BOUNDS FOR SINGULAR FRACTIONAL INTEGRALS
AND RELATED FOURIER INTEGRAL OPERATORS

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1. Introduction

Let $\Omega \subset \widetilde{\Omega}$ be open sets in $\mathbb{R}^d$, $I \subset \mathbb{R}^{d-\ell}$ be an open neighborhood of the origin and let η be a compactly supported smooth function on $\Omega \times I$; we assume that $\eta(\cdot, 0)$ does not vanish identically. For each $x \in \Omega$ let $t \mapsto \Gamma(x,t) \subset \widetilde{\Omega}$ be a regular parametrization of a submanifold $\mathcal{M}_x \subset \widetilde{\Omega}$ with codimension ℓ. We assume that $\Gamma(x,t) \subset \Omega$ if $(x,t) \in \operatorname{supp} \eta$, and that Γ satisfies $\Gamma(x,0) = x$ and depends smoothly on (x,t).

We shall consider the singular fractional integral operator (or weakly singular Radon transform) $\mathcal{R}_\sigma$, defined by

$$(1.1) \qquad \mathcal{R}_\sigma f(x) = \int \eta(x,t) f(\Gamma(x,t)) |t|^{-(d-\ell-\sigma)} \, dt,$$

under suitable "curvature" assumptions on the singular support and the wavefront sets of the distribution kernel of the integral operator.

To formulate these assumptions we shall work with a submanifold $\mathcal{M}$ of codimension ℓ in $\Omega \times \Omega$; so that

$$(1.2) \qquad \Delta := \{(x,x) : x \in \Omega\} \subset \mathcal{M}.$$

To relate this to the operator in (1.1) we assume that $\eta(x,t)$ vanishes unless $|t| < \delta$ for small δ and note that the differential of the map $(x,t) \mapsto \gamma(x,t)$ has maximal rank $d + \ell$; then we take

$$\mathcal{M} = \{(x,y) : x \in \Omega, y = \Gamma(x,t) \text{ for some } |t| < \delta\}.$$

Moreover we assume the following standard hypotheses in the theory of Fourier integral operators:

Nondegeneracy assumptions.

(1.3) *The natural projections $(x,y) \mapsto x$ and $(x,y) \mapsto y$ are submersions when restricted to $\mathcal{M}$.*

(1.4) *The twisted normal bundle $N^*\mathcal{M}' \subset T^*\Omega \times T^*\Omega$ is locally the graph of a canonical transformation. Here $N^*\mathcal{M}'$ consists of all $(x,\xi,y,-\eta)$ where $(x,y) \in \mathcal{M}$ and $(\xi,\eta) \in T^*_{(x,y)}\mathcal{M}$ annihilates the tangent vectors in $T_{(x,y)}\mathcal{M}$.*

Assumption (1.3) implies that the sections

$$\mathcal{M}_x = \{y \in \Omega : (x,y) \in \mathcal{M}\}$$
$$\mathcal{M}^y = \{x \in \Omega : (x,y) \in \mathcal{M}\}$$

are immersed submanifolds of Ω, of codimension ℓ. We may assume that $\mathcal{M}$ is given by a defining function

$$(1.5) \qquad\qquad \mathcal{M} = \{(x,y) : \Phi(x,y) = 0\}$$

where Φ is $\mathbb{R}^\ell$-valued satisfying $\Phi(x,x) = 0$ so that (1.2) is satisfied and rank Φ_x = rank $\Phi_y = \ell$ so that (1.3) is satisfied.

Assumption (1.4) can be reformulated as follows. Let $\Psi(x,y,\tau) = \tau \cdot \Phi(x,y)$. Then the assumption (1.4) on $N^*\mathcal{M}'$ is equivalent with

$$(1.6) \qquad\qquad \det \begin{pmatrix} \Psi_{xy} & \Psi_{x\tau} \\ \Psi_{\tau y} & \Psi_{\tau\tau} \end{pmatrix} = \det \begin{pmatrix} \tau \cdot \Phi_{xy} & \Phi_x \\ \Phi_y & 0 \end{pmatrix} \neq 0 \quad \text{for all } \tau \in S^{\ell-1},$$

see [12]; in (1.6) Φ_x should be read as a $d \times \ell$ - matrix and Φ_y as an $\ell \times d$ - matrix. For $\ell = 1$ hypothesis (1.4) is just the rotational curvature assumption of Phong and Stein [16]. We note that for (1.4) to hold the codimension ℓ has to be sufficiently small, and we are mainly interested in the case of hypersurfaces.

Theorem 1.1. *Suppose that* $1 \leq p \leq q \leq \infty$, $0 < \sigma < d - \ell$ *and suppose that* $\mathcal{M}$ *satisfies the nondegeneracy assumptions* (1.3), (1.4). *Then* $\mathcal{R}_\sigma$ *maps* $L^p(\Omega) \to L^q(\widetilde{\Omega})$ *if and only if the following conditions are satisfied:*

a) $(1/p, 1/q)$ *belongs to the triangle with corners* $(0,0)$, $(1,1)$ *and* $(\frac{d}{d+\ell}, \frac{\ell}{d+\ell})$.

b) $(1/p, 1/q)$ *belongs to the halfplane defined by* $(d+\ell)(\frac{1}{p} - \frac{1}{q}) \leq \sigma$.

A special translation invariant case is due to M. Christ [3], extending earlier results by Ricci and Stein [18]. These authors consider the translation invariant case where $\Phi(x,y) = x_d - y_d - |x' - y'|^2$ and a related model case on the Heisenberg group. For these dilation invariant examples one actually proves global results which one could deduce from local ones by scaling arguments.

The weakly singular Radon transforms are special cases of oscillatory integrals with singular symbols as considered by Melrose [13], Greenleaf and Uhlmann [11] and others. Let $I^{\rho,-\sigma}(\Omega \times \Omega; \mathcal{M}, \Delta)$ denote the class of distribution kernels introduced in [11]; we denote by $\mathcal{I}^{\rho,-\sigma}(\Omega \times \Omega; \mathcal{M}, \Delta)$ the associated class of operators and refer for a general discussion and other references to previous work to [11].

Possibly after a change of variable we may locally parametrize $\mathcal{M}$ as a graph of an $\mathbb{R}^\ell$ valued function,

$$(1.7) \qquad\qquad y'' = S(x,y')$$

with $y' = (y_1, \ldots, y_{d-\ell})$, $y'' = (y_{d-\ell+1}, \ldots, y_d)$, $S = (S_{d-\ell+1}, \ldots, S_d)$. so that

$$(1.8) \qquad\qquad \text{rank } S_{x''} = \ell$$

and

$$(1.9) \qquad\qquad \det \begin{pmatrix} \theta \cdot S_{x'y'} & S_{x'} \\ \theta \cdot S_{x''y'} & S_{x''} \end{pmatrix} \neq 0$$

for all $\theta \in \mathbb{R}^\ell \setminus \{0\}$.

We recall from [11], [5] that a distribution kernel K belongs to $I^{\rho,-\sigma}(\Omega \times \Omega; \mathcal{M}, \Delta)$ if it is a locally finite sum of K_ν, so that each K_ν can be written after a change of variable in Ω as an oscillatory integral

$$(1.10) \qquad\qquad \iint_{\mathbb{R}^{d-\ell} \times \mathbb{R}^\ell} e^{i[\langle \tau, y'' - S(x,y') \rangle + \langle \xi, x' - y' \rangle]} a(x,y,\tau,\xi) d\tau d\xi.$$

Here S satisfies (1.8), (1.9) and the symbol a satisfies the differential inequalities

$$(1.11) \qquad\qquad |\partial_{x,y}^\gamma \partial_\tau^\alpha \partial_\xi^\beta a(x,y,\tau,\xi)| \leq C_{\alpha,\beta,\gamma}(1 + |\tau| + |\xi|)^{\rho - |\alpha|}(1 + |\xi|)^{-\sigma - |\beta|}.$$

We refer to the class of symbols satisfying (1.11) as $S^{\rho,-\sigma}(\Omega \times \Omega, \mathbb{R}^\ell, \mathbb{R}^{d-\ell})$. We shall sometimes denote the operator with kernel (1.10) as $\mathcal{T}[a]$.

It is well known that the weakly singular Radon transform as considered in Theorem 1.1 is an operator in $\mathcal{I}^{0,-\sigma}(\Omega \times \Omega; \mathcal{M}, \Delta)$ (see *e.g.* [11]). Namely after an appropriate localization it suffices to work with

$$(1.12) \qquad \mathcal{R}^\sigma f(x) = \int f(y', S(x,y'))|x' - y'|^{\sigma+\ell-d}\chi(x', S(x,y'), y')dy'$$

where χ has small support. Then the distribution kernel is given by

$$\delta(y'' - S(x,y'))|x' - y'|^{\sigma-d+\ell}g(x,y')$$

where δ is the Dirac measure at the origin in $\mathbb{R}^\ell$ and g is smooth and compactly supported. We expand the Dirac measure using the Fourier inversion formula in $\mathbb{R}^\ell$ and apply the Fourier inversion formula in $\mathbb{R}^{d-\ell}$ to the function $h \to |h|^{\sigma-d+\ell}g(x, x' + h)$. As a result we can write the distribution kernel in the form (1.10) where the symbol a is given by

$$a(x,y,\tau,\xi) = (2\pi)^{-d}\int |w'|^{\sigma-d+\ell}g(x, x' + w')e^{-\imath\langle\xi,w'\rangle}dw'.$$

We now formulate estimates for general operators of class $\mathcal{I}^{\rho,-\sigma}$. Since the composition of a standard pseudo-differential operator of order m with an operator in $\mathcal{I}^{\rho,-\sigma}(\Omega \times \Omega; \mathcal{M}, \Delta)$ belongs to $\mathcal{I}^{\rho+m,-\sigma}(\Omega \times \Omega; \mathcal{M}, \Delta)$ (see [5], [11]) the following results yield $L_a^p \to L_{a+m}^q$ Sobolev estimates for weakly singular Radon transforms.

Theorem 1.2. *Suppose that $1 \le p \le q \le \infty$. Let $T \in \mathcal{I}^{\rho,-\sigma}(\Omega \times \Omega; \mathcal{M}, \Delta)$, with compactly supported distribution kernel, and assume that the nondegeneracy assumptions (1.3), (1.4) hold.*

1.2.1. *Suppose $0 < \rho < \frac{d-\ell}{2}$ and $2\rho < \sigma < d - \ell$. Then T maps L^p to L^q if the following two conditions are satisfied.*

   a) $(1/p, 1/q)$ belongs to the closed triangle with corners $(\frac{\rho}{d-\ell}, \frac{\rho}{d-\ell})$, $(\frac{d-\ell-\rho}{d-\ell}, \frac{d-\ell-\rho}{d-\ell})$ and $(\frac{d-\rho}{d+\ell}, \frac{\rho+\ell}{d+\ell})$.
   b) $(1/p, 1/q)$ belongs to the halfspace defined by $(d+\ell)(\frac{1}{p} - \frac{1}{q}) \le \sigma - 2\rho$.

1.2.2. *Suppose $\rho = 0$ and $0 < \sigma < d - \ell$. Then T maps L^p to L^q if the following two conditions are satisfied.*

   a) $(1/p, 1/q)$ belongs to the closed triangle with corners $(0,0)$, $(1,1)$ and $(\frac{d}{d+\ell}, \frac{\ell}{d+\ell})$, with the possible exception of the points $(0,0)$ and $(1,1)$.
   b) $(1/p, 1/q)$ belongs to the halfspace defined by $(d+\ell)(\frac{1}{p} - \frac{1}{q}) \le \sigma$.

   Moreover, T is bounded from the Hardy space H^1 to L^1 and from L^∞ to BMO.

1.2.3. *Suppose $-\ell < \rho < 0$ and $-\rho\frac{d-\ell}{\ell} < \sigma < d - \ell$. Then T maps L^p to L^q if the following two conditions are satisfied.*

   a) $(1/p, 1/q)$ belongs to the pentagon with corners $(1,1)$, $(0,0)$, $(1, \frac{\rho+\ell}{\ell})$, $(\frac{-\rho}{\ell}, 0)$ and $(\frac{d-\rho}{d+\ell}, \frac{\rho+\ell}{d+\ell})$, with the possible exceptions of the points $(1, \frac{\rho+\ell}{\ell})$, $(\frac{-\rho}{\ell}, 0)$.
   b) $(1/p, 1/q)$ belongs to the halfspace defined by $(d+\ell)(\frac{1}{p} - \frac{1}{q}) \le \sigma - 2\rho$.

1.2.4. *Suppose $-\ell < \rho < 0$ and $0 < \sigma \le -\rho\frac{d-\ell}{\ell}$. Then T maps L^p to L^q if $(1/p, 1/q)$ belongs to the quadrilateral with corners $(1,1)$, $(0,0)$, $(1, \frac{d-\sigma-\ell}{d-\ell})$ and $(\frac{\sigma}{d-\ell}, 0)$, with the possible exception of the points $(1, \frac{d-\sigma-\ell}{d-\ell})$ and $(\frac{\sigma}{d-\ell}, 0)$.*

We remark that the analytic family of fractional integrals considered by Grafakos [9] in the translation invariant case can be considered as a model family of operators of class $\mathcal{I}^{\rho,-\sigma}$, however the L^2 endpoint case in this family belongs to $\mathcal{I}^{\frac{d-\ell}{2}, \ell-d}$ but satisfies better L^2 estimates than the general operator in $\mathcal{I}^{\frac{d-\ell}{2}, \ell-d}$.

Operators in $\mathcal{I}^{0,0}$ are bounded on L^p for $1 < p < \infty$, see Greenleaf and Uhlmann [11], and for the main special case of singular Radon transforms Phong and Stein [16], [17]. The endpoint $L^p \to L^p$ estimates for the case $2\rho = \sigma$, $p_\rho = (d - \ell - \rho)/(d - \ell)$ or $p'_\rho = \rho/(d - \ell)$ may fail as demonstrated by Christ [4]. It is likely that the best possible Lorentz-space endpoint estimate, namely an $L^{p_\rho} \to L^{p_\rho,2}$ bound holds; a proof of this estimate in the translation-invariant case was given by Tao and one of the authors [20].

A variant of the methods in this paper has been used by the authors [21] to prove new L^p theorems for variable-coefficient maximal and singular integral operators associated to families of curves in $\mathbb{R}^2$ (extending results in [2], [19]).

It is well known that at least under the assumption of nonvanishing rotational curvature certain parabolic cutoffs can be used to write a singular integral along a hypersurface as a sum of two operators, where one of them is a pseudodifferential operator of type $(1/2, 1/2)$ and the other one a Fourier integral operator, of type $(1/2, 1/2)$. This decomposition is due to Melrose (see [13], [11]), but related arguments had been used by Nagel, Stein and Wainger [15], see also Phong and Stein [17] for a different version. In the course of this paper we shall make use of (variants of) all these decompositions.

The paper is organized as follows: §2 contains some preparations and the discussion of a crucial change of variables. §3 contains preliminary estimates for dyadic pieces of fractional Radon transforms. After appropriate localizations these are reduced to standard estimates for Fourier integral operators via parabolic scalings. In §4 we consider some variants of fractional integrals which are relevant for the estimation of the pseudodifferential contribution to operators in $\mathcal{I}^{\rho,-\sigma}$ when $\rho \leq 0$. Here we shall also see that part 1.2.4 follows in a straightforward way from estimates for a class for certain product-type fractional integrals. In §5 we give the proof of Theorem 1.1. It turns out that after some changes of variables angular Littlewood-Paley decompositions may be applied just as in the previously known translation-invariant case ([3]). As in that case a positivity argument is crucial; however the estimates for the error terms are more involved. In §5 we also bound a family of less singular positive operators which dominate operators in $\mathcal{I}^{\rho,-\sigma}$ when $\rho < 0$; thus we can then give a proof of 1.2.3. Finally, in §5, we discuss standard examples which show th sharpness of the results. In §6 we establish $L^p \to L^p$ bounds by suitable interpolation between $L^2 \to L^2$ and Hardy-space estimates. §7 contains estimates for general operators in $\mathcal{I}^{0,-\sigma}$ and additional interpolation arguments to finish the proof of Theorem 1.2.

2. Preliminaries

2.0. Notation.

2.0.1. B_ε will denote the open ball in $\mathbb{R}^d$ of radius ε centered at the origin.

2.0.2. $m(D)$ denotes the convolution operator with Fourier multiplier $m(\xi)$. We split variables in $\mathbb{R}^d = \mathbb{R}^{d-\ell} \times \mathbb{R}^\ell$ as $x = (x', x'')$ and denote by $h(D'')$ the convolution operator with Fourier multiplier $h(\xi'')$.

2.0.3. A function F on $\{z : 0 \leq \mathrm{Re}(z) \leq 1\}$ is called of admissible growth if $|F(z)| \leq C e^{A|z|}$ for some $A > 0$, $C \geq 0$.

2.0.4. The differentiability inequalities (1.11) are supposed to hold for all multiindices of length $\leq M_0$ where M_0 is large, say $M_0 = 10^{100}d$, those multiindices are termed admissible. Exponents $N, N_0, ..., N_4$ in §4 and §7 are assumed to be $\geq d + 1$ and $\leq 10^{10}d$.

