

# **$L^p$ SOBOLEV REGULARITY OF A RESTRICTED X-RAY TRANSFORM IN $\mathbb{R}^3$**

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ABSTRACT. We consider the  $L^p \rightarrow L^p_\sigma$  boundedness of a model restricted X-ray transform in  $\mathbb{R}^3$ , associated to a rigid line complex. We discuss some necessary conditions and, assuming a finite type condition, we show that the sharp  $L^p \rightarrow L^p_{1-1/p}$  result holds for  $p > 1$  close to 1.

## 1. INTRODUCTION

Let  $I$  be a compact interval and suppose that  $\gamma : I \rightarrow \mathbb{R}^2$  be a smooth regular curve (i.e. we assume  $\gamma'(s) \neq 0$ ). We say for  $m \geq 2$  that  $\gamma$  is of type  $m$  at  $s$  if  $\gamma$  has contact of order  $m$  with its tangent line at  $\gamma(s)$ . The maximal order of contact in  $I$  is referred to as the maximal type of  $\gamma$  in  $I$ . For a Schwartz function  $f \in \mathcal{S}(\mathbb{R}^3)$  and  $\alpha \in I$  define

$$(1.1) \quad \mathcal{R}f(x', \alpha) = \chi_1(\alpha) \int_1^2 f(x' + s\gamma(\alpha), s) \chi_2(s) ds,$$

where  $x' = (x_1, x_2)$ , and  $\chi_1$  and  $\chi_2$  are smooth real valued functions supported in the interior of  $I$  and  $[1, 2]$  respectively; we shall assume that  $\chi_2$  is nonnegative. The operator  $\mathcal{R}$  exhibits a partial translation invariance; i.e.  $\mathcal{R}f(x' + z', \alpha) = \mathcal{R}[f(\cdot + z', \cdot)](x', \alpha)$ . It serves as a model case for a more general class of restricted X-ray transforms considered in [3], [6], [7], [8], [4] and elsewhere. In the well-curved case (i.e.  $m = 2$ ) these operators are Fourier integral operators with one-sided fold singularities.

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We are interested in the  $L^p \rightarrow L^p_\sigma$  mapping properties of  $\mathcal{R}$  (here  $L^p_\sigma$  denotes the standard  $L^p$ -Sobolev space). Necessary and sufficient conditions for  $L^2$ -Sobolev inequalities of general restricted X-ray transforms in the well-curved case are contained in [7], [8], [4];  $\mathcal{R}$  maps  $L^2$  to  $L^2_{1/4}$  and this is best possible. For results on  $L^2$  boundedness in the finite type case see [11].

Concerning the  $L^p$  Sobolev regularity we first state some necessary conditions.

**Proposition 1.1.** *Suppose that the cutoff functions  $\chi_1$  and  $\chi_2$  are not identically zero and suppose that there is  $\alpha \in I$  with  $\chi_1(\alpha) \neq 0$  and  $\gamma$  is of type  $m$  at  $\alpha$ . Suppose that  $\mathcal{R}$  maps  $L^p$  boundedly to  $L^p_\sigma$ . Then*

$$(1.2) \quad \sigma \leq \min\left\{1 - \frac{1}{p}, \frac{1}{4}, \frac{1}{mp}\right\}.$$

The proposition will be proved in §2. We remark that the condition  $\sigma \leq \min\{1 - 1/p, 1/mp\}$  is essentially known; see [7] for an example in the well curved case (where  $\gamma'' \neq 0$ ) and [11] for the finite type case; a simpler argument is given in §2.2 below. The condition  $\sigma \leq 1/4$  seems to be not have been observed in the nonzero curvature case  $m = 2$  (it is redundant if  $m \geq 3$ ). The relevant example is related to one by Oberlin and Smith [12], for a family of Bessel multipliers in  $\mathbb{R}^2$  and for convolutions with arclength measure on a helix in  $\mathbb{R}^3$ ; it is also closely related to examples for a deep inequality on decompositions of cone multipliers due to Wolff.

We describe this result of Wolff [24] as it is crucial for deriving sufficient conditions for the  $L^p$ -Sobolev regularity of  $\mathcal{R}$ . One considers a collection  $\{\Psi_\nu\}$  of smooth functions supported in disjoint  $1 \times \delta^{1/2} \times \delta$ -plates that are tangential to the light cone  $\{\xi : \xi_3^2 = \xi_1^2 + \xi_2^2\}$  with the long side pointing in the radial direction; we assume that the  $\Psi_\nu$  satisfy the natural size estimates and differentiability properties (see §3.1 below). Let  $f_\nu$  be a family of tempered distributions. Wolff's theorem states that for sufficiently large  $q$  and for all  $\epsilon > 0$  there exists a finite  $C_{\epsilon,q}$  such that

$$(1.3) \quad \left\| \sum_\nu \widehat{\Psi}_\nu * f_\nu \right\|_q \leq C_{\epsilon,q} \delta^{-\frac{1}{2} + \frac{2}{q} - \epsilon} \left( \sum_\nu \|f_\nu\|_q^q \right)^{\frac{1}{q}}.$$

In [24] this inequality is established for all  $q \geq 74$ . In what follows we shall assume the validity of (1.3) for  $q \geq q_W$ . We show that the sharp  $L^p \rightarrow L^p_{1-1/p}$  inequality holds for  $p$  close to 1; the precise range is determined by the value of  $q_W$ .

**Theorem 1.2.** *Suppose that  $\gamma \in C^{m+3}(I)$  is of maximal type at most  $m$ , and suppose that  $1 < p < \min((q_W + 2)/q_W, (m + 1)/m)$ . Then  $\mathcal{R}$  maps  $L^p(\mathbb{R}^3)$  boundedly into  $L^p_{1-1/p}(\mathbb{R}^3)$ .*

This result is somewhat analogous to a recent result by the authors [16] on convolutions with measures supported on curves with nonvanishing curvature and torsion (see [13] for a prior partial result based on [1], [22]). A counterexample in [24] shows that necessarily  $q_W \geq 6$  (i.e. (1.3) does not hold for  $q < 6$ ). A proof of (1.3) for all  $q > 6$  would imply the sharp endpoint  $L^p \rightarrow L^p_{1-1/p}$  for all  $1 < p < \min\{4/3, (m + 1)/m\}$  and sharp results up to endpoints for larger  $p$ . In particular if  $m > q_W/2$  (i.e.  $m > 37$  according to [24]) then one obtains an almost complete result (except for endpoint bounds in the range  $(m + 1)/m \leq p < 2$ ).

