SINGULAR INTEGRAL AND MAXIMAL OPERATORS ASSOCIATED TO HYPERSURFACES: L^p THEORY

ANDREAS SEEGER, STEPHEN WAINGER
JAMES WRIGHT, SARAH ZIESLER

ABSTRACT. We consider singular integral and maximal operators associated to hypersurfaces given by the graph of a function whose level sets are defined by a convex function of finite type. We investigate the L^p theory for these operators which depend on geometric properties of the hypersurface.

1. INTRODUCTION

In this paper we continue the study of singular integral and maximal operators associated to hypersurfaces in $\mathbb{R}^{N+1}$. Given a real-valued $\Psi \in C^\infty(\mathbb{R}^N)$ with $\Psi(0) = 0$, we consider the following operators associated to the hypersurface defined by the graph $y_{N+1} = \Psi(y)$, $y \in \mathbb{R}^N$:

$$\mathcal{H}_\Psi f(x, x_{N+1}) \;=\; p.v. \int_{|y|\le 1} f(x-y, x_{N+1} - \Psi(y)) K(y)\, dy,$$

and

$$\mathcal{M}_\Psi f(x, x_{N+1}) \;=\; \sup_{0<h<1} \frac{1}{\varpi_N h^N} \int_{|y|\le h} |f(x-y, x_{N+1} - \Psi(y))|\, dy.$$

The singular integral operator $\mathcal{H}_\Psi$ is defined with respect to a Calderón-Zygmund kernel K; that is, $K \in C^\infty(\mathbb{R}^N \setminus \{0\})$ is homogeneous of degree $-N$ with mean value zero over the unit sphere. The maximal function $\mathcal{M}_\Psi$ is defined with respect to averages over Euclidean balls in $\mathbb{R}^N$ where ϖ_N denotes the volume of the unit ball.

It is well known that if a principal curvature of the hypersurface defined by Ψ does not vanish to infinite order at the origin, then $\mathcal{H}_\Psi$ and $\mathcal{M}_\Psi$ are bounded operators on the Lebesgue spaces $L^p(\mathbb{R}^{N+1}), 1 < p < \infty$; see for

1991 *Mathematics Subject Classification.* 42B20, 42B25.
Key words and phrases. Singular integrals along hypersurfaces, maximal functions, cancellation conditions and counterexamples.
The first and second authors were supported in part by NSF grants.

example, [11]. A basic question is to understand these operators when this curvature condition fails.

In this direction, there have been extensive investigations when $N = 1$; that is, when $y_2 = \Psi(y)$, $y \in \mathbb{R}$, defines a curve in the plane. For instance, there are smooth Ψ such that the operators $\mathcal{H}_\Psi$ and $\mathcal{M}_\Psi$ are unbounded on $L^p(\mathbb{R}^2), 1 \leq p < \infty$, see [12]; even in the case when Ψ is convex, [8] and [13] (see also Remarks 1.4 below). On the other hand, there are a number of results giving sufficient conditions on a convex Ψ so that the corresponding operators are bounded on $L^p(\mathbb{R}^2), 1 < p < \infty$; see for example, [2], [4] and [8].

In this paper we are interested in studying the operators $\mathcal{H}_\Psi$ and $\mathcal{M}_\Psi$ in higher dimensions, when $N \geq 2$. We are mainly interested in the case when Ψ is a convex function on $\mathbb{R}^N$. By the method of rotations, one can extend the positive one dimensional results, say in the convex case referred to above, by imposing the established sufficient conditions on each function $t \to \Psi(t\omega)$, uniformly in $\omega \in \mathbb{S}^{N-1}$; see for example, [1]. However there are better results for certain classes of hypersurfaces; for instance when $\Psi(y) = \phi(|y|)$ defines a radial hypersurface (in this case, Ψ is convex precisely when the function ϕ defined on $\mathbb{R}^+$ is convex). Here one can exploit the nonvanishing curvature of the level sets of Ψ (which are Euclidean spheres – when $N = 1$ these spheres become two point sets and the underlying curvature is lost) to show that the corresponding operators $\mathcal{H}_\Psi$ and $\mathcal{M}_\Psi$ are bounded on $L^p, 1 < p < \infty$; see [7]. We stress that this is valid for *any* convex radial hypersurface as long as $N \geq 2$.

When the convexity assumption on ϕ is dropped, $\mathcal{H}_\Psi$ and $\mathcal{M}_\Psi$ can be unbounded on L^p for nontrivial ranges of p (for all finite p in the case of the maximal operator $\mathcal{M}_\Psi$); see [7] and [10]. However there is a further interesting phenomenon related to the singular integral operator $\mathcal{H}_\Psi$ in the radial hypersurface case; namely, that $\mathcal{H}_\Psi$ is bounded on L^2 for *any* measurable ϕ as long as $N \geq 2$; see [7]. This does not extend to $L^p, p \neq 2$, see [10].

It remains an interesting problem to explore to what extent the above two phenomena for radial hypersurfaces (L^p boundedness in the convex case and L^2 boundedness of the singular integral operator in the general case) persists for more general hypersurfaces. As a step in this direction, a study was initiated in [14] to understand the L^2 phenomenon for the singular integral operator $\mathcal{H}_\Psi$ where the level sets of Ψ are parameterized by a fixed convex function G of finite type. More precisely, instead of Euclidean spheres parameterized by $G(y) = |y|^2$, one considers a general convex function G of finite type at the origin (that is, the graph defined by G has no lines tangent to infinite order at 0) such that $G(0) = \nabla G(0) = 0$. The corresponding hypersurface is then given by the graph of

$$\Psi(x) = \phi(G(x))$$

where ϕ is a real-valued function defined on $\mathbb{R}^+$. In [14] it was shown that for general $\phi \in C^1$, the L^2 boundedness of $\mathcal{H}_\Psi$ depends on the codimension of E_{ℓ_0} where ℓ_0 is the smallest positive integer such that

$$E_\ell \equiv E_\ell[G] \ = \ \{v \in \mathbb{R}^N \ : \ G(sv) = O(s^{\ell+1}) \text{ for small } s > 0\}$$

is not all of $\mathbb{R}^N$. In fact if the codimension of E_{ℓ_0} is at least 2, then $\mathcal{H}_\Psi$ is bounded on $L^2(\mathbb{R}^{N+1})$ for *any* $\phi \in C^1$. Furthermore if the codimension of E_{ℓ_0} is 1, then $\mathcal{H}_\Psi$ is bounded on $L^2(\mathbb{R}^{N+1})$ for *any* $\phi \in C^1$ if and only if the Calderón-Zygmund kernel K satisfies the additional cancellation condition

$$\int_{v\cdot\theta \geq 0} K(\theta)\, d\sigma(\theta) \ = \ 0, \tag{1}$$

where v is any nonzero vector in $E_{\ell_0}^{\perp}$.

Applying this result in the radial hypersurface setting where $\Psi(x) = \phi(|x|^2)$, $G(x) = |x|^2$ and so $\ell_0 = 2$ and the codimension of $E_2 = \{0\}$ is N, we recover the result in [7] regarding $\mathcal{H}_\Psi$ when $N \geq 2$. In the radial case as we mentioned earlier, these L^2 bounds do not in general extend to L^p bounds, although when $\gamma(s) := \phi(s^2)$ is convex (i.e., $x_{N+1} = \Psi(x)$ is a convex hypersurface), $\mathcal{H}_\Psi$ and $\mathcal{M}_\Psi$ are bounded on all $L^p, 1 < p < \infty$.

The main purpose of this paper is to examine the convex case in the above setting where the level sets are given by a (fixed) general smooth convex function of finite type. First of all we have positive results when the codimension of E_{ℓ_0} is at least 2.

Theorem 1.1. *Let G be a smooth convex function on $\mathbb{R}^N$ of finite type at the origin such that $G(0) = \nabla G(0) = 0$ and let $\gamma(s) := \phi(s^{\ell_0})$ where ϕ is a C^1 function in a neighborhood of the origin and ℓ_0 is the smallest positive integer such that E_ℓ is not all of $\mathbb{R}^N$ as described above. If the codimension of E_{ℓ_0} is at least 2 and $\gamma(s)$ is convex, then $\mathcal{H}_\Psi$ and $\mathcal{M}_\Psi$ are bounded on $L^p(\mathbb{R}^{N+1}), 1 < p < \infty$.*

Remarks 1.2.

