LEBESGUE SPACE ESTIMATES FOR A CLASS OF FOURIER INTEGRAL OPERATORS ASSOCIATED WITH WAVE PROPAGATION

SANGHYUK LEE ANDREAS SEEGER

Dedicated to Professor Hans Triebel

Abstract. We prove L^q estimates related to Sogge's conjecture for a class of Fourier integral operators associated with wave equations.

1. Introduction

In this note we prove a variable coefficient version of a recent result in [6] on the local L^q space-time regularity results for solutions of wave equations. The solution operators are Fourier integral operators satisfying the 'cinematic curvature' hypothesis introduced in [17] (see also [14]).

For the general setup let $\mathcal{Y}$ and $\mathcal{Z}$ be paracompact C^∞ manifolds, $\dim(\mathcal{Y}) = d$, $\dim(\mathcal{Z}) = d+1$; in the current paper we shall need to assume $d \ge 4$. We are interested in sharp local regularity estimates for Fourier integral operators $\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})$ (associated with the Fourier integral distributions defined in [8]). Here the canonical relation

$$\mathcal{C} \subset T^*\mathcal{Z} \setminus 0_L \times T^*\mathcal{Y} \setminus 0_R$$

is a conic manifold of dimension $2d+1$, which is Lagrangian with respect to the symplectic form $d\zeta \wedge dz - d\eta \wedge dy$. We denote by 0_L and 0_R the zero-sections in $T^*\mathcal{Z}$ and $T^*\mathcal{Y}$, respectively.

We formulate a curvature hypothesis which appeared in [3], [10] for classes of oscillatory integral operators (see [13], [2] for current results on these classes). We follow the exposition in [14] and impose conditions on the following projection maps.

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow & \downarrow & \searrow & \\ T^*\mathcal{Y}\setminus 0 & & \mathcal{Z} & & T^*_z\mathcal{Z}\setminus 0 \end{array}$$

We require that the projection $\pi_L : \mathcal{C} \to T^*\mathcal{Y}$ is a submersion (*i.e.* the differential has maximal rank $2d$). We also require that the space projection $\Pi_\mathcal{Z} : \mathcal{C} \to \mathcal{Z}$ is a submersion (*i.e.* its differential has maximal rank $d+1$). As discussed in §2 of [14] this implies that for fixed $z \in \Pi_Z\mathcal{C}$ the image of the projection to the fiber,

$$\Gamma_z = \{\zeta :\in T^*_z\mathcal{Z} : \exists (y,\eta) \text{ such that } (z,\zeta,y,\eta) \in \mathcal{C}\},$$

Date: December 12, 2013.

Key words and phrases. Space-time estimates for wave equations on compact manifolds, cinematic curvature, Fourier integral operators.

Supported in part by NRF grant 2011-0001251 (Korea) and and NSF grant 0652890 (USA).

is an immersed conic $(d-1)$-dimensional hypersurface in $T^*_z\mathcal{Z}\setminus 0_L$. We then make an assumption on the curvature of the cones Γ_z:

Curvature hypothesis $\mathcal{H}(\ell)$**:** $\mathcal{C}\subset T^*\mathcal{Z}\setminus 0_L\times T^*\mathcal{Y}\setminus 0_R$, *the projections* π_R *and* $\Pi_{\mathcal{Z}}$ *are submersions and for each* z *the cone* Γ_z *has at least* ℓ *nonvanishing principal curvatures at any* $\zeta\in\Gamma_z$.

Theorem 1.1. *Let* $\ell\ge 3$, *let* $\mathcal{C}$ *satisfy hypothesis* $\mathcal{H}(\ell)$ *and let* $\mathcal{F}\in I^{\mu-1/4}(\mathcal{Z},\mathcal{Y};\mathcal{C})$. *Suppose* $\frac{2\ell}{\ell-2}<q<\infty$ *and* $\mu\le\frac{d}{q}-\frac{d-1}{2}$. *Then* $\mathcal{F}$ *maps* $L^q_{\mathrm{comp}}(\mathcal{Y})$ *to* $L^q_{\mathrm{loc}}(\mathcal{Z})$.

We may apply the theorem with $\ell=d-1$ to solutions of the wave equation on a compact Riemannian manifold M, with initial data in L^q-Sobolev spaces $L^q_\alpha(M)$. Let Δ be the Laplace-Beltrami operator on M. If one combines Theorem 1.1 with the usual parametrix construction (*cf.* [4]) one obtains (arguing as in [14])

Corollary 1.2. *Let* $d\ge 4$, $\frac{2(d-1)}{d-3}<q<\infty$, *and let* I *be a compact time interval. There is* $C>0$ *such that*

$$\Big(\int_I \big\|e^{it\sqrt{-\Delta}}f\big\|^q_{L^q(M)}\,dt\Big)^{1/q}\le C\|f\|_{L^q_\alpha(M)},\qquad \alpha=\frac{d-1}{2}-\frac{d}{q},$$

for all $f\in L^q_\alpha(M)$.

Note that the constant may strongly depend on the choice of I. There are further regularity improvements in the scale of Triebel-Lizorkin spaces (*cf.* §3 below); in particular $L^q_\alpha(M)$ can be replaced by the Besov space $B^q_{\alpha,q}(M)$.

In §2 we prove a frequency localized version of Theorem 1.1 and combine the estimates corresponding to different frequencies in §3. In §4 we discuss some generalizations in the constant coefficient case.

Remarks. In the constant coefficient case one can recover from Theorem 1.1 the space time estimates of [6] which correspond to an endpoint version of Sogge's conjecture in the range given for $\ell=d-1$, see also §4 for other generalizations. For previous partial results on Sogge's conjecture, also in lower dimensions, see the groundbreaking paper of Wolff [20] and the subsequent papers [12], [5].

The case $\ell=d-1$ essentially corresponds to the assumption of cinematic curvature in [17]. We use Hörmander's convention for the definition of order, *i.e.*, in view of the different dimensions of $\mathcal{Z}$ and $\mathcal{Y}$ operators of class $I^{\mu-1/4}(\mathcal{Z},\mathcal{Y};\mathcal{C})$ correspond to locally finite sums of operators with integral kernels in the standard representation (1) below, involving d frequency variables and standard symbols of order μ. One can use a partition of unity and finite decompositions in the fiber variable to reduce matters to the estimation of an integral operator with compactly supported kernel $\mathcal{K}$ which is given as an oscillatory integral distribution in the sense of [8]. Namely if Z is an open set in $\mathbb{R}^{d+1}$ and Y is an open set in $\mathbb{R}^d$ we may assume that

$$\mathcal{K}(z,y)=\int a(z,y,\xi)e^{i(\phi(z,\xi)-\langle y,\xi\rangle)}d\xi \tag{1}$$

where a is a standard symbol of order μ, a is supported for z, y in compact subsets of Z and Y, resp., and ϕ is smooth away from the origin and homogeneous of order one with respect to the variable ξ, and supported in an open set which is conic in ξ. We then have $\nabla_z\phi(z,\xi)\ne 0$ for $\xi\ne 0$ and the mixed second derivative $(d+1)\times d$ matrix $\phi''_{z\xi}(z,\xi)$ has rank d. For fixed (z,ξ), if the vector u is in the cokernel of $\phi''_{z\xi}(z,\xi)$ then the Hessian matrix $\nabla^2_{\xi\xi}\langle u,\nabla_z\phi\rangle(z,\xi)$ has rank at least ℓ, by our curvature assumption.