## 2. NECESSARY CONDITIONS

We begin with two preliminary observations. Consider a multiplier  $m(\xi')$  depending only on  $\xi' = (\xi_1, \xi_2)$  and observe that  $\mathcal{R}$  commutes with the operator  $m(D')$ . Now if  $\mathcal{R}$  maps  $L^p$  to  $L^p_\sigma$  for some  $p \in (1, \infty)$  and if  $m_k$  is a standard symbol of order zero in  $\mathbb{R}^2$  which vanishes for  $|\xi'| \leq 2^k$ , then it follows

$$(2.1) \quad \|m_k(D')\mathcal{R}f\|_p \leq C_p 2^{-k\sigma} \|f\|_p.$$

Secondly, let  $\mathcal{Z}_k^\infty$  be the set of tempered distributions whose Fourier transform is supported in  $\{\xi : |\xi| \geq 2^k\}$  (thus distributions in  $\mathcal{Z}_k^\infty$  have cancellation). Let  $\Phi$  be smooth and supported in  $\{\xi : |\xi| \leq 2\}$  and let  $\Phi(\xi) = 1$  for  $|\xi| \leq 1$ . Define  $P_l$  by  $\widehat{P_l f}(\xi) = \Phi(2^{-l}\xi)\widehat{f}(\xi)$ . It is straightforward to see (using integration by parts arguments for generalized Fourier integrals with so called operator phase functions, see [10]) that there is the estimate

$$\|P_l \mathcal{R}f\|_p \leq C_N 2^{-Nk} \|f\|_p \quad \text{if } l < k - C_1, \quad f \in L^p \cap \mathcal{Z}_k^\infty,$$

and  $C_1$  is large. This implies that if  $\mathcal{R} : L^p \rightarrow L^p_\sigma$  is bounded then one also has for large  $k$

$$(2.2) \quad \|\mathcal{R}f\|_p \lesssim 2^{-k\sigma} \|f\|_p, \quad f \in \mathcal{Z}_k^\infty.$$

In what follows  $J$  will always denote a compact interval in  $(1, 2)$  so that  $\chi_2(s) > 0$  on  $J$ . We shall also choose a fixed  $a_0 \in I$  so that  $\chi_1(a_0) \neq 0$ .

### 2.1. $\sigma \leq 1 - 1/p$ .

Let  $\eta$  be an even Schwartz function in  $\mathbb{R}^2$  with  $\widehat{\eta}(0) = 1$  and with  $\eta$  supported in  $\{\xi' : 1/2 \leq |\xi'| \leq 2\}$ . Let  $c_0$  so that  $\widehat{\eta}(x') > 1/2$  for  $|x'| \leq c_0$ . Let  $m_k(\xi') = \eta(2^{-k}\xi')$ , then

$$(2.3) \quad \begin{aligned} & m_k(D')\mathcal{R}f(x', \alpha) \\ &= (2\pi)^{-2} \chi_1(\alpha) \int 2^{2k} \widehat{\eta}(2^k(x' - y' + y_3\gamma(\alpha))) f(y) \chi_2(y_3) dy. \end{aligned}$$

Now let  $s_0$  be in the interior of  $J$  so that  $\chi_2(s_0) > 0$  and choose  $k$  large so that  $\chi_2(s) \geq c' > 0$  for  $|s - s_0| \leq 2^{-k}$ . Let  $f_k$  be the characteristic function of a ball of radius  $\varepsilon 2^{-k}$  centered at  $(0, 0, s_0)$  and let  $\ell(\alpha)$  be the line segment  $\{-s\gamma(\alpha) : s \in J\}$ . For small  $\varepsilon$  let  $E_\alpha$  be the set of all  $x'$  for which  $\text{dist}(x'; \ell(\alpha)) \leq \varepsilon 2^{-k}$ . As  $\widehat{\eta}$  is positive near the origin we see that the integrand in (2.3) is  $\geq c 2^{2k}$  if  $x' \in E_\alpha$ . Thus the integral (2.3) is bounded below by  $2^{-k}$ . Now  $E_\alpha$  is of measure  $\approx 2^{-2k}$  and after integrating in  $\alpha$  we see that  $\|m_k(D')\mathcal{R}f_k\|_p \gtrsim 2^{-k} 2^{-2k/p}$ . Since  $\|f_k\|_p \lesssim 2^{-3k/p}$  we see that the  $L^p$  operator norm of  $\mathcal{R}$  is at least  $2^{-k(1-1/p)}$  and since  $\eta$  vanishes for  $|\tau| \lesssim 2^k$  inequality (2.1) shows that the  $L^p \rightarrow L^p_\sigma$  boundedness of  $\mathcal{R}$  implies  $\sigma \leq 1 - 1/p$ .

### 2.2. $\sigma \leq (mp)^{-1}$ .

Let us assume that  $\gamma$  has contact of order  $m$  with its tangent line at  $\alpha = a_0$ . Let  $\zeta_1$  be an even Schwartz function in  $\mathbb{R}$  so that  $\zeta_1$  is supported in  $\{\beta : 1/2 \leq |\beta| \leq 2\}$  and with the property that  $\widehat{\zeta}_1(u) \geq 1/2$ ,  $|u| \leq c_0$ . Let  $\zeta_0$  be a Schwartz function in  $\mathbb{R}$  for which  $\widehat{\zeta}_0$  is nonnegative everywhere and positive in  $[-1/2, 1/2]$ . Let  $\eta_k$  be defined by

$$\eta_k(\tau) = \zeta_0(2^{-k}\langle \tau, \gamma'(a_0) \rangle) \zeta_1(2^{-k}\langle \tau, \mathbf{n}(a_0) \rangle)$$

where  $\mathbf{n} = (-\gamma'_2, \gamma'_1)$ . The function  $\eta_k$  vanishes for  $|\xi| \leq c 2^k$  and by (2.1) it suffices to prove that the  $L^p$  operator norm of  $\eta_k(D')\mathcal{R}$  is  $\gtrsim 2^{-k/(mp)}$ .

Now let  $g_k$  be the characteristic function of the set defined by

$$|\langle y' - y_3\gamma(a_0), \gamma'(a_0) \rangle| \leq 2^{-k/m}, |\langle y' - y_3\gamma(a_0), \mathbf{n}(a_0) \rangle| \leq 2^{-k}, y_3 \in J.$$

We evaluate  $\eta_k(D')\mathcal{R}g_k(x', \alpha)$  on the set  $P_k$  defined by

$$|\langle x', \mathbf{n}(a_0) \rangle| \leq c\varepsilon 2^{-k}, |\langle x', \gamma'(a_0) \rangle| \leq c\varepsilon 2^{-k/m}, |\alpha - a_0| \leq c\varepsilon 2^{-k/m}.$$