- We will see later that $G(x) = P(x) + R(x)$ where P is a positive polynomial and R is smaller than P in a certain sense (see [9]). The significance of the power ℓ_0 is that $1/\ell_0$ is the smallest number such that $P(x)^{1/\ell_0}$ is convex and thus $\phi(P(x))$ is convex when $\gamma(s) = \phi(s^{\ell_0})$ is convex.
- The case when $E_{\ell_0} = \{0\}$ has already been treated; see Theorem 4 in [14]. In fact, the strategy of the proof of Theorem 1.1 is to reduce ourselves to this case.

When the codimension of E_{ℓ_0} is 1, there are no positive results in general and we construct counterexamples:

Theorem 1.3. *There exists a C^∞, convex function γ defined on $[0,1]$ such that if G is any convex function of finite type at the origin with the codimension of E_{ℓ_0} equal to 1, then for $\Psi(x) = \gamma(G(x)^{1/\ell_0})$, the following holds.*

(i) The associated maximal operator $\mathcal{M}_\Psi$ is unbounded on all $L^p(\mathbb{R}^{N+1}), 1 \le p < \infty$ whenever $N \ge 1$.

(ii) The associated singular integral operator $\mathcal{H}_\Psi$ is bounded only on $L^2(\mathbb{R}^{N+1})$, $N \ge 2$, for some Calderón-Zygmund kernel K satisfying the additional cancellation condition (1).

Remarks 1.4.

- Our construction of γ follows a construction due to J.O. Strömberg [13] of a convex function where the maximal function along the corresponding curve in the plane is unbounded on all $L^p(\mathbb{R}^2)$. We extend this construction to produce a C^∞ convex function; this is the case $N = 1$ with $G(t) = t^{\ell_0}$ in Theorem 1.3.
- When $N = 1$ and $G(t) = t^{\ell_0}$, the singular integral operator $\mathcal{H}_\Psi$ is the Hilbert transform along an even convex curve in the plane and a necessary and sufficient condition for $\mathcal{H}_\Psi$ to be bounded on L^2 in this case is known, see [8] (this was later extended to all $L^p, 1 < p < \infty$ in [4]). The curve we construct will not satisfy this condition and so the associated $\mathcal{H}_\Psi$ will not be bounded on any $L^p(\mathbb{R}^2)$ in the case $N = 1$.
- The counterexample for the singular integral operators is related to a counterexample for singular integrals along nonconvex curves in the plane in [10]. After several reductions we essentially reduce matters to the fact that $\xi_1^{i\xi_2}$ is not a Fourier multiplier of $L^p(\mathbb{R}^2)$ for any $p \ne 2$; more precisely we reduce to the main estimate which is used in the proof of this fact (see [10]).
- This example works for a large class of homogeneous Calderón-Zygmund kernels K with the addition cancellation (1). Specifically we assume that the integral of K over a half ray in $E_{\ell_0}^\perp$ defines a function which is not identically zero; see (43) below.

Notation: Let A, B be complex-valued quantities. We use the notation $A \lesssim B$ or $A = O(B)$ to denote the estimate $|A| \le C|B|$ where C denotes an absolute constant which may depend on the Calderón-Zygmund kernel K, the hypersurface given by Ψ, or the dimension N. We use $A \sim B$ to denote the estimates $A \lesssim B \lesssim A$.

In the next section we give the proof of Theorem 1.1. In the following section we will construct the smooth convex function γ needed for the counterexamples in Theorem 1.3. The final two sections will be devoted to the proof of Theorem 1.3.

2. Proof of Theorem 1.1

In the proof that follows we make the assumption that $\phi \not\equiv 0$ in a neighborhood of the origin (that is, $\phi(t) > 0$ when $t > 0$) in order to apply the Calderón-Zygmund theory adapted to a general family of dilations as developed in [2]. If ϕ vanishes in a neighborhood of the origin, then we replace $\phi(t)$ with $\phi(t) + \varepsilon t^2$ and note that the estimates in the proof given below are uniform in $\varepsilon > 0$.

We begin with the treatment of the maximal operator. We denote points in $\mathbb{R}^{N+1}$ by (x, x_{N+1}) where $x \in \mathbb{R}^N$ and observe that to bound the maximal operator $\mathcal{M}_\Psi$ it suffices to bound $\sup_{j\geq 0} |M_j f|$ where

$$M_j f(x, x_{N+1}) = 2^{jN} \int_{2^{-j}\leq |t| \leq 2^{-j+1}} f(x - t, x_{N+1} - \phi(G(t))\, dt.$$

According to Schulz [9], after a rotation of coordinates, we may write

$$G(t) = P(t) + R(t)$$

where

$$P(t) = \sum_{j=1}^{r} a_j t_j^{\ell_0} + \sum_{j=r+1}^{N} a_j t_j^{m_j} + P_1(t). \tag{2}$$

Here $P(t) > 0$ is a convex polynomial for $t \neq 0$, $a_j > 0$ for $1 \leq j \leq N$, and ℓ_0, m_j are positive even integers satisfying $\ell_0 < m_j$ for $r + 1 \leq j \leq N$. Furthermore, $P_1(t)$ has no pure powers of t, and if $At_1^{\alpha_1} \dots t_N^{\alpha_N}$ is a monomial of $P_1(t)$,

$$\frac{1}{\ell_0} \sum_{j=1}^{r} \alpha_j + \sum_{j=r+1}^{N} \frac{\alpha_j}{m_j} = 1. \tag{3}$$

We see that r is the codimension of E_{ℓ_0} and therefore $r \geq 2$ by hypothesis. The function $R(t)$ is smooth and if $At_1^{\alpha_1} \dots t_N^{\alpha_N}$ is a term in the Taylor expansion of $R(t)$

$$\frac{1}{\ell_0} \sum_{j=1}^{r} \alpha_j + \sum_{j=r+1}^{N} \frac{\alpha_j}{m_j} > 1, \tag{4}$$

and in particular the degree $\alpha_1 + \dots + \alpha_N \geq \ell_0 + 1$.

Let $H_P(t)$ be the part of $P(t)$ which is homogeneous of degree ℓ_0. Then $H_P(t)$ is a function of only $t_1, \dots, t_r$. In fact if

$$A\, t_1^{\alpha_1} \cdots t_r^{\alpha_r} \cdots t_N^{\alpha_N}$$

were a monomial of H_P, then $\sum_{j=1}^N \alpha_j = \ell_0$ whereas

$$1 = \frac{1}{\ell_0} \sum_{j=1}^{r} \alpha_j + \sum_{j=r+1}^{N} \frac{\alpha_j}{m_j} < \frac{1}{\ell_0} \sum_{j=1}^{N} \alpha_j = 1$$

if $\sum_{j=r+1}^N \alpha_j > 0$ which is impossible. Similarly every monomial of $P(t)$ which depends only on $t_1, \dots, t_r$ belongs to H_P. So

$$P(t_1, \dots, t_r, 0, \dots, 0) = H_P(t_1, \dots, t_r, 0, \dots, 0) := H(t_1, \dots, t_r) \tag{5}$$

is convex and positive if some t_j is nonzero.