2.0.5. We denote by $\zeta_0 \equiv \omega_0$ an even $C_0^\infty(\mathbb{R})$ function with $\zeta_0(s) = 1$ for $|s| \leq 1/2$ and $\zeta_0(s) = 0$ for $|s| \geq 1$. Also let $\zeta(s) = \zeta_0(s/2) - \zeta_0(s)$, $\omega(s) = \omega_0(s/4) - \omega_0(s)$ so that ζ is supported in $[1/2, 2]$ and ω is

supported in $[1/4, 4]$; moreover

$$\zeta_0(s) + \sum_{j=1}^{\infty} \zeta(2^{-j}s) = 1$$

$$\omega_0(s) + \sum_{j=1}^{\infty} \omega(4^{-j}s) = 1$$

for all $s \in \mathbb{R}$.

2.0.6. For two quantities A and B we write $A \lesssim B$ or $B \gtrsim A$ if there exists an absolute positive constant C so that $a \leq Cb$. We write $A \approx B$ if both $A \lesssim B$ and $A \gtrsim B$ hold.

2.1 Standard assumptions. For our Fourier integrals (1.10) and for the weakly singular Radon transforms any contribution away from the diagonal is handled by standard estimates for Fourier integral operators, see Lemma 3.1 below. Therefore, in view of the compact support assumption on the kernel it is sufficient to prove Theorems 1.1 and 1.2 under the assumption that the kernels of our operators are supported in a small neighborhood of a given point $(P, P) \in \Delta$. We shall introduce coordinates that vanish at P, and assume that in these coordinates the kernels are supported where $|x|, |y| \leq \varepsilon^{10}$; ε is chosen in (2.16) below.

For further preparation choose $\varepsilon_0 > 0$ so that in a neighborhood of the closure of $B_{\varepsilon_0} \times B_{\varepsilon_0}$ the manifold $\mathcal{M}$ is given as a graph

$$(2.1) \qquad\qquad\qquad\qquad y'' = S(x, y');$$

by performing a linear transformation we can also assume that

$$(2.2) \qquad\qquad\qquad\qquad S_{x'}(0,0) = O_{\ell, d-\ell}$$

(the $\ell \times (d - \ell)$ zero-matrix).

Since $\Delta \subset \mathcal{M}$ we have

$$(2.3) \qquad\qquad\qquad\qquad x'' = S(x, x')$$

for all $x \in B_{\varepsilon_0}$ and consequently

$$(2.4) \qquad\qquad\qquad S_{x'}(x, x') + S_{y'}(x, x') = 0$$

$$(2.5) \qquad\qquad S_{x'x'}(x, x') + 2S_{x'y'}(x, x') + S_{y'y'}(x, x') = 0$$

$$(2.6) \qquad\qquad\qquad\qquad S_{x''}(x, x') = I_{\ell, \ell}$$

where $I_{\ell, \ell}$ denotes the $\ell \times \ell$ identity matrix.

We shall also assume that for some constant $C_0 \geq 1$

$$(2.7) \qquad\qquad \sum_{|\alpha| \leq 10^{100} d} \sup_{\substack{|x| \leq \varepsilon_0 \\ |y'| \leq \varepsilon_0}} |\partial_{x, y'}^{\alpha} S(x, y')| \leq C_0.$$

Moreover, by the assumption (1.9) and by (2.2) we have for some positive $c_0 < 1$

$$(2.8) \qquad\qquad\qquad \|(\theta \cdot S_{x'y'}(0,0))^{-1}\| \leq c_0^{-1},$$

for all unit vectors $\theta \in \mathbb{S}^{\ell-1}$; here $\| \cdot \|$ denotes the Hilbert-Schmidt norm.

2.2 Straightening near the diagonal.

We now introduce a family of changes of variables, depending on unit vectors u in $\mathbb{R}^{d-\ell}$

$$w \mapsto \mathcal{Q}(w; u) := (w', w'' + F(w; u))$$

so that

$$(2.9.1) \qquad\qquad\qquad\qquad F'_w(0; u) = 0$$
$$(2.9.2) \qquad\qquad\qquad\qquad F(0; u) = 0$$

and so that

$$(2.10) \qquad \begin{aligned} y'' = S(x, y') &\iff z'' = \widetilde{S}(w, z'; u) \\ \text{if} \quad y &= \mathcal{Q}(z; u), \quad x = \mathcal{Q}(w; u) \end{aligned}$$

and

$$(2.11) \qquad\qquad \langle u, \nabla_{w'} \rangle \widetilde{S}^i(w, w'; u) = 0, \quad i = d - \ell + 1, \ldots, d.$$

To describe this change of variables let $B = B(u)$ be a rotation on $\mathbb{R}^{d-\ell}$ depending smoothly on u such that $Be_1 = u$ (with $e_1 = (1, 0, \ldots, 0)$). We define an $\mathbb{R}^\ell$-valued function $G = G(\cdot; u)$ by requiring that G satisfies the following system of ordinary differential equations, with respect to the variable w_1 and initial data depending on the parameters $w_2, \ldots, w_d$:

$$\frac{\partial G}{\partial w_1}(w) = \langle u, S_{y'} \rangle (Bw', w'' + G(w), Bw')$$
$$G(0, w_2, \ldots, w_d) = 0$$

Set

$$F(w) \equiv F(w; u) = G(B^{-1}w', w''; u);$$

then F satisfies (2.9) and

$$(2.12) \qquad\qquad \langle u, \nabla_{w'} \rangle F(w) = \langle u, S_{y'}(w', w'' + F(w), w') \rangle.$$

For the following discussion fix u. Since the functions S and $\widetilde{S}$ are related by (2.10) we have

$$(2.13) \qquad\qquad \widetilde{S}(w, z') + F(z', \widetilde{S}(w, z)) = S(w', w'' + F(w), z').$$

Denote by $D_u = \langle u, \nabla_{z'} \rangle$ the directional derivative with respect to u. Differentiation of (2.13) yields

$$D_u \widetilde{S}(w, z') + D_u F(z', \widetilde{S}(w, z)) + F_{z''}(z', \widetilde{S}(w, z)) D_u \widetilde{S}(w, z) = \langle u, \nabla_{y'} \rangle S(w', w'' + F(w), z')$$

and by (2.12) we obtain

$$D_u \widetilde{S}(w, z') + \langle u, S_{y'}(z', \widetilde{S}(w, z') + F(z', \widetilde{S}(w, z')), z') \rangle + F_{z''}(z', \widetilde{S}(w, z')) D_u \widetilde{S}(w, z')$$
$$= \langle u, S_{y'}(w', w'' + F(w), z') \rangle.$$

Now we evaluate for $w = z$ and take into account that $\widetilde{S}(z, z') = z''$. This yields

$$(I + F_{z''}(z)) D_u \widetilde{S}(z, z') = 0$$

Since $F_{z''}(0) = 0$ by (2.9), we obtain $\langle u, \nabla_{z'}\rangle \widetilde{S}(z, z') = 0$ in a neighborhood of $(0, 0)$, and since also $\widetilde{S}_{w'}(w, w') + \widetilde{S}_{z'}(w, w') = 0$ this yields (2.11).

In view of (2.9) we may fix a number $\delta_1 \ll \varepsilon_0$ so that

$$(2.14) \qquad B_{\delta/2} \subset \mathcal{Q}(w, u)B_\delta \subset B_{2\delta} \quad \text{for} \quad \delta \leq \delta_1, w \in B_\delta.$$

Let

$$(2.15) \qquad C_1 = \sup_{|\alpha| \leq 10^{100}d} \sup_{|w| \leq \delta_1} |F^{(\alpha)}(w)| + C_0$$

where C_0 is as in (2.7). We may assume throughout this paper that the cutoff function χ in (1.12) satisfies

$$(2.16) \qquad \operatorname{supp} \chi \subset \{(x, y') : |x| + |y'| \leq \varepsilon\} \text{ where } 0 < \varepsilon < (100dC_1/c_0)^{-1}\delta_1.$$

Moreover the distribution kernels of the the Fourier integrals defined by (1.10) are assumed to be supported in $B_{\varepsilon^{10}} \times B_{\varepsilon^{10}}$.

Note also that for $|x|, |y| \leq \varepsilon$

$$(2.17) \qquad \|S_{x'}\| + \|S_{x''} - I_{\ell,\ell}\| \ll \varepsilon^9 \ll \delta_1$$

$$(2.18) \qquad \|F_w\| \ll \varepsilon^9 \ll \delta_1.$$

2.3. Adjoint operators. Suppose that $\mathcal{M}$ is given as a graph (1.7)with (1.8) and the symbol has small (x, y) support then we may solve the equation $y'' = S(x, y')$ in x'' so that $y'' = S(x', \mathfrak{S}(y, x'), y')$ and

$$(2.19) \qquad y'' - S(x', x'', y') = \mathcal{C}(x, y)(x'' - \mathfrak{S}(y', y'', x'))$$

in a neighborhood of $\mathcal{M}$, with $\mathcal{C}(x, y)$ is an invertible $\ell \times \ell$ matrix depending smoothly on (x, y). If in the oscillatory integral (1.10) we make a linear change in the τ-variables, $\widetilde{\tau} = \mathcal{C}(x, y)^T\tau$, then we see that (1.10) can be rewritten as a linear combination of integrals with phase function $\langle \tau, x'' - \mathfrak{S}(y, x')\rangle$. This shows that for an operator in $\mathcal{I}^{\rho, -\sigma}(\Omega \times \Omega, \mathcal{M}, \Delta)$ the adjoint operator belongs to $\mathcal{I}^{\rho, -\sigma}(\Omega \times \Omega, \mathcal{M}^*, \Delta)$ where $\mathcal{M}^* = \{(x, y) : (y, x) \in \mathcal{M}\}$ (and $\mathcal{M}^*$ satisfies (1.3), (1.4)).

3. Nonsingular Radon transforms and scaling

We first recall a well-known result on $L^p \to L^q$ estimates for Fourier integral operators associated to a canonical graph. These estimates take care of contributions of the kernels away from the diagonal. In the formulation of this Lemma the order of a Fourier integral opertator is as in the standard theory of Fourier integral operators; thus the standard Radon-type operators is of order $-(d - \ell)/2$.

Lemma 3.1. *Suppose $-\ell < \rho < \frac{d-\ell}{2}$.*

Let T be a Fourier integral operator of order $\rho - \frac{d-\ell}{2}$ associated to a local canonical graph $\mathcal{C} \subset T^*\Omega \backslash \{0\} \times T^*\Omega \backslash \{0\}$. Suppose that the restrictions $\mathcal{C}$ of the projections $(x, y) \to x$ and $(x, y) \to y$ have differentials with maximal rank d and that the projection $\mathcal{C} \to \Omega \times \Omega$ has a differential with constant rank $\leq 2d - \ell$. Suppose that the distribution kernel of T has compact support.

(i) If $\rho > 0$ then T maps L^p to L^q if $(1/p, 1/q)$ belongs to the closed triangle with corners $(\frac{\rho}{d-\ell}, \frac{\rho}{d-\ell})$, $(\frac{d-\ell-\rho}{d-\ell}, \frac{d-\ell-\rho}{d-\ell})$ and $(\frac{d-\rho}{d+\ell}, \frac{\rho+\ell}{d+\ell})$.

(ii) If $\rho = 0$ then T maps L^p to L^q if $(1/p, 1/q)$ belongs to the closed triangle with corners $(0, 0)$, $(1, 1)$ and $(\frac{d}{d+\ell}, \frac{\ell}{d+\ell})$, with the possible exception of the corners $(0, 0)$ and $(1, 1)$; then an $H^1 \to H^1$ or $L^\infty \to BMO$ bound holds.

(iii) If $-\ell < \rho < 0$ then T maps L^p to L^q if $(1/p, 1/q)$ belongs to the pentagon with corners $(1,1)$, $(0,0)$, $(1, \frac{\ell+\rho}{\ell})$, $(\frac{-\rho}{\ell}, 0)$ and $(\frac{d-\rho}{d+\ell}, \frac{\rho+\ell}{d+\ell})$, with the possible exceptions of the points $(1, \frac{\ell+\rho}{\rho})$, $(\frac{-\rho}{\ell}, 0)$.

Sketch of the argument. The main $L^p \to L^{p'}$ estimates are essentially proved in [1]. We sketch the argument. Consider first the main endpoint $L^{\frac{d+\ell}{d-\rho}} \to L^{\frac{d+\ell}{\rho+\ell}}$ estimate. In view of the constant rank assumptions on the projection of $\mathcal{C}$ to the base space we may after appropriate localization and choice of coordinates write the kernel as the sum $\sum_{k \geq 1} K_k(x, y)$ and a C_0^∞ function; here

$$K_k(x,y) = 2^{k\rho} \int e^{i\langle \tau, y'' - S(x,y')\rangle} a_k(x, y, \tau) d\tau$$

where the integral is extended over a conic open set of $\mathbb{R}^\ell$, S is as in the introduction, the symbols a_k are of order 0 with uniform bounds in $k \geq 1$, and $a_k(x, y, \theta) = 0$ if $|\theta| \notin (2^{k-1}, 2^{k+1})$.

Let T_k be the operator with kernel K_k. Standard L^2 theory (see [12], [22]) shows that T_k is bounded on L^2, with norm $O(2^{k(\rho - \frac{d-\ell}{2})})$. Clearly $|K_k(x,y)| \lesssim 2^{k(\rho+\ell)}$. Thus T_k maps L^1 to L^∞ with norm $\lesssim 2^{k(\rho+\ell)}$. Interpolation yields that T_k maps $L^{\frac{d+\ell}{d-\rho}}$ to $L^{\frac{d+\ell}{\rho+\ell}}$ with bounds uniform in k. Since we assume that the canonical relation $\mathcal{C}$ does not meet $\{0\} \times T^*\Omega$ and $T^*\Omega \times \{0\}$ one can use standard integration by parts arguments ([12]) and Littlewood-Paley theory to put the pieces together and one obtains the desired $L^{\frac{d+\ell}{d-\rho}} \to L^{\frac{d+\ell}{\rho+\ell}}$ estimate, *cf.* also [1]. For the endpoint $L^p \to L^p$ (or $H^1 \to L^1$ estimate) and more references see [22, ch. IX].

Finally assume $-\ell < \rho < 0$. Then an integration by parts argument shows that

$$|K_k(x,y)| \lesssim 2^{k(\ell+\rho)}(1 + 2^k|y'' - S(x,y')|)^{-N}$$

and therefore the sum in k is bounded by $|y'' - S(x,y')|^{-(\rho+\ell)}$. In view of the compact support of the kernel we see that $K(x, \cdot)$ and $K(\cdot, y)$ are uniformly in Weak-$L^{\frac{\ell}{\ell+\rho}}$. Thus the operator maps L^1 to Weak-$L^{\frac{\ell}{\ell+\rho}}$. A similar argument applies to the adjoint operator. Now one uses the Marcinkiewicz interpolation to interpolate with the endpoint $L^{\frac{d+\ell}{d-\rho}} \to L^{\frac{\rho+\ell}{d+\ell}}$ estimate and further interpolation with the trivial L^1 and L^∞ estimates to conclude. $\square$

Let $\chi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a nonnegative function. Now let $-\ell < \rho \leq 0$, $0 < \sigma < d - \ell$. If also $\rho < 0$ we define the distribution kernel $G^{\rho,\sigma}$ by

$$(3.1) \qquad G^{\rho,\sigma}(x,y) = \chi(x,y)|x' - y'|^{-(d-\ell-\sigma)} c_{\ell,\rho}|y'' - S(x,y')|^{-(\rho+\ell)} \qquad \text{if } -\ell < \rho < 0$$

where

$$c_{\ell,\rho} = 2^\rho \pi^{-\ell/2} \frac{\Gamma(\frac{\ell+\rho}{2})}{\Gamma(\frac{-\rho}{2})}$$

so that the Fourier transform on $\mathbb{R}^\ell$ of $c_{\ell,\rho}|\cdot|^{-(\ell+\rho)}$ is $|\xi|^\rho$, see [8]. Define $G^{0,\sigma} = \lim_{\rho \to 0-} G^{\rho,\sigma}$ where the limit is taken in the sense of distributions; clearly

$$(3.2) \qquad G^{0,\sigma}(x,y) = \delta(y'' - S(x,y'))|x' - y'|^{-(d-\ell-\sigma)}\chi(x,y).$$

Define the operator $\mathcal{R}^{\rho,\sigma}$ by

$$(3.3) \qquad \mathcal{R}^{\rho,\sigma} f(x) = \langle G^{\rho,\sigma}(x,\cdot), f\rangle$$

so that for $\rho = 0$ we recover the weakly singular Radon transform. We wish to apply Lemma 3.1 to dyadic pieces localized in $x' - y'$, after a suitable rescaling. Therefore we decompose dyadically

$$(3.4) \qquad \mathcal{R}^{\rho,\sigma} = \sum_j (\mathcal{R}_j^{\rho,\sigma} + \mathcal{E}_j^{\rho,\sigma})$$

with

$$(3.4) \qquad \mathcal{R}_j^{\rho,\sigma} f(x) = 2^{j(d-\ell-\sigma)} \langle b_j^{\rho,\sigma}(x,\cdot), f \rangle$$

$$(3.5) \qquad \mathcal{E}_j^{\rho,\sigma} f(x) = 2^{j(d-\ell-\sigma)} \langle h_j^{\rho,\sigma}(x,\cdot), f \rangle$$

where

$$(3.6) \qquad b_j^{\rho,\sigma}(x,y) = 2^{-j(d-\ell-\sigma)} \zeta(2^j |x'-y'|) \zeta_0\left(\frac{|y''-S(x,y')|}{|x'-y'|^2}\right) G^{\rho,\sigma}(x,y)$$

and

$$(3.7) \qquad h_j^{\rho,\sigma}(x,y) = 2^{-j(d-\ell-\sigma)} \zeta(2^j |x'-y'|)\left(1 - \zeta_0\left(\frac{|y''-S(x,y')|}{|x'-y'|^2}\right)\right) G^{\rho,\sigma}(x,y).$$

Note that this implies $h^{0,\sigma} \equiv 0$.