Notice that if  $y \in \text{supp}(g_k)$  and  $(x', \alpha) \in P_k$  then

$$|\langle x' - y' + \gamma(\alpha)y_3, \mathbf{n}(a_0) \rangle| \geq c'2^{-k}$$

since  $\langle \gamma(\alpha) - \gamma(a_0), \mathbf{n}'(a_0) \rangle = O(2^{-k})$  by the contact of order  $m$  assumption. Thus  $\zeta_1(2^k \langle x' - y' + \gamma(\alpha)y_3, \mathbf{n}(a_0) \rangle) \geq c'$  if  $y \in \text{supp}(g_k)$  and  $(x', \alpha) \in P_k$ . Because of the positivity assumption on  $\zeta_0$  we see that for  $(x', \alpha) \in P_k$  and for fixed  $y_3, \alpha$  the integral

$$\int \widehat{\eta}_k(x' - y' - \gamma(\alpha)y_3) g_k(y', y_3) dy'$$

is nonnegative, and if  $y_3 \in J$  it is bounded below by a positive constant. Thus  $|\eta_k(D')\mathcal{R}g_k(x', \alpha)| \geq c_1$  on  $P_k$  and therefore  $\|\eta_k(D')\mathcal{R}g_k\|_p \geq c_2 2^{-k(m+2)/(mp)}$ . Since  $\|g_k\|_p \lesssim 2^{-k(m+1)/mp}$  we deduce our necessary condition  $\sigma \leq (mp)^{-1}$ .

### 2.3. $\sigma \leq 1/4$ .

By a change of variable we may assume that  $\gamma$  is parametrized by arclength. We pick a closed interval  $I_0$  in the support of  $\chi_1$  so that the curvature is bounded below in  $I_0$ , *i.e.*

$$(2.4) \quad |\kappa(\alpha)| = |\gamma_1''(\alpha)\gamma_2'(\alpha) - \gamma_1'(\alpha)\gamma_2''(\alpha)| \geq c > 0.$$

for  $\alpha \in I_0$ . Suppose that  $|\gamma(\alpha)| \leq B$  for all  $\alpha$ . Let  $\rho$  be an even  $C^\infty$  function with the property that  $\rho(x) \geq 1$  for  $|x| \leq 4B$  and so that  $\widehat{\rho}$  is compactly supported.

We fix a positive integer  $n$  which will be chosen large (depending on the geometry). Let  $k \gg 2n$ . Let  $\{\alpha_\nu\}$  be a maximal set of points in  $I_0$  which have mutual distance  $2^{n-k/2}$ . Define

$$(2.5) \quad \xi_\nu = (\gamma_2'(\alpha_\nu), -\gamma_1'(\alpha_\nu), \gamma_1'(\alpha_\nu)\gamma_2(\alpha_\nu) - \gamma_1(\alpha_\nu)\gamma_2'(\alpha_\nu))$$

and define

$$f_{k,\nu}(x) = \rho(x) e^{i2^k \langle \xi_\nu, x \rangle}$$

so that  $\widehat{f_{k,\nu}}(\xi) = \widehat{\rho}(\xi - 2^k \xi_\nu)$ . Let  $\{r_\nu\}$  be the sequence of Rademacher functions and define for  $\omega \in [0, 1]$

$$f_k^\omega(x) = \sum_\nu r_\nu(\omega) f_{k,\nu}(x);$$

then by the usual  $L^p$  inequalities for the Rademacher functions [20]

$$(2.6) \quad \left( \int_0^1 \|f_k^\omega\|_p^p d\omega \right)^{1/p} \approx \left\| \left( \sum_\nu |f_{k,\nu}|^2 \right)^{1/2} \right\|_p \approx 2^{k/4-n/2} \|\rho\|_p.$$

Observe that

$$(2.7) \quad \mathcal{R}f_{k,\mu}(x', \alpha) = \chi_1(\alpha) \int \rho(x' + s\gamma(\alpha), s) e^{i2^k \langle \xi'_\mu, x' + s\gamma(\alpha) \rangle + s\xi_{\mu 3}} \chi_2(s) ds.$$

Now define  $v_\nu^{(1)} = (\gamma(\alpha_\nu), 1)$ ,  $v_\nu^{(2)} = (\gamma'(\alpha_\nu), 0)$  and check that both  $\langle \xi_\nu, v_\nu^{(1)} \rangle = 0$ ,  $\langle \xi_\nu, v_\nu^{(2)} \rangle = 0$ . Moreover  $\langle \xi'_\nu, \gamma''(\alpha_\nu) \rangle = \kappa(\alpha_\nu)$  and thus

$$(2.8) \quad \langle \xi'_\nu, \gamma(\alpha) \rangle + \xi_{\nu 3} = \frac{\kappa}{2}(\alpha - \alpha_\nu)^2 + O(\alpha - \alpha_\nu)^3.$$

Observe that if  $|x'| \leq 1$  and  $s \in J$  then  $|x' + s\gamma(\alpha)| \leq 4B$ ; moreover, by (2.8), we have  $|e^{i(\langle \xi'_\nu, \gamma(\alpha) \rangle + \xi_{\nu 3})} - 1| \leq 1/2$  if  $|\alpha - \alpha_\nu| \leq c2^{-k/2}$  and  $c$  is small. By (2.7),

$$\operatorname{Re}(\mathcal{R}f_{k,\nu}(x', \alpha)) \geq c_1 \quad \text{if } |\alpha - \alpha_\nu| \leq c2^{-k/2}, |x'| \leq 1,$$

and consequently

$$(2.9) \quad \left( \sum_\nu \int_{|\alpha - \alpha_\nu| \leq 2^{-k/2}} \int_{|x'| \leq 1} |\mathcal{R}f_{k,\nu}(x', \alpha)|^p d\alpha dx' \right)^{1/p} \geq c_2 2^{-n/p}.$$

Now we find an upper bound for  $|\mathcal{R}f_{k,\mu}(x', \alpha)|$  when  $|\alpha - \alpha_\nu| \lesssim 2^{-k/2}$  and  $\mu \neq \nu$ . We use (2.7), (2.8) and apply integration by parts to see that

$$|\mathcal{R}f_{k,\mu}(x', \alpha)| \leq C_N (2^k |\alpha_\nu - \alpha_\mu|^2)^{-N}, \quad \text{if } |\alpha - \alpha_\nu| \leq c2^{-k/2}, \mu \neq \nu.$$

Thus by the separation property of the  $\alpha_\mu$

$$(2.10) \quad \left( \sum_\nu \int_{|\alpha - \alpha_\nu| \leq 2^{-k/2}} \int_{|x'| \leq 1} \left[ \sum_{\mu \neq \nu} |\mathcal{R}f_{k,\mu}(x', \alpha)| \right]^p d\alpha dx' \right)^{1/p} \leq C_N 2^{-nN}.$$

If  $n$  is chosen sufficiently large a combination of (2.9) and (2.10) yields that  $\|\mathcal{R}f_k^\omega\|_p \geq c(p) > 0$  uniformly in  $\omega$ . Since  $\widehat{\rho}$  has compact support

and  $|\xi_\nu| \geq 1$  the Fourier transforms of the functions  $f_k^\omega$  are supported in  $\{\xi : |\xi| \geq c_3 2^k\}$  if  $k$  is sufficiently large. Using (2.6) we see that for large  $k$

$$\sup\{\|\mathcal{R}f\|_p : \|f\|_p \leq 1, f \in \mathcal{Z}_k^\infty \cap L^p\} \gtrsim 2^{-k/4}$$

and thus by the consideration leading to (2.2) the operator  $\mathcal{R}$  does not map  $L^p$  to  $L_\sigma^p$  if  $\sigma > 1/4$ .