We write $u = (t_1, \dots, t_r) \in \mathbb{R}^r$ and $v = (t_{r+1}, \dots, t_N) \in \mathbb{R}^{N-r}$. We shall suppose $N - r \geq 1$, otherwise the proof is similar but simpler. We then write

$$P(u, v) = H(u) + P_2(u, v). \tag{6}$$

From (3) and (5) we see that the monomials in P_2, like those in R, have degree $\geq \ell_0 + 1$. With this notation, the multiplier for the averaging operator M_j is

$$m_j(\xi, \eta, \gamma) \;=\; 2^{jN} \int\limits_{2^{-2j} \leq |u|^2 + |v|^2 \leq 2^{-2j+2}} e^{i[\xi \cdot u + \eta \cdot v + \gamma \phi(G(u,v))]} du\, dv, \tag{7}$$

where $\xi \in \mathbb{R}^r, \eta \in \mathbb{R}^{N-r}$ and $\gamma \in \mathbb{R}$. We compare m_j to the multiplier

$$n_j(\xi, \eta, \gamma) \;=\; 2^{jN} \int\limits_{2^{-2j} \leq |u|^2 + |v|^2 \leq 2^{-2j+2}} e^{i[\xi \cdot u + \eta \cdot v + \gamma \phi(H(u))]} dudv \tag{8}$$

whose corresponding operator N_j is given by

$$\begin{aligned} &N_j f(x, x_{N+1}) \\ &\quad = 2^{jN} \int\limits_{2^{-2j} \leq |u|^2 + |v|^2 \leq 2^{-2j+2}} f(x' - u, x'' - v, x_{N+1} - \phi(H(u)))\, dudv. \end{aligned}$$

Here we have written $x = (x', x'') \in \mathbb{R}^r \times \mathbb{R}^{N-r}$. We will prove

$$\|m_j - n_j\|_{L^\infty(\mathbb{R}^{N+1})} \lesssim 2^{-\epsilon j} \tag{9}$$

for some $\epsilon > 0$. Since the L^q operator norms of M_j and N_j are uniformly bounded in j for any $1 \leq q \leq \infty$, then (9) implies that the L^p operator norm of $M_j - N_j$ is $O(2^{-\epsilon_p j})$ for some $\epsilon_p > 0$ whenever $1 < p < \infty$ and this in turn shows that the L^p bounds for $\sup_{j \geq 0} |M_j f|$ follow from the corresponding L^p bounds for the maximal operator $Nf = \sup_{j \geq 0} |N_j f|$.

However, we have the pointwise estimate

$$Nf(x, x_{N+1}) \lesssim M_{HL}\big[\widetilde{M}(f_{(\cdot)})(x', x_{N+1})\big](x'') \tag{10}$$

where

$$\widetilde{M}g(x', x_{N+1}) = \sup_{0<h\leq 1} \frac{1}{h^r} \int_{|u|\leq h} |g(x'-u, x_{N+1} - \phi(H(u)))|\, du, \tag{11}$$

and M_{HL} denotes the classical Hardy-Littlewood maximal operator on $\mathbb{R}^{N-r}$. Here $f_{x''}(x', x_{N+1}) = f(x', x'', x_{N+1}) = f(x, x_{N+1})$. The space $E_{\ell_0}, \ell_0 = r$, corresponding to the convex polynomial H in $\mathbb{R}^r$ consists of only the zero vector 0 and hence Theorem 4 in [14] shows that $\widetilde{M}$ is bounded on all $L^p, 1 < p \leq \infty$. This will complete the proof of Theorem 1.1 for $\mathcal{M}_\Psi$ once (9) has been established.

To prove (9), we introduce polar coordinates in the u variables. That is, we write $u = s\omega$ where $s > 0$ and ω runs over the surface $H(\omega) = 1$. The integral defining m_j in (7) becomes

$$2^{jN} \int_{\substack{2^{-2j}\leq s^2|\omega|^2+|v|^2\leq 2^{-2j+2} \\ H(\omega)=1}} e^{i[\xi\cdot s\omega+\eta\cdot v+\gamma\phi(G(s\omega,v))]} s^{r-1} h(\omega) ds d\omega dv \tag{12}$$

where h is a smooth function. From (5) and (6), we see that $G(s\omega, v) = s^{\ell_0} + P_2(s\omega, v) + R(s\omega, v)$. Similarly, the integral in (8) defining n_j becomes

$$2^{jN} \int_{\substack{2^{-2j}\leq s^2|\omega|^2+|v|^2\leq 2^{-2j+2} \\ H(\omega)=1}} e^{i[\xi\cdot s\omega+\eta\cdot v+\gamma\phi(s^{\ell_0})]} s^{r-1} h(\omega) ds d\omega dv. \tag{13}$$

We note that in both (12) and (13) the region of integration may be further restricted in the integrals to the region where $s \geq 2^{-(1+\epsilon)j}$ and $|v| \geq 2^{-(1+\epsilon)j}$, making an error $O(2^{-\epsilon j})$ which is allowable. We will denote the restricted integrals still by m_j and n_j.

Next in the integral for m_j, we make the change of variables

$$\sigma \equiv \sigma(s, \omega, v) \;=\; (s^{\ell_0} + P_2(s\omega, v) + R(s\omega, v))^{1/\ell_0} \tag{14}$$

in the s integral. Since $2^{-(1+\epsilon)j} \leq s, |v| \leq 2^{-j+1}$ and the terms in the Taylor expansion of P_2 and R have degrees $\geq \ell_0 + 1$, we see that $\sigma = s + O(s^{3/2})$ if $\epsilon > 0$ is small enough. Similarly, $d\sigma/ds = 1 + O(s^{1/2})$ and so (14) is a valid change of variables for small $s > 0$. Let $s(\sigma) \equiv s(\sigma, \omega, v)$ denote the inverse function. Then $m_j \;=$

$$2^{jN} \int_{\substack{2^{-2j}\leq s^2|\omega|^2+|v|^2\leq 2^{-2j+2} \\ |v|, s(\sigma)\geq 2^{-(1+\epsilon)j},\;\; H(\omega)=1}} e^{i[\xi\cdot s(\sigma)\omega+\eta\cdot v+\gamma\phi(\sigma)]} \sigma^{r-1} h(\omega) d\sigma d\omega dv + O(2^{-\epsilon j}).$$

With a change of notation we see that the expression for n_j is precisely the above integral except that in the oscillation, $\xi \cdot s(\sigma)\omega$ is replaced by $\xi \cdot \sigma\omega$.

Since $|s(\sigma)-\sigma| \lesssim 2^{-3j/2}$, we see that in the region where $|\xi| \le 2^{5j/4}$,

$$|m_j(\xi,\eta,\gamma) - n_j(\xi,\eta,\gamma)| \lesssim 2^{-\epsilon j}.$$

Therefore it suffices to show that both m_j and n_j are $O(2^{-\epsilon j})$ in the region where $|\xi| \ge 2^{5j/4}$. We will use the hypothesis that the codimension of E_{ℓ_0} is at least 2 ($r \ge 2$) so that the surface $H(\omega)=1$ is at least one dimensional (the dimension being $r-1$) and so we could possibly hope for a decay estimate in the ω integral. In fact we will use the finite type hypothesis of the level sets for the hypersurface to show there is enough oscillation in the ω integral to guarantee such a decay estimate.

To do this we use the fact that $H(\omega)=1$ is of finite type so that for each ω_0 on $H(\omega)=1$ we may parameterize $H(\omega)=1$ in a small neighborhood of ω_0 as

$$\omega_0 + (\tau_1,\dots,\tau_{r-1}, g(\tau_1,\dots,\tau_{r-1}))$$

where $g(0)=0$, $\nabla g(0)=0$, and for some $j_0 \ge 2, \partial^j g/\partial\tau_1^j(0) = 0$ for $1 \le j \le j_0-1$ and $\partial^{j_0} g/\partial\tau_1^{j_0}(0) \ne 0$. It follows that we may assume $\partial^{j_0} g/\partial\tau_1^{j_0} \ne 0$ for all τ in this small neighborhood of 0. Therefore since $s(\sigma,\omega,v) \sim \sigma$, one can use van der Corput's lemma (see e.g., [11]) to estimate

$$\int_{\substack{|\omega-\omega_0|\le\delta \\ H(\omega)=1}} e^{is(\sigma,\omega,v)\xi\cdot\omega} h(\omega)\, d\omega \lesssim \frac{1}{(\sigma|\xi|)^\delta}$$

for some positive δ. Integrating this estimate in the other variables shows that the contribution to the integral defining either m_j or n_j is $O(2^{-\epsilon j})$ when $|\xi| \ge 2^{5j/4}$, if $\epsilon > 0$ is chosen small enough. This establishes the estimate (9) and hence completes the proof of Theorem 1.1 for $\mathcal{M}_\Psi$.