2. The frequency localized case

By making further localizations, changing variables in z and y, and ignoring error terms which are smoothing of high order we may assume that our kernel is given by

$$K(z,y)=\sum_{k=1}^{\infty}2^{k\mu}K_k(z,y),$$

where

$$K_k(z,y)=\int\chi_k(z,y,2^{-k}\xi)e^{i(\varphi(z,\xi)-\langle y,\xi\rangle)}d\xi, \tag{2}$$

and the functions χ_k are smooth and supported in a compact subset of $Z\times Y\times\Xi$. Here Z is a small neighborhood of the origin in $\mathbb{R}^{d+1}$, Y is a small neighborhood of the origin in $\mathbb{R}^d$ and Ξ is a small neighborhood of the vector $e_1:=(1,0,\dots,0)$ in $\mathbb{R}^d$. Moreover

$$\varphi''_{z\xi}(0,e_1)=\begin{pmatrix}I_d\\0\end{pmatrix} \tag{3}$$

$$\text{rank }\nabla^2_{\xi\xi}\varphi'_{z_{d+1}}(0,e_1)\geq\ell; \tag{4}$$

and in view of the small choice of Z, Y, Ξ we may assume that for all $(z,\xi)\in Z\times\Xi$ the gradient $\varphi'_z(z,\xi)$ is close to e_1, and, with $z=(z',z_{d+1})$, we may assume that $\varphi''_{z'\xi}(z,\xi)$ is close to the identity matrix I_d and $\varphi''_{z_{d+1}\xi}(z,\xi)$ is small. We may further perform a rotation and assume that in coordinates $\xi=(\xi_1,\xi',\xi'')$ with $\xi'=(\xi_2,\dots,\xi_{\ell+1})$ we have

$$\text{rank }\nabla^2_{\xi'\xi'}\varphi'_{z_{d+1}}(0,e_1)=\ell. \tag{5}$$

Finally, $|\partial^\alpha_{z,y,\xi}\chi_k(z,y,\xi)|\leq C_\alpha$ for any multiindex α, uniformly in k, $(z,y,\xi)\in Z\times Y\times\Xi$.

Let $T_kf(z)=\int K_k(z,y)f(y)dy$. Here we prove that the $L^q(\mathbb{R}^d)\to L^q(\mathbb{R}^{d+1})$ operator norm of T_k is $O(2^{k(\frac{d-1}{2}-\frac{d}{q})})$ for $q>\frac{2\ell}{\ell-2}$, and in the next section we discuss how to put the estimates for T_k together. The L^∞ estimate

$$\|T_kf\|_{L^\infty}\lesssim 2^{k\frac{d-1}{2}}\|f\|_\infty$$

can be found in [16]. By interpolation it is enough to prove

Theorem 2.1. *Let $\ell\geq3$ and $q_\ell=\frac{2\ell}{\ell-2}$. The operator T_k is of restricted weak type (q_ℓ,q_ℓ), with operator norm*

$$\|T_k\|_{L^{q_\ell,1}(\mathbb{R}^{d+1})\to L^{q_\ell,\infty}(\mathbb{R}^d)}\lesssim 2^{k(d/\ell-1/2)}\,.$$

By duality we need to prove the restricted weak type inequality for the adjoint operator T_k^*, given by

$$T_k^*F(y)=\int\int\chi_k(z,y,2^{-k}\xi)e^{i(\langle y,\xi\rangle-\varphi(z,\xi))}d\xi\,F(z)dz;$$

i.e. for each measurable set E contained in $[-1/2,1/2]^{d+1}$,

$$\|T_k^*\chi_E\|^{p_\ell}_{L^{p_\ell,\infty}}\lesssim 2^{k\frac{2d-\ell}{\ell+2}}|E|,\quad p_\ell=\frac{2\ell}{\ell+2}\,. \tag{6}$$

The estimate (6) will be derived from the following Proposition 2.2, which is a discretized version of (6) and will be proved in §2.3.

Proposition 2.2. *Let* $p_\ell = \frac{2\ell}{\ell+2}$, $\ell \ge 3$. *For* $k > 2$ *let* $\mathcal{Z}_k = 2^{-k}\mathbb{Z}^{d+1} \cap [-\varepsilon^2, \varepsilon^2]^{d+1}$, *for sufficiently small* $\varepsilon > 0$. *Suppose that for each* $\mathfrak{z} \in \mathcal{Z}_k$ *we are given a symbol* $a_{k,\mathfrak{z}}$ *supported in* $\{\xi : 2^{k-1} < |\xi| < 2^{k+1}, |\frac{\xi}{|\xi|} - e_1| \le \varepsilon^2\}$ *so that*

$$|\partial_\xi^\alpha a_{k,\mathfrak{z}}(\xi)| \le 2^{-k|\alpha|}, \quad |\alpha| \le 10d. \tag{7}$$

Define $S_\mathfrak{z} \equiv S_\mathfrak{z}^k$ *by*

$$S_\mathfrak{z}(y) = \int a_{k,\mathfrak{z}}(\xi) e^{i(\langle y,\xi\rangle - \varphi(\mathfrak{z},\xi))} d\xi. \tag{8}$$

Then for each $\mathcal{E} \subset \mathcal{Z}_k$ *we have*

$$\operatorname{meas}\Big(\Big\{y \in \mathbb{R}^d : \Big|\sum_{\mathfrak{z}\in\mathcal{E}} S_\mathfrak{z}\Big| > \alpha\Big\}\Big) \le C 2^{k(\frac{d+1}{2}p_\ell - 1)} \alpha^{-p_\ell} \#\mathcal{E}. \tag{9}$$

In the following subsections we prove some preparatory L^1 and L^2 estimates, then prove Proposition 2.2, and that Proposition 2.2 implies (6). In §3 we combine the dyadic estimates in Theorem 2.1.

2.1. L^1 **estimates.** L^1-estimates for the expressions $S_\mathfrak{z}$ can be found in [16]. In what follows we let Θ_k be a maximal $2^{-k/2}$-separated set of unit vectors. Using a homogeneous extension of a partition of unity on the sphere one can split

$$a_{k,\mathfrak{z}}(\xi) = \sum_{\theta\in\Theta_k} a_{k,\mathfrak{z},\theta}(\xi)$$

where $a_{k,\mathfrak{z},\theta}$ is supported on the intersection of the cone $\{\xi : \big|\frac{\xi}{|\xi|} - \theta\big| \le 2^{-k/2}\}$ with the support of $a_{k,\mathfrak{z}}$; moreover if u_i are unit vectors perpendicular to θ we have the estimates

$$\big|\langle\theta, \nabla_\xi\rangle^{M_1} \prod_{i=1}^{M_2} \langle u_i, \nabla_\xi\rangle a_{k,\mathfrak{z},\theta}(\xi)\big| \le C(M_1, M_2) 2^{-kM_1} 2^{-kM_2/2}$$

whenever $M_1 + M_2 \le 10d$. Let

$$S_{\mathfrak{z},\theta}(y) = \int a_{k,\mathfrak{z},\theta}(\xi) e^{i(\langle y,\xi\rangle - \varphi(\mathfrak{z},\xi))} d\xi. \tag{10}$$