### 3. $L^p$ REGULARITY

**3.1. Preliminaries.** We begin by describing an extension of Wolff's inequality proved in [16]. Let  $\alpha \mapsto g(\alpha) = (g_1(\alpha), g_2(\alpha)) \in \mathbb{R}^2$  be a  $C^3$  curve on the plane defined on a closed subinterval  $I$  of  $[-1, 1]$ . We assume that for positive constants  $b_0, b_1, b_2$ ,

$$(3.1) \quad \|g\|_{C^3(I)} \leq b_0, \quad |g'(\alpha)| \geq b_1, \quad |g'_1(\alpha)g''_2(\alpha) - g'_2(\alpha)g''_1(\alpha)| \geq b_2.$$

Given  $\alpha \in I$ , we define three vectors

$$(3.2) \quad u_1(\alpha) = (g(\alpha), 1), \quad u_2(\alpha) = (g'(\alpha), 0), \quad u_3(\alpha) = u_1(\alpha) \times u_2(\alpha),$$

so that a basis of the tangent space of the cone  $\mathcal{C}_g = \{rg(\alpha)\}$  is given by  $\{u_1(\alpha), u_2(\alpha)\}$ . Then for given  $\lambda > 0$  and  $0 < \delta \ll 1$ , the  $(\delta, \lambda)$ -plate at  $\alpha$ , denoted by  $P_{\delta, \lambda}^\alpha$  is defined to be the parallelepiped

$$P_{\delta, \lambda}^\alpha = \{\xi : \lambda/2 \leq |\langle u_1(\alpha), \xi \rangle| \leq 2\lambda, |\langle u_2(\alpha), \xi - \xi_3 u_1(\alpha) \rangle| \leq \lambda \delta^{1/2}, \\ |\langle u_3(\alpha), \xi \rangle| \leq \lambda \delta\}.$$

Note that  $P_{\delta, \lambda}^\alpha$  has dimension  $\approx \lambda$  in the radial direction tangent to the cone  $\mathcal{C}_g$ , dimension  $\approx \lambda \delta^{1/2}$  in the tangential direction perpendicular to the radial direction, and is supported in a neighborhood of width  $\approx \lambda \delta$  of the cone. An  $A$ -extension of the plate  $P_{\delta, \lambda}^\alpha$  is a parallelepiped that is localized between heights  $\xi_3 = \lambda/(2A)$  and  $\xi_3 = 2A\lambda$  of  $\mathcal{C}_g$ , and whose widths along  $(g'(\alpha), -\langle u_1(\alpha), u_2(\alpha) \rangle)$  and  $u_3(\alpha)$  are  $A\lambda \delta^{1/2}$  and  $A\lambda \delta$  respectively. For  $\theta$  and  $\sigma$  with  $\sigma \leq \delta^{1/2} \leq \theta$ , a  $(\delta, \lambda, \theta)$ -plate family associated to  $g$  is a finite collection of  $(\delta, \lambda)$ -plates  $\mathcal{P} = \{P_{\delta, \lambda}^{\alpha_\nu}\}_{\nu=1}^N$  satisfying (i)  $|\alpha_\nu - \alpha_{\nu'}| \geq \delta^{1/2}$  and (ii)  $\max_\nu \{\alpha_\nu\} - \min_\nu \{\alpha_\nu\} \leq \theta$ . An *admissible bump function* associated to  $P_{\delta, \lambda}^\alpha$  is a  $C^\infty$  function  $\phi$

supported in  $P_{\delta,\lambda}^\alpha$  satisfying the estimates

$$(3.3) \quad \begin{aligned} |\langle u_1(\alpha), \nabla \rangle^{n_1} \langle u_2(\alpha), \nabla \rangle^{n_2} \langle u_3(\alpha), \nabla \rangle^{n_3} \phi(\xi)| &\leq \lambda^{-n_1-n_2-n_3} \delta^{-n_2/2-n_3}, \\ 0 &\leq n_1 + n_2 + n_3 \leq 4. \end{aligned}$$

The Wolff inequality in this general context says that

$$(3.4) \quad \left\| \sum_{P \in \mathcal{P}} \mathcal{F}^{-1}[\phi_P \widehat{f_P}] \right\|_q \leq C(\epsilon) \delta^{\frac{2}{q} - \frac{1}{2} - \epsilon} \left( \sum_{P \in \mathcal{P}} \|f_P\|_q^q \right)^{1/q},$$

where  $\{\phi_P\}$  is a collection of admissible bump function associated to the plates in  $\mathcal{P}$ . Wolff [24] proved this for the light cone, *i.e.*  $g(\alpha) = (\cos \alpha, \sin \alpha)$  (when  $q > 74$ ) but the authors showed in §2 of [16] that if (3.4) holds for the light-cone and some  $q$  then it holds for the same  $q$  for every curved cone generated by  $g$  as in (3.1) (with a different constant  $\tilde{A}(\epsilon)$ ). The proof involves various rescaling and an induction on scales argument.

We now describe the structure of the wavefront set of the Schwartz kernel of the operator  $\mathcal{R}$ . We shall assume that our curve  $\gamma$  is parametrized by arclength. Moreover we deal with the case  $m = 2$  of Theorem 1.2 and assume the lower bound (2.4) for the curvature everywhere in  $\text{supp}(\chi_1)$ . It will be convenient to work with the adjoint operator  $\mathcal{R}^*$ , and we write out the convolution kernel by expanding a Dirac measure in two dimensions by a Fourier integral; thus

$$(3.5) \quad \begin{aligned} \mathcal{R}^* f(x) &= \chi_2(x_3) \int f(x' - x_3 \gamma(y_3), y_3) \chi_1(y_3) dy_3 \\ &= \chi_2(x_3) \int e^{i\varphi(x,y,\tau)} \chi_1(y_3) f(y) d\tau dy, \end{aligned}$$

where  $x = (x', x_3)$ ,  $y = (y', y_3)$ ,  $\tau = (\tau_1, \tau_2)$ , and

$$(3.6) \quad \varphi(x, y, \tau) = \sum_{i=1}^2 \tau_i (y_i - x_i + x_3 \gamma(y_3)).$$

The theorem will be proved if we can show that  $\mathcal{R}^*$  maps  $L_{-1/q}^q$  to  $L^q$  (or more generally  $L_{\beta-1/q}^q$  to  $L_\beta^q$  for all  $\beta$ ), for  $q > (q_W + 2)/2$ .