To treat the singular integral operator we decompose $\mathcal{H}_\Psi = \sum_{j\ge 0} H_j$ where

$$H_j f(x', x'', x_{N+1}) = \int_{2^{-2j} \le |u|^2+|v|^2 \le 2^{-2j+2}} f(x'-u, x''-v, x_{N+1} - \phi(G(u,v)))K(u,v)\, du dv$$

and compare H_j to S_j where

$$S_j f(x', x'', x_{N+1}) = \int_{2^{-2j} \le |u|^2+|v|^2 \le 2^{-2j+2}} f(x'-u, x''-v, x_{N+1} - \phi(H(u)))K(u,v)\, du dv.$$

The same argument which establishes (9) shows that

$$\|H_j f - S_j f\|_p \lesssim 2^{-\epsilon j}\|f\|_p$$

holds for $p=2$ and hence for any $1 < p < \infty$ since $\|H_j\|_{q\to q}$, $\|S_j\|_{q\to q} = 0(1)$ whenever $1 \le q \le \infty$. Therefore it suffices to bound the operator $S :=$

$\sum_{j\geq 0} S_j$. Unfortunately we cannot use the simple pointwise factorization estimate (10), as we did for the maximal operator N, to separate the u and v integration defining each S_j and reduce matters to an application of Theorem 4 in [14]. Instead we use Littlewood-Paley arguments as in [5] which rely on corresponding maximal function estimates where the above pointwise factorization (10) can be employed. In fact we will repeatedly use a generalization of Theorem D′ in [5] which we explicitly state for the convenience of the reader. On a fixed subspace of $W \subset \mathbb{R}^d$ (in our situation $d = N+1$) let $\{\delta(t)\}_{t>0}$ be a family of linear operators acting on W and satisfying the Rivière condition, namely for $s \leq t$,

$$\|\delta^{-1}(t)\delta(s)\| \leq C(s/t)^{\epsilon} \tag{15}$$

for some $C, \epsilon > 0$. For notational convenience we set $\delta_k = \delta(2^{-k})$. The Calderón-Zygmund theory with respect to the dilations $\delta(t)$ has been developed in [2].

Proposition 2.1. *Let $\{\sigma_k\}_{k\geq 1}$ be Borel measures satisfying $\sup_k \|\sigma_k\| < \infty$. Let $\zeta = \zeta_W + \zeta_{W^\perp} \in W \oplus W^\perp$ and suppose*

$$|\widehat{\sigma}_k(\zeta)| \leq C \min(|\delta^*_{k+1}\zeta_W|, |\delta^*_k \zeta_W|^{-1})^{\epsilon} \tag{16}$$

for some $\epsilon > 0$ (here A^* denotes the transpose of the linear operator A). Furthermore suppose that the maximal operator defined by $\sigma(f) = \sup_k |f * |\sigma_k||$ is bounded on $L^q(\mathbb{R}^d), 1 < q \leq \infty$. Then the mapping $f \to \sum_k f * \sigma_k$ is bounded on $L^p(\mathbb{R}^d), 1 < p < \infty$.

The proof of Proposition 2.1 follows exactly along the lines in [5] together with the Calderón-Zygmund theory with respect to a general family of dilations $\{\delta(t)\}_{t>0}$ satisfying (15). In fact one fixes a nonnegative $\Phi \in C_c^\infty(W)$ such that $\widehat{\Phi}(\zeta_W) = 1$ for $|\zeta_W| \leq 1$ and $\widehat{\Phi}(\zeta_W) = 0$ for $|\zeta_W| \geq 2$. Then

$$f \;=\; \sum_j S_j f \;=\; \sum_j S_j \widetilde{S}_j f$$

where $\widehat{S_j f}(\zeta) = \left[\widehat{\Phi}(\delta^*_{j-L}\zeta_W) - \widehat{\Phi}(\delta^*_{j+L}\zeta_W)\right] \widehat{f}(\zeta)$ and $L > \log_2(C)/2\epsilon$ with C and $\epsilon > 0$ as in the Rivière condition (15). Furthermore $\widetilde{S}_j$ is defined in the same manner except L is replace by $L' > L + \log_2(C)/\epsilon$; see [2] for a proof of this reproducing formula. Hence

$$\sum_k f * \sigma_k \;=\; \sum_j T_j f$$

where $T_j f = \sum_k S_{j+k}\widetilde{S}_{j+k}(\sigma_k * f)$ and thus for any $p_0 > 1$,

$$\|T_j f\|_{p_0} \lesssim \|\big(\sum_k |\sigma_k * (S_{j+k} f)|^2\big)^{1/2}\|_{p_0} \lesssim \|\big(\sum_k |S_{j+k} f|^2\big)^{1/2}\|_{p_0} \lesssim \|f\|_{p_0}.$$

The first and third inequalities use the corresponding Littlewood-Paley theory (see [2] and [3]) and the second inequality uses the hypothesis that the maximal operator σ is bounded on all $L^p, p > 1$, exactly as in [5].

Finally one can use (16), together with the Rivière condition (15), to show

$$\|T_j f\|_2 \;\le\; C\, 2^{-\varepsilon|j|}\|f\|_2$$

for some $\varepsilon > 0$ (see [2] or [3]). By interpolation with the above L^{p_0} estimate on T_j shows that $f \to \sum_k f * \sigma_k$ is bounded on all $L^p, p > 1$.

In what follows we will employ Proposition 2.1 several times with possibly different subspaces W and dilations $\{\delta(t)\}$ in each instance.

To continue with the proof of the L^p boundedness of S, fix a smooth ρ on $\mathbb{R}^r$ so that $\rho(u) = 1$ for $|u| \le 1$ and $\rho(u) = 0$ for $|u| \ge 2$. Then when $u \ne 0$ we have

$$\rho(u) = \sum_{j=0}^{\infty}[\rho(2^j u) - \rho(2^{j+1}u)] := \sum_{j=0}^{\infty}\psi_j(u)$$

and we rewrite (up to an operator which is convolution with an L^1 kernel) $Sf = \sum_{j=0}^{\infty} f * \sigma_j$ where

$$\sigma_j(g) = \int\int g(u, v, \phi(H(u)))K(u, v)\psi_j(u)\varrho(v)\, dudv.$$

Here ϱ is defined in the same way as ρ but on the space $\mathbb{R}^{N-r}$. We apply Proposition 2.1 with respect to the subspace

$$W = \{(\xi, \eta, \lambda) \in \mathbb{R}^{N+1} : \xi = \lambda = 0\}$$

and dilations

$$\delta_k = 2^{-k} I$$

(I denoting the identity on W) but not directly to S. We first compare S to another operator T which we define as follows: fix a nonnegative $\Phi \in C_c^\infty(W)$ with $\int \Phi = 1$ and set $Tf = \sum_{j=0}^{\infty} f * [\Phi_j \otimes \sigma_j^{W^\perp}]$ where we define the measure $\sigma_j^{W^\perp}$ on $W^\perp$ by

$$\sigma_j^{W^\perp}(g) = \int g(u, \phi(H(u)))\widetilde{K}(u)\psi_j(u)du.$$

Here $\widetilde{K}(u) = \int K(u, v)\varrho(v)\, dv$ and $\Phi_j(v) = 2^{j(N-r)}\Phi(2^j v)$. We now apply Proposition 2.1 to see that $S - T$ is bounded on all $L^p(\mathbb{R}^{N+1}), 1 < p < \infty$. If $\nu_j = \sigma_j - \Phi_j \otimes \sigma_j^{W^\perp}$ so that $[S - T]f = \sum_j f * \nu_j$, one needs to verify

$$|\widehat{\nu}_j(\xi, \eta, \lambda)| \;\lesssim\; \min(2^{-j}|\eta|, [2^{-j}|\eta|]^{-1})^\epsilon \tag{17}$$

for some $\epsilon > 0$ and that the corresponding maximal operator ν is bounded on all $L^q(\mathbb{R}^{N+1}), 1 < q \leq \infty$. On the one hand,

$$\widehat{\sigma}_j(\xi, \eta, \lambda) = \int e^{i[\xi \cdot u + \lambda \phi(H(u))]} \psi_j(u) \Big[\int K(u, v) e^{i\eta \cdot v} \varrho(v) \, dv \Big] du$$

whereas

$$\widehat{\sigma_j^{W^\perp}}(\xi, \lambda) = \int e^{i[\xi \cdot u + \lambda \phi(H(u))]} \widetilde{K}(u) \psi_j(u) \, du, \tag{18}$$

and the estimates (17) follow easily from the basic properties of the Calderón-Zygmund kernel K and a straightforward integration by parts argument. On the other hand, $\nu(f)$ satisfies the pointwise factorization estimate (10) and so Theorem 4 in [14] can be invoked to show that ν is bounded on all $L^q(\mathbb{R}^{N+1}), 1 < q \leq \infty$.