By homogeneity we have

$$\phi''_{\xi\xi}(z,\theta)\theta = 0. \tag{11}$$

Using this observation we get, as in [16], by an integration by parts

$$|S_{\mathfrak{z},\theta}(y)| \le C_d 2^{k\frac{d+1}{2}} \big(1 + 2^k |\langle \varphi'_\xi(\mathfrak{z},\theta) - y, \theta\rangle| + 2^{k/2} |\Pi_{\theta^\perp}(\varphi'_\xi(\mathfrak{z},\theta) - y)|\big)^{-10d};$$

here $\Pi_{\theta^\perp}$ denotes the projection to the orthogonal complement of θ. This estimate implies $\|S_{\mathfrak{z},\theta}\|_1 = O(1)$ and therefore

$$\|S_\mathfrak{z}\|_1 \lesssim 2^{k\frac{d-1}{2}}. \tag{12}$$

Moreover we get for $1 \le R \le 2^k$,

$$\int_{\substack{|\Pi_{\theta^\perp}(\varphi'_\xi(\mathfrak{z},\theta)-y)| \\ \ge (2^{-k}R)^{1/2}}} |S_{\mathfrak{z},\theta}(y)|\, dy \lesssim \int_{\substack{w'\in\mathbb{R}^{d-1} \\ |w'|\ge(2^{-k}R)^{1/2}}} \frac{2^{k(d-1)/2}}{(1 + 2^{k/2}|w'|)^{10d-2}} dw' \lesssim R^{\frac{1-9d}{2}},$$

and similarly

$$\int_{\substack{|\langle\varphi'_\xi(\mathfrak{z},\theta)-y,\theta\rangle|\\ \ge 2^{-k}R}} |S_{\mathfrak{z},\theta}(y)|\,dy \lesssim \int_{|w_d|\ge 2^{-k}R} \frac{2^k}{(1+2^k|w_d|)^{9d-1}}dw_d \lesssim R^{2-9d}.$$

Now clearly

$$\left|\varphi'_\xi(z,\theta)-\varphi'_\xi(\widetilde{z},\widetilde{\theta})\right| \lesssim |z-\widetilde{z}|+|\theta-\widetilde{\theta}|\,,$$

and by (11) also

$$\left|\langle\varphi'_\xi(z,\theta)-\varphi'_\xi(\widetilde{z},\widetilde{\theta}),\theta\rangle\right| \lesssim |z-\widetilde{z}|+|\theta-\widetilde{\theta}|^2\,.$$

Thus if

$$V^k_\theta(z,R)=\big\{y:|\langle\varphi'_\xi(z,\theta)-y,\theta\rangle|\le R2^{-k},\quad |\Pi_{\theta^\perp}(\varphi'_\xi(z,\theta)-y)|\le (R2^{-k})^{1/2}\big\} \tag{13}$$

then the above calculations give

$$\|S_{\widetilde{\mathfrak{z}},\widetilde{\theta}}\|_{L^1(\mathbb{R}^d\setminus V^k_{\theta(\mathfrak{z},R)})}\le C(C_1)R^{-4d} \quad \text{if } |\widetilde{\mathfrak{z}}-\mathfrak{z}|\le C_1R2^{-k},\quad |\widetilde{\theta}-\theta|\le C_1(R2^{-k})^{1/2} \tag{14}$$

for $C_1\ge 1$.

2.2. **Estimates for scalar products.** Based on standard calculations for oscillatory integrals ([10], [18], [1], [11], [14]) we prove some estimates for scalar products $\langle S_{\mathfrak{z}},S_{\mathfrak{z}'}\rangle$; these results are closely related to the scalar product estimates in [6]. For the Fourier transforms we have

$$\widehat{S}_{\mathfrak{z}}(\xi)=a_{k,\mathfrak{z}}(\xi)e^{-i\varphi(\mathfrak{z},\xi)}$$

and

$$(2\pi)^d\langle S_{\mathfrak{z}},S_{\widetilde{\mathfrak{z}}}\rangle=\langle \widehat{S}_{\mathfrak{z}},\widehat{S}_{\widetilde{\mathfrak{z}}}\rangle=\int a_{k,\mathfrak{z}}(\eta)\overline{a_{k,\widetilde{\mathfrak{z}}}(\eta)}e^{i(\varphi(\widetilde{\mathfrak{z}},\eta)-\varphi(\mathfrak{z},\eta))}d\eta \tag{15}$$

$$=2^{kd}\int b_{k,\mathfrak{z},\widetilde{\mathfrak{z}}}(\xi)e^{i2^k(\varphi(\widetilde{\mathfrak{z}},\xi)-\varphi(\mathfrak{z},\xi))}d\xi \tag{16}$$

where $b_{k,\mathfrak{z},\widetilde{\mathfrak{z}}}$ is supported on a subset of diameter $O(\varepsilon^2)$ of the annulus $\{|\xi|\approx 1\}$, near e_1, with ε sufficiently small. We may assume in what follows that $\mathfrak{z}$, $\widetilde{\mathfrak{z}}$ are in a neighborhood of the origin in $\mathbb{R}^{d+1}$, of diameter $\lesssim\varepsilon^2$. We split coordinates $z=(z',z_{d+1})$, take advantage of (3) and get

$$|\varphi'_\xi(\mathfrak{z},\xi)-\varphi'_\xi(\widetilde{\mathfrak{z}},\xi)|\ge c|\mathfrak{z}'-\widetilde{\mathfrak{z}}'|-C\varepsilon|\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|$$

and after an integration by parts we get

$$\left|\langle S_{\mathfrak{z}},S_{\widetilde{\mathfrak{z}}}\rangle\right|\lesssim\frac{2^{kd}}{(1+2^k|\mathfrak{z}-\widetilde{\mathfrak{z}}|)^{9d}} \quad \text{if } |\mathfrak{z}'-\widetilde{\mathfrak{z}}'|\ge C_1\varepsilon|\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|\,. \tag{17}$$

For $s\in[0,1]$ set $\mathfrak{z}_s=\mathfrak{z}+s(\widetilde{\mathfrak{z}}-\mathfrak{z})$. If

$$|\mathfrak{z}'-\widetilde{\mathfrak{z}}'|\le C_2\varepsilon|\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|$$

(with suitable $C_1\ll C_2\ll\varepsilon^{-1}$) we consider

$$\frac{\varphi(\widetilde{\mathfrak{z}},\xi)-\varphi(\mathfrak{z},\xi)}{\widetilde{\mathfrak{z}}_{d+1}-\mathfrak{z}_{d+1}}=\int_0^1\Big[\varphi'_{z_{d+1}}(\mathfrak{z}_s,\xi)+\Big\langle\frac{\widetilde{\mathfrak{z}}'-\mathfrak{z}'}{\widetilde{\mathfrak{z}}_{d+1}-\mathfrak{z}_{d+1}},\varphi'_{z'}(\mathfrak{z}_s,\xi)\Big\rangle\Big]\,ds\,.$$

Note that $\frac{\varphi(\widetilde{\mathfrak{z}},\xi)-\varphi(\mathfrak{z},\xi)}{\widetilde{\mathfrak{z}}_{d+1}-\mathfrak{z}_{d+1}}$ is a small perturbation of $\varphi'_{z_{d+1}}(0,\xi)$ if ε is sufficiently small. We apply the method of stationary phase (with parameters, [8]) in the ξ'-variables, using (5). This yields

$$\left|\langle S_{\mathfrak{z}},S_{\widetilde{\mathfrak{z}}}\rangle\right|\lesssim\frac{2^{kd}}{(1+2^k|\widetilde{\mathfrak{z}}_{d+1}-\mathfrak{z}_{d+1}|)^{\ell/2}},\quad \text{if } |\mathfrak{z}'-\widetilde{\mathfrak{z}}'|\le C_2\varepsilon|\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|,$$

and combining this with (17) we get

$$|\langle S_{\mathfrak{z}}, S_{\widetilde{\mathfrak{z}}}\rangle| \lesssim \frac{2^{kd}}{(1+2^k|\widetilde{\mathfrak{z}}-\mathfrak{z}|)^{\ell/2}} \tag{18}$$

whenever $|\mathfrak{z}-\widetilde{\mathfrak{z}}| = O(\varepsilon^2)$.

2.3. **Proof of Proposition 2.2.** If $\alpha \le 2^{k\frac{d+1}{2}}$ then the desired inequality follows from (12). Indeed by Tshebyshev's inequality the left hand side of (9) is $\lesssim \alpha^{-1}2^{k(d-1)/2}\#\mathcal{E}$ which is dominated by the right hand side of (9) if $\alpha \le 2^{k\frac{d+1}{2}}$.