We record a few standard facts about  $\mathcal{R}^*$  that will be used in the analysis. We denote  $(x, \xi)$ -space by  $T^*\mathbb{R}_L^3$  and  $(y, \eta)$ -space by  $T^*\mathbb{R}_R^3$  and



the canonical relation associated to  $\mathcal{R}^*$  is given by

$$\begin{aligned} \mathfrak{C} &= \{(x, \varphi_x, y, -\varphi_y) : \varphi_\tau = 0\} \\ &= \{(x, \xi, y, -\eta) : \xi' = -\tau, \xi_3 = \langle \tau, \gamma(y_3) \rangle, y' = x' - x_3 \gamma'(y_3), \\ &\quad \eta' = \tau, \eta_3 = x_3 \langle \tau, \gamma'(y_3) \rangle\}. \end{aligned}$$

Let  $\pi_L, \pi_R$  be the projections of  $\mathfrak{C}$  to  $T^*\mathbb{R}_L^3$  and  $T^*\mathbb{R}_R^3$ , respectively. The structure of the projections  $\pi_L, \pi_R$  for more general X-ray transform satisfying a version of the Gelfand admissibility condition has been investigated in [6], [7], [9] (see also the survey [15]), namely  $\pi_L$  is a fold and  $\pi_R$  is a blowdown. In particular the  $L_a^2 \rightarrow L_{a+1/4}^2$  estimates are shown in these references (and in our model case this result is rather straightforward as  $\mathcal{R}\mathcal{R}^*$  is a convolution operator).

Let  $\mathfrak{C}^{\text{deg}}$  be the variety where  $\det \pi_L = 0$  (equivalently  $\det d\pi_R = 0$ ); then  $\mathfrak{C}^{\text{deg}}$  is a conic submanifold of  $\mathfrak{C}$  and the restriction of  $\pi_L$  to  $\mathfrak{C}^{\text{deg}}$  is locally a diffeomorphism onto a conic hypersurface of  $T^*\mathbb{R}_L^3$ . Moreover the sets  $\Sigma_x = \{\xi : (x, \xi) \in \pi_L \mathfrak{C}^{\text{deg}}\}$  are smooth two-dimensional cones in each fiber. In our special case the condition  $\det d\pi_{L/R} = 0$  reduces to

$$(3.7) \quad \langle \tau, \gamma'(y_3) \rangle = 0$$

and the cones  $\Sigma_x$  are given by

$$\Sigma = \{\xi \in \mathbb{R}^3 : \xi = \lambda(\gamma'_2, -\gamma'_1, -\gamma_1 \gamma'_2 + \gamma'_1 \gamma_2), \lambda \in \mathbb{R}\}.$$

Recall that for our example in §2.3 the points  $\xi_\nu$  from (2.5) were chosen to lie on  $\Sigma$ .

A simple computation shows that the cone  $\Sigma$  has one principal non-vanishing curvature. Indeed after suitable localization and rotation we may assume  $|\gamma'_1(y_3)| > 1/2$  for all  $y_3 \in I$ . Then

$$(3.8) \quad g(\alpha) = \left( -\frac{\gamma'_2(\alpha)}{\gamma'_1(\alpha)}, \frac{\gamma_1(\alpha)\gamma'_2(\alpha)}{\gamma'_1(\alpha)} - \gamma_2(\alpha) \right)$$

parametrizes the curve that is the cross-section  $\xi_2 = 1$  of  $\Sigma$ , and the curvature property of  $\Sigma$  can be expressed in terms of the curvature of  $g$ . A computation shows that

$$\det \begin{pmatrix} g'_1(\alpha) & g'_2(\alpha) \\ g''_1(\alpha) & g''_2(\alpha) \end{pmatrix} = -\frac{\kappa^2(\alpha)}{(\gamma'_1(\alpha))^3} \neq 0.$$

Thus  $g$  satisfies all the conditions of (3.1), with  $b_0$ ,  $b_1$  and  $b_2$  depending only on  $\|\gamma\|_{C^2}$  and lower bounds of  $|\kappa|$ . We also observe that  $\Sigma$  is the cone dual to  $\mathcal{C}_\gamma$ ; indeed a normal vector to  $\Sigma$  at  $r(\gamma'_2, -\gamma'_1, \gamma'_1\gamma'_2 - \gamma_1\gamma'_2) = (\gamma', 0) \times (\gamma, 1)$  is given by  $(\gamma(\alpha), 1)$ .

**3.2. Dyadic estimates.** We first decompose the oscillatory integral (3.5) dyadically in  $\tau$  and then we introduce a further decomposition in terms of the size of  $|\det \pi_L| \approx |\langle \tau, \gamma'(y_3) \rangle|$ . In what follows  $m_k(\tau)$  will be a standard multiplier symbol of order 0 supported where  $|\tau| \approx 2^k$ , and

$$R_k f(x) = \chi_2(x_3) \int e^{i\varphi(x,y,\tau)} \chi_1(y_3) m_k(\tau) d\tau f(y) dy,$$

where  $\varphi$  is as in (3.6).

We shall prove here (under the assumption (2.4)) that

$$(3.9) \quad \|R_k f\|_q \leq C_q 2^{-k/q} \|f\|_q, \quad q > (q_W + 2)/2,$$

which implies an estimate for  $\mathcal{R}^*$  on Besov spaces. The Sobolev estimates will be briefly discussed in §3.3.

To describe our further decomposition let  $\eta_0 \in C_0^\infty(\mathbb{R})$  be an even function so that  $\eta_0(s) = 1$  if  $|s| \leq 1/2$  and  $\text{supp}(\eta_0) \subset (-1, 1)$ , and let  $\eta_1(s) = \eta_0(s/2) - \eta_0(s)$ . Define

$$(3.10) \quad a_{k,l}(y_3, \tau) = m_k(\tau) \eta_1(2^{l-k} \langle \gamma'(y_3), \tau \rangle)$$

and

$$b_k(y_3, \tau) = m_k(\tau) \left(1 - \sum_{l < k/2} \eta_1(2^{l-k} \langle \gamma'(y_3), \tau \rangle)\right).$$

Let

$$R_{k,l} f(x) = \chi_2(x_3) \int e^{i\varphi(x,y,\tau)} \chi_1(y_3) a_{k,l}(\tau) d\tau f(y) dy,$$

and define  $\tilde{R}_k$  similarly, with  $a_{k,l}$  replaced by  $b_k$ .