Proposition 3.2. *Let* $0 < \sigma < d - \ell$, $-\ell < \rho \le 0$ *and let* $\mathcal{R}_j^{\rho,\sigma}$ *be as in (3.1).*

(i) Suppose that $(1/p, 1/q)$ *belongs to the triangle with corners* $(0,0)$, $(1,1)$ *and* $(\frac{d}{d+\ell}, \frac{\ell}{d+\ell})$. *Then*

$$\|\mathcal{R}_j^{0,\sigma} f\|_q \lesssim 2^{j[(d+\ell)(\frac{1}{p} - \frac{1}{q}) - \sigma]} \|f\|_p.$$

(ii) Suppose that $-\ell < \rho < 0$. *Then the inequality*

$$\|\mathcal{R}_j^{\rho,\sigma} f\|_q \lesssim 2^{j[(d+\ell)(\frac{1}{p} - \frac{1}{q}) + 2\rho - \sigma]} \|f\|_p$$

holds if $(1/p, 1/q)$ *belongs to the pentagon with corners* $(1,1)$, $(0,0)$, $(1, \frac{\ell+\rho}{\ell})$, $(\frac{-\rho}{\ell}, 0)$ *and* $(\frac{d-\rho}{d+\ell}, \frac{\rho+\ell}{d+\ell})$, *with the possible exception of the points* $(1, \frac{\ell+\rho}{\ell})$, $(\frac{-\rho}{\ell}, 0)$. *Then*

Proof. Let $\delta > 0$ and

$$(3.8) \qquad B(a,\delta) = \{y : |y'-a'| \le \delta, |y''-a'' - \langle S_{y'}(a,a'), y'-a' \rangle| \le \delta^2\}.$$

A sufficiently small neighborhood U of the origin is then made into a space of homogeneous space with the balls $B(x,\delta)$ (see [16], [22] at least for the case $\ell = 1$), and for sufficiently large j we can cover U with a family of balls $B(x_\nu, 2^{-j})$ which have bounded overlap.

Fix j and observe that if f is supported in $B(x_\nu, 2^{-j})$ then $\mathcal{R}_j f$ is supported in $B(x_\nu, C2^{-j})$ for a fixed C. Therefore in order to prove the asserted inequality it suffices to verify it under the assumption that f is supported in a ball $B(a,\delta)$ where $a \in \Omega$ is near the origin.

Fix a. Then we perform an affine change of variables, so that in the new coordinates we can write $R_j^{\rho,\sigma}$ as in (3.4), (3.6) with $S(x,y_1)$ replaced by $s(x,y_1)$ satisfying

$$(3.9) \qquad s_{x'}(a,a') = 0, \qquad s_{y'}(a,a') = 0.$$

(3.9) implies that the ball $B(a, 2^{-j})$ is contained in

$$\{y : |y'-a'| \le A2^{-j}, |y''-a''| \le A2^{-2j}\}$$

for suitable A. Moreover we also see the rotational curvature in (1.9) at (a,a') is given by $\det \theta \cdot s_{x'y'}(a,a')$ since we still have $s_{x''}(a,a') = I_{\ell,\ell}$, cf. (2.6).

We now perform a scaling argument and write

$$\mathcal{R}_j^{\rho,\sigma} f(a' + 2^{-j} v', a'' + 2^{-2j} v'') = 2^{j(2\rho-\sigma)} \widetilde{\mathcal{R}}_j^{\rho,\sigma} f_j(v)$$

where

$$\widetilde{\mathcal{R}}_j^{\rho,\sigma} g(v) = \langle \widetilde{b}_j^{\rho,\sigma}(v,\cdot), g \rangle,$$
$$f_j(w',w'') = f(a' + 2^{-j}w', a'' + 2^{-2j}w''),$$
$$S_{j,a}(v,w') = 2^{2j}(-a'' + s(a' + 2^{-j}v', a'' + 2^{-2j}v'', a' + 2^{-j}w')),$$
$$\widetilde{b}_j^{\rho,\sigma}(v,w) = b_j^{\rho,\sigma}(a' + 2^{-j}v', a'' + 2^{-2j}v'', a' + 2^{-j}w', a'' + 2^{-2j}w'').$$

In view of $s(a,a') = a''$ and $s_{x'}(a,a') = 0$ we check that the derivatives of $S_{j,a}$ are uniformly bounded (in a fixed neighborhood of $(0,0)$, which can be chosen independently of j and a) and also that the rotational curvature is bounded below.

The rescaled operators $\widetilde{\mathcal{R}}_j^{\rho,\sigma}$ are standard Fourier integral operators, to which Lemma 3.1 (ii), (iii) can be applied, the resulting $L^p \to L^q$ bounds are uniform in j, and in a. We apply Lemma 3.1 with the relevant choice of p and q and it follows that

$$2^{j\frac{d+\ell}{q}} \|\mathcal{R}_j^{\rho,\sigma} f\|_q = 2^{j(2\rho-\sigma)} \|\widetilde{\mathcal{R}}_j f_j\|_q \lesssim 2^{j(2\rho-\sigma)} \|f_j\|_p \lesssim 2^{j(2\rho-\sigma)} 2^{j\frac{d+\ell}{p}} \|f\|_p$$

which proves the Proposition. $\square$

For the estimation of the error term involving the terms $\mathcal{E}_j^{\rho,\sigma}$ see Proposition 4.2 below.

4. Regular and product type fractional integrals

In this section we study nonisotropic and product type pseudodifferential operators, which come up as low frequency contributions to operators in $\mathcal{I}^{\rho,-\sigma}$; in particular we prove $L^p \to L^q$ estimates for the error term in (3.4). We recall a sharp version of Young's inequality (see Theorem (6.35) in [6]) which states that the conditions $1 < p < q < \infty$ and

$$(4.1) \qquad \sup_x \|K(x,\cdot)\|_{L^{r,\infty}} + \sup_y \|K(\cdot,y)\|_{L^{r,\infty}} < \infty, \qquad \frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q},$$

imply that the integral operator with kernel $K(x,y)$ is bounded from $L^p \to L^q$.

Lemma 4.1. *Suppose* $1 < p < q < \infty$. *Define*

$$(4.2) \qquad K_1^{\rho,\sigma}(x,y) = \chi(x,y)|x'-y'|^{\sigma-d+\ell}|y''-S(x,y')|^{-\rho-\ell}$$

$$(4.3) \qquad K_2^{\rho,\sigma}(x,y) = \chi(x,y)(|x'-y'| + |y''-S(x,y')|^{1/2})^{\sigma-2\rho-d-\ell}$$

and

$$(4.4) \qquad K_3^{\rho,\sigma}(x,y) = \begin{cases} \chi(x,y)|x'-y'|^{\sigma-d+\ell}|y''-S(x,y')|^{-\rho-\ell} & \text{if } |x'-y'|^2 \leq 10|y''-S(x,y')| \\ 0 & \text{if } |x'-y'|^2 \geq 10|y''-S(x,y')| \end{cases}.$$

(i) *Assume* $0 < \sigma < d-\ell$, $-\ell < \rho < 0$, $(d-\ell)(1/p - 1/q) \leq \sigma$, $\ell(1/p - 1/q) \leq -\rho$. *Then the integral operator with kernel* $K_1^{\rho,\sigma}$ *maps* L^p *to* L^q.

(ii) *Assume* $-\ell < \rho \leq 0$, $0 < \sigma < d-\ell$ *and* $(d+\ell)(1/p - 1/q) \leq \sigma - 2\rho$. *Then the integral operator with kernel* $K_2^{\rho,\sigma}$ *maps* L^p *to* L^q.

(iii) *Assume* $-\ell < \rho \leq 0$, $-\rho(d-\ell)/\ell < \sigma < d-\ell$ *and* $(d+\ell)(1/p - 1/q) \leq \sigma - 2\rho$. *Then the integral operator with kernel* $K_3^{\rho,\sigma}$ *maps* L^p *to* L^q.

Proof. We first consider (i). Let $\mathcal{J}_{x',y'}$ denote the integral operator acting on functions in $\mathbb{R}^\ell$, with kernel

$$J_{x',y'}(x'',y'') = \chi(x',x'',y',y'')|y''-S(x,y')|^{-\rho-\ell}.$$

If $\ell(1/p-1/q) \leq -\rho$ then $\sup_{x''} \|J_{x',y'}(x'',\cdot)\|_{L^{r,\infty}} \leq C$ for $1/r = 1-1/p+1/q$, uniformly in x', y'. Since the quantities $|y''-S(x,y')|$ and $|x''-\mathfrak{S}(y,x')|$ are comparable ($cf.$ §2.3) we also have $\sup_{y''} \|J_{x',y'}(\cdot,y'')\|_{L^{r,\infty}} \leq C$. Thus by the sharp form of Young's inequality stated above the condition $\ell(1/p-1/q) \leq -\rho$ implies that $J_{x',y'}$ maps $L^p(\mathbb{R}^\ell)$ to $L^q(\mathbb{R}^\ell)$, with bounds independent of x', y'. Likewise, since $(d-\ell)(1/p-1/q) \leq \sigma$ the integral operator with kernel $\widetilde{\chi}(x',y')|x'-y'|^{\sigma-d+\ell}$ maps $L^p(\mathbb{R}^{d-\ell})$ to $L^q(\mathbb{R}^{d-\ell})$ if $\widetilde{\chi}$ is compactly supported. Thus by Minkowski's inequality (if $T_1^{\rho,\sigma}$ is the integral operator with kernel $K_1^{\rho,\sigma}$)

$$\|T_1^{\rho,\sigma}f\|_q \leq \Big(\int \Big[\int \widetilde{\chi}(x',y')|x'-y'|^{\sigma-d+\ell} \big\| J_{x',y'}[f(y',\cdot)] \big\|_{L^q(\mathbb{R}^\ell)} dy' \Big]^q dx' \Big)^{1/q}$$

$$\lesssim \Big(\int \Big[\int \widetilde{\chi}(x',y')|x'-y'|^{\sigma-d+\ell} \|f(y',\cdot)\|_{L^p(\mathbb{R}^\ell)} dy' \Big]^q dx' \Big)^{1/q}$$

$$\lesssim \Big(\int \|f(x',\cdot)\|_{L^p(\mathbb{R}^\ell)}^p dx' \Big)^{1/p}$$

and hence $T_1^{\rho,\sigma}$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. This proves (i).

(ii) is proved by checking directly the condition (4.1) for $r \leq \frac{d+\ell}{d+\ell+2\rho-\sigma}$; the calculation is standard and therefore omitted.

It remains to consider the operator with kernel $K_3^{\rho,\sigma}$. We now fix x and prove $\|K_3^{\rho,\sigma}(x,\cdot)\|_{L^{r,\infty}} \leq C$ with C independent of x; here again $r = \frac{d+\ell}{d+\ell+2\rho-\sigma}$. Let $v' = y' - x'$ and $v'' = y'' - S(x,y')$.

For $\alpha > 0$ let

$$\Omega(\alpha) = \{(v',v'') : |v'|^{\sigma-d+\ell}|v''|^{-\rho-\ell} > \alpha, \ |v'|^2 \leq 10|v''|^{1/2}, \ |v''| \leq C_2\}.$$

We have to show that the set $\Omega(\alpha)$ has measure $O(\alpha^{-r})$. If $v \in \Omega(\alpha)$ then $|v'|^2 \leq 10|v''| \leq 10\alpha^{-\frac{1}{\rho+\ell}}|v'|^{-\frac{d-\ell-\sigma}{\rho+\ell}}$ and this implies $|v'|^{2+\frac{d-\ell-\sigma}{\ell+\rho}} \lesssim \alpha^{-\frac{1}{\rho+\ell}}$ or $|v'| \lesssim \alpha^{-\frac{1}{d+\ell+2\rho-\sigma}}$. Thus

$$|\Omega(\alpha)| \lesssim \int_{|v'| \lesssim \alpha^{-\frac{1}{d+\ell+2\rho-\sigma}}} \alpha^{-\frac{\ell}{\rho+\ell}}|v'|^{-\frac{d-\ell-\sigma}{\rho+\ell}\ell} dv'$$

Now the condition $-\rho\frac{d-\ell}{\ell} < \sigma$ is equivalent with $-\frac{d-\ell-\sigma}{\rho+\ell}\ell > -(d-\ell)$ and therefore one can verify

$$|\Omega(\alpha)| \leq C\alpha^{-\frac{\ell}{\ell+\rho}}\alpha^{-\frac{1}{d+\ell+2\rho-\sigma}(d-\ell-\frac{(d-\ell-\sigma)\ell}{\ell+\rho})} = C\alpha^{-\frac{d+\ell}{d+\ell+2\rho-\sigma}}$$

and thus $\sup_x \|K_3^{\rho,\sigma}(x,\cdot)\|_{L^{r,\infty}} < \infty$. The verification of the condition $\sup_y \|K_3^{\rho,\sigma}(\cdot,y)\|_{L^{r,\infty}} < \infty$ is similar. $\square$

Proposition 4.2. *Suppose that* $1 < p \leq q < \infty$, $-\ell < \rho < 0$ *and* $0 < \sigma < d - \ell$. *Let* $\mathcal{E}^{\rho,\sigma} = \sum_j \mathcal{E}_j^{\rho,\sigma}$ *(as defined in (3.5)) Then* $\mathcal{E}^{\rho,\sigma}$ *is bounded from* L^p *to* L^q *if either one of the following two conditions is satisfied.*

(i) $-\rho\frac{d-\ell}{\ell} < \sigma < d-\ell$ *and* $(d+\ell)(1/p-1/q) \leq \sigma - 2\rho$.

(ii) $0 < \sigma \leq -\rho\frac{d-\ell}{\ell}$ *and* $(d-\ell)(1/p-1/q) \leq \sigma$.

Proof. The kernel of $\mathcal{E}^{\rho,\sigma}$ can be estimated by both $K_1^{\rho,\sigma}$ and $K_3^{\rho,\sigma}$ in Lemma 4.1. For (i) apply the estimate for the integral operator with kernel $K_3^{\rho,\sigma}$. To prove (ii) from Lemma 4.1 observe that inequality $\ell(1/p-1/q) \leq -\rho$ is implied by $0 < \sigma \leq -\rho\frac{d-\ell}{\ell}$ and $(d-\ell)(1/p-1/q) \leq \sigma$. $\square$

We shall now look at the basic dyadic pieces in decompositions of operators in $\mathcal{I}^{\rho,-\sigma}$. Let

$$(4.5) \qquad \beta_{k,m}(x,y,\tau,\xi) = \begin{cases} \omega(2^{-2k}|\tau|)\zeta(2^{-m}|\xi|) & \text{if } k > 0, m > 0, \\ \omega_0(|\tau|)\zeta(2^{-m}|\xi|) & \text{if } m > 0, k = 0, \\ \omega(2^{-2k}|\tau|)\zeta_0(|\xi|) & \text{if } k > 0, m = 0, \\ \omega_0(|\tau|)\zeta_0(|\xi|) & \text{if } k = m = 0. \end{cases}$$

Suppose $a \in S^{\rho,-\sigma}$. Let

$$(4.6) \qquad K_{k,m}(x,y) = \iint_{\mathbb{R}^{d-\ell} \times \mathbb{R}^\ell} e^{i[\langle \tau, y'' - S(x,y') \rangle + \langle \xi, x' - y' \rangle]} (a\beta_{k,m})(x,y,\tau,\xi) d\tau d\xi.$$

Let $T_{k,m}$ be the integral operator with kernel $K_{k,m}(x,y)$.

Lemma 4.3. *If $a \in S^{\rho,-\sigma}$ then*

(i)

$$(4.7) \qquad |K_{k,m}(x,y)| \lesssim 2^{2k\rho-m\sigma} \frac{2^{2k\ell}}{(1+2^{2k}|y''-S(x,y')|)^N} \frac{2^{m(d-\ell)}}{(1+2^m|x'-y'|)^N};$$

moreover

$$(4.8) \qquad |\nabla K_{k,m}(x,y)| \lesssim \max\{4^k, 2^m\} 2^{2k\rho-m\sigma} \frac{2^{2k\ell}}{(1+2^{2k}|y''-S(x,y')|)^N} \frac{2^{m(d-\ell)}}{(1+2^m|x'-y'|)^N}.$$

(ii) *Let K be the Schwartz kernel of an operator in $\mathcal{I}^{\rho,-\sigma}$ given by (1.10), and assume that $-\ell < \rho < 0$, $0 < \sigma < d-\ell$. Then K satisfies*

$$|K(x,y)| \lesssim |y''-S(x,y')|^{-\rho-\ell}|x'-y'|^{\sigma-d+\ell}.$$

Proof. (i) follows by integration by parts. (ii) is deduced from (i) by summing geometric series. $\square$

Proof of Theorem 1.2.4. Immediate from Lemma 4.3 (ii) and Lemma 4.1. $\square$

We shall now look at a general operator in $\mathcal{I}^{\rho,-\sigma}$ and consider the contribution which gives rise to a nonisotropic pseudo-differential operator .