In what follows we shall therefore assume that $\alpha > 2^{k\frac{d+1}{2}}$ and set

$$u_k(\alpha) := \big(\alpha 2^{-k\frac{d+1}{2}}\big)^{p_\ell} > 1\,. \tag{19}$$

The argument is a variant of one in [6]; it is based on a Calderón-Zygmund type decomposition at height $u_k(\alpha)$ where volume is replaced by diameter.

By the usual Vitali procedure there is a finite (possibly empty) family $\mathfrak{B}^k$ of disjoint balls so that

$$u_k(\alpha)2^k \mathrm{diam}(B) \le \#(\mathcal{E}\cap B) \quad \text{for } B\in\mathfrak{B}^k; \tag{20}$$

moreover if we remove the balls in $\mathfrak{B}^k$ and set

$$\mathcal{E}_* = \mathcal{E}\setminus \bigcup_{B\in\mathfrak{B}^k} B,$$

then

$$\#(\mathcal{E}_*\cap B) \le C_d u_k(\alpha)2^k\mathrm{diam}(B) \quad \text{for every ball } B. \tag{21}$$

Since $\mathcal{E}\subset\mathcal{Z}_k$ which is 2^{-k}-separated, we may assume that $\mathrm{diam}(B)\ge 2^{-k}$ if $B\in\mathfrak{B}^k$.

We need to establish the following two inequalities:

$$\mathrm{meas}\Big(\Big\{y\in\mathbb{R}^d : \Big|\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} S_{\mathfrak{z}}\Big| > \alpha/2\Big\}\Big) \le C2^{k(\frac{d+1}{2}p_\ell-1)}\alpha^{-p_\ell}\#\mathcal{E}, \tag{22}$$

$$\mathrm{meas}\Big(\Big\{y\in\mathbb{R}^d : \Big|\sum_{\mathfrak{z}\in\mathcal{E}_*} S_{\mathfrak{z}}\Big| > \alpha/2\Big\}\Big) \le C2^{k(\frac{d+1}{2}p_\ell-1)}\alpha^{-p_\ell}\#\mathcal{E}. \tag{23}$$

Proof of (22). We first form an exceptional set as follows. Let z_B denote the center of a ball $B\in\mathfrak{B}^k$ and let $R_B = 10d2^k\mathrm{diam}(B)\gtrsim 1$. Let $\Theta(k,B)$ be a maximal $C_1(2^{-k}R_B)^{1/2}$ separated subset of S^{d-1}. Here C_1 is the constant in (14). Define (using the notation in (13))

$$\mathcal{V}^k = \bigcup_{B\in\mathfrak{B}^k}\ \bigcup_{\vartheta\in\Theta(k,B)} V^k_\vartheta(z_B,R_B). \tag{24}$$

Observe that $\mathrm{meas}(V^k_\vartheta(z_B,R_B))$ is $O((R_B2^{-k})^{(d+1)/2})$ and $\#\Theta(k,B) = O((2^kR_B^{-1})^{(d-1)/2})$. Thus

$$\begin{aligned}\mathrm{meas}(\mathcal{V}^k) &\lesssim \sum_{B\in\mathfrak{B}^k}\sum_{\vartheta\in\Theta(k,B)}\mathrm{meas}(V^k_\vartheta(z_B,R_B)) \lesssim \sum_{B\in\mathfrak{B}^k} R_B2^{-k}\\ &\lesssim \sum_{B\in\mathfrak{B}^k}\mathrm{diam}(B) \lesssim \sum_{B\in\mathfrak{B}^k} 2^{-k}\frac{\#(\mathcal{E}\cap B)}{u_k(\alpha)} \lesssim 2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}\#\mathcal{E}\,,\end{aligned}$$

by the disjointness of the balls in $\mathfrak{B}^k$, (20), and the definition of $u_k(\alpha)$.

To conclude the proof of (22) we have to estimate the contribution in the complement of $\mathcal{V}^k$. For this we bound

$$\operatorname{meas}\Big(\Big\{y\in\mathbb{R}^d\setminus\mathcal{V}^k : \Big|\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} S_{\mathfrak{z}}\Big| > \frac{\alpha}{2}\Big\}\Big) \lesssim \alpha^{-1}\Big\|\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} S_{\mathfrak{z}}\Big\|_{L^1(\mathbb{R}^d\setminus\mathcal{V}^k)}.$$

Now fix B. For every $\theta\in\Theta_k$ we may choose a $\vartheta=\vartheta_B(\theta)\in\Theta(k,B)$ so that $|\vartheta_B(\theta)-\theta|\le C_1(R_B2^{-k})^{-1/2}$. Recalling $S_{\mathfrak{z}}=\sum_{\theta\in\Theta_k}S_{\mathfrak{z},\theta}$, we see

$$\begin{aligned}
&\Big\|\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} S_{\mathfrak{z}}\Big\|_{L^1(\mathbb{R}^d\setminus\mathcal{V}^k)}\\
&\le \sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B}\sum_{\theta\in\Theta_k}\|S_{\mathfrak{z},\theta}\|_{L^1(\mathbb{R}^d\setminus V^k_{\vartheta_B(\theta)}(z_B,R_B))}\\
&\lesssim \sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B}\sum_{\theta\in\Theta_k} R_B^{-4d} \lesssim \sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} 2^{k(d-1)/2}R_B^{-4d}.
\end{aligned}$$

For the second inequality we use (14) and the last one follows from $\#\Theta_k=O(2^{k(d-1)/2})$. Now we note that $2^{k\frac{d-1}{2}}\alpha^{-1}=2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}u_k(\alpha)^{1-\frac{1}{p_\ell}}$. Thus

$$\begin{aligned}
&\operatorname{meas}\Big(\Big\{y\in\mathbb{R}^d\setminus\mathcal{V}^k : \Big|\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} S_{\mathfrak{z}}\Big| > \frac{\alpha}{2}\Big\}\Big) \lesssim \alpha^{-1}\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} 2^{k(d-1)/2}R_B^{-4d}\\
&\lesssim 2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}u_k(\alpha)^{1-\frac{1}{p_\ell}}\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} R_B^{-4d}.
\end{aligned}$$

By (20) we have for $B\in\mathfrak{B}^k$

$$u_k(\alpha)\lesssim\frac{\#(\mathcal{E}\cap B)}{R_B}\lesssim R_B^d$$

and therefore

$$\begin{aligned}
&\operatorname{meas}\Big(\Big\{y\in\mathbb{R}^d\setminus\mathcal{V}^k : \Big|\sum_{B\in\mathfrak{B}^k}\sum_{\mathfrak{z}\in\mathcal{E}\cap B} S_{\mathfrak{z}}\Big| > \frac{\alpha}{2}\Big\}\Big)\\
&\lesssim 2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}u_k(\alpha)^{1-\frac{1}{p_\ell}-4}\sum_{B\in\mathfrak{B}^k}\#(\mathcal{E}\cap B)\lesssim 2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}\#\mathcal{E}
\end{aligned}$$

since $u_k(\alpha)\ge 1$ and $R_B\gtrsim 1$. □

Proof of (23). We check from (19) that

$$2^{kd}u_k(\alpha)^{2/\ell}\alpha^{-2}=2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}.$$

Thus by Tshebyshev's inequality it suffices to prove

$$\Big\|\sum_{\mathfrak{z}\in\mathcal{E}_*} S_{\mathfrak{z}}\Big\|_2^2\lesssim 2^{kd}u_k(\alpha)^{2/\ell}\#\mathcal{E}_*\,. \tag{25}$$

We set

$$\begin{aligned} L &:= u_k(\alpha)^{2/\ell}, \\ I(n,L) &:= [n2^{-k}L, (n+1)2^{-k}L), \\ \mathcal{E}(n,L) &:= \{\mathfrak{z} \in \mathcal{E}_* : \mathfrak{z}_{d+1} \in I(n,L)\}, \\ \mathfrak{S}_n &:= \sum_{\mathfrak{z}\in\mathcal{E}(n,L)} S_{\mathfrak{z}}\,. \end{aligned}$$