This leaves us with bounding $Tf = \sum_j f * [\Phi_j \otimes \sigma_j^{W^\perp}]$. By using polar coordinates with respect to the surface $H(\omega) = 1$ in the integral appearing in (18) and observing that one has decay in λ in the ω integral because $r \geq 2$, one can argue exactly as in [14] or [7] to obtain the following decay estimate:

$$|\widehat{\sigma_j^{W^\perp}}(\xi, \lambda)| \leq C |\delta_j(\xi, \lambda)|^{-\epsilon} \tag{19}$$

for some $\epsilon > 0$ where

$$\delta(t)(\xi, \lambda) = (t\xi, \gamma(t)\lambda).$$

Recall that $\gamma(t) = \phi(t^{\ell_0})$ and without loss of generality we may assume that $\gamma(0) = 0$ and since ϕ is smooth, we also have $\gamma'(0) = 0$. Thus one easily sees that the Rivière condition (15) holds for the dilations $\{\delta(t)\}$. If we had the strong cancellation condition

$$\int \widetilde{K}(u) \psi_j(u) du = \int \int K(u, v) \psi_j(u) \varrho(v) \, du dv = 0 \tag{20}$$

for all $j \geq 0$, then the corresponding estimates to (19) for small frequencies (ξ, λ) would hold and we would be in a position to employ Proposition 2.1 once again; now we take W to be the subspace

$$W = \{(\xi, \eta, \lambda) \in \mathbb{R}^{N+1} : \ \eta = 0\}$$

and above dilations defined in terms of γ. However in general, only the following weaker cancellation condition

$$\Big| \sum_{A \leq j \leq B} \int \widetilde{K}(u) \psi_j(u) du \Big| = \Big| \int \int K(u, v) \sum_{A \leq j \leq B} \psi_j(u) \rho(v) \, du dv \Big| \leq C \tag{21}$$

holds uniformly in A and B since $K(u, v)$ is integrable over the region $\{(u, v) \in \mathbb{R}^N : \ |u|^2 + |v|^2 \geq r^2; \ |u| \leq r\}$, uniformly in r.

We will follow a procedure outlined in [11] (chapter XIII, section 5.3) that will allow us to pass from the weaker cancellation condition (21) to the stronger cancellation condition (20). Let $c_j = \int \widetilde{K}(u)\psi_j(u)du$, set $\widetilde{\psi}_j(u) = (\int \psi)^{-1}2^{jr}\psi(2^j u)$ where $\psi(u) = \psi_0(u) = \rho(u) - \rho(2u)$ and write $\widetilde{K}'_j(u) = \widetilde{K}(u)\psi_j(u) - c_j\widetilde{\psi}_j(u)$ so that $\int \widetilde{K}'_j(u)du = 0$ for all j. Now by summation by parts,

$$\sum_{j=0}^{\infty} \widetilde{K}(u)\psi_j(u)\Phi_j(v) = \sum_{j=0}^{\infty} \widetilde{K}'_j(u)\Phi_j(v) + \sum_{j=0}^{\infty} s_j[\widetilde{\psi}_j(u)\Phi_j(v) - \widetilde{\psi}_{j+1}(u)\Phi_{j+1}(v)]$$

where $s_j = \sum_{\ell=0}^{j} c_\ell$. Furthermore we split the last sum into two sums:

$$\sum_{j=0}^{\infty} s_j[\widetilde{\psi}_j(u) - \widetilde{\psi}_{j+1}(u)]\Phi_j(v) \; + \; \sum_{j=0}^{\infty} s_j\widetilde{\psi}_{j+1}(u)[\Phi_j(v) - \Phi_{j+1}(v)].$$

This in turn divides the operator $Tf = \sum_j f * [\Phi_j \otimes \sigma_j^{W^\perp}] = \sum_j f * [\omega_{1,j} + \omega_{2,j} + \omega_{3,j}]$ into three operators;

$$\omega_{L,j}(f) = \int\int f(u, v, \phi(H(u)))K_{L,j}(u)\Phi_{L,j}(v)\, dudv$$

where $K_{1,j}(u) = \widetilde{K}'_j(u)$, $K_{2,j}(u) = s_j[\widetilde{\psi}_j(u) - \widetilde{\psi}_{j+1}(u)]$, $K_{3,j}(u) = s_j\widetilde{\psi}_{j+1}(u)$ and $\Phi_{L,j}(v) = \Phi_j(v)$ for $L = 1, 2$ but $\Phi_{3,j}(v) = \Phi_j(v) - \Phi_{j+1}(v)$.

By (21) we see that $\sup_j |s_j| < \infty$ which allows us to deduce as before that (19) also holds for the Fourier transforms of $\omega_{L,j}$ for each $L = 1, 2$ or 3. Furthermore when $L = 1$ or 2, the strong cancellation condition (20) holds for $K_{L,j}$, i.e., $\int K_{L,j}(u)du = 0$. Thus we also obtain the corresponding estimates to (19) for $\omega_{1,j}$ and $\omega_{2,j}$; namely for $L \in \{1, 2\}$,

$$|\widehat{\omega_{L,j}}(\xi, \eta, \lambda)| \le C \min(|\delta_j(\xi, \lambda)|, |\delta_j(\xi, \lambda)|^{-1})^{\epsilon} \tag{22}$$

for some $\epsilon > 0$. Now the maximal function $\omega_L^*(f)$ satisfies the factorization estimate (10) and hence Theorem 4 in [14] shows that this maximal operator is bounded on all $L^p, 1 < p \le \infty$ which, together with (22), implies that when $L = 1$ or 2 the operators $T_L f = \sum_j f * \omega_{L,j}$ are bounded on $L^p, 1 < p < \infty$ by Proposition 2.1.

The third operator $T_3 f = \sum_j f * \omega_{3,j}$ is a much simpler operator to handle and can be shown to be bounded on $L^p, 1 < p < \infty$, by a final application of Proposition 2.1. The subspace W in this instance is $\{(0, \eta, 0)\}$ and the dilations are $\{2^{-j}I\}$ (here I is the identity operator on W). The verification of the hypotheses of Proposition 2.1 is straightforward. This completes the proof of Theorem 1.1.

3. Construction of a convex function

In this section we construct a C^∞ convex function γ on $[0,1]$ with $\gamma^{(j)}(0) = 0$, $j = 0, 1, \ldots$ which will be the desired function giving us the counterexamples we seek in Theorem 1.3. We use a basic idea of Strömberg [13] to construct a piecewise linear curve resembling the graph of $t - \log t$ on very long intervals for large t and then rescale to the interval $[0,1]$. We shall just construct $\gamma(t)$ on an interval $[0, t_0]$ for some $t_0 > 0$. The extension to $[0,1]$ from $[0, t_0]$ is routine.

We start with two sequences of strictly increasing integers N_k and M_k such that

$$2M_k \le N_k \tag{23}$$

and

$$M_k \ge 10N_{k-1}, \tag{24}$$

with all N_k and $M_k \ge 100$.