Proposition 4.4. *Let $a \in S^{\rho,-\sigma}$ and suppose that $1 < p \leq q < \infty$. Suppose that $-\ell < \rho \leq 0$ and that $-\rho\frac{d-\ell}{\ell} < \sigma < d-\ell$ and $(d+\ell)(1/p-1/q) \leq \sigma-2\rho$. Then the operator $\sum_{k \geq 0} \sum_{m \geq k} T_{k,m}$ is bounded from L^p to L^q.*

Proof. We use the kernel estimates (4.7) and sum. We find that the kernel $P(x,y)$ of $\sum_{k \geq 0} \sum_{m \geq k} T_{k,m}$ satisfies the estimate

$$|P(x,y)| \lesssim \begin{cases} |x'-y'|^{\sigma-2\rho-d-\ell} & \text{if } |y''-S(x,y)|^{1/2} \lesssim |x'-y'| \\ |x'-y'|^{\ell-d+\sigma}|y''-S(x,y')|^{-\rho-\ell} & \text{if } |y''-S(x,y)|^{1/2} \gtrsim |x'-y'| \end{cases}.$$

Thus

$$|P(x,y)| \lesssim K_2^{\rho,\sigma}(x,y) + K_3^{\rho,\sigma}(x,y)$$

and the assertion follows from Lemma 4.1. $\square$

For later use we also write down a similar estimate for an operator with localization in $|x'-y'|$.

Lemma 4.5. *Let $a \in S^{0,-\sigma}$ and $K_{k,m}$ as in (4.6), with $\rho = 0$. Denote by $W_{k,m}$ the operator with kernel $K_{k,m}(x,y)\zeta_0(2^k(|x'-y'|))$. Suppose $1 < p \leq q < \infty$ and $(d+\ell)(1/p-1/q) \leq \sigma$, $0 < \sigma < d-\ell$. Then for $s > 0$ the operator $\sum_{k>s} W_{k,k-s}$ is bounded from from L^p to L^q, with operator norm $O(2^{-s(d-\ell-\sigma)})$.*

Proof. This follows in a straightforward manner from (4.7) and Lemma 4.1. We have the estimate

$$|K_{k,k-s}(x,y)| \lesssim \frac{2^{k(d+\ell-\sigma)}2^{-s(d-\ell-\sigma)}}{(1+2^{2k}|y''-S(x,y')| + 2^{k-s}|x'-y'|)^N}|\zeta_0(2^k(x'-y'))|;$$

here we choose $N > (d+\ell-\sigma)$. If $|y''-S(x,y')| \leq |x'-y'| \approx 2^{-k}$ we simply dominate by $2^{k(d+\ell-\sigma)}2^{-s(d-\ell-\sigma)}$ which is in the present case controlled by $2^{-s(d-\ell-\sigma)}K_2^{\rho,\sigma}(x,y)$ (cf. (4.3)).

If $|y''-S(x,y')| \geq |x'-y'| \approx 2^{-k}$ then $|K_{k,k-s}(x,y)| \lesssim 2^{-s(d-\ell-\sigma)}|y''-S(x,y')|^{-(d+\ell-\sigma)/2}$ and in the case under consideration this is also controlled by $2^{-s(d-\ell-\sigma)}K_2^{\rho,\sigma}(x,y)$. Since for fixed (x,y) the sum $\sum_{k>s} K_{k,k-s}(x,y)$ contains at most three terms, we see that the assertion follows from Lemma 4.1. $\square$

5. Weakly singular Radon transforms and some variants

In this section we give a proof of Theorem 1 and part 1.2.3 of Theorem 1.2. We first introduce an additional angular localization in the angular variable.

Let $v \in \mathbb{R}^{d-\ell}$ be a unit vector. Let

$$\kappa(x,y) = \chi(x,y)\zeta_0\big(\varepsilon^{-10}\big|\frac{x'-y'}{|x'-y'|} - v\big|\big)\zeta_0\big(\frac{|y''-S(x,y')|}{|x'-y'|^2}\big)$$

$$\kappa_j(x,y) = \kappa(x,y)\zeta(2^j(|x'-y'|))$$

here χ is a nonnegative smooth function supported where $|x| + |y'| \le \varepsilon^{10}$ (see (2.16)). Thus

$$(5.1) \qquad \text{supp } \kappa \subset \big\{(x,y) : \big|\frac{x'-y'}{|x'-y'|} - v\big| \ll \varepsilon^{10}, |x| \le \varepsilon^{10}, |y| \le \varepsilon^{10}, |y''-S(x,y')| \le |x'-y'|^2\big\}.$$

Let $G^{\rho,\sigma}$ be as in (3.1) and define

$$(5.2) \qquad R^{\rho,\sigma}f(x) = \langle G^{\rho,\sigma}(x,\cdot)\kappa(x,\cdot), f\rangle.$$

The operator $\mathcal{R}^{0,\sigma}$ introduced in §3 is a finite sum of operators of type $R^{0,\sigma}$ (with suitable choices of χ and v). Moreover, for $\rho < 0$ we recover the operators $\mathcal{R}^{\rho,\sigma}$ modulo error terms which are already estimated by Proposition 4.2. The case $\rho = 0$ of the following result implies the assertion of Theorem 1.1.

Theorem 5.1. *Let* $1 \le p \le q \le \infty$.

(i) Suppose that $(1/p, 1/q)$ *belongs to the intersection of the halfspace defined by* $(d+\ell)(\frac{1}{p} - \frac{1}{q}) \le \sigma$ *with the triangle with corners* $(0,0)$, $(1,1)$ *and* $(\frac{d}{d+\ell}, \frac{\ell}{d+\ell})$. *Then* $R^{0,\sigma}$ *maps* L^p *to* L^q.

(ii) Suppose $-\ell < \rho < 0$ *and* $-\rho\frac{d-\ell}{\ell} < \sigma < d - \ell$. *Suppose that* $(1/p, 1/q)$ *belongs to the intersection of the halfspace defined by* $(d+\ell)(\frac{1}{p} - \frac{1}{q}) \le \sigma - 2\rho$ *with the pentagon with corners* $(1,1)$, $(0,0)$, $(1, \frac{\ell+\rho}{\ell})$, $(\frac{-\rho}{\ell}, 0)$ *and* $(\frac{d-\rho}{d+\ell}, \frac{\rho+\ell}{d+\ell})$, *with the exception of the points* $(1, \frac{\ell+\rho}{\ell})$, $(\frac{-\rho}{\ell}, 0)$. *Then* $R^{\rho,\sigma}$ *maps* L^p *to* L^q.

For the rest of this section we fix ρ, σ and will not explicitly indicate the dependence on these parameters. If $p = q$ the assertion is easily verified by Minkowski's inequality. This also applies to the cases $p = 1$ and $q < \ell/(\ell+\rho)$, and $q = \infty$ and $p < -\ell/\rho$ (when $-\ell < \rho < 0$). Thus we may assume $1 < p < q < \infty$, and that $(1/p, 1/q)$ satisfies the restrictions in Theorem 5.1; moreover we may assume $p \le 2$ since the case $p > 2$ follows by considering the adjoint operator. It is always assumed that the function f is supported where $|y| \le \varepsilon^{10}$ and ε is as in (2.16). These assumptions are always assumed but not explicitly stated in various lemmas throughout this section.

Define

$$(5.3) \qquad \mathcal{R}_j f(x) = \langle G^{\rho,\sigma}(x,\cdot)\kappa_j(x,\cdot), f\rangle.$$

Then $\mathcal{R}_j$ is bounded from L^p to L^q with a bound independent of j, by Proposition 3.2. Let M be such that $2^M \ge (\varepsilon c_0)^{-10}$ (with c_0 as in (2.8)) and let J be a finite set of integers, all of them $\ge M$. Let

$$(5.4) \qquad \mathcal{R}f = \sum_{j \in J} \mathcal{R}_j f.$$

A priori we know that $\mathcal{R}$ is bounded from $L^p \to L^q$ with norm $O(\text{card}(J))$, and our task is to improve this to show that the $L^p \to L^q$ bound is independent of the cardinality of J. Once this is proved the $L^p \to L^q$ boundedness of $\mathcal{R}^{\rho,\sigma}$ follows immediately from applications of the monotone convergence theorem.

We begin by cutting out the low frequencies (here we follow essentially [2], [11]) and split $\mathcal{R} = \mathcal{A} + \mathcal{B}$ with

$$(5.5.1) \qquad \mathcal{A} = \sum_{j \in J} \omega_0(2^{-2j}|D''|)\mathcal{R}_j,$$

$$(5.5.2) \qquad \mathcal{B} = \sum_{j \in J} (I - \omega_0(2^{-2j}|D''|))\mathcal{R}_j.$$

We first prove

Lemma 5.2. *The operator $\mathcal{A}$ is bounded from L^p to L^q, with norm independent of the family J.*

Proof. Since the convolution kernel $\omega_0(2^{-2j}|D''|)$ is $O(2^{2j\ell}(1 + 2^{2j}|x''|)^{-N}$ we see that for $\rho < 0$

$$\left| \omega_0(2^{-2j}|D''|)\mathcal{R}_j f(x) \right|$$

$$\lesssim \iiint_{\substack{|y'' - S(x',w'',y')| \lesssim 2^{-2j} \\ |x'-y'| \approx 2^{-j}}} \frac{2^{2j\ell}}{(1 + 2^{2j}|x'' - w''|)^N} G^{\rho,\sigma}(x',w'',y',y'')|f(y',y'')|dy'dy''dw''$$

$$\lesssim \iint_{\substack{\{(y',y''): \\ |x'-y'| \approx 2^{-j}\}}} |f(y',y'')|2^{j(d-\ell-\sigma)} \int_{\substack{|w'' - \mathfrak{S}(y',y'',x')| \\ \lesssim 2^{-2j}}} \frac{2^{2j\ell}}{(1 + 2^{2j}|x'' - w''|)^N}|w'' - \mathfrak{S}(y',y'',x')|^{-\rho-\ell}dw'' \, dy'dy''$$

$$\lesssim \iint_{\substack{|x'-y'| \approx 2^{-j}}} \frac{2^{j(d+\ell+2\rho-\sigma)}}{(1 + 2^{2j}|x'' - \mathfrak{S}(y',y'',x')|)^N}|f(y',y'')|dy'dy'';$$

here $\mathfrak{S}$ is as in §2.3. The same estimate applies to the case $\rho = 0$ (with only notational changes in the argument).

We see that the kernel of $\omega_0(2^{-2j}D'')\mathcal{R}_j$ can be estimated by $K_2^{\rho,\sigma}$ (as in (4.3)), uniformly in j. This bound also applies to the sum $\sum_{j \in J} \omega_0(2^{-2j}D'')\mathcal{R}_j$ since the kernel of $\omega_0(2^{-2j}D'')\mathcal{R}_j$ is supported where $|x' - y'| \approx 2^{-j}$. Thus the assertion follows from Lemma 4.1. $\square$

We now turn to the operator $\mathcal{B}$ and we shall first prove estimates for a frequency localized variant.

Proposition 5.3. *Let ϑ be a fixed unit vector in $\mathbb{R}^\ell$ and let u be unit vector in $\mathbb{R}^{d-\ell}$ so that*

$$(5.6) \qquad |\langle u, \nabla_{x'}\rangle\langle v, \nabla_{y'}\rangle\vartheta \cdot S(0,0)| = \max_{U \in S^{d-\ell}} |\langle U, \nabla_{x'}\rangle\langle v, \nabla_{y'}\rangle\vartheta \cdot S(0,0)|.$$

Suppose further that the standard assumptions of §2.1 and (2.16) hold and

$$(5.7) \qquad \langle u, \nabla_{x'}\rangle S(x,x') = 0$$

for all $|x| \leq \varepsilon$. Let $a(\eta'')$ be supported in $\{\eta'' : |\frac{\eta''}{|\eta''|} - \vartheta| \leq \varepsilon^5\}$ and satisfy $|\partial^\alpha a(\eta'')| \lesssim |\eta''|^{-|\alpha|}$ for all admissible multiindices α. Let

$$\Theta = a(D).$$

Then the operator $\Theta\mathcal{B}$ is bounded from L^p to L^q and its operator norm satisfies the estimate

$$\|\Theta\mathcal{B}\|_{L^p \to L^q} \lesssim 1 + \|\mathcal{R}\|_{L^p \to L^q}^{1-\frac{p}{2}}.$$

Proof of Proposition 5.3.

We can rewrite $\mathcal{B}$ as

$$\mathcal{B} = \sum_{j \in J} \sum_{k > j} \omega(2^{-2k}|D''|)\mathcal{R}_j.$$

Let $\mathcal{L}_k$ be defined by

$$\widehat{\mathcal{L}_k f}(\eta) = \omega(2^{-2k}|\eta''|)a(\eta'').$$

then $\Theta\mathcal{B} = \sum_{j \in J} \sum_{k > j} \mathcal{L}_k \mathcal{R}_j$.

We shall now introduce an angular Littlewood-Paley decomposition (as in [14]) and proceed for the proof of our endpoint estimate using a well known argument by M. Christ (his preprint [3] is unpublished but the argument has been used in various related articles on L^p improving properties of convolution operators; for a rather general formulation see [10]). Define operators $P_{k,j}$, $\widetilde{P}_{k,j}$ by

$$(5.8.1) \qquad\qquad P_{k,j} = \sum_{i=-M}^{M} \zeta(2^{-2k+j+i}|\langle u, D'\rangle|)$$

$$(5.8.2) \qquad\qquad \widetilde{P}_{k,j} = \sum_{i=-M-10}^{M+10} \zeta(2^{-2k+j+i}|\langle u, D'\rangle|)$$

(we have chosen $2^M \geq c_0^{-1}\varepsilon^{-10}$). Define also

$$\widetilde{L}_k = \sum_{i=-10}^{10} \omega(2^{-2k+i}|D''|)$$

The operator $\Theta\mathcal{B}$ is then decomposed as

$$\Theta\mathcal{B} = \sum_{j \in J} \sum_{k > j} \mathcal{L}_k \mathcal{R}_j = \mathcal{T} + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

where

$$(5.9) \qquad\qquad \mathcal{T} = \sum_{j \in J} \sum_{k > j} \mathcal{L}_k P_{k,j} \mathcal{R}_j \widetilde{P}_{k,j} \widetilde{L}_k$$

$$(5.10) \qquad\qquad \mathcal{E}^1 = \sum_{j \in J} \sum_{k > j} \mathcal{L}_k (I - P_{k,j})\mathcal{R}_j \widetilde{P}_{k,j} \widetilde{L}_k$$

$$(5.11) \qquad\qquad \mathcal{E}^2 = \sum_{j \in J} \sum_{k > j} \mathcal{L}_k \mathcal{R}_j (I - \widetilde{P}_{k,j}) \widetilde{L}_k$$

$$(5.12) \qquad\qquad \mathcal{E}^3 = \sum_{j \in J} \sum_{k > j} \mathcal{L}_k \mathcal{R}_j (I - \widetilde{L}_k).$$

The main term is represented by $\mathcal{T}$, and we shall show that the operators $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3$ have quantitative properties similar to or better than the operator considered in Lemma 5.2.

For the main term we use the known argument in the translation invariant case [3]. Let $\mathcal{T}_{\text{vect}}$ denote the operator acting on $L^p(\ell^2(\mathbb{Z}^2))$ functions $F = \{F_{j,k}\}$ by

$$[\mathcal{T}_{\text{vect}}F]_{j,k} = \mathcal{R}_j F_{j,k}.$$

By Littlewood-Paley theory and complex interpolation (note that $p \leq 2$)

$$\|\mathcal{T}\|_{L^p \to L^q} \lesssim \|\mathcal{T}_{\text{vect}}\|_{L^p(\ell^2) \to L^q(\ell^2)}$$

(5.13)
$$\lesssim \|\mathcal{T}_{\text{vect}}\|_{L^p(\ell^p) \to L^q(\ell^p)}^{p/2} \|\mathcal{T}_{\text{vect}}\|_{L^p(\ell^\infty) \to L^q(\ell^\infty)}^{1-p/2}.$$

From Proposition 3.2 and Minkowski's inequality it follows that

(5.14)
$$\|\mathcal{T}_{\text{vect}}\|_{L^p(\ell^p) \to L^q(\ell^p)} \lesssim 1.$$

Also by the pointwise inequality $|\mathcal{R}_j(f)| \leq \mathcal{R}(|f|)$ and the positivity of $\mathcal{R}$ we have

$$\sup_{j,k} |\mathcal{R}_j F_{j,k}(x)| \leq \mathcal{R}[\sup_{j,k} |F_{j,k}|](x)$$

so that

(5.15)
$$\|T_{\text{vect}}\|_{L^p(\ell^\infty) \to L^q(\ell^\infty)} \lesssim \|\mathcal{R}\|_{L^p \to L^q}.$$

Therefore in view of Lemma 5.2 and (5.13-15)

(5.16)
$$\|\mathcal{T}\|_{L^p \to L^q} \leq C(1 + \|\mathcal{R}\|_{L^p \to L^q}^{1-p/2} + \sum_{i=1}^{3} \|\mathcal{E}_i\|_{L^p \to L^q})$$

Consequently the proof of Proposition 5.3 will be complete once we verify the uniform $L^p \to L^q$ boundedness of the operators $\mathcal{E}^1$, $\mathcal{E}^2$, $\mathcal{E}^3$.