Now

$$\Big\| \sum_{\mathfrak{z}\in\mathcal{E}_*} S_{\mathfrak{z}} \Big\|_2^2 \lesssim \sum_n \sum_{\tilde n : |n-\tilde n|\le 4} \langle \mathfrak{S}_n, \mathfrak{S}_{\tilde n}\rangle + \sum_n \sum_{\tilde n : |n-\tilde n| > 4} \langle \mathfrak{S}_n, \mathfrak{S}_{\tilde n}\rangle =: I + II\,.$$

For I we use the Schwarz inequality and then (17) to get

$$\begin{aligned} |I| &\lesssim \sum_n \|\mathfrak{S}_n\|_2^2 = \sum_n \Big\| \sum_{\mathfrak{z}_{d+1}\in I(n,L)} \sum_{\mathfrak{z}'} S_{(\mathfrak{z}',\mathfrak{z}_{d+1})} \Big\|_2^2 \\ &\lesssim L \sum_n \sum_{\mathfrak{z}_{d+1}\in I(n,L)} \Big\| \sum_{\substack{\mathfrak{z}' : \\ (\mathfrak{z}',\mathfrak{z}_{d+1})\in\mathcal{E}_*}} S_{(\mathfrak{z}',\mathfrak{z}_{d+1})} \Big\|_2^2 \\ &\lesssim L \sum_n \sum_{\mathfrak{z}_{d+1}\in I(n,L)} \sum_{\substack{\mathfrak{z}' : \\ (\mathfrak{z}',\mathfrak{z}_{d+1})\in\mathcal{E}}} \sum_{\widetilde{\mathfrak{z}}'} 2^{kd}(1+2^k|\mathfrak{z}'-\widetilde{\mathfrak{z}}'|)^{-8d} \lesssim L2^{kd}\#\mathcal{E}\,. \end{aligned}$$

For II we use (18) and estimate

$$|II| \lesssim \sum_{\mathfrak{z}\in\mathcal{E}_*} \sum_{\substack{\widetilde{\mathfrak{z}}\in\mathcal{E}_* \\ |\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|\ge 2^{-k}L}} |\langle S_{\mathfrak{z}}, S_{\widetilde{\mathfrak{z}}}\rangle| \lesssim \sum_{\mathfrak{z}\in\mathcal{E}_*} \sum_{\substack{\widetilde{\mathfrak{z}}\in\mathcal{E}_* : \\ |\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|\ge 2^{-k}L}} \frac{2^{kd}}{(1+2^k|\mathfrak{z}-\widetilde{\mathfrak{z}}|)^{\ell/2}}\,.$$

By (21) we have for $R\ge 1$ and fixed $\mathfrak{z}$

$$\sum_{\substack{\widetilde{\mathfrak{z}}\in\mathcal{E}_* \\ 2^{-k}R\le|\mathfrak{z}-\widetilde{\mathfrak{z}}|\le 2^{1-k}R}} (1+2^k|\mathfrak{z}-\widetilde{\mathfrak{z}}|)^{-\ell/2} \lesssim u_k(\alpha) R^{1-\frac{\ell}{2}}$$

and since $\ell/2>1$ we get (after setting $R=2^m$ and summing over m with $2^m\ge L$)

$$|II| \lesssim 2^{kd} u_k(\alpha) L^{1-\frac{\ell}{2}} \#\mathcal{E}\,.$$

Hence

$$I + II \lesssim 2^{kd}\#\mathcal{E}\big(L + u_k(\alpha)L^{1-\frac{\ell}{2}}\big)$$

and with the optimal choice of $L=[u_k(\alpha)]^{2/\ell}$ we obtain (25). □

2.4. **Proof that Proposition 2.2 implies** (6)**.** We shall first assume that in (2)

$$\chi_k(z,y,2^{-k}\xi) = \eta_k(z,2^{-k}\xi)\chi_\circ(y) \tag{26}$$

where $\chi_\circ \in C_c^\infty(\mathbb{R}^d)$ is supported on a small neighborhood of the origin but so that $\chi_\circ(y)=1$ for $y\in E$; moreover η_k is compactly supported in a set of diameter $O(\varepsilon^2)$ near $(z,\xi)=(0,e_1)$, and the derivatives of η_k up to order $10d$ are uniformly bounded.

Let $Q_{\mathfrak{z}} = \prod_{i=1}^{d+1}[\mathfrak{z}_i, \mathfrak{z}_i + 2^{-k}]$. For $m \geq 0$ let

$$\begin{aligned} \mathcal{E}_m &= \{\mathfrak{z} \in \mathcal{Z}_k : 2^{-k(d+1)-m-1} < |Q_{\mathfrak{z}} \cap E| \leq 2^{-k(d+1)-m}\}, \\ E_m &= \bigcup_{\mathfrak{z}\in\mathcal{E}_m} Q_{\mathfrak{z}} \cap E. \end{aligned} \tag{27}$$

And we also set

$$\begin{aligned} a_{k,\mathfrak{z},m}(\xi) &= 2^{m+(k+1)d} \int_{Q_{\mathfrak{z}}\cap E_m} \eta_k(z, 2^{-k}\xi) e^{i(\varphi(\mathfrak{z},\xi)-\varphi(z,\xi))} dz\,, \\ S_{\mathfrak{z},m}(y) &= \int a_{k,\mathfrak{z},m}(\xi) e^{i(\langle y,\xi\rangle - \varphi(\mathfrak{z},\xi))} d\xi\,. \end{aligned}$$

Then it follows that

$$T_k^*\chi_E(y) = \sum_{m=0}^{\infty} 2^{-m-(k+1)d} S_{\mathfrak{z},m}(y).$$

Since $\partial_\xi^\alpha(\varphi(\mathfrak{z},\xi) - \varphi(z,\xi)) = O(2^{-k})$ for any multiindex α it is easy to see that $a_{\cdot,\cdot,m}$ satisfies (7) uniformly in m. Hence, the result of Proposition 2.2 can be applied to $\sum_{\mathfrak{z}\in\mathcal{E}_m} S_{\mathfrak{z},m}(y)$ and we get uniform bounds. Thus

$$\Big\| \sum_{\mathfrak{z}\in\mathcal{E}_m} S_{\mathfrak{z},m} \Big\|_{L^{p_\ell,\infty}} \lesssim 2^{k(\frac{d+1}{2}-\frac{1}{p_\ell})} (\#\mathcal{E}_m)^{1/p_\ell} \lesssim 2^{k(\frac{d+1}{2}-\frac{1}{p_\ell})} 2^{m/p_\ell} \big(2^{k(d+1)}\mathrm{meas}(E_m)\big)^{1/p_\ell}$$

with implicit constants independent of m. For the second inequality we use

$$\#\mathcal{E}_m \lesssim 2^m 2^{k(d+1)} \mathrm{meas}(E_m)$$

which follows from (27). Consequently we get

$$\begin{aligned} \big\|T_k^*\chi_E\big\|_{L^{p_\ell,\infty}} &\lesssim \sum_{m=0}^{\infty} 2^{-m-(k+1)d} \|S_{\mathfrak{z},m}\|_{L^{p_\ell,\infty}} \\ &\lesssim \sum_{m=0}^{\infty} 2^{-m(1-\frac{1}{p_\ell})} 2^{k(\frac{d}{p_\ell}-\frac{d+1}{2})} (\mathrm{meas}(E_m))^{1/p_\ell} \lesssim 2^{k(\frac{d}{p_\ell}-\frac{d+1}{2})} (\mathrm{meas}(E))^{1/p_\ell} \end{aligned}$$

which is the desired estimate.