For $2^{-N_k} \le t \le 2^{-M_k}$ set

$$\gamma_k(t) = B_k(2^{N_k}t - \log 2^{N_k}t + t) - D_k$$

with

$$D_k = B_k(1 + 2^{-N_k} - 2^{\frac{-M_{k+1}}{2}}),$$

where the small positive numbers B_k will be determined later.

On the intervals $[2^{-N_k}, 2^{-M_k}]$ our function γ will be defined as $\gamma(t) = \gamma_k(t)$. On the intermediate intervals $[2^{-M_k}, 2^{-N_{k-1}}]$ we shall set $\gamma(t) = \eta_k(t)$ where η_k will be defined for $2^{-M_k} \le t \le 2^{-N_{k-1}}$ as a smooth convex function such that

$$\begin{aligned} \eta_k^{(j)}(2^{-M_k}) &= \gamma_k^{(j)}(2^{-M_k}), \\ \eta_k^{(j)}(2^{-N_{k-1}}) &= \gamma_{k-1}^{(j)}(2^{-N_{k-1}}), \end{aligned} \tag{25}$$

for $j = 0, 1, 2, \ldots$

To show smoothness at the origin we shall also need

$$\gamma^{(j)}(t) \to 0 \quad \text{as} \quad t \to 0 \tag{26}$$

for every fixed j.

Our definition of η_k will use a fixed non-negative C^∞ function χ supported in $[-3/4, 3/4]$ so that $\chi(t) = 1,$ for $|t| \le 1/4$ and

$$\int \chi(t)dt = 1; \tag{27}$$

In order to satisfy (25) we assume

$$B_k \le \frac{1}{10} 2^{-6N_{k+1}} B_{k-1} \tag{28}$$

and in order to ensure (26) we impose the condition

$$B_k \le (200)^{-k} \frac{1}{(k+1)!} 2^{-kN_{k+1}} \frac{1}{\sum_{\nu=0}^{k+1} \|\chi^{(\nu)}\|_\infty}. \tag{29}$$

Note that

$$\gamma_k(2^{-N_k}) = B_k 2^{-\frac{M_{k+1}}{2}}$$
$$\gamma_k(2^{-M_k}) = B_k\big(2^{N_k-M_k} - (N_k - M_k)\log 2 + 2^{-M_k} - 1 - 2^{-N_k} + 2^{-M_{k+1}/2}\big).$$

The assumptions (28) imply then

$$\frac{9}{10} B_{k-1} 2^{-\frac{M_k}{2}} \le \gamma_{k-1}(2^{-N_{k-1}}) - \gamma_k(2^{-M_k}) \le B_{k-1} 2^{-\frac{M_k}{2}}. \tag{30}$$

We see further that $\gamma_k'(2^{-N_k}) = B_k$ and $\gamma_k'(2^{-M_k}) = B_k(2^{N_k} - 2^{M_k} + 1)$, and thus

$$\frac{9}{10} B_{k-1} \le \gamma_{k-1}'(2^{-N_{k-1}}) - \gamma_k'(2^{-M_k}) \le B_{k-1}. \tag{31}$$

We shall now construct the intermediate functions η_k. This is done by first constructing the second derivative of η_k to first get the inequalities (25) for $j = 2, 3, \dots$.

Note that for $j \ge 2$

$$\gamma_k^{(j)}(2^{-M_k}) = B_k(-1)^j (j-1)! 2^{M_k j}$$

and

$$\gamma_{k-1}^{(j)}(2^{-N_{k-1}}) = B_{k-1}(-1)^j (j-1)! 2^{N_{k-1} j}.$$

We shall construct the second derivative as a sum of three functions ζ_k^L, ζ_k^R and $\zeta_{k,\alpha}^M$, where ζ_k^L is concentrated near the left endpoint of the interval $[2^{-M_k}, 2^{-N_{k-1}}]$, the function ζ_k^R is concentrated near the right endpoint and $\zeta_{k,\alpha}^M$ lives in the middle (away from the endpoints).

With χ as described above, we form

$$\zeta_k^L(t) = B_k 2^{2M_k} \frac{1}{(1 + (2^{M_k}(s - 2^{-M_k}))^2)} \cdot \chi(100 \cdot 2^{M_k}(s - 2^{-M_k}))$$

and

$$\zeta_k^R(t) = B_{k-1} 2^{2N_{k-1}} \frac{1}{(1 + (2^{N_{k-1}}(s - 2^{-N_{k-1}}))^2)} \cdot \chi(100 \cdot 2^{M_k}(s - 2^{-N_{k-1}})).$$

Also for $a_k > 0$ and α with $2^{-M_k+1} \le \alpha \le 2^{-N_{k-1}} - 2^{-M_k+1}$, set

$$\zeta^M_{k,\alpha}(t) = a_k 2^{M_k} \chi(2^{M_k}(t-\alpha)).$$

Note that for $\nu \ge 2$

$$\gamma_k^{(\nu)}(2^{-M_k}) = \zeta_k^{L(\nu-2)}(2^{-M_k})$$

and

$$\gamma_{k-1}^{(\nu)}(2^{-N_{k-1}}) = \zeta_k^{R(\nu-2)}(2^{-N_{k-1}}).$$

Moreover $\zeta^M_\alpha(t)$ vanishes near 2^{-M_k} and $2^{-N_{k-1}}$, $\zeta^L_k(t)$ vanishes near $2^{-N_{k-1}}$, and $\zeta^R_k(t)$ vanishes near 2^{-M_k}. Thus if we define η_k, so that for $2^{-M_k} \le t \le 2^{-N_{k-1}}$

$$\eta_k''(t) = \zeta^L_k(t) + \zeta^R_k(t) + \zeta^M_{k,\alpha}(t),$$

the conditions (25) will be satisfied for $j \ge 2$. It is natural to define η_k by setting

$$\begin{aligned} \eta_k(t) \quad &= \quad \gamma_k(2^{-M_k}) + \gamma_k'(2^{-M_k})(t - 2^{-M_k}) \\ &\quad + \int_{s=2^{-M_k}}^t \int_{u=2^{-M_k}}^s \big(\zeta^L_k(u) + \zeta^R_k(u) + \zeta^M_{k,\alpha}(u)\big) du\, ds. \end{aligned} \tag{32}$$

Then clearly $\eta_k(2^{-M_k}) = \gamma_k(2^{-M_k})$ and $\eta_k'(2^{-M_k}) = \gamma_k'(2^{-M_k})$ and it remains to show that we can pick a_k and α in the definition of $\zeta^M_{k,\alpha}$ so that

$$\eta_k'(2^{-N_{k-1}}) = \gamma_{k-1}'(2^{-N_{k-1}}) \tag{33}$$

and

$$\eta_k(2^{-N_{k-1}}) = \gamma_{k-1}(2^{-N_{k-1}}). \tag{34}$$

Note that

$$\int_{2^{-N_k}}^{2^{-N_{k-1}}} \zeta^M_\alpha(u) du = a_k$$

by (27) and thus (33) will be satisfied if we choose

$$a_k = \gamma_{k-1}'(2^{-N_{k-1}}) - \gamma_k'(2^{-M_k}) - \int_{2^{-M_k}}^{2^{-N_{k-1}}} (\zeta^L_k(u) + \zeta^R_k(u)) du.$$