It will be convenient to work with oscillatory integral representations of the kernels of $\mathcal{R}_j$. Since the Fourier transform of $c_{\ell,\rho} | \cdot |^{\rho+\ell}$ is $|\xi|^\rho$ (see [8]) we can write the kernel R_j of $\mathcal{R}_j$ as an oscillatory integral

$$R_j(x, y) = \kappa_j(x, y) |x' - y'|^{\sigma - d + \ell} \int e^{i\langle \tau, y'' - S(x, y') \rangle} |\tau|^\rho d\tau.$$

For $k \geq 1$ we denote by $\mathcal{R}_j^k$ the operator with integral kernel

$$R_j^k(x, y) = \kappa_j(x, y) |x' - y'|^{-(d - \ell - \sigma)} \int e^{i\langle \tau, y'' - S(x, y') \rangle} \omega(2^{-2k} |\tau|) \zeta_0(\varepsilon^{-4} |\frac{\tau}{|\tau|} - \vartheta|) |\tau|^\rho d\tau;$$

the operator $\mathcal{R}_j^0$ is defined similarly but with $\omega(2^{-2k}|\tau|)$ replaced by $\omega_0(|\tau|)$.

Lemma 5.4. *(i) The operator $\sum_j \mathcal{R}_j^0$ maps L^p to L^q.*

(ii) Let $s \geq 0$. Let $Z_s(x, y)$ denote the distribution kernel of the operator $\sum_j \mathcal{L}_{j+s}(\mathcal{R}_j - \sum_{i=-4}^{4} \mathcal{R}_j^{j+s+i})$. Then

$$|Z_s(x, y)| \lesssim 4^{-s} |K_2^{\rho,\sigma}(x, y)|$$

where $K_2^{\rho,\sigma}$ is defined in (4.3). Thus this operator maps $L^p \to L^q$ with operator norm $O(4^{-s})$.

Proof. (i) It is easy to see that by the theorem on fractional integration the operator $\sum_j \mathcal{R}_j^0$ maps L^p to L^q, provided that $1 < p < q < \infty$ and $(d - \ell)(1/p - 1/q) \leq \sigma$. However the condition $(d - \ell)(1/p - 1/q) \leq \sigma$ is implied by $(d + \ell)(1/p - 1/q) \leq \sigma - 2\rho$ and $-\rho(d - \ell)/\ell \leq \sigma$ which is assumed throughout this section.

(ii) Note that

$$\mathcal{R}_j - \sum_{i=-4}^{4} \mathcal{R}_j^{j+s+i} = \sum_{r \geq 5} \mathcal{R}_j^{j+s+r} + \mathcal{S}_{j,j+s}^0 + \sum_{r \geq -4} \mathcal{V}_{j,j+s+r} + \mathcal{V}_{j,j+s}^0$$

where the kernels $S_{j,k}^0$, $V_{j,k}$ and $V_{j,k}^0$ of $\mathcal{S}_{j,k}^0$, $\mathcal{V}_{j,k}$ and $\mathcal{V}_{j,k}^0$ are given by

$$S_{j,k}^0(x,y) = \kappa_j(x,y)|x'-y'|^{-(d-\ell-\sigma)} \int e^{i\langle \tau, y''-S(x,y')\rangle} \omega_0(2^{-2(k-5)}|\tau|)|\tau|^\rho \zeta_0(\varepsilon^{-4}|\frac{\tau}{|\tau|}-\vartheta|)d\tau$$

$$V_{j,k}(x,y) = \kappa_j(x,y)|x'-y'|^{-(d-\ell-\sigma)} \int e^{i\langle \tau, y''-S(x,y')\rangle} \omega(2^{-2k}|\tau|)|\tau|^\rho \left(1-\zeta_0(\varepsilon^{-4}|\frac{\tau}{|\tau|}-\vartheta|)\right)d\tau$$

$$V_{j,k}^0(x,y) = \kappa_j(x,y)|x'-y'|^{-(d-\ell-\sigma)} \int e^{i\langle \tau, y''-S(x,y')\rangle} \omega_0(2^{-2(k-5)}|\tau|)|\tau|^\rho \left(1-\zeta_0(\varepsilon^{-4}|\frac{\tau}{|\tau|}-\vartheta|)\right)d\tau.$$

We shall now show that the distribution kernel of $\sum_j \mathcal{L}_{j+s}\mathcal{R}_j^{j+s+r}$ is for $r \geq 5$ controlled by $4^{-(s+r)}K_2^{\rho,\sigma}$ (cf. (4.3)). Also the kernels of $\sum_j \mathcal{L}_{j+s}\mathcal{S}_{j,j+s}^0$ and $\sum_j \mathcal{L}_{j+s}\mathcal{V}_{j,j+s}^0$ are bounded by $4^{-s}K_2^{\rho,\sigma}$; we shall omit the entirely analogous argument.

The kernel of $\mathcal{L}_n\mathcal{R}_j^k$ is given by

$$K_{j,k,n}(x,y) = (2\pi)^{-\ell} \iiint e^{i[\langle x''-w'',\eta''\rangle + \langle \tau, y''-S(x',w'',y')\rangle]} \times$$

$$\omega(2^{-2k}|\tau|)|\tau|^\rho \omega(2^{-2n}|\eta''|)a(\eta'')\zeta_0(\varepsilon^{-5}|\frac{\tau}{|\tau|}-\vartheta|)\frac{\kappa_j(x',w'',y',y'')}{|x'-y'|^{d-\ell-\sigma}}dw''\,d\eta''\,d\tau.$$

We need to estimate this kernel when $k \geq n+5$, and $n \geq j$. The w''-gradient of the phase function is $-\eta'' - \nabla_{w''}(\tau \cdot S(w,y'))$ and since $\|S_{w''} - I_{\ell,\ell}\| \ll \varepsilon^{1/2}$ this gradient is now $\approx 2^{2k}$ (note that it would be $\approx 2^{2n}$ if we worked with $\mathcal{L}_n\mathcal{S}_{j,n}^0$).

We use integration by parts with respect to w'' followed by integration by parts with respect to τ and η. Observe that with each differentiation of $\kappa_j(x',w'',y)$ we loose a factor of 2^{2j}, the main contribution coming from differentiating $\zeta_0(|w''-S(x',w'',y')|/|w'-y'|^2)$. Thus we gain 2^{-2k+2j} with each integration by parts in w''. As a result we obtain that the kernel of $\mathcal{L}_n\mathcal{R}_j^k$ is dominated by a constant times

$$2^{-(2k-2j)N_0} \int |x'-y'|^{\sigma-d+\ell} \frac{2^{2k(\ell+\rho)}}{(1+2^{2k}|y''-S(x',w'',y')|)^{N_1}} \frac{2^{2n\ell}}{(1+2^{2n}|x''-w''|)^{N_1}}dw''$$

$$\lesssim \min\{2^{-(2n-2j)(N_0-N_1)}, 2^{-(2k-2j)(N_0-N_1)}\}|x'-y'|^{\sigma-d+\ell} \frac{2^{2k(\ell+\rho)}}{(1+2^{2k}|y''-S(x',x'',y')|)^{N_1}};$$

here we choose $N_0 \gg N_1$. Moreover the kernel of the operator $\mathcal{L}_n\mathcal{S}_{j,k}$ is of course supported where $|x'-y'| \approx 2^{-j}$. The asserted pointwise estimate for $\sum_j \mathcal{L}_{j+s}\mathcal{R}_j^{j+s+r}$ is now a consequence of summing geometric series.

The same argument applies to the operators $\sum_j \mathcal{L}_{j+s}\mathcal{V}_{j,j+s+r}$, $r \geq -4$. Note that the above restriction $r > 4$ (or $k > n+4$) is not necessary now in view of the factor $(1-\zeta_0(\varepsilon^{-4}|\frac{\tau}{|\tau|}-\vartheta|))$; namely the assumptions $\eta'' \in \text{supp}\, a$ (hence $|\eta''/|\eta''|-\vartheta| \leq \varepsilon^5$) and $|\tau/|\tau|-\vartheta| \geq \varepsilon^4/2 \gg \varepsilon^5$ guarantee that $|-\eta''-\tau\cdot S_{w''}(w,y')| \approx \max\{|\eta''|,|\tau|\}$ which is sufficient to carry out the above integration by parts arguments. $\square$

We shall now bound the operators $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3$ in (5.10-12). However we first modify these operators by replacing $\mathcal{L}_k\mathcal{R}_j$ in the definitions (5.10-12) by $\sum_{i=-4}^4 \mathcal{L}_k\mathcal{R}_j^{k+i}$. Let for $i = -4, \ldots, 4$

(5.17.1) $$\mathcal{E}_{j,k,i}^1 = \mathcal{L}_k(I-P_{k,j})\mathcal{R}_j^{k+i}\widetilde{P}_{k,j}\widetilde{L}_k$$

(5.17.2) $$\mathcal{E}_{j,k,i}^2 = \mathcal{L}_k\mathcal{R}_j^{k+i}(I-\widetilde{P}_{k,j})\widetilde{L}_k$$

and

(5.18) $$\mathcal{E}_{j,k,i}^3 = \mathcal{L}_k\mathcal{R}_j^{k+i}(I-\widetilde{L}_k)$$

and let

(5.19) $$\widetilde{\mathcal{E}}^{1,i} = \sum_{j\in J}\sum_{k>j}\mathcal{E}_{j,k,i}^1, \quad i = -4, \ldots, 4;$$

similarly define $\widetilde{\mathcal{E}}^{2,i}$, $\widetilde{\mathcal{E}}^{3,i}$.

Lemma 5.5. *The operators $\mathcal{E}^1 - \sum_{i=-4}^{4} \widetilde{\mathcal{E}}^{1,i}$, $\mathcal{E}^2 - \sum_{i=-4}^{4} \widetilde{\mathcal{E}}^{2,i}$, and $\mathcal{E}^3 - \sum_{i=-4}^{4} \widetilde{\mathcal{E}}^{3,i}$ are bounded from L^p to L^q.*

Proof. This is a consequence of Lemma 5.4. We use it in conjunction with Littlewood-Paley theory, the iterated version of the Fefferman-Stein vector-valued maximal function and the Marcinkiewicz-Zygmund theorem on vector-valued extensions of $L^p \to L^q$ bounded operators ([7], [22]). We use the pointwise estimate $|P_{k,j}g| \leq \mathfrak{M}g$ where $\mathfrak{M}$ denotes the strong maximal function. Let $F^{\rho,\sigma}$ be the fractional integral operator with distribution kernel $K_2^{\rho,\sigma}$. Then

$$\Big\|\mathcal{E}^1 f - \sum_{i=-4}^{4} \widetilde{\mathcal{E}}^{1,i} f\Big\|_q \lesssim \sum_{s \geq 0} \Big\| \Big(\sum_{j \in J} |(I - P_{j+s,j})\mathcal{L}_{j+s}\big(\mathcal{R}_j - \sum_{i=-4}^{4} \mathcal{R}_j^{j+s+i}\big) \widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f|^2 \Big)^{1/2} \Big\|_q$$

$$\lesssim \sum_{s \geq 0} 4^{-s} \Big\| \Big(\sum_{j \in J} \big[\mathfrak{M}F^{\rho,\sigma}[|\widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f|]\big]^2 \Big)^{1/2} \Big\|_q \lesssim \sum_{s \geq 0} 4^{-s} \Big\| \Big(\sum_{j \in J} \big[F^{\rho,\sigma}[|\widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f|]\big]^2 \Big)^{1/2} \Big\|_q$$

$$\lesssim \sum_{s \geq 0} 4^{-s} \Big\| \Big(\sum_{j \in J} |\widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f|^2 \Big)^{1/2} \Big\|_p \lesssim \|f\|_p.$$

The other estimates are proved in a similar way. $\square$

As a consequence of Lemma 5.5 it remains, in order to conclude the proof of Proposition 5.3, to show that the operators $\widetilde{\mathcal{E}}^{1,i}$, $\widetilde{\mathcal{E}}^{2,i}$, $\widetilde{\mathcal{E}}^{3,i}$ are bounded from L^p to L^q. We shall show that $\widetilde{\mathcal{E}}^{1,i}$ maps L^p to L^q. The proof of the boundedness of $\widetilde{\mathcal{E}}^{2,i}$ is very similar and will therefore be omitted. Finally, the arguments in the proof of Lemma 5.4 show the $L^p \to L^q$ boundedness of $\widetilde{\mathcal{E}}^{3,i}$; the details will be omitted as well.

Boundedness of $\widetilde{\mathcal{E}}^{1,i}$. We analyze the kernel of $\mathcal{L}_k(I - P_{k,j})\mathcal{R}_j^{k+i}$ which is given by

$$(5.20) \qquad K_{k,j,i}(x,y) = (2\pi)^{-\ell-1} \iint \iiint e^{i\varphi(x,t,h'',y,\tau,\lambda,\eta'')} a_{k,j,i}(x,t,h'',y,\tau,\lambda,\eta'')\, d\tau\, d\eta''\, d\lambda\, dh''\, dt$$

where

$$(5.21) \qquad \varphi(x,t,h'',y,\tau,\lambda,\eta'') = -t\lambda - \langle \eta'', h'' \rangle - \langle \tau, S(x'+tu, x''+h'', y') - y'' \rangle$$

and

$$(5.22) \quad a_{k,j,i}(x,t,h'',y,\tau,\lambda,\eta'') = a(\eta'')\omega(2^{-2k}|\eta''|)\omega(2^{-2(k+i)}|\tau|)|\tau|^\rho$$

$$\zeta_0(\varepsilon_2^{-1}|\tau/|\tau| - \vartheta|)\chi(x'+tu, x''+h'', y)\kappa_j(x'+tu, x''+h'', y)|x'+tu-y'|^{\sigma-d+\ell}(1 - \zeta_M(2^{-2k+j}|\lambda|))$$

with $\zeta_M = \sum_{s=-M}^{M} \zeta(2^s \cdot)$.

Claim. For $s \geq 0$, $i = -4, \ldots, 4$ we have

$$|K_{j+s,j,i}(x,y)| \lesssim 4^{-s} |K_2^{\rho,\sigma}(x,y)|$$

uniformly in j. Here the right hand side is defined in (4.3).

Taking the claim for granted we can argue as in the proof of Lemma 5.5 and obtain using Littlewood-Paley theory and the boundedness of the operator $F^{\rho,\sigma}$ with kernel $K_2^{\rho,\sigma}$

$$\big\|\widetilde{E}^{1,i} f\big\|_q = \Big\| \sum_{s>0} \sum_{j \in J} \mathcal{L}_{j+s}(I - P_{j+s,j})\mathcal{R}_j^{j+s+i} \widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f \Big\|_q$$

$$\lesssim \sum_{s>0} \Big\| \Big(\sum_{j \in J} \big|\mathcal{L}_{j+s}(I - P_{j+s,j})\mathcal{R}_j^{j+s+i} \widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f\big|^2 \Big)^{1/2} \Big\|_q$$

$$\lesssim \sum_{s>0} 4^{-s} \Big\| \Big(\sum_{j \in J} \big|F^{\rho,\sigma}[|\widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f|]\big|^2 \Big)^{1/2} \Big\|_q$$

$$\lesssim \sum_{s>0} 4^{-s} \Big\| \Big(\sum_{j \in J} \big|\widetilde{P}_{j+s,j} \widetilde{L}_{j+s} f\big|^2 \Big)^{1/2} \Big\|_p \lesssim \|f\|_p$$

We proceed to prove the pointwise estimate claimed above. We note that

$$(5.23) \qquad a_{k,j,i}(x,t,h'',y,\tau,\lambda,\eta'') = 0 \quad \text{if } |\lambda| \in [2^{2k-j-M+4}, 2^{2k-j+M-4}].$$

Now we first integrate by parts many times in (5.20) with respect to t; this is then followed by an integration by parts in the (λ, η'', τ) variables.