Finally we have to remove the assumption (26). Here one uses Fourier series in y and expands $\chi_k(z, y, 2^{-k}\xi) = \sum_{\nu\in\mathbb{Z}^d} c_{k,\nu}\eta_{k,\nu}(z, 2^{-k}\xi)e^{i\langle y,\nu\rangle}$ where the functions $\eta_{k,\nu}(z, 2^{-k}\xi)$ are as before but now with a bound that decays fast in ν. We note that multiplication with $e^{i\langle y,\nu\rangle}$ does not affect the $L^{p_\ell,1}$ norm, apply the previous bounds to the summands and sum in ν using the rapid decay in ν. □

3. Combining the frequency localized pieces

We now combine the previous estimates on the operators T_k and prove the following result in Triebel-Lizorkin spaces $F^q_{a,s}$. Recall ([19]) that if $\{L_k\}_{k=0}^\infty$ is a standard inhomogeneous dyadic frequency decomposition then the norm $\|f\|_{F^q_{a,s}}$ can be defined as the $L^q(\ell^s)$ norm of the sequence $\{2^{ka}L_k f\}$. In view of the embeddings $L^q = F^q_{0,2} \subset F^q_{0,q}$ for $q \geq 2$, $L^q \supset F^q_{0,r}$ for $r \leq 2$ the following result sharpens Theorem 1.1. For the case $r \geq 1$ one could argue by duality and follow [6] but we shall rely on a result in [15] which gives an estimate for all $r > 0$.

Theorem 3.1. *Let $\ell \ge 3$, let $\mathcal{Z}$, $\mathcal{Y}$ be coordinate patches in $\mathbb{R}^{d+1}$, $\mathbb{R}^d$, resp., let $\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})$, with Schwartz kernel compactly supported in $\mathcal{Z} \times \mathcal{Y}$, and let $\mathcal{C}$ satisfy hypothesis $\mathcal{H}(\ell)$. Suppose $\frac{2\ell}{\ell-2} < q < \infty$, $a = \mu + b + d(1/2 - 1/q) - 1/2$ and $r > 0$. Then $\mathcal{F}$ maps $F^q_{a,q}(\mathbb{R}^d)$ boundedly to $F^q_{b,r}(\mathbb{R}^{d+1})$.*

We state (a slight variant of) the result from [15]. In this setting one is given operators $\mathcal{T}_k$ defined on the Schwartz space $\mathcal{S}(\mathbb{R}^{d_1})$,

$$\mathcal{T}_k f(z) = \int \mathcal{K}_k(z,y) f(y)\, dy, \quad z \in \mathbb{R}^{d_2}$$

and each $\mathcal{K}_k$ is continuous and bounded. Let $\zeta \in \mathcal{S}(\mathbb{R}^{d_2})$ and $\zeta_k(z) = 2^{kd_2}\zeta(2^k z)$, and define $P_k g = \zeta_k * g$.

Theorem 3.2 ([15]). *Let $d_1 \le d_2$, $0 < \gamma < d_2$, $\varepsilon > 0$, $1 < q_0 < q < \infty$, and assume*

$$\sup_{k>0} 2^{k\gamma/q_0} \|\mathcal{T}_k\|_{L^{q_0}(\mathbb{R}^{d_1}) \to L^{q_0}(\mathbb{R}^{d_2})} < \infty\,, \tag{28}$$

$$\sup_{k>0} \|\mathcal{T}_k\|_{L^{\infty}(\mathbb{R}^{d_1}) \to L^{\infty}(\mathbb{R}^{d_2})} < \infty\,. \tag{29}$$

Furthermore assume that for each cube Q there is a measurable set $\mathcal{W}_Q \subset \mathbb{R}^{d_2}$ so that

$$\operatorname{meas}(\mathcal{W}_Q) \le C \max\{|Q|^{1-\gamma/d_2}, |Q|\} \tag{30}$$

and there is $\delta > 0$ such that for every $k \in \mathbb{N}$ and every cube Q with $2^k \operatorname{diam}(Q) \ge 1$

$$\sup_{x \in Q} \int_{\mathbb{R}^{d_2} \setminus \mathcal{W}_Q} |\mathcal{K}_k(x,y)| dy \le C \max\{(2^k \operatorname{diam}(Q))^{-\delta}, 2^{-k\delta}\}\,. \tag{31}$$

Then for $q_0 < q < \infty$, $r > 0$

$$\Big\| \Big(\sum_k 2^{k\gamma r/q} |P_k \mathcal{T}_k f_k|^r \Big)^{1/r} \Big\|_q \lesssim \Big\| \Big(\sum_k \|f_k\|_q^q \Big)^{1/q} \Big\|_q.$$

This (or a slightly sharper version) was formulated in [15] only for the case $d_1 = d_2$, but the result there implies the version cited above. Indeed if $d_1 < d_2$ we can define an operator $\widetilde{\mathcal{T}}_k$ on functions F on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2 - d_1}$ by

$$\widetilde{\mathcal{T}}_k F(z) = \iint K_k(z,y) \chi(w) F(y,w) dy\, dw$$

where χ is a nontrivial $C^\infty_c(\mathbb{R}^{d_2-d_1})$ function. The assumptions on $\mathcal{T}_k$ imply the corresponding assumptions on $\widetilde{\mathcal{T}}_k$, by Minkowski's and Hölder's inequalities. Thus the equidimensional case in [15] may be applied and if in the conclusion we specialize to tensor products, $F(y,w) = f(y)\chi_1(w)$ we get the above generalization.

In order to prepare for our application of Theorem 3.2 we let $T_k f(z) = \int K_k(z,y) f(y) dy$ with K_k as in (2). Let β_0 be a C^∞-function supported in $\{\eta \in \mathbb{R}^{d+1} : |\eta| \le 3/2\}$ so that $\beta_0(\eta) = 1$ for $|\eta| \le 1$; and, for $k \ge 1$ let $\beta_k(\eta) = \beta_0(2^{-k}(\eta)) - \beta_0(2^{1-k}(\eta))$. Define L_k on functions on $\mathbb{R}^{d+1}$ by $\widehat{L_k g} = \beta_k(\eta)\widehat{g}(\eta)$. We use calculations in [8] and first observe that there is a constant $A_0 > 1$ so that

$$\|L_{\widetilde{k}} T_k\|_{L^q \to L^q} \le C_N \min\{2^{-kN}, 2^{-\widetilde{k}N}\} \quad \text{if } |k - \widetilde{k}| \ge A_0. \tag{32}$$

This follows from the assumption that $\mathcal{C}$ does not meet the zero sections, which, by homogeneity implies that $c_1|\xi| \le |\varphi'_z(z,\xi)| \le C_1|\xi|$. The kernel $\mathcal{K}_{k\widetilde{k}}$ of $L_{\widetilde{k}}T_k$ is given by

$$\mathcal{K}_{k\widetilde{k}}(z,y) = \iiint \beta_{\widetilde{k}}\eta)\chi_k(z,y,2^{-k}\xi)e^{i(\langle z-w,\eta\rangle+\varphi(w,\xi)-\langle y,\xi\rangle)}dw\,d\eta\,d\xi$$

and if $k-\widetilde{k}$ are sufficiently large then $|\eta+\varphi'_w(w,\xi)| \approx \max\{|\xi|,|\eta|\}$ on the support of the amplitude. Thus in this case we may use an integration by parts in w (followed by an integration by parts in ξ when z is large) to show that the kernels $\mathcal{K}_{k\widetilde{k}}$ of $L_{\widetilde{k}}T_k$ satisfy the estimate

$$|\mathcal{K}_{k\widetilde{k}}(z,y)| \le C_N 2^{-kN}(1+|z|)^{-N}$$

and vanish for y in the complement of a fixed compact set. Thus (32) follows. Similarly if $\{\mathcal{L}_k\}_{k=0}^{\infty}$ is the corresponding frequency decomposition in $\mathbb{R}^d$ we also see that $T_k\mathcal{L}_{\widetilde{k}}$ has $L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^{d+1})$ operator norm $\le C_N \min\{2^{-kN}, 2^{-\widetilde{k}N}\}$ if $|k-\widetilde{k}| > 4$.