To see that a_k is positive, we first note that since $M_k \ge 2N_{k-1}$

$$\int_{2^{-M_k}}^{2^{-N_{k-1}}} \zeta^R_k(t) dt \le 2^{2N_{k-1}-M_k} \frac{B_{k-1}}{10} \le \frac{B_{k-1}}{10} \tag{35}$$

and also

$$\int_{2^{-M_k}}^{2^{-N_{k-1}}} \zeta^L_k(t) dt \le B_k 2^{M_k}. \tag{36}$$

So $\int_{2^{-M_k}}^{2^{-N_{k-1}}} \zeta^L_k(t)dt \le B_{k-1}/10$ while $\gamma'(2^{-N_{k-1}}) - \gamma_k'(2^{-M_k}) \ge \frac{9}{10} B_{k-1}$. Hence a_k defined as above will be positive, indeed

$$\frac{7}{10} B_{k-1} \le a_k \le B_{k-1}. \tag{37}$$

It remains to choose α so that (34) is satisfied. Interchanging the order of integration in (32) we see

$$\begin{aligned}\eta_k(2^{-N_{k-1}}) &= \gamma(2^{-M_k}) + \gamma'(2^{-M_k})(2^{-N_{k-1}} - 2^{-M_k}) \\ &\quad + \int_{2^{-M_k}}^{2^{-N_{k-1}}} (2^{-N_{k-1}} - s)\big(\zeta_k^L(s) + \zeta_k^R(s) + \zeta_{k,\alpha}^M(s)\big)ds,\end{aligned}$$

so we want to find α so that

$$\begin{aligned}&\int_{2^{-M_k}}^{2^{-N_{k-1}}} (2^{-N_{k-1}} - s)\zeta_\alpha^M(s)ds \\ &\qquad = \gamma_{k-1}(2^{-N_{k-1}}) - \gamma_k(2^{-M_k}) - \gamma_k'(2^{-M_k})(2^{-N_{k-1}} - 2^{-M_k}) \\ &\qquad\quad - \int_{2^{-M_k}}^{2^{-N_{k-1}}} (2^{-N_{k-1}} - s)(\zeta_k^L(s) + \zeta_k^R(s))ds.\end{aligned} \tag{38}$$

If we use (28) and (30) we see that the expressions

$$\gamma_{k-1}(2^{-N_{k-1}}) - \gamma_k(2^{-M_k}) - \gamma_k'(2^{-M_k})(2^{-N_{k-1}} - 2^{-M_k})$$

lie between $B_{k-1}2^{-\frac{M_k}{2}-1}$ and $B_{k-1}2^{-\frac{M_k}{2}}$. Furthermore (35), (36), (24) , and (28) show that

$$\int_{2^{-M_k}}^{2^{-N_{k-1}}} (2^{-N_{k-1}} - s)\big(\zeta_k^R(s) + \zeta_k^L(s)\big)ds \le B_{k-1}2^{-\frac{3}{4}M_k}.$$

Thus the right hand side of (38) is between $B_{k-1}2^{-\frac{M_k}{2}}/10$ and $B_{k-1}2^{-\frac{M_k}{2}}$ and the left hand side depends continuously on α. We can choose α to achieve (38) by the intermediate value theorem provided that we can show

$$\int_{2^{-M_k}}^{2^{-N_{k-1}}} (2^{-N_{k-1}} - s)\zeta_\alpha^M(s)ds \begin{cases} \ge B_{k-1}2^{-\frac{M_k}{2}} & \text{if } \alpha = 2^{-M_k+1} \\ \le \frac{1}{10}B_{k-1}2^{-\frac{M_k}{2}} & \text{if } \alpha = 2^{-N_{k-1}} - 2^{-M_k+1}. \end{cases} \tag{39}$$

This follows easily using $B_{k-1}/2 \le a_k \le B_{k-1}$ and $2^{-N_{k-1}} > 10 \cdot 2^{-M_k/2}$.

To summarize we define

$$\gamma(t) = \begin{cases} \gamma_k(t), & 2^{-N_k} \le t \le 2^{-M_k} \\ \eta_k(t), & 2^{-M_k} \le t \le 2^{-N_{k-1}}. \end{cases}$$

The conditions (25) guarantee that $\gamma(t)$ is smooth on $(0, t_0]$. The condition (26) implies that $\gamma^{(j)}(t) \to 0$ as $t \to 0$, for each j. So if we set $\gamma(0) = 0$, $\gamma(t)$ will be smooth on $[0, t_0]$. Finally the construction of γ_k and η_k gives the nonnegativity of γ_k'' and η_k'', so γ is convex. It is now easy to extend γ to a smooth function on $[0, 1]$ so that γ is convex and we also have $\gamma^{(j)}(0) = 0$ for all $j = 0, 1, \dots$.

4. Unboundedness of the maximal operator

In this section we consider an arbitrary convex function $G(t)$ of finite type at the origin on $\mathbb{R}^N$ with $G(0) = \nabla G(0) = 0$, and so that E_{ℓ_0} has codimension 1. We write $t = (u, v) \in \mathbb{R}^N$ where $u \in \mathbb{R}$ and $v \in \mathbb{R}^{N-1}$ and consider the maximal operator

$$\mathcal{M}f(x, x_{N+1}) = \sup_{0<h\leq 1} \frac{1}{\varpi_N h^N} \int_{u^2+|v|^2\leq h^2} |f(x' - u, x'' - v, x_{N+1} - \gamma(G(u, v)^{1/\ell_0})| dudv$$

where γ is the C^∞ convex function constructed in section 3. To show that $\mathcal{M}$ is unbounded on all L^p, it suffices to show that the operator norm of $\mathcal{M}_k f(x, x_{N+1}) =$

$$\sup_{j\in J_k} 2^{jN} \int_{\substack{\frac{5}{4}2^{-j}\leq u\leq \frac{3}{2}2^{-j} \\ |v|\leq 2^{-j}}} |f(x' - u, x'' - v, x_{N+1} - \gamma(G(u, v)^{1/\ell_0})| dudv$$

on L^p is unbounded when $k \to \infty$. Here $J_k = [(1 - \sigma)N_k, N_k - 100]$ and $\sigma > 0$ will be chosen to be sufficiently small. Recall that $N_k \geq 2M_k$ and so $J_k \subset [M_k, N_k]$.

We will need to understand $G(u, v) = P(u, v) + R(u, v)$ as a function of u and to do this we set

$$s = [G(u, v)]^{1/\ell_0}. \tag{40}$$

Using (2), (3) and (4) when $r = 1$, it is easy to see that in the region $|v|^{1+\epsilon} \leq |u|$ with $\epsilon > 0$ small enough,

i) $s = |u| + O(|u|^{1+\delta})$, *ii)* $ds/du = \mathrm{sgn}(u) + O(|u|^\delta)$, *iii)* $d^2s/du^2 = O(|u|^{\delta-1})$

for some $\delta > 0$ (the σ appearing the definition of J_k will depend on δ).

First of all we test $\mathcal{M}_k$ on χ_S where S is the set

$$\{(u, v, w) \in \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R} :$$
$$- 1 \leq u \leq 0, |v| \leq 1, B_k(2^{N_k} + 1)u \leq w \leq B_k(2^{N_k} + 1)u + 2B_k\}.$$

For $j \in J_k$, we see that $\mathcal{M}_k\chi_S(2^{-j}, 0, \gamma(2^{-j})) \gtrsim 1$. In fact this follows from the inequalities

$$B_k(2^{N_k} + 1)(2^{-j} - u) \leq \gamma(2^{-j}) - \gamma(s) \leq B_k(2^{N_k} + 1)(2^{-j} - u) + 2B_k \tag{41}$$

where $5 \cdot 2^{-j-2} \leq u \leq 3 \cdot 2^{-j-1}$ and $|v| \leq 2^{-j}$ (See Fig. 1).

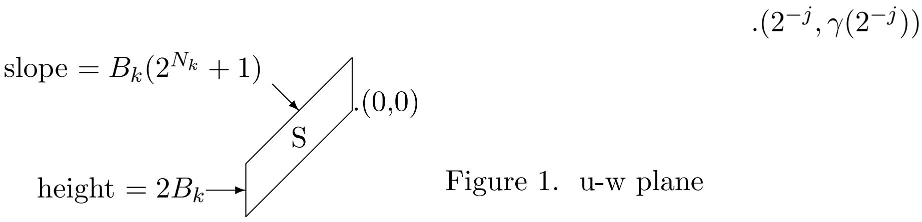

Figure 1. u-w plane

To see this, recall that $\gamma(s) = B_k((2^{N_k}+1)s - \log(2^{N_k}s)) - D_k$ when $2^{-N_k} \le s \le 2^{-M_k}$. By property i) for s, we have

$$\begin{aligned}(42)\quad &\gamma(2^{-j}) - \gamma(s) = \\ &\qquad B_k((2^{N_k}+1)(2^{-j}-u) + \log(2^j u)) + O(2^{-(1+\delta)j}2^{N_k}B_k + 2^{-\delta j}B_k)\end{aligned}$$

and the O term is less than a small multiple of B_k for large j if σ is chosen small enough in the definition of J_k. Therefore (41) is established since $\log(2^j u)$ is bounded between $\log(5/4)$ and $\log(3/2)$ when $5 \cdot 2^{-j-2} \le u \le 3 \cdot 2^{-j-1}$.