Note that because of $\langle u, \nabla_{w'} S(y', w'', y') \rangle = 0$ we may expand

$$\partial_t \varphi(x,t,h'',y',\tau,\lambda,\eta) = -\lambda - \langle u, \tau \cdot S_{x'}(x'+tu, x''+h'', y') \rangle$$
$$= -\lambda + \langle u, \tau \cdot S_{x'x'}(y', x''+h'', y')(x'+tu-y') \rangle + \tau \cdot r_1(x,y',t,h'')$$
$$(5.24) \qquad = -\lambda + \langle u, \tau \cdot S_{x'x'}(0,0,0)(x'+tu-y') \rangle + \tau \cdot \Big(\sum_{\nu=1,2} r_\nu(x,y',t,h'') \Big)$$

where

$$|r_1(x,y',t,h'')| \le C_1 |y'-x'-tu|^2$$
$$|r_2(x,y',t,h'')| \le C_1 \varepsilon^{10} |y'-x'-tu|.$$

Differentiating (5.7) we see that

$$\langle u, \widetilde{S}_{x'x'}(x,x') + \widetilde{S}_{x'y'}(x,x') \rangle = 0$$

and by (2.7-8) and the choice of u we deduce that

$$c_0 2^{2k-j-2} \le \big| \langle u, \tau \cdot S_{x'x'}(0,0,0)(x'+tu-y') \rangle \big| \le c_0^{-1} 2^{2k-j+3}$$

and consequently, by our choice of M

$$2^{2k-j-M+5} \le c_0 2^{2k-j-2} \le \big| \partial_t \varphi(x,t,h'',y',\tau,\lambda,\eta'') + \lambda \big| \le c_0^{-1} 2^{2k-j+3} \le 2^{2k-j+M-5}$$

on the support of the symbol; hence by (5.23)

$$|\partial_t \varphi(x,t,h'',y',\tau,\lambda,\eta'')| \gtrsim \max\{\lambda, 2^{2k-j}\}.$$

Moreover the higher derivatives of the phase functions are $O(2^{2k-j})$. Taking s derivatives of κ_j with respect to w' (in any direction) causes a blowup of size $O(2^{2js})$ which would be too much for our argument. Fortunately, in view of the assumption $\langle u, \nabla_{x'} S(y', w'', y') \rangle = 0$ we have the better estimate

$$(\langle u, \nabla_{w'} \rangle^s \kappa_j(w,y)) \rangle = O(2^{js}).$$

Thus we may perform integration by parts in the t variables and gain factors of size $2^{(2j-2k)N}$. This is then followed by an integration by parts in the frequency variables and we obtain

$$|K_{k,j,i}(x,y)| \lesssim 2^{j(d-\ell-\sigma)} 2^{2k\rho} 2^{-(2k-2j)N_1} \int \chi_j(x'+tu-y') \chi(x'+tu, x''+h'', y) \times$$
$$\frac{2^{2k-j}}{(1+2^{2k-j}|t|)^{N_2}} \frac{2^{2k\ell}}{(1+2^{2k}|h''|)^{N_2}} \frac{2^{2k\ell}}{(1+2^{2k}|y''-S(x'+tu,x''+h'',y')|)^{N_3}} dt dh''.$$

Now observe that

$$|S(x'+tu, x''+h'', y') - S(x,y')| \lesssim |h''| + 2^{-j}|t| + |t|^2$$

and therefore

$$\frac{2^{2k\ell}}{(1+2^{2k}|y''-S(x'+tu,x''+h'',y')|)^{N_3}} \lesssim \frac{2^{2k\ell}}{(1+2^{2k}|y''-S(x,y')|)^{N_3}}(1+2^{2k-j}|t|+|t|^2+2^{2k}|h''|)^{N_3}$$

This yields

$$|K_{k,j,i}(x,y)| \lesssim 2^{-(2k-2j)(N_1-\rho-\ell)}2^{j(d+\ell-\sigma+2\rho)}(1+2^{2k}|y''-S(x,y')|)^{-N_3} \times$$

$$\iint_{\mathbb{R}\times\mathbb{R}^\ell}\chi_j(x'-y'-tu)\chi(x''+tu,x''+h'',y)\frac{2^{2k-j}}{(1+2^{2k-j}|t|)^{N_2-N_3}}\frac{2^{2k\ell}}{(1+2^{2k}|h''|))^{N_2-N_3}}dt dh''$$

where χ_j denotes the characteristic function of $[2^{-j-1},2^{-j+1}]\cup[-2^{-j+1},-2^{-j-1}]$.

This integral is straightforward to estimate. Observe that $2^{j(d+\ell-\sigma+2\rho)}(1+2^{2k}|y''-S(x,y')|)^{-N_3}$ is bounded by $|y''-S(x,y')|^{-(d+\ell-\sigma+2\rho)/2}$; thus if $|x'-y'|\le C2^{-j}$ we use either this bound or the bound $2^{j(d+\ell-\sigma+2\rho)}$ and estimate $|K_{k,j,i}(x,y)|$ by $C2^{-(2k-2j)(N_1-\rho-\ell)}K_2^{\rho,\sigma}(x,y)$.

Next, if $C2^{-j}\le|x'-y'|\le\varepsilon$ and $|y''-S(x,y)|\le\varepsilon$ then $\chi_j(x'-y'-tu)$ vanishes unless $|t|\ge c|x'-y'|$. In this case the contribution of the t integral above is

$$O\big((2^{j-2k}|x'-y'|^{-1})^{N_2-N_3-1}\big)+O\big((2^{-2k}|x'-y'|^{-1})^{N_2-N_3-d+\ell}\big).$$

Thus in this case

$$|K_{k,j,i}(x,y)| \lesssim 2^{-(2k-2j)(N_1-\rho-\ell)}2^{j(d+\ell-\sigma+2\rho)}(1+2^j|x'-y'|)^{-2N}(1+2^{2k}|y''-S(x,y')|)^{-N}$$

where $2N=\min\{N_2-N_3-d+\ell,N_3\}$. We may choose $2N\le N_1+2d$ and $N\gg d$ and again the bound $|K_{k,j,i}(x,y)|$ by $C2^{-(2k-2j)(N_1-\rho-\ell)}K_2^{\rho,\sigma}(x,y)$ is straightforward. Thus we have established the pointwise estimate claimed above. This concludes the proof of Proposition 5.3.

Proof of Theorem 5.1, conclusion. We have to prove that $\mathcal{R}$ in (5.4) maps L^p to L^q; assuming the angular localization (5.1) in the $x'-y'$ variables. We split the identity operator as $E_0+\sum_\nu\Theta_\nu$ where $E_0=\eta_0(D'')$ and η_0 is compactly supported in $\{\eta'':|\eta''|\le 1000\}$. Moreover let $\Theta_\nu=a_\nu(D'')$ where a_ν is a constant coefficient symbol of order 0 supported in

$$\{\eta'':|\frac{\eta''}{|\eta''|}-\vartheta_\nu|\le\varepsilon^5,|\eta''|\ge 100\};$$

we can arrange this decomposition so that the sum in ν is extended over $O(\varepsilon^{-5(\ell-1)})$ terms. Clearly it suffices to bound $E_0\mathcal{R}$ and $\Theta_\nu\mathcal{R}$ for all ν. We first note that the argument of Lemma 5.2 shows that $E_0\mathcal{R}$ maps $L^p\to L^q$ if $(d+\ell)(1/p-1/q)\le\sigma-2\rho$.

It remains to consider $\Theta_\nu\mathcal{R}^\sigma$ for fixed ν. Let u_ν be a unit vector in $\mathbb{R}^\ell$ so that

$$|\langle u_\nu,\nabla_{x'}\rangle\langle v,\nabla_{y'}\rangle\vartheta_\nu\cdot S(0,0)|=\max_{U\in S^{d-\ell}}|\langle U,\nabla_{x'}\rangle\langle v,\nabla_{y'}\rangle\vartheta_\nu\cdot S(0,0)|.$$

Now denote by $\mathcal{Q}^\nu$ the change of variable $\mathcal{Q}(\cdot,u_\nu)$ as defined in §2.2, moreover define $\mathfrak{Q}_\nu h(w)=h(\mathcal{Q}_\nu w)$ for functions supported in B_{ε^9}. Let $\mathcal{R}^\nu=\mathfrak{Q}_\nu\mathcal{R}\mathfrak{Q}_\nu^{-1}$; then the assumptions of Proposition 5.3 apply to $\mathcal{R}^\nu$ (with $u=u_\nu$).

Define $\widetilde{\Theta}_\nu=\widetilde{a}_\nu(D'')$ so that $\widetilde{a}_\nu$ is supported in $\{\eta'':|\frac{\eta''}{|\eta''|}-\vartheta_\nu|\le\varepsilon^2,|\eta''|\ge 10\}$; and $\widetilde{a}_\nu(\eta'')=1$ if $|\frac{\eta''}{|\eta''|}-\vartheta_\nu|\le\varepsilon^2$ and $|\eta''|\ge 20\}$. Then by Proposition 5.3 and Lemma 5.2

(5.26) $$\|\widetilde{\Theta}_\nu\mathcal{R}^\nu\|_{L^p\to L^q}\le C(1+\|\mathcal{R}^\nu\|_{L^p\to L^q}^{1-\frac{p}{2}})$$

But in view of the support properties of the kernel of $\mathcal{R}$ and the local L^p and L^q boundedness of the operators $\mathfrak{Q}_\nu$ and $\mathfrak{Q}_\nu^{-1}$ we get

$$\|\mathcal{R}^\nu\|_{L^p \to L^q} \lesssim \|\mathcal{R}\|_{L^p \to L^q}.$$

To conclude the proof we split

$$\mathcal{R} = E_0 \mathcal{R} + \sum_\nu \Theta_\nu \mathfrak{Q}_\nu^{-1} \mathcal{R}^\nu \mathfrak{Q}_\nu$$

$$= E_0 \mathcal{R} + \sum_\nu \Theta_\nu \mathfrak{Q}_\nu^{-1} \widetilde{\Theta}_\nu \mathcal{R}^\nu \mathfrak{Q}_\nu + \sum_\nu \Theta_\nu \mathfrak{Q}_\nu^{-1} (I - \widetilde{\Theta}_\nu) \mathcal{R}^\nu \mathfrak{Q}_\nu.$$

By (5.26)

$$(5.27) \qquad \|\Theta_\nu \mathfrak{Q}_\nu^{-1} \widetilde{\Theta}_\nu \mathcal{R}^\nu \mathfrak{Q}_\nu\|_{L^p \to L^q} \lesssim 1 + \|\mathcal{R}\|_{L^p \to L^q}^{1-p/2}$$

and it remains to show that

$$(5.28) \qquad \|\Theta_\nu \mathfrak{Q}_\nu^{-1} (I - \widetilde{\Theta}_\nu) \mathcal{R}^\nu \mathfrak{Q}_\nu\|_{L^p \to L^q} \lesssim 1.$$

Now let $L_0 = \omega_0(|D''|)$ and $L_k = \omega(4^{-k}|D''|)$. We analyze the kernel of $L_k \Theta_\nu \mathfrak{Q}_\nu^{-1} (I - \widetilde{\Theta}_\nu) L_{k'}$, denoted by $H_{k,k',\nu}(x', x'', y'')$. The inverse change of variable $\mathcal{Q}_\nu^{-1}$ is of the form $x \mapsto (x', \mathcal{G}_\nu(x))$, with $\|(\mathcal{G}_\nu)_{x''} - I_{\ell,\ell}\| \le \varepsilon^7$ (cf. (2.17/18)). Thus $H_{k,k',\nu}$ is given by

$$H_{k,k',\nu}(x', x'', y'') =$$
$$\iiint e^{i(\langle x'' - z'', \eta'' \rangle + \langle \mathcal{G}_\nu(x', z'') - y'', \xi'' \rangle)} \omega(4^{-k}|\eta''|)\omega(4^{-k'}|\xi''|) a_\nu(\eta'')(1 - \widetilde{a}_\nu(\xi'')) dz'' d\xi'' d\eta''$$

The z''-gradient of the phase function is of size $\approx \max\{4^k, 4^{k'}\}$, therefore we may argue as in the proof of Lemma 5.4 above. In particular, after additional integration by parts in ξ'', η'' when x is large we obtain that

$$|H_{k,k',\nu}(x', x'', y'')| \lesssim \min\{4^{-kN_1}, 4^{-k'N_1}\}(1 + |x''|)^{-N_2}.$$

In view of the localization properties of $\mathcal{R}^\nu$ and the L^p boundedness of $\mathcal{R}^\nu$ it follows that

$$\|L_k \Theta_\nu \mathfrak{Q}_\nu^{-1} (I - \widetilde{\Theta}_\nu) L_{k'} \mathcal{R}^\nu \mathfrak{Q}_\nu\|_{L^p \to L^q} \lesssim \min\{4^{-k}, 4^{-k'}\}$$

and as a consequence (5.28) holds.

Putting all the estimates together we obtain that

$$(5.29) \qquad \|\mathcal{R}\|_{L^p \to L^q} \lesssim 1 + \|\mathcal{R}\|_{L^p \to L^q}^{1-p/2}$$

and since we already know the finiteness of $\|\mathcal{R}\|_{L^p \to L^q}$ the estimate (5.29) implies a bound uniform in the family J. $\square$

We can now give the

Proof of Theorem 1.2.3. By summing geometrical series we see from Lemma 4.3 that the operator $\sum_{m \ge 0} \sum_{k > m} T_{k,m}$ can be pointwise bounded by a combination of operators handled in Theorem 5.1; in this calculation we use that ρ is negative. Moreover the operator $\sum_{m \ge 0} \sum_{k \le m} T_{k,m}$ is bounded by Proposition 4.4. The assertion 1.2.3 follows. $\square$

Necessary conditions. The necessity of the conditions in Theorems 1.1 and 5.1 follows from standard examples. For the sake of completeness we shall briefly describe them. We assume that $\rho \leq 0$ and $1 \leq p \leq q \leq \infty$ and consider the operator $R^{\rho,\sigma}$. We remark that for the case $\rho > 0$, the conditions in 1.2.1 also cannot be improved. This is because any strict improvement would yield to an improvement in the case $\rho = 0$, by interpolation with the estimates for a negative ρ_1 close to 0.

Let B_δ be the ball of radius $\delta \ll \varepsilon^{10}$, centered at the origin, and let χ_δ be the characteristic function of B_δ. Then $\|\chi_\delta\|_p \gtrsim \delta^{d/p}$ and $R^{\rho,\sigma}\chi_\delta \gtrsim \delta^{d-\ell-\rho}$ on the set $\{x : |x'| \leq \delta^2, |x'' - \mathfrak{S}(0,x')| \leq c\delta\}$ for small c. Thus $\|R^{\rho,\sigma}\chi_\delta\|_q \gtrsim \delta^{d-\ell-\rho-(d-\ell)/q}$ and we see that the condition $d/p - \ell/q \leq d - \ell - \rho$ is necessary. By applying the same example to the adjoint operator we get the necessary condition $\ell/p - d/q \leq -\rho$. Thus $(1/p, 1/q)$ belongs to the pentagon with corners $(1,1)$, $(0,0)$, $(1, \frac{\rho+\ell}{\ell})$, $(\frac{-\rho}{\ell}, 0)$ and $(\frac{d-\rho}{d+\ell}, \frac{\rho+\ell}{d+\ell})$ and this pentagon becomes the triangle in Theorem 1.1 when $\rho = 0$.

If $\rho < 0$ then the operator $R^{\rho,\sigma}$ is not bounded from L^1 to $L^{\ell/\ell+\rho}$ as one checks that one has the lower bound $R^{\rho,\sigma}\chi_\delta \gtrsim \delta^d |x'' - \mathfrak{S}(0,x')|^{-\rho-\ell}$ if $C\delta \leq |x'' - \mathfrak{S}(0,x)| \leq \varepsilon$, with C large. By applying this to the adjoint operator it follwos that $R^{\rho,\sigma}$ is not bounded from $L^{-\rho/\ell}$ to L^∞.

Next let P_δ be the plate $\{y : |y'| \leq \delta, |y''| \leq \delta\}$ and let f_δ be the characteristic function of P_δ, thus $\|f_\delta\|_p \lesssim \delta^{(d+\ell)/p}$. One checks that in a fixed fraction of P_δ one has the lower bound $R^{\rho,\sigma}f_\delta(x) \gtrsim \delta^{\sigma-2\rho}$; in this calculation we use (2.2) and (2.6). Thus $\|R^{\rho,\sigma}f_\delta\|_q \gtrsim \delta^{\sigma-2\rho+(d+\ell)/q}$ and the condition $(d+\ell)(1/p - 1/q) \leq \sigma - 2\rho$ is necessary. This concludes the proof of necessity in Theorem 1.1 and Theorem 5.1.

A third necessary condition for the $L^p \to L^q$ boundedness of $R^{\rho,\sigma}$ is $(d - \ell)(1/p - 1/q) \leq \sigma$. To see this let g_δ be the characteristic function of $\{y : |y'| \leq \delta, |y''| \leq \varepsilon\}$. Then $R^{\rho,\sigma}g_\delta \geq \delta^\sigma$ for all x in a fixed fraction of this set and from this one deduces the necessity of the condition $(d - \ell)(1/p - 1/q) \leq \sigma$. Notice that the condition $(d - \ell)(1/p - 1/q) \leq \sigma$ is more restrictive than $(d + \ell)(1/p - 1/q) \leq \sigma - 2\rho$ if and only if $\sigma < -\rho(d - \ell)/\ell$; thus this example is only relevant to show the sharpness of 1.2.4.

6. L^p estimates for Fourier integral operators

It will be convenient to introduce some normalized classes of symbols.

Let $k > 0$ and $0 < m < k$. Then we denote by $\mathcal{S}_{k,m}$ the class of symbols $a(x, y, \xi, \tau)$ supported in

$$(6.1) \quad \begin{cases} \{(x, y, \tau, \xi) : |x| + |y| \leq \varepsilon, 2^{2k-1} \leq |\tau| \leq 2^{2k+1}, 2^{m-1} \leq |\xi| \leq 2^{m+1}\} & \text{if } 0 < m < k, \\ \{(x, y, \tau, \xi) : |x| + |y| \leq \varepsilon, 2^{2k-1} \leq |\tau| \leq 2^{2k+1}, |\xi| \leq 2\} & \text{if } m = 0. \end{cases}$$

for which (1.11) holds, with $\rho = \sigma = 0$. Moreover, if $m > 0$ let Σ_m be the class of symbols $a(x, y, \xi, \tau)$ supported in

$$(6.2) \quad \{(x, y, \tau, \xi) : |x| + |y| \leq \varepsilon, |\tau| \leq 2^{2m+1}, 2^{m-1} \leq |\xi| \leq 2^{m+1}\}$$

such that (1.11) holds with $\rho = \sigma = 0$.

We recall that $\mathcal{T}[a]$ denotes the integral operator with kernel (1.10).

L^2 estimates.

We shall assume that $a \in \mathcal{I}^{\rho,-\sigma}$ and begin by proving L^2 estimates. These are quick consequences of what is already proved in [11], and we shall be brief. It is shown in in [11] that L^2 boundedness holds if $2\rho - \sigma \leq 0$, $0 \leq \sigma < d - \ell$. While the endpoint estimate corresponding to $(\rho, \sigma) = ((d - \ell)/2, d - \ell)$ may fail the proof of the estimates in [11] still provides useful information which will be used in an interpolation argument in §7.