From these preliminary remarks it follows quickly that for the proof of Theorem 3.1 it suffices to prove the inequalities

$$\Big\|\Big(\sum_k 2^{kbr}\big|2^{k\mu}L_{k+i_1}T_k\mathcal{L}_{k+i_2}g\big|^r\Big)^{1/r}\Big\|_q \lesssim \Big\|\Big(\sum_k 2^{kaq}\|\mathcal{L}_{k+i_2}g\|_q^q\Big)^{1/q}\Big\|_q, \; |i_1| \le A_0, \; |i_2| \le 4$$

with $a = \mu + b + d(1/2 - 1/q) - 1/2$. Setting $\mathcal{T}_k = 2^{-k\frac{d-1}{2}}T_k$ and $f_k = 2^{ka}\mathcal{L}_{k+i_2}g$, the preceding inequality follows from

$$\Big\|\Big(\sum_k 2^{k\frac{d}{q}r}\big|L_{k+i_1}\mathcal{T}_k f_k\big|^r\Big)^{1/r}\Big\|_q \lesssim \Big\|\Big(\sum_k \|f_k\|_q^q\Big)^{1/q}\Big\|_q, \quad |i_1| \le A_0\,, \tag{33}$$

for $q_0 < q < \infty$, where $q_0 > \frac{2\ell}{\ell-2}$. This in turn follows from an application of Theorem 3.2 with $d_1 = d$, $d_2 = d+1$, $\gamma = d$, and $P_k = L_{k+i_1}$. The hypothesis (29) follows from the calculations in [16] (*cf.* also §2.1 above). The hypothesis (28) for $\frac{2\ell}{\ell-2} < q < \infty$ follows from Theorem 2.1 and (29) by interpolation.

If $\operatorname{diam}(Q) > \varepsilon$ then $\mathcal{W}_Q$ is simply an expanded cube and (31) follows by the support assumption of K_k. If $\operatorname{diam}(Q) < \varepsilon$ the exceptional sets are formed as in §2.1. For $\theta \in S^{d-1}$ we set

$$W_\theta(z_Q,C) = \big\{y : |\langle\varphi'_\xi(z_Q,\theta)-y,\theta\rangle| \le C\operatorname{diam}(Q),\; |\Pi_{\theta^\perp}(\varphi'_\xi(z,\theta)-y)| \le C(\operatorname{diam}(Q))^{1/2}\big\}$$

and if Θ_Q is a maximal set of $(\operatorname{diam}(Q))^{1/2}$ separated unit vectors we set

$$\mathcal{W}_Q = \bigcup_{\theta\in\Theta_Q} W_\theta(z_Q,C).$$

Then the measure of $\mathcal{W}_Q$ is $O(\operatorname{diam}(Q)) = O(|Q|^{1-\frac{d}{d+1}})$ so that (30) holds. The hypothesis (31) (even with large δ) holds by the calculations in §2.1 (*cf.* (14)).

4. Remarks on the constant coefficient case

We now let ρ be a $C^\infty(\mathbb{R}^d \setminus \{0\})$ function which is homogeneous of degree 1, so that $\rho(\xi) \ne 0$ for $\xi \ne 0$. We are interested in space time estimates for $U_t \equiv e^{-it\rho(D)}$ defined by

$$\widehat{U_t f}(\xi) = e^{it\rho(\xi)}\widehat{f}(\xi)$$

and obtain a result under a decay assumption for the Fourier transform of surface carried measure on

$$\Sigma_\rho = \{\xi : \rho(\xi) = 1\}.$$

Theorem 4.1. *Let $\kappa>1$ and let ρ be as above such that the surface measure $d\sigma$ of Σ_ρ satisfies*

$$\sup_\xi (1+|\xi|)^\kappa\, |\widehat{d\sigma}(\xi)| \,<\, \infty\,. \tag{34}$$

Let $\frac{2\kappa}{\kappa-1}<q<\infty$ and let I be a compact time interval. There is $C>0$ such that

$$\Big(\int_I \|e^{it\rho(D)}f\|^q_{L^q(\mathbb{R}^d)}\,dt\Big)^{1/q} \le C\|f\|_{B^q_{\alpha,q}(\mathbb{R}^d)}, \quad \alpha=\frac{d-1}{2}-\frac{d}{q}\,,$$

for all $f\in B^q_{\alpha,q}(\mathbb{R}^d)$.

Remark. Under the assumption that Σ_ρ has nonvanishing curvature everywhere this follows from Theorem 3.1 above. We note that in this particular case a weaker result with $\alpha>(d-1)/2-d/q$ is already in [7].

Sketch of proof. We will assume that $I=[-1,1]$, as one can use rescaling to reduce to this case. Fix k and define $S_{x,t}\equiv S^k_{x,t}$ by

$$S_{x,t}(y)=\int e^{i\langle x-y,\xi\rangle+it\rho(\xi)}\chi(2^{-k}\xi)\,d\xi$$

and as before we may assume that the support of χ has diameter $\le\varepsilon^2$, for sufficently small $\varepsilon>0$, and is contained in $\{1/2<|\xi|\le 2\}$. Fix $\xi_\circ\in\Sigma_\rho$ and $\rho_\circ>0$ so that $\rho_\circ\xi_\circ\in\operatorname{supp}(\chi)$. Let $u\mapsto\Xi(u)$ be a parametrization of Σ_ρ near $\xi_\circ$ with parameter $u\in\mathbb{R}^{d-1}$ near $u_\circ$, and $\Xi(u_\circ)=\xi_\circ$. Let $\mathfrak{n}_\circ$ be the outer normal unit vector to Σ_ρ at $\xi_\circ$. Let Γ be the cone formed by the $(\rho\Xi(u),\rho)$ with $\rho>0$ and u near $u_\circ$. Let $\vec N_\circ=\mathfrak{n}_\circ-\langle\xi_\circ,\mathfrak{n}_\circ\rangle\vec e_{d+1}$, which is a normal vector to Γ at $(\rho\Xi(u_\circ),\rho)$. By finite decompositions of χ we may further assume that

$$S_{x,t}(y)=\int\beta(2^{-k}\xi)e^{i(\langle x-y,\xi\rangle+it\rho(\xi))}d\xi$$

where $\beta\in C^\infty_c$ supported in an ε^2-neighborhood of $\rho_\circ\Xi(u_\circ)$.

The proof of Theorem 4.1 is a straightforward variant of the proof of Theorem 3.1 once we have established the two appropriate replacements for the scalar product bounds (18) and (17), namely

$$|\langle S_{x,t},S_{\tilde x,\tilde t}\rangle|\lesssim 2^{kd}(1+2^k|x-\tilde x|+2^k|t-\tilde t|)^{-\kappa}\,, \tag{35}$$

and a better estimate when $(x-\tilde x,t-\tilde t)$ is orthogonal (or near orthogonal) to $\vec N_\circ$:

$$|\langle S_{x,t},S_{\tilde x,\tilde t}\rangle|\le C_M 2^{kd}(1+2^k|x-\tilde x|+2^k|t-\tilde t|)^{-M} \quad \text{if } |\langle x-\tilde x,\mathfrak{n}_\circ\rangle-(t-\tilde t)\langle\xi_\circ,\mathfrak{n}_\circ\rangle|\le\epsilon_\circ|(x-\widetilde x,t-\widetilde t)| \tag{36}$$

for a small $\epsilon_\circ>0$.