**Proposition 3.1.** *For  $q_W < q < \infty$ ,*

$$(3.11) \quad \|R_{k,l} f\|_q \leq C_\epsilon 2^{-k/q} 2^{-l/q + l\epsilon} \|f\|_q, \quad l < k/2,$$

$$(3.12) \quad \|\tilde{R}_k f\|_q \leq C_\epsilon 2^{-3k/(2q) + k\epsilon} \|f\|_q.$$

*The constants  $C_\epsilon$  depend only on  $\epsilon$ ,  $\|\gamma\|_{C^2}$  and the lower bound in (2.4).*

*Proof.* We give the proof of (3.11). The proof of (3.12) is similar, with mainly notational changes. We note that  $R_{k,l} = 0$  for  $l < -C$  and that the asserted bound for small  $l$  follows from standard estimates for generalized Radon transforms or Fourier integral operators ([10]).

By a calculation with Fourier transforms we get

$$(3.13) \quad \mathcal{F}_{\mathbb{R}^3} [R_{k,l}f](\xi', \xi_3) = \int \widehat{\chi}_2(\xi_3 + \langle \gamma(y_3), \xi' \rangle) a_{k,l}(y_3, \xi') \mathcal{F}_{\mathbb{R}^2} f(-\xi', y_3) dy_3$$

In view of the fast decay of  $\widehat{\chi}_2$  we further split  $\chi_2$  in a low and a high frequency part. Define  $\vartheta_{k,l}$  by

$$\widehat{\vartheta_{k,l}}(\beta) = \eta_0(2^{-k+2l(1-\epsilon)}\beta)\widehat{\chi}_2(\beta)$$

and split

$$(3.14) \quad R_{k,l} = T_{k,l} + E_{k,l}$$

where  $T_{k,l}$  is similarly defined as  $R_{k,l}$  but with  $\chi_2(x_3)$  replaced by  $\vartheta_{k,l}(x_3)$ .

The error term  $E_{k,l}$  is easily handled; we claim that given any  $q \geq 2$  and  $N \geq 1$ , there exists  $C_{q,N} > 0$  such that

$$(3.15) \quad \|E_{k,l}f\|_q \leq C_{q,N} 2^{-(k-2l+2l\epsilon)N/q} \|f\|_q.$$

Indeed by an integration by parts it is easy to see that  $E_{k,l}$  is bounded on  $L^\infty$ . Thus, by interpolation it suffices to consider the case  $q = 2$ . We use the formula analogous to (3.13), with  $\chi_2 - \vartheta_{k,l}$  in place of  $\chi_2$ , and the fact that the Fourier transform of  $\chi_2 - \vartheta_{k,l}$  vanishes for  $|\beta| > c2^{k-2l+2l\epsilon}$ . Consequently, by the Cauchy-Schwarz inequality and Plancherel's theorem

$$\begin{aligned} \|E_{k,l}f\|_2^2 &\leq C_N \int \int_{\substack{|\xi_3 + \langle \gamma(y_3), \xi' \rangle| \\ \gtrsim 2^{k-2l+2l\epsilon}}} \frac{|\mathcal{F}_{\mathbb{R}^2} f(-\xi', y_3)|^2}{|\xi_3 + \langle \gamma(y_3), \xi' \rangle|^{2N}} d\xi dy_3 \\ &\leq C_N 2^{-l} 2^{-(k-2l+2l\epsilon)(2N-1)} \|f\|_2^2 \end{aligned}$$

which is (3.15)

In order to estimate the main term  $T_{k,l}$  in (3.14) we further decompose  $a_{k,l}$  into pieces supported on  $2^{-l}$  subintervals of  $I$ . Let  $\zeta \in C_0^\infty$  be supported in  $(-1, 1)$  so that  $\sum_{\nu \in \mathbb{Z}} \zeta(\cdot - \nu) \equiv 1$ . We set

$$a_{k,l,\nu}(y_3, \tau) = \zeta(2^l y_3 - \nu) a_{k,l}(y_3, \tau)$$

so that  $a_{k,l} \equiv \sum_{\nu} a_{k,l,\nu}$ , and let

$$T_{k,l,\nu} f(x) = \vartheta_{k,l}(x_3) \int e^{i\varphi(x,y,\tau)} \chi_1(y_3) a_{k,l,\nu}(\tau) d\tau f(y) dy,$$

Notice the natural estimates for the derivatives of  $a_{k,l,\nu}$ , namely if  $|\alpha - 2^{-l}\nu| \leq 2^{-l}$  and if  $\mathbf{n}(\alpha) = (-\gamma'_2(\alpha), \gamma'_1(\alpha))$  then

$$(3.16) \quad |\langle \gamma'(\alpha), \nabla_{\tau} \rangle^{n_1} \langle \mathbf{n}(\alpha), \nabla_{\tau} \rangle^{n_2} a_{k,l,\nu}(y_3, \tau)| \leq C_{n_1, n_2} 2^{(l-k)n_1} 2^{-kn_2}.$$

Moreover, the Fourier transforms of the functions  $T_{k,l,\nu} f$  have support properties which are favorable for the application of Wolff's inequality. Let  $\alpha_{\nu} = 2^{-l}\nu$ , and let  $\{u_{\nu}^{(1)}, u_{\nu}^{(2)}, u_{\nu}^{(3)}\}$  be an orthonormal basis where  $u_{\nu}^{(3)}$  is parallel to  $(\gamma(\alpha_{\nu}), 1)$ ,  $u_{\nu}^{(1)}$  is orthogonal to both  $(\gamma(\alpha_{\nu}), 1)$  and  $(\gamma'(\alpha_{\nu}), 0)$ . Thus if  $\xi = r(\gamma'_2, -\gamma'_1, \gamma'_1\gamma_2 - \gamma_1\gamma'_2)(\alpha_{\nu})$  then  $u_{\nu}^{(3)}$  is normal to  $\Sigma$  at  $\xi$  and  $u_{\nu}^{(1)}, u_{\nu}^{(2)}$  are tangent vectors. One now verifies that the Fourier transform of  $T_{k,l,\nu} f$  is supported in a set where

$$(3.17a) \quad C^{-1} \leq |\langle \frac{\xi}{|\xi|}, u_{\nu}^{(1)} \rangle| \leq C,$$

$$(3.17b) \quad |\langle \frac{\xi}{|\xi|}, u_{\nu}^{(2)} \rangle| \leq C2^{-l},$$

$$(3.17c) \quad |\langle \frac{\xi}{|\xi|}, u_{\nu}^{(3)} \rangle| \leq C2^{-2l+2l\epsilon}.$$

Indeed we have

$$(3.18) \quad \begin{aligned} & \mathcal{F}_{\mathbb{R}^3} [T_{k,l,\nu} f](\xi', \xi_3) \\ &= \int \widehat{\vartheta_{k,l}}(\xi_3 + \langle \gamma(y_3), \xi' \rangle) a_{k,l,\nu}(y_3, \xi') \mathcal{F}_{\mathbb{R}^2} f(-\xi', y_3) dy_3. \end{aligned}$$

To see the assertion on the support we first note that (3.17a) follows since our symbols are supported near  $\Sigma$  and away from the origin. Next  $|\langle \xi, u_{\nu}^{(3)} \rangle| \approx |\xi_3 + \langle \gamma(\alpha_{\nu}), \xi' \rangle|$  so that (3.17c) is an immediate consequence of the support property of  $\widehat{\vartheta_{k,l}}$  and the fact that  $\alpha_{\nu} \in \text{supp } a_{k,l,\nu}(\cdot, \xi')$ . To show (3.17b) let  $\mathbf{t}_{\nu} = (\gamma'(\alpha_{\nu}), 0)$  and observe that  $u_{\nu}^{(2)}$  belongs to the span of  $\mathbf{t}_{\nu}$  and  $u_{\nu}^{(3)}$ . By the definition of  $a_{k,l,\nu}$  we have  $\langle \mathbf{t}_{\nu}, \xi \rangle = O(2^{k-l})$  and this together with (3.17c) implies (3.17b).