Lemma 6.1. *(i) Let $a_m \in \Sigma_m$ and suppose that $\sup_{m \geq 1} |c_m| \leq 1$. Then $\sum_{m=1}^\infty c_m \mathcal{T}[a_m]$ is bounded on L^2.*

(ii) Let $a \in S^{\frac{d-\ell}{2}, \ell-d}$ and suppose that $a(x, y, \tau, \xi) = 0$ if $|\tau| \geq |\xi|^2$. Then $\mathcal{T}[a]$ is bounded on L^2.

Proof. We note that the phase function $\Phi(x, y, \xi, \tau) = \langle \xi, x' - y' \rangle + \langle \tau, y'' - S(x, y') \rangle$ parametrizes the diagonal in $T^*\Omega \times T^*\Omega$ as a Lagrangian manifold; that is $\{(x, \Phi_x, y, -\Phi_y) : \Phi_\xi = 0, \Phi_\tau = 0\}$ is a subset of $\{(x, \xi, x\xi)\}$.

Because of the support restriction of a_m the symbol $\sum_{m>0} c_m a_m$ belongs to the Calderón-Vaillancourt symbol class $S^0_{1/2, 1/2}$. It is shown in the proof of Proposition 2.7 in [11] that Hörmander's equivalence of phase function theorem remains valid with $S^0_{1/2, 1/2}$ symbols and that consequently $\sum_{m=1}^{\infty} c_m \mathcal{T}[a_m]$ is a pseudodifferential operator of order 0, with symbols of type $(1/2, 1/2)$. Thus the L^2 boundedness follows from the Calderón-Vaillancourt theorem. (ii) is an immediate consequence of (i). $\square$

Lemma 6.2. (i) Let $m_0 \geq 0$ be fixed and for $k > m_0$ let $m(k)$ be an integer such that $m_0 \leq m(k) < k$. Suppose that $\sup_{k \geq 1} |c_k| \leq 1$ and that $\alpha_k \in \mathcal{S}_{k, m(k)}$.

Then the operator $\sum_{k>m_0} c_k 4^{k\frac{d-\ell}{2}} 2^{-m(k)(d-\ell)} \mathcal{T}[\alpha_k]$ is bounded on L^2, with norm independent of the chosen sequence $\{m(k)\}$.

(ii) Suppose $a \in S^{\frac{d-\ell}{2}, \ell-d}$ and suppose that $a(x, y, \tau, \xi) = 0$ if $|\tau| \leq C|\xi|^{1/2}$, and, for $m > 0$, let $a_m(x, y, \tau, \xi) = \zeta(2^{-m}|\xi|) a(x, y, \tau, \xi)$. Then $\mathcal{T}[a_m]$ is bounded on L^2 with operator norm independently on m.

(iii) Let $\{\alpha_k\}$ be as in (i) and let $\eta \in S^0_{1/2, 1/2}(\Omega \times \Omega, \mathbb{R}^d)$. Then the statement in (ii) remains valid if α_k is replaced by $\eta \alpha_k$.

Proof. For (i) we note that the kernel of $\mathcal{T}[\alpha_k]$ is given by

(6.3)
$$\int e^{i\langle \tau, y'' - S(x, y') \rangle} b_k(x, y, \tau) \, d\tau$$

where

(6.4)
$$b_k(x, y, \tau) = \int \alpha_k(x, y, \tau, \xi) e^{i\langle x' - y', \xi \rangle} \, d\xi.$$

Note that for every k the ξ integration is extended over a dyadic annulus $\{\xi : |\xi| \approx 2^{m(k)}\}$ and thus $|b_{k,m}(x, y, \tau)| \lesssim 4^{k(d-\ell)/2} \approx |\tau|^{(d-\ell)/2}$. Moreover, by examining the derivatives of $b_{k,m}$ one checks as in [11] that b_k is a symbol of order $(d-\ell)/2$ and type $(1/2, 1/2)$. Since the phase function involves ℓ frequency variables one may argue as in [11] and deduce that $\sum_{k \geq m_0} c_k \mathcal{T}[\alpha_k]$ are Fourier integral operators of order 0 and type $(1/2, 1/2)$, hence bounded in L^2 (with bounds independent of the sequence $\{\alpha_k\}$).

Part (ii) follows from part (i) with the choice $m(k) = m$ if we observe that the symbols a_m with the assumed support property can be decomposed as $C \sum_{k>m} 2^{k(d-\ell)/2} 2^{m(\ell-d)} c_{k,m} a_{k,m}$ where $c_{k,m} \leq 1$ and $a_{k,m} \in \mathcal{S}_{k,m}$. Clearly the above argument also proves (iii). $\square$

Remark. The variant (iii) is included in order to cover localizations of the form $a_{k,m}(x, y, \tau, \xi) \zeta(2^j(|x' - y'|))$ if $j \leq k$; these are of type $(1/2, 1/2)$ since $a_{k,m}$ is supported where $\tau \approx 2^{2k}$.

$H^1 \to L^1$ estimates.

Lemma 6.3. Suppose $0 \leq \sigma < d - \ell$, $a \in S^{0, -\sigma}$, and suppose that $a(x, y, \tau, \xi)$ is supported where $|\xi| \geq \frac{1}{2}|\tau|^{1/2}$. Let

(6.5)
$$a_m(x, y, \tau, \xi) = \begin{cases} a(x, y, \tau, \xi) \zeta(2^{-m}|\xi|) & \text{if } m > 0 \\ a(x, y, \tau, \xi) \zeta_0(|\xi|) & \text{if } m = 0 \end{cases}.$$

Then $\mathcal{T}[a_m]$ maps L^1 boundedly to L^1, with operator norm $O((1+m)2^{-m\sigma})$.

Proof. The kernel K_m can be written as $\sum_{k \leq m} K_{k,m}$ where $K_{k,m}$ is as in (4.6) and satisfies (4.7) with $\rho = 0$. The operator with kernel $K_{k,m}$ is clearly bounded on L^1, with norm $O(2^{-m\sigma})$. $\square$

Lemma 6.4. *Suppose* $a \in S^{0,-\sigma}$, $0 \leq \sigma < d - \ell$ *and suppose that* $a(x, y, \tau, \xi)$ *is supported where* $|\xi| \leq 2|\tau|^{1/2}$. *Let* a_m *be as in (6.5). Then* $\mathcal{T}[a_m]$ *maps* H^1 *boundedly to* L^1, *with operator norm dominated by* $C2^{-m\sigma}$.

Proof. By the theorem on the atomic decomposition ([7], [22]) it suffices to estimate $\mathcal{T}[a_m]f_Q$ where f_Q is an L^2 function supported on a cube Q with center y_Q and sidelength $\delta_Q \ll 1$ so that $\|f_Q\|_2 \leq \delta_Q^{-d/2}$ and $\int f_Q dx = 0$.

We define the exceptional set

$$W_Q = \{x : |x' - (y_Q)'| \leq \varepsilon, |x'' - \mathfrak{S}(y_Q, x')| \leq C\delta_Q\};$$

for large but fixed C; on this set we shall use a mixed norm $L^1(L^2)$ estimate.

We define phase functions and amplitudes on $\mathbb{R}^\ell$ depending on the parameters x', y'. Let

$$b_m^{x',y'}(x'', y'', \tau) = \int a_m(x', x'', y', y'', \tau, \xi)e^{i\langle x' - y', \xi\rangle} d\xi$$

and

$$\Phi^{x',y'}(x'', y'', \tau) = \langle \tau, S(x', x'', y') - y''\rangle.$$

Denote by $\mathcal{T}_m^{x',y'}$ the operator with kernel

$$K_m^{x',y'}(x'', y'') = \int e^{i\Phi^{x',y'}(x'',y'',\tau)} b_m^{x',y'}(x'', y'', \tau)d\tau.$$

By an integration by parts one sees that

$$|\partial_{x'',y''}^\alpha \partial_\tau^\beta b_m^{x',y'}| \leq C_{\alpha,\beta} \frac{2^{m(d-\ell-\sigma)}}{(1 + 2^m|x' - y'|)^N}$$

and by the standard theory for pseudodifferential operators and their behavior under changes of variables it follows that

$$\left\|\mathcal{T}_m^{x',y'}\right\|_{L^2(\mathbb{R}^\ell)\to L^2(\mathbb{R}^\ell)} \lesssim \frac{2^{m(d-\ell-\sigma)}}{(1 + 2^m|x' - y'|)^N}.$$

We now estimate the contribution on W_Q. For fixed x' set $W_Q^{x'} = \{x'' : (x', x'') \in W_Q\}$. Let $f^{y'}(y'') = f(y', y'')$, then

$$\mathcal{T}_m f_Q(x', x'') = \int_{y'} \mathcal{T}_m^{x',y'} f^{y'} dy'.$$

On W_Q we bound

$$\int_{W_Q} |\mathcal{T}_m f_Q(x)|dx \leq \int_{\substack{|x'-(y_Q)'| \\ \leq \varepsilon}} \int_{\substack{|x''-\mathfrak{S}(y_Q,x')| \\ \leq C\delta_Q}} \int |\mathcal{T}_m^{x',y'} f_Q^{y'}(x')|dy'dx''dx'$$

$$\lesssim \delta_Q^{\frac{\ell}{2}} \int \left(\int \left|\int |\mathcal{T}_m^{x',y'} f^{y'}(x'')|dy'\right|^2 dx''\right)^{1/2} dx'$$

$$\lesssim \delta_Q^{\ell/2} \int\int \left(\int |\mathcal{T}_m^{x',y'} f_Q^{y'}(x'')|^2 dx''\right)^{1/2} dx'dy'$$

$$\lesssim \delta_Q^{\ell/2} \int\int \frac{2^{m(d-\ell-\sigma)}}{(1 + 2^m|x' - y'|)^N}\left(\int_{y''} |f_Q^{y'}(y'')|^2 dy''\right)^{1/2} dx'dy'$$

$$\lesssim 2^{-m\sigma}\delta_Q^{\ell/2} \int_{x'} \left(\int_{y''} |f_Q^{x'}(y'')|^2 dy''\right)^{1/2} dx'$$

(6.6)
$$\lesssim 2^{-m\sigma}\delta_Q^{d/2}\|f_Q\|_2 \lesssim 2^{-m\sigma}.$$

On the complement of W_Q we use the kernel estimates of Lemma 4.3.

We split $a_m = \sum_{k \geq m-1} a_{k,m}$ where the kernel $K_{k,m}$ of $\mathcal{T}[a_{k,m}]$ satisfies the estimate (4.7) with $\rho = 0$. Consequently since $|x'' - \mathfrak{S}(y, x')| \approx |y'' - S(x, y')|$ we have

$$(6.7) \qquad \int_{W_Q^c} |\mathcal{T}_m^k f_Q(x)| dx \lesssim 4^{-k} \delta_Q^{-1} 2^{-m\sigma} \|f_Q\|_1 \quad \text{if } 4^k \delta_Q \geq 1.$$

From the gradient estimates in (4.8) and by using the cancellation property of the atom f_Q we get

$$(6.8) \qquad \int |\mathcal{T}_m^k f_Q(x)| dx \lesssim 4^k \delta_Q 2^{-m\sigma} \|f_Q\|_1 \quad \text{if } 4^k \delta_Q \leq 1,$$

and the asserted $H^1 \to L^1$ bound follows from (6.6), (6.7) and (6.8). $\square$

Corollary 6.5. *Suppose that $0 < \rho < (d - \ell)/2$ and $\sigma > 2\rho$. Then $T \in \mathcal{I}^{\rho, -\sigma}$ is bounded on $L^{\frac{d-\ell}{d-\ell-\rho}}$ and bounded on $L^{\frac{d-\ell}{\rho}}$.*

Proof. We shall prove the $L^{\frac{d-\ell}{d-\ell-\rho}}$ boundedness; by §2.3 this also implies the $L^{\frac{d-\ell}{\rho}}$ boundedness.

Let $a \in S^{\rho, -\sigma}$ and let a_m be as in (6.5). Define

$$a_{m,z}(x, y, \tau, \xi) = a_m(x, y, \tau, \xi)(1 + |\tau|^2 + |\xi|^2)^{(\rho_0(1-z)+\rho_1 z - \rho)/2}(1 + |\xi|^2)^{(\sigma - \sigma_0(1-z) - \sigma_1 z)/2}$$

where $\sigma_0 = \frac{d-\ell}{d-\ell-2\rho}(\sigma - 2\rho)$, $\sigma_1 = d - \ell$, $\rho_0 = 0$ and $\rho_1 = (d - \ell)/2$. Then $a_{m,\theta} = a_m$ for $\theta = 2\rho/(d - \ell)$. For $\mathrm{Re}(z) = 0$ the symbol $a_{m,z}$ belongs to S^{ρ_0, σ_0} and for $\mathrm{Re}(z) = 1$ it belongs to S^{ρ_1, σ_1}. By Lemma 6.3 and Lemma 6.4 the operator $\mathcal{T}[a_{m,z}]$ is bounded from H^1 to L^1, with norm $(1 + m)2^{-m\sigma_0}$ if $\mathrm{Re}(z) = 0$. By Lemma 6.1 and Lemma 6.2 it is bounded on L^2 with norm $O(1)$ if $\mathrm{Re}(z) = 1$. By interpolation we find that $\mathcal{T}[a_m]$ is bounded on $L^{\frac{d-\ell}{d-\ell-\rho}}$ with norm $O((1+m)2^{-m\sigma_0(1-\theta)}) = O((1+m)2^{-m(\sigma-2\rho)})$. The assertion follows by summing in m. $\square$

7. $L^p \to L^q$ estimates for Fourier integral operators

We begin by giving a different formulation of parts 1.2.1 and 1.2.2 of Theorem 1.2. Suppose that $0 < \rho < (d - \ell)/2$ and $2\rho < \sigma < d - \ell$. Then statement 1.2.1 of Theorem 1.2 says that $T \in \mathcal{I}^{\rho, -\sigma}$ maps $L^p \to L^q$ if $(1/p, 1/q)$ belongs to the closed trapezoid with corners $(\frac{\rho}{d-\ell}, \frac{\rho}{d-\ell})$, $(\frac{d-\ell-\rho}{d-\ell}, \frac{d-\ell-\rho}{d-\ell})$, $(1/p_{\rho,\sigma}, 1/q_{\rho,\sigma}), (1/q'_{\rho,\sigma}, 1/p'_{\rho,\sigma})$ where

$$(7.1) \qquad \begin{aligned} \frac{1}{p_{\rho,\sigma}} &= \frac{d-\ell-\rho}{d-\ell} - \frac{(\sigma - 2\rho)\ell}{(d+\ell)(d-\ell)} \\ \frac{1}{q_{\rho,\sigma}} &= \frac{d-\ell-\rho}{d-\ell} - \frac{(\sigma - 2\rho)d}{(d+\ell)(d-\ell)}. \end{aligned}$$

Observe that

$$(7.2) \qquad \frac{1}{p_{\rho_0,\sigma_0}} = \frac{1}{q_{\rho_0,\sigma_0}} = \frac{1}{2} \quad \text{if} \quad \rho_0 = \frac{d-\ell}{2}, \quad \sigma_0 = d - \ell,$$

and if

$$(7.3) \qquad \rho_1 = 0, \quad \sigma_1 = (\sigma - 2\rho)\frac{d-\ell}{d-\ell-2\rho}, \quad \theta = \frac{d-\ell-2\rho}{d-\ell}$$

then $2\rho < \sigma < d - \ell$ implies $0 < \sigma_1 < d - \ell$ and we compute that

$$(7.4) \qquad (1-\theta)\Big(\frac{1}{p_{\rho_0,\sigma_0}}, \frac{1}{q_{\rho_0,\sigma_0}}\Big) + \theta\Big(\frac{1}{p_{\rho_1,\sigma_1}}, \frac{1}{q_{\rho_1,\sigma_1}}\Big) = \Big(\frac{1}{p_{\rho,\sigma}}, \frac{1}{q_{\rho,\sigma}}\Big).$$

Therefore, one would like to prove Theorem 1.2 by interpolation from an $L^{p_1} \to L^{q_1}$ result for operators in $\mathcal{I}^{0,-\sigma_1}$ (already proved only for the case of weakly singular Radon transforms) and an L^2 result for operators in $\mathcal{I}^{\frac{d-\ell}{2},\ell-d}$. Unfortunately, operators in the latter class may fail to be bounded on L^2; this somewhat complicates the interpolation argument.

Performing a finite finite conic partition of unity in the τ variables we may assume that

$$\text{supp } a \subset \big\{(x,y,\tau,\xi) : |x| + |y| \leq \varepsilon^{10}, |\tau| + |\xi| \geq 2^{M+10}, \big|\tfrac{\tau}{|\tau|} - \vartheta\big| \leq \varepsilon + |\tau|^{-1}\big\},$$

for some given unit vector ϑ in $\mathbb{R}^\ell$, and M is chosen as in §5.