As before $(2\pi)^d\langle S_{x,t},S_{\tilde x,\tilde t}\rangle=\langle\widehat S_{x,t},\widehat S_{\tilde x,\tilde t}\rangle$. We scale and then use generalized polar coordinates $\xi=\rho\Xi(u)$ to write

$$\langle\widehat S_{x,t},\widehat S_{\tilde x,\tilde t}\rangle \;=2^{kd}\iint b(\rho,u)e^{i2^k\rho(\langle x-\tilde x,\Xi(u)\rangle+t-\tilde t)}du\,d\rho \tag{37}$$

where b is smooth and supported near $(\rho_\circ,u_\circ)$. Now for any $\chi\in C^\infty_c$ we have $|\widehat{\chi d\sigma}(\xi)|\lesssim(1+|\xi|)^{-\kappa}$, by assumption (34). We apply this to the inner integral of (37) and obtain

$$|\langle S_{x,t},S_{\tilde x,\tilde t}\rangle|\lesssim 2^{kd}(1+2^k|x-\tilde x|)^{-\kappa}. \tag{38}$$

Let $B = 2\max\{|\xi| : \xi \in \Sigma_\rho\}$. By an integration by parts in ρ (after interchanging the order of integration in (37)) we obtain

$$|\langle S_{x,t}, S_{\tilde{x},\tilde{t}}\rangle| \le C_N 2^{kd}(1 + 2^k|t - \tilde{t}|)^{-N} \quad \text{if } |t - \tilde{t}| \ge B|x - \tilde{x}|\,, \tag{39}$$

and (35) follows from (38) and (39).

We now prove (36) and in view of (39) we may assume $|t - \tilde{t}| < B|x - \tilde{x}|$. We distinguish two cases. In the first case we assume $\big|\langle \frac{x-\tilde{x}}{|x-\tilde{x}|}, \mathfrak{n}_\circ\rangle\big| \le 1 - \varepsilon$. We note that $|\Xi(u) - \Xi(u_\circ)| = O(\varepsilon^2)$ and $|(\nabla_u \langle x - \tilde{x}, \Xi(u)\rangle)_{u=u_\circ}| \sim |\Pi_{\mathfrak{n}_\circ^\perp}(x - \tilde{x})|$. Hence we have $|\nabla_u \langle x - \tilde{x}, \Xi(u)\rangle| \ge c\varepsilon|x - \tilde{x}|$ on the support of b provided that ε is sufficiently small, and higher derivatives of $\langle x - \tilde{x}, \Xi(u)\rangle$ are $O(|x - \tilde{x}|)$. Thus, integrating by parts in the inner u-integral in (37), we get (36). We now consider the second case $\big|\langle \frac{x-\tilde{x}}{|x-\tilde{x}|}, \mathfrak{n}_\circ\rangle\big| \ge 1 - \varepsilon$. This means that $\frac{x-\tilde{x}}{|x-\tilde{x}|} = s\mathfrak{n}_\circ + O(\varepsilon)$ where $s = 1$ or $s = -1$. From the condition on $(x - \tilde{x}, t - \tilde{t})$ in (36) we see that the ρ-derivative of the phase is

$$\begin{aligned}\langle x - \tilde{x}, \Xi(u)\rangle + t - \tilde{t} &= \langle x - \tilde{x}, \xi_\circ\rangle + \frac{\langle x - \tilde{x}, \mathfrak{n}_\circ\rangle}{\langle \xi_\circ, \mathfrak{n}_\circ\rangle} + O((\varepsilon^2 + \epsilon_\circ)|x - \tilde{x}|)\\ &= s|x - \tilde{x}|\Big(\langle \xi_\circ, \mathfrak{n}_\circ\rangle + \frac{1}{\langle \xi_\circ, \mathfrak{n}_\circ\rangle}\Big) + O((\varepsilon + \epsilon_\circ)\|x - \tilde{x}|)\,.\end{aligned}$$

Now $|\langle \xi_\circ, \mathfrak{n}_\circ\rangle| \ge c > 0$ which is a consequence of the homogeneity relation $\rho(\xi) = \langle \xi, \nabla\rho(\xi)\rangle$. Hence, $|\langle x - \tilde{x}, \Xi(u)\rangle + t - \tilde{t}| \gtrsim |x - \tilde{x}|$ if ε and $\epsilon_\circ$ are small enough, and another integration by parts in ρ gives (36). □

Acknowledgement. The second named author would like to thank Fedya Nazarov.

References

[1] M. Bézard, *Une version générale de l'inégalité de Strichartz,* C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), no. 12, 1241–1244.

[2] J.Bourgain and L.Guth, *Bounds on oscillatory integral operators based on multilinear estimates,* Geom. Funct. Anal., to appear.

[3] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc,* Studia Math. **44** (1972), 287–299.

[4] J.J. Duistermaat, Fourier integral operators. Progress in Mathematics, 130. Birkhäuser, 1996.

[5] G. Garrigós, A. Seeger, W. Schlag, *Improvements in Wolff inequality for decompositions of cone multipliers.* Manuscript.

[6] Y. Heo, F. Nazarov and A. Seeger, *Radial Fourier multipliers in high dimensions,* Acta Math. **206**, (2011), no.1, 55-92.

[7] Y. Heo and C. Yang, *Local smoothing estimates for some half-wave operators,* J. Math. Anal. Appl., **381** (2011), 573–581.

[8] L. Hörmander, *Fourier integral operators I,* Acta Math. **127** (1971), 79–183.

[9] L. Hörmander, The analysis of linear partial differential operators Vols. I-IV, Springer-Verlag, Berlin, 1983, 1985.

[10] L. Hörmander, *Oscillatory integrals and multipliers on FL^p,* Ark. Mat. **11** (1971), 1–11.

[11] L. V. Kapitanskiĭ, *Estimates for norms in Besov and Lizorkin-Triebel spaces for solutions of second-order linear hyperbolic equations,* (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 171 (1989), Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 20, 106–162, 185–186; translation in J. Soviet Math. **56** (1991), no. 2, 2348–2389.

[12] I. Łaba and T. Wolff. *A local smoothing estimate in higher dimensions,* J. Anal. Math. **88** (2002), 149–171.

[13] S. Lee, *Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces,* J. Funct. Anal. **241** (2006), no. 1, 56–98.

[14] G. Mockenhaupt, A. Seeger, C.D. Sogge, *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates.* J. Amer. Math. Soc. **6** (1993), no. 1, 65–130.

[15] M. Pramanik, K.M. Rogers, A. Seeger, *A Calderón-Zygmund estimate with applications to generalized Radon transforms and Fourier integral operators,* Studia Math. **202** (2011), 1-15.

[16] A. Seeger, C.D. Sogge and E.M. Stein, *Regularity properties of Fourier integral operators,* Annals of Math. **134** (1991) 231–251.

[17] C.D. Sogge, *Propagation of singularities and maximal functions in the plane,* Invent. Math. **104** (1991), no. 2, 349–376.

[18] E. M. Stein, *Oscillatory integrals in Fourier analysis,* Beijing Lectures in Harmonic Analysis, Princeton Univ. Press, Princeton, N.J. (1986), 307–356.

[19] H. Triebel, Theory of function spaces. Monographs in Mathematics, 78. Birkhäuser Verlag, Basel, 1983.

[20] T. Wolff, *Local smoothing type estimates on L^p for large p,* Geom. Funct. Anal. **10** (2000), no. 5, 1237–1288.

Sanghyuk Lee, School of Mathematical Sciences, Seoul National University, Seoul 151-742, Korea

E-mail address: `shlee@math.snu.ac.kr`

Andreas Seeger, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI, 53706, USA

E-mail address: `seeger@math.wisc.edu`