By (3.17), the Fourier transforms of  $\widehat{T_{k,l,\nu} f}$  are supported in  $C$ -extensions of plates  $P_{\nu}$  which form a family of  $(2^k, 2^{-2l+2l\epsilon})$  plates generated by  $g$  as in (3.8). After a straightforward reduction we may apply our

variant of Wolff's inequality (3.4) and get for  $q > q_W$

$$(3.19) \quad \left\| \sum_{\nu} T_{k,l,\nu} f \right\|_q \leq C_{\epsilon} 2^{2l\left(\frac{1}{2} - \frac{2}{q} + \epsilon\right)} \left( \sum_{\nu} \|T_{k,l,\nu} f\|_q^q \right)^{1/q}.$$

In order to finish the proof we need to prove that

$$(3.20) \quad \|T_{k,l,\nu}\|_{L^2 \rightarrow L^2} \leq C 2^{(l-k)/2},$$

$$(3.21) \quad \|T_{k,l,\nu}\|_{L^\infty \rightarrow L^\infty} \leq C 2^{-l}.$$

These inequalities imply the stronger bound

$$(3.22) \quad \left( \sum_{\nu} \|T_{k,l,\nu} f\|_q^q \right)^{1/q} \lesssim 2^{-l(1-3/q)} 2^{-k/q} \|f\|_q.$$

For  $q = 2$  (3.22) follows from (3.20) and the almost disjointness of the supports of  $y_3 \mapsto a_{k,l,\nu}(\tau, y_3)$ . For  $2 \leq q < \infty$  it follows by interpolation with the  $\ell^\infty(L^\infty)$  bound from (3.21). The asserted estimate for  $R_{k,l}$  follows then by combining (3.19) and (3.22).

We conclude by proving (3.20) and (3.21). To see (3.20) we recall that  $|\langle \gamma'(y_3), \tau \rangle| \approx 2^{k-l}$  on the support of  $a_{k,l,\nu}$  and write  $a_{k,l,\nu} = a_{k,l,\nu}^+ + a_{k,l,\nu}^-$ , where  $a_{k,l,\nu}^\pm$  further localize to the regions where  $\langle \gamma'(y_3), \tau \rangle$  is positive and negative, respectively. Consequently we get a decomposition  $T_{k,l,\nu} = T_{k,l,\nu}^+ + T_{k,l,\nu}^-$ . We only work with  $\mathcal{T} \equiv T_{k,l,\nu}^+$ , the other case being similar. Let

$$\begin{aligned} K(x, \tau, y_3) &= \vartheta_{k,l}(x_3) \int e^{i\langle x-z, \xi \rangle + \langle \tau, -z' + z_3 \gamma(y_3) \rangle} a_{k,l,\nu}(y_3, \xi') dz d\xi, \\ &= e^{-i\langle \tau, x' - x_3 \gamma(y_3) \rangle} \vartheta_{k,l}(x_3) a_{k,l,\nu}(y_3, -\tau); \end{aligned}$$

then  $\mathcal{T}f(x) = \int K(x, y_3, \tau) \mathcal{F}_{\mathbb{R}^2} f(\tau, y_3) d\tau dy_3$  and by Schur's lemma and Plancherel's theorem

$$\begin{aligned} \|\mathcal{T}\|_{L^2 \rightarrow L^2}^2 &\leq \sup_{\tau, y_3} \iint \left| \int K(x, \tau, y_3) \overline{K(x, \zeta', z_3)} dx \right| d\zeta' dz_3 \\ &\lesssim \|a_{k,l,\nu}^+\|_\infty \int \left| \int e^{ix_3 \langle \gamma(y_3) - \gamma(z_3), \tau \rangle} |\vartheta_{k,l}(x_3)|^2 \overline{a_{k,l,\nu}^+(z_3, -\tau)} dx_3 \right| dz_3 \end{aligned}$$

which is controlled by

$$(3.23) \quad \begin{aligned} & \sup_{\tau} \int \frac{|a_{k,l,\nu}^+(z_3, -\tau)|}{(1 + |\langle \gamma(y_3) - \gamma(z_3), \tau \rangle|)^2} dz_3 \\ & \leq C \sup_{y_3} \int \frac{dz_3}{(1 + 2^{k-l}|y_3 - z_3|)^2} \lesssim 2^{l-k}. \end{aligned}$$

Here we integrated by parts twice in  $x_3$  and for the last estimate we used  $|\langle \gamma(y_3) - \gamma(z_3), \tau \rangle| = |y_3 - z_3| |\langle \gamma'(w), \tau \rangle| \approx 2^{k-l}|y_3 - z_3|$ ; the point  $w$  lies between  $y_3$  and  $z_3$ , and since we work with  $a_{k,l,\nu}^+$  the quantity  $\langle \gamma'(\cdot), \tau \rangle$  does not change sign in  $[y_3, z_3]$ . (3.23) yields the  $L^2$  bound (3.20).

For the  $L^\infty$  estimate, we integrate by parts in  $\tau$  using the directional derivatives  $\langle \gamma'(\alpha_\nu), \nabla_\tau \rangle$  and  $\langle \mathbf{n}(\alpha_\nu), \nabla_\tau \rangle$  and the symbol estimates (3.16). This gives

$$\begin{aligned} \|\mathcal{T}f\|_\infty & \leq \sup_x \left| \int \left[ \int e^{i\langle \tau, y' - x' + x_3 \gamma(y_3) \rangle} \vartheta_{k,l}(x_3) a(y_3, -\tau) d\tau \right] f(y) dy \right| \\ & \lesssim \|f\|_\infty \sup_x \int_{\text{supp}(a)} (1 + 2^k |\langle \mathbf{n}(\alpha_\nu), y' - x' + x_3 \gamma(y_3) \rangle|)^{-2} \times \\ & \quad (1 + 2^{k-l} |\langle \gamma'(\alpha_\nu), y' - x' + x_3 \gamma(y_3) \rangle|)^{-2} d\tau dy \end{aligned}$$

and we integrate first in  $y'$  and then in  $y_3, \tau$  over the set where

$$|\langle \gamma'(\alpha_\nu), \tau \rangle| \lesssim 2^{k-l}, \quad |\langle \mathbf{n}(\alpha_\nu), \tau \rangle| \lesssim 2^k, \quad |y_3 - \alpha_\nu| \lesssim 2^{-l}.$$

