,|^2}{\mathrm{R}^2}\right) |\mathring{\mathrm{Rm}}|^2$$

$$\leq \frac{4}{\mathrm{R}^{1-\sigma}} \left[-\frac{\varepsilon}{n} + \frac{\sigma}{2} \left(f_0 + \frac{1}{n} \right) \right] |\mathring{\mathrm{Rm}}|^2$$
$$\leq \frac{4}{\mathrm{R}^{1-\sigma}} \left[-\frac{\varepsilon}{n} + \frac{\sigma}{2n} \left(\frac{2\delta_n}{n-1} + 1 \right) \right] |\mathring{\mathrm{Rm}}|^2 .$$

Choosing $\sigma > 0$ small enough depending only on n and ε the right-hand-side becomes negative and the claim follows by the maximum principle. □

This completes the proof of Theorem 3.1.

REFERENCES

[1] M. Berger. Les variétés riemanniennes $\frac{1}{4}$-pincées. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.*, 14:161–170, 1960.

[2] S. Brendle. *Ricci flow and the sphere theorem*, volume 111 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2010.

[3] S. Brendle and R. Schoen. Manifolds with $1/4$-pinched curvature are space forms. *J. Am. Math. Soc.*, 22(1):287–307, 2009.

[4] S. Brendle and R. Schoen. Curvature, sphere theorems, and the Ricci flow. *Bull. Am. Math. Soc., New Ser.*, 48(1):1–32, 2011.

[5] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differ. Geom.*, 17:255–306, 1982.

[6] R. S. Hamilton. Four-manifolds with positive curvature operator. *J. Differ. Geom.*, 24:153–179, 1986.

[7] H. Hopf. Zum Clifford-Kleinschen Raumproblem. *Math. Ann.*, 95:313–339, 1925.

[8] G. Huisken. Ricci deformation of the metric on a Riemannian manifold. *J. Differ. Geom.*, 21:47–62, 1985.

[9] W. Klingenberg. Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung. *Comment. Math. Helv.*, 35:47–54, 1961.

[10] H. E. Rauch. A contribution to differential geometry in the large. *Ann. Math. (2)*, 54:38–55, 1951.