We shall now set up the various interpolation arguments. We fix ρ and σ and use the abbreviation

$$(p,q) = (p_{\rho,\sigma}, q_{\rho,\sigma}), \quad (p_i,q_i) = (p_{\rho_i,\sigma_i}, q_{\rho_i,\sigma_i}), \quad i = 1,2.$$

We may split $T = T_{FIO} + T_{PsDO}$ where T_{FIO} corresponds to a symbol which is supported where $|\tau|^{1/2} \geq |\xi|/2 + 2^{M+5}$ and T_{PsDO} corresponds to a symbol supported in the complementary region. Thus $T_{PsDO} = \mathcal{T}[b]$ where b vanishes if $|\tau|^{1/2} \geq 2|\xi| + 10$. Let

$$W_z(\xi,\tau) = (1 + |\tau|^2 + |\xi|^2)^{(\rho_0(1-z)+\rho_1 z - \rho)/2}(1 + |\xi|^2)^{(\sigma - \sigma_0(1-z) - \sigma_1 z)/2}$$

and $b_z(x,y,\tau,\xi) = b(x,y,\tau,\xi)W_z(\xi,\tau)$, so that $W_\theta = 1$. By Lemma 6.1 the operator $\mathcal{T}[b_z]$ is bounded on L^2 if $\text{Re}(z) = 0$ and by Proposition 4.4 it is bounded from L^{p_1} to L^{q_1} if $\text{Re}(z) = 1$; all bounds are of admissible growth in z. Thus T_{PsDO} maps L^p to L^q by analytic interpolation.

Now we consider $T_{FIO} = \mathcal{T}[a]$ where a vanishes if $|\tau|^{1/2} \leq \max\{2^M, |\xi|/2\}$. We first split off another operator which behaves like T_{PsDO}. Let $a_z = aW_z$ and $a_{k,m,z} = \beta_{k,m}a_z$ where $\beta_{k,m}$ is as in (4.5). Also let

$$a_{k,m,j,z}(x,y,\tau,\xi) = a_{k,m,z}(x,y,\tau,\xi)\zeta(2^j|x'-y'|)$$
$$\widetilde{a}_{k,m,z}(x,y,\tau,\xi) = a_{k,m,z}(x,y,\tau,\xi)\zeta_0(2^k|x'-y'|)$$

Let

$$\mathcal{V}_{s,z} := \sum_{k \geq s} \mathcal{T}[\widetilde{a}_{k,k-s,z}]$$

By Lemma 6.2 (i), with the choice $m(k) = k - s$, the operator $\mathcal{V}_{s,z}$ is bounded on L^2, uniformly in s, if $\text{Re}(z) = 0$. By Lemma 4.5 it is bounded from $L^{p_1} \to L^{q_1}$ if $\text{Re}z = 1$; the bound is $O(2^{-s(d-\ell-\sigma_1)})$; all bounds are admissible in z. Interpolating we see that $\mathcal{V}_{s,\theta}$ maps $L^p \to L^q$ with norm $O(2^{-s(d-\ell-\sigma_1)\theta}) = O(2^{-s(d-\ell-\sigma)})$; hence $\sum_{k,m} \mathcal{T}[\widetilde{a}_{k,m}]$ maps L^p to L^q.

It remains to estimate the operator $\sum_{k>0} \sum_{m<k} \sum_{j<k} \mathcal{T}[a_{k,m,j,z}]$. We wish to use an angular Littlewood-Paley decomposition as in the proof of Proposition 5.3. Given a unit vector v in $\mathbb{R}^{d-\ell}$ we make an angular localization in $x' - y'$. By employing a finite partition of unity it then suffices to bound $\sum_{k>0} \sum_{m<k} \sum_{j<k} \mathcal{T}[\alpha_{k,m,j,z}]$ where

$$\alpha_{k,m,j,z}(x,y,\tau,\xi) = a_{k,m,j,z}(x,y,\tau,\xi)\zeta_0\Big(\varepsilon^{-5}\Big|\frac{x'-y'}{|x'-y'|} - v\Big|\Big).$$

We choose u as in (5.6) and perform the change of variable $w \mapsto (w', w'' + F(w;u)) \equiv \mathcal{Q}(w)$ in §2.2, and define $\mathfrak{Q}h(z) = h(\mathcal{Q}(z))$.

As a result we have to show the $L^p \to L^q$ bound for the operator

$$(7.5) \qquad \sum_{k>0} \sum_{m<k} \sum_{j<k} \mathfrak{Q}\mathcal{T}[\alpha_{k,m,j,z}]\mathfrak{Q}^{-1} = \sum_{k>0} \sum_{m<k} \sum_{j<k} \mathcal{T}_{k,m,j}^z$$

which has kernel

$$\sum_{k>0} \sum_{m<k} \sum_{j<k} \iint e^{i[\langle \tau, y'' - \widetilde{S}(x,y') \rangle + \langle x' - y', \xi \rangle]} \widetilde{\alpha}_{k,m,j,z}(x,y,\tau,\xi)\, d\tau d\xi$$

where $\langle u, \widetilde{S}_{x'}(x,x') \rangle = \langle u, \widetilde{S}_{y'}(x,x') \rangle = 0$ and $\widetilde{\alpha}_{k,m,j,z}(x,y,\tau,\xi) = \alpha_{k,m,j,z}(\mathcal{Q}(x), \mathcal{Q}(y), \tau, \xi)\, g(x)/g(w)$, and g is smooth and positive.

We now use a Littlewood-Paley operators L_k defined by $L_k = \sum_{i=-4}^4 \omega(4^{-k+i}|D''|)$ and also the angular the Littlewood-Paley operator $P_{k,j}$ defined in (5.8). Let

$$\mathcal{T}_{k,m,j} = \mathcal{T}[\widetilde{\alpha}_{k,m,j,\theta}].$$

We split

$$\sum_{k,m,j} \mathcal{T}_{k,m,j} = \sum_{k,m,j} L_k \mathcal{T}_{k,m,j} L_k + \sum_{k,m,j} (I - L_k)\mathcal{T}_{k,m,j} L_k + \sum_{k,m,j} \mathcal{T}_{k,m,j}(I - L_k)$$

and then

$$\sum_{k,m,j} L_k \mathcal{T}_{k,m,j} L_k = (I + II) + (III + IV) + (V + VI)$$

where

$$I + II = \Big[\sum_{\substack{k,m,j \\ m \leq j}} + \sum_{\substack{k,m,j \\ m > j}} \Big] L_k P_{k,j} \mathcal{T}_{k,m,j} P_{k,j} L_k$$

$$III + IV = \Big[\sum_{\substack{k,m,j \\ m \leq j}} + \sum_{\substack{k,m,j \\ m > j}} \Big] L_k (I - P_{k,j})\mathcal{T}_{k,m,j} P_{k,j} L_k$$

$$V + VI = \Big[\sum_{\substack{k,m,j \\ m \leq j}} + \sum_{\substack{k,m,j \\ m > j}} \Big] L_k \mathcal{T}_{k,m,j}(I - P_{k,j}) L_k.$$

We then split $I = \sum_{s \geq 0} I_s$ by linking $m = j - s$ for $s \geq 0$ and prove bounds for the expressions I_s which decay in s. Similarly we split II setting $j = m - s$. The expressions III, IV, V, VI are split into a double series depending on nonnegative parameters r, s; we prove then decay in r, s. We set $j = k - r$, $m = k - r - s$ when estimating III and V and $j = k - r - s$, $m = k - r$ when estimating IV and VI. In the following proposition we state the relevant estimates for the pieces.

Proposition 7.1. *Let* $0 \leq \rho < (d - \ell)/2$ *and* $2\rho < \sigma < d - \ell$ *and let* $p = p_{\rho,\sigma}$, $q = q_{\rho,\sigma}$. *There is* $\delta = \delta(\rho, \sigma) > 0$ *so that the following estimates hold.*

(i) *For* $s \geq 0$

$$(7.6) \qquad \Big\| \sum_{k>s} \sum_{s \leq j < k} L_k P_{k,j} \mathcal{T}_{k,j-s,j} P_{k,j} L_k \Big\|_{L^p \to L^q} \lesssim 2^{-s\delta}$$

(ii) *For* $s \geq 0$

$$(7.7) \qquad \Big\| \sum_{k>s} \sum_{s \leq m < k} L_k P_{k,m-s} \mathcal{T}_{k,m,m-s} P_{k,m-s} L_k \Big\|_{L^p \to L^q} \lesssim 2^{-s\delta}$$

(iii) For $s \geq 0$, $r \geq 0$,

$$(7.8) \qquad \Big\| \sum_{k > s+r} L_k(I - P_{k,k-r})\mathcal{T}_{k,k-r-s,k-r}P_{k,k-r}L_k \Big\|_{L^p \to L^q} \lesssim 2^{-(r+s)\delta}$$

$$(7.9) \qquad \Big\| \sum_{k > s} L_k \mathcal{T}_{k,k-r-s,k-r}(I - P_{k,k-r})L_k \Big\|_{L^p \to L^q} \lesssim 2^{-(r+s)\delta}.$$

(iv) For $s \geq 0$, $r \geq 0$,

$$(7.10) \qquad \Big\| \sum_{k > s+r} L_k(I - P_{k,k-r-s})\mathcal{T}_{k,k-r,k-r-s}P_{k,k-r-s}L_k \Big\|_{L^p \to L^q} \lesssim 2^{-r-s}$$

$$(7.11) \qquad \Big\| \sum_{k > 0} L_k \mathcal{T}_{k,k-r,k-r-s}(I - P_{k,k-r-s})L_k \Big\|_{L^p \to L^q} \lesssim 2^{-r-s}.$$

(v) For $j < k$, $m < k$

$$(7.12) \qquad \|(I - L_k)\mathcal{T}_{k,m,j}L_k\|_{L^p \to L^q} \lesssim 2^{-k}$$

$$(7.13) \qquad \|\mathcal{T}_{k,m,j}(I - L_k)\|_{L^p \to L^q} \lesssim 2^{-k}.$$

Taking Proposition 7.1 for granted we can complete the

Proof of Theorem 1.2. Let $p_{\rho,\sigma}$ and $q_{\rho,\sigma}$ be as in (7.1). A combination of the estimates in Proposition 7.1 shows that the operator in (7.5) is bounded from $L^{p_{\rho,\sigma}}$ to $L^{q_{\rho,\sigma}}$. Together with the discussion preceding (7.5) this yields the $L^{p_{\rho,\sigma}} \to L^{q_{\rho,\sigma}}$ bound of the operator $\mathcal{T}[a]$ where $a \in S^{\rho,-\sigma}$. If we apply this to the adjoint operator we obtain the $L^{q'_{\rho,\sigma}} \to L^{p'_{\rho,\sigma}}$ bound. If $\rho > 0$ we interpolate with the $L^p \to L^p$ estimate in §6, and if $\rho = 0$ we interpolate instead with the $H^1 \to L^1$ bound in §6. This yields the proof of statements 1.2.1 and 1.2.2. Statements 1.2.4 and 1.2.3 have already been proved in §4 and §5, respectively. $\quad\square$

We now give a sketch of the

Proof of Proposition 7.1.

We begin by estimating the main terms (7.6), (7.7) and use

Lemma 7.2. *Let $\mathcal{R}^{\sigma_1}$ be as in (1.12) and let $Re(z) = 1$. Then*

$$|\mathcal{T}^z_{k,m,j}f(x)| \lesssim \min\{2^{-(j-m)(d-\ell-\sigma_1)}, 2^{-(m-j)}\}\mathfrak{M}(\mathfrak{Q}\mathcal{R}^{\sigma_1}\mathfrak{Q}^{-1}[f])$$

where $\mathfrak{M}$ denotes the strong maximal function.

Proof. This follows from the kernel estimates (4.7) in a straightforward way. $\quad\square$

Proof of (7.6), (7.7). By Theorem 5.1 we know that $\mathcal{R}^{\sigma_1}$ maps L^{p_1} to L^{q_1} and so does $\widetilde{\mathcal{R}}^{\sigma_1}\mathfrak{Q}\mathcal{R}^{\sigma_1}\mathfrak{Q}^{-1}$. Arguing as in the proof of Lemma 5.5, by the Fefferman-Stein and Marcinkiewicz-Zygmund theorems we therefore have the vector-valued inequality

$$\Big\| \Big(\sum_{j,k} |\mathfrak{M}\widetilde{\mathcal{R}}^{\sigma_1}f_{j,k}|^2 \Big)^{1/2} \Big\|_{q_1} \lesssim \Big\| \Big(\sum_{j,k} |f_{j,k}|^2 \Big)^{1/2} \Big\|_{p_1}.$$

We apply the $L^{q_1} \to L^{q_1}$ and $L^{p_1} \to L^{p_1}$ Littlewood-Paley inequalities for the Littlewood-Paley decompositions $\{L_k P_{k,j}\}_{j,k}$ and Lemma 7.2 and obtain

$$(7.14) \qquad \Big\| \sum_{k > s} \sum_{s \leq j < k} L_k P_{k,j}\mathcal{T}^z_{k,j-s,j}P_{k,j}L_k \Big\|_{L^{p_1} \to L^{q_1}} \lesssim 2^{-s(d-\ell)\sigma_1} \quad \text{if } Re(z) = 1.$$

By Lemma 6.2 and the almost orthogonality of the Littlewood-Paley operators

$$
(7.15) \qquad \Big\| \sum_{k>s} \sum_{s \le j < k} L_k P_{k,j} \mathcal{T}^z_{k,j-s,j} P_{k,j} L_k \Big\|_{L^2 \to L^2} \lesssim 1 \quad \text{if } \mathrm{Re}(z) = 0.
$$

(7.14) and (7.15) prove (7.6) by interpolation and (7.7) is proved in the same way.

Proof of (7.8), (7.9), (7.10), (7.11). We analyze the kernel of $L_k(I - P_{k,j})T^z_{k,m,j}$ which is given by

$$
\iint \iiiint e^{i\psi(x,t,h'',y,,\lambda,\eta'',\tau,\xi)} \gamma_{k,m,j,z}(x,t,h'',y,\lambda,\eta,\tau,\xi)\, d\eta'' d\lambda d\tau d\xi\, dt dh''
$$

where

$$
\psi(x,t,h'',y,\lambda,\eta'',\tau,\xi) = -t\lambda - \langle h'', w'' \rangle + \langle \tau, y'' - \widetilde{S}(x'+tu, x''+h'', y') \rangle + \langle x'+tu-y', \xi \rangle
$$

and

$$
\gamma_{k,m,j,z}(x,t,h'',y,,\lambda,\eta'',\tau,\xi)
$$
$$
= \sum_{i_1=-4}^{4} \zeta(4^{-k+i}|\eta''|) \sum \Big(1 - \sum_{i_2=-M}^{M} \zeta(2^{-2k+j+i_2}|\lambda|)\Big) \widetilde{\alpha}_{k,m,j,z}(x'+tu, x''+h'', y, \tau, \xi).
$$

Arguing as in §5 we first integrate by parts with respect to t. This yields the pointwise estimate

$$
2^{(2j-2k)N_2} \int_w \frac{2^{2k-j}}{(1+2^{2k-j}|t|)^N} \frac{2^{2k\ell}}{(1+2^{2k}|h''|)^N}
$$
$$
\times \chi_j(x'+tu-y') \frac{2^{m(d-\ell-\sigma)}}{(1+2^m|x'+tu-y'|)^N} \frac{2^{2k\ell}}{(1+2^{2k}|y''-\widetilde{S}(x'+tu, x''+h'', y')|)^{N_1}} dt dh''
$$

here $N_2 \gg N_1, N$ and χ_j is the characteristic function of $\cup_\pm \pm [2^{-j-1}, 2^{-j+1}]$. A somewhat lengthy but straightforward calculation similar to the one for the term $\widetilde{\mathcal{E}}^{1,i}$ in §5 shows that for $s \le j \le k$

$$
|L_k(I - P_{k,j})\mathcal{T}^z_{k,j-s,j} f(x)| \lesssim \int 4^{j-k} 2^{-s(d-\ell-\sigma_1)} (|x'-y'| + |y''-\widetilde{S}(x,y')|^{1/2})^{\sigma_1-d-\ell} |f(y)| dy, \quad \mathrm{Re}(z) = 1,
$$

if $|x| \le \varepsilon$ and better (trivial) decay estimates for $|x| \ge \varepsilon$.

By using the $L^{p_1} \to L^{q_1}$ mapping property of the standard fractional integral operator and its vector-valued extension, together with the L^p inequalities for the Littlewood-Paley operator defined by L_k (or $\widetilde{L}_k$ with $\widetilde{L}_k L_k = L_k$) we obtain the estimate

$$
\Big\| \sum_{k>s+r} L_k(I - P_{k,k-r})\mathcal{T}^z_{k,k-r-s,k-r} P_{k,k-r} L_k \Big\|_{L^{p_1} \to L^{q_1}} \lesssim 2^{-r} 2^{-s(d-\ell-\sigma)}, \quad \mathrm{Re}(z) = 1.
$$

By Lemma 6.2, $\mathcal{T}^z_{k,k-r-s,k-r}$ is bounded on L^2 if $\mathrm{Re}(z) = 0$, uniformly in s, r and k, and by the almost orthogonality of the L_k (or $\widetilde{L}_k$) we get

$$
\Big\| \sum_{k>s+r} L_k(I - P_{k,k-r})\mathcal{T}^z_{k,k-r-s,k-r} P_{k,k-r} L_k \Big\|_{L^2 \to L^2} \lesssim 1, \quad \mathrm{Re}(z) = 0.
$$

Analytic interpolation yields (7.8). The estimates (7.9), (7.10) and (7.11) are proved in the same way.

Proof of (7.12), (7.13). One writes out the integrals defining the kernels of the decompositions of $L_l T^z_{k,m,j}$ and, if $|l-k| > 2$ one gains factors $\min\{2^{-kN}, 2^{-lN}\}$ by integrating in the ''-variables. $\quad \square$

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