The result is the asserted bound  $\|\mathcal{T}f\|_\infty \lesssim 2^{-l} \|f\|_\infty$ . This concludes the proof of the proposition.  $\square$

The proposition, and a further interpolation yield

**Corollary 3.2.** *For  $q > (q_W + 2)/2$ , there is  $\epsilon_0 = \epsilon_0(q) > 0$  such that*

$$(3.24) \quad \|R_{k,l}f\|_q \leq C_q 2^{-k/q} 2^{-\epsilon_0 l/q} \|f\|_q.$$

*Proof.* By the almost disjointness of the plate families and the  $L^2$  bounds in (3.20) we see that the  $L^2$  operator norm of  $T_{k,l}$  is  $O(2^{(l-k)/2})$ , and by (3.15) this also holds for  $R_{k,l}$ . Interpolating this with the  $L^q$  bounds of Proposition 3.1 yields the assertion.  $\square$

**3.3. Bounds in Sobolev spaces.** We now complete the proof of Theorem 1.2 in the nonvanishing curvature case; here we still have to put the estimates (3.24) for different  $k$  together. We wish to show that  $\mathcal{R}^*$  maps  $L_\beta^q$  to  $L_{\beta+1/q}^q$  for  $q > (q_W + 2)/2$ , and by duality (with  $\beta = -1/q$ ) this will imply the asserted  $L^p \rightarrow L_{1-1/p}^p$  for  $\mathcal{R}$ . By a vector-valued version of the Fefferman-Stein inequality for the  $\#$ -function, Littlewood-Paley theory and standard integration by parts arguments (as in [10] or [18], p. 695) one reduces matters to an estimate for

$$(3.25) \quad S_l F(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left( \sum_{k \geq 2l} \left| 2^{k/q} [R_{k,l} f_k(w) - \frac{1}{|Q|} \int R_{k,l} f_k(z) dz] \right|^2 \right)^{1/2}$$

where  $F = \{f_k\}_{k \in \mathbb{N}}$  and the supremum is taken over all cubes  $Q$  containing  $x$ . The assertion follows from

$$(3.26) \quad \|S_l F\|_q \leq 2^{-l\epsilon(q)} \|F\|_{L^q(\ell^2)}, \quad \frac{q_W + 2}{2} < q < \infty.$$

One splits  $S_l F(x)$  into three parts. The main part is concerned with the terms where  $2^{-Cl} \leq |2^k \text{diam}(Q)| \leq 2^{Cl}$  for some large  $C$ ; here one applies Hölder's inequality in  $k$  and uses the dyadic  $L^q$  estimates above (cf. Corollary 3.2). The terms with  $|2^k \text{diam}(Q)| > 2^{Cl}$  or  $< 2^{-Cl}$  are dealt with by standard  $L^2$  and  $L^\infty \rightarrow BMO$  bounds for generalized Radon transforms. We omit the details which are very similar to those in §3 of [16] (based on arguments in [17] in a different context).

**3.4. Extension to curves of finite type.** We now relax the curvature assumption on  $\gamma$  and assume that  $\gamma$  is of finite maximal type  $\leq m$ . In the terminology of [18] the operator  $\mathcal{R}$  satisfies a *right finite type condition* of order  $\leq m + 1$ , while in the terminology of [19] the underlying incidence relation is of type  $\preceq (m, 1)$ . Finally in Comech's terminology [2] the projection  $\pi_R$  (for the canonical relation associated to  $\mathcal{R}$ ) is of type  $\leq m - 1$  (while  $\pi_L$  as a blowdown is not of finite type).

We may relax the finite type assumption a bit by not necessarily assuming that  $\gamma' \neq 0$ . We shall fix  $a_0 \in I$  and estimate  $R$  under the assumption that  $\chi_1$  is supported in a small neighborhood of  $\alpha_0$ . Assume

$$\gamma(a_0 + \alpha) = \gamma(a_0) + (\beta_1 \alpha^{n_1} \varphi_1(\alpha), \beta_2 \alpha^{n_2} \varphi_2(\alpha)),$$

where  $1 \leq n_1 < n_2 \leq m$ ,  $\beta_1$  and  $\beta_2$  are nonzero constants and  $\varphi_i \in C^{m+5-n_i}$  with  $\varphi_i(0) = 1$ . We may reduce Theorem 1.2 to this case (with  $n_1 = 1$ ) by localization and perhaps a rotation. Furthermore, we may assume that  $1/2 \leq \varphi_i \leq 3/2$ ,  $i = 1, 2$  on  $\text{supp}(\chi_1)$ .

We work with a dyadic partition of unity  $\zeta_j(\alpha) = \zeta(2^j(\alpha - a_0))$  where  $\zeta_j$  is supported in the two intervals where  $|\alpha| \approx 2^{-j}$ . Let  $\mathcal{R}_j f(x', \alpha) = \zeta_j(\alpha) \mathcal{R}f(x', a)$ . We claim that for  $1 < p < (q_W + 2)/q_W$

$$(3.27) \quad \|\mathcal{R}_j f\|_{L_{1-1/p}^p} \leq C 2^{j(m - \frac{m+1}{p})} \|f\|_p;$$

this clearly yields the assertion of the theorem. We use a simple scaling argument. For  $|u| \approx 1$  define

$$\Gamma_j(u) = (\beta_1 u^{n_1} \phi_1(a_0 + 2^{-j}u), \beta_2 u^{n_2} \phi_2(a_0 + 2^{-j}u));$$

and let

$$T_j f(x', u) = \zeta(u) \int \chi_2(s) f(x' + s\Gamma_j(u), s) ds.$$

Notice that  $\det(\Gamma_j'(u), \Gamma_j''(u)) \approx \beta_1 \beta_2 u^{n_1+n_2} (1 + O(2^{-j}))$  and that the derivatives of  $\Gamma_j$  are uniformly bounded above. Thus by our estimate for the nonvanishing curvature case the operators  $T_j$  map  $L^p$  to  $L_{1-1/p}^p$ ,  $1 < p < (q_W + 2)/q_W$ , with bounds uniform in  $j$ . A short computation shows that  $\mathcal{R}_j f(2^{-jn_1}x_1, 2^{-jn_2}x_2, a_0 + 2^{-j}u) = \chi_1(a_0 + 2^{-j}u) T_j f_j(x', u)$  where  $f_j(y) = f(2^{-jn_1}y_1, 2^{-jn_2}y_2, y_3)$ . Since  $\max\{n_1, n_2\} \leq m$  the inequality (3.27) follows quickly.  $\square$

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