By using (42) and translation, we have the bound $\mathcal{M}_k \chi_{2S}(x', x'', x_{N+1}) \gtrsim 1$ on the set $(2^{-j}, 0, \gamma(2^{-j})) + T$ where by $2S$ we mean the set

$$\begin{aligned}&\{(u,v,w) \in \mathbb{R}^{N+1} : \\ &\qquad -2 \le u \le 0,\ |v| \le 2,\ B_k(2^{N_k}+1)u \le w \le B_k(2^{N_k}+1)u + 2B_k\}\end{aligned}$$

and

$$\begin{aligned}&T = \{(u,v,w) : \\ &\quad -1 \le u \le 0, |v| \le 1, B_k(2^{N_k}+1)u \le w \le B_k(2^{N_k}+1)u + \log(2)B_k\}.\end{aligned}$$

This boils down to the fact that $\log(3) < 2$. We claim that the sets $\{(2^{-j}, 0, \gamma(2^{-j})) + T\}$ are disjoint as j varies over J_k. Since the measure of T is a fixed multiple of $2S$, we see then that the L^p operator norm of $\mathcal{M}_k$ is larger than the p'th root of the cardinality of J_k which is σN_k, proving that the original maximal operator $\mathcal{M}$ is unbounded on L^p.

To see that the sets $\{(2^{-j}, 0, \gamma(2^{-j})) + T\}$ are disjoint, it suffices to check that the line $w = \gamma(2^{-j+1}) + \log(2)B_k + B_k(2^{N_k}+1)(u - 2^{-j+1})$ in the $u-w$ plane associated with the set $\{(2^{-j+1}, 0, \gamma(2^{-j+1})) + T$ lies below the corresponding line $w = \gamma(2^{-j}) + B_k(2^{N_k}+1)(u - 2^{-j})$ associated with the set $\{(2^{-j}, 0, \gamma(2^{-j})) + T\}$. That is, we need to verify that

$$\gamma(2^{-j+1}) + \log(2)B_k + B_k(2^{N_k}+1)(2^{-j} - 2^{-j+1}) \le \gamma(2^{-j}).$$

In fact we have equality since $\gamma(2^{-j+1})-\gamma(2^{-j}) = B_k(2^{N_k}+1)2^{-j}-\log(2)B_k$. This completes the part of Theorem 1.3 showing that the maximal operator $\mathcal{M}$ is unbounded on all L^p.

5. Unboundedness of the singular integral operator

We use the same C^∞ convex γ constructed in section 3 to show that only L^2 bounds hold for the singular integral operator defined by

$$\mathcal{H}f(x,x_{N+1}) = \\ p.v. \int_{|u|^2+|v|^2\le 1} f(x'-u,x''-v,x_{N+1}-\gamma(G(u,v)^{1/\ell_0}))K(u,v)\,dudv,$$

where G is an arbitrary convex function of finite type at the origin on $\mathbb{R}^N$ with the codimension of E_{ℓ_0} equal to 1 and K is an appropriate Calderón-Zygmund kernel satisfying the extra cancellation condition (1). Here we are using the same notation as in section 4 and we may assume that G has the form given in (2), and therefore the one dimensional subspace $E_{\ell_0}^\perp$ may be taken to be $\{(x_1,0,\dots,0)\in\mathbb{R}^N\}$. We consider any Calderón-Zygmund kernel K satisfying the nondegeneracy condition

$$\kappa(v) := \int_0^\infty K(u,v)du \not\equiv 0, \tag{43}$$

together with the additional cancellation condition (1) which in this setting takes the form

$$\int_{\mathbb{S}_+^{N-1}} K(\theta)\,d\sigma(\theta) \;=\; 0 \tag{44}$$

where $\mathbb{S}_+^{N-1} = \{x=(x_1,\dots,x_N)|\,|x|=1, x_1\ge 0\}$. Condition (43) implies that $\widehat{\kappa}\not\equiv 0$ and since κ is homogeneous of degree $-(N-1)$, we can find a $\Sigma\in\mathbb{S}^{N-2}$ such that

$$0\ne\widehat{\kappa}(\Sigma) = \int_0^\infty\int_{\mathbb{R}^{N-1}} K(u,v)e^{iv\cdot\Sigma}dvdu = \int_0^\infty \widehat{K_+}(u\Sigma)\frac{du}{u} \tag{45}$$

where we set

$$K_+(v) = K(1,v) \quad\text{and}\quad K_-(v) = K(-1,v). \tag{46}$$

On the other hand, the integral over the hemisphere in condition (44) can be written as an integral over $\mathbb{R}^{N-1}$ (see [14]) and together with the fact that K has mean value zero over the unit sphere, we have

$$0 = \int_{\mathbb{S}_\pm^{N-1}} K(\theta)d\sigma(\theta) = \int_{\mathbb{R}^{N-1}} K(\pm1,v)dv = \widehat{K_\pm}(0). \tag{47}$$

We will assume that $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^{N+1})$ for some $p < 2$ and arrive at a contradiction. The idea is to first pass to a truncation of $\mathcal{H}$ where we will be able to compare $\mathcal{H}$ to a simpler operator in which $\gamma(G(u,v)^{1/\ell_0})$ is replaced by $\gamma(|u|)$. Next, by using de Leeuw's theorem (see e.g., [6]) twice, we will reduce matters to examining the multiplier $\chi(\rho,\lambda)e^{i\lambda\log(\rho)}$ on $\mathbb{R}^2$ where χ localizes (ρ,λ) to the region $\lambda \ll 1$ and $\rho \gg 1$. In [10] such multipliers were shown to be $L^p(\mathbb{R}^2)$ multipliers only for $p = 2$.

We begin by truncating $\mathcal{H}$ in both the u and v variables separately. That is, if $\phi_1 \in C_0^\infty(\mathbb{R})$ and $\phi_2 \in C_0^\infty(\mathbb{R}^{N-1})$, then the truncated operators $\mathcal{H}_{\epsilon,\eta}$, defined on f by

$$\int \phi_1(u/\epsilon)\phi_2(v/\eta) f(x'-u, x''-v, x_{N+1} - \gamma(G(u,v)^{1/\ell_0}))K(u,v)\,dudv,$$

are uniformly bounded on L^p if the same is true for $\mathcal{H}$ (simply use the Fourier inversion formula for ϕ_1 and ϕ_2). As in the proof of Theorem 1.1 we may truncate further, restricting the integration to the region $|v|^{1+\epsilon} \le |u| \le |v|^{1-\epsilon}$. Thus, if we set

$$\mathcal{R}_k = \{(u,v) : 10^{10}2^{-N_k} \le |u|, |v| \le 2^{-(1-\sigma)N_k}, |v|^{1+\epsilon} \le |u| \le |v|^{1-\epsilon}\}$$

for small $\sigma, \epsilon > 0$ and define an operator $\mathcal{H}_k$ by

$$\mathcal{H}_k f(x, x_{N+1}) = \int_{\mathcal{R}_k} f(x'-u, x''-v, x_{N+1} - \gamma(G(u,v)^{1/\ell_0}))K(u,v)\,dudv$$

then the $\mathcal{H}_k$ are uniformly bounded on L^p. In other words, the multipliers for the $\mathcal{H}_k$,

$$h_k(\xi,\eta,\lambda) = \int_{(u,v)\in\mathcal{R}_k} e^{i[\xi u + \eta\cdot v + \lambda\gamma(s(u,v))]}K(u,v)\,dudv,$$

are Fourier multipliers of $L^p(\mathbb{R}^{N+1})$, with multiplier norm $\|h_k\|_{M^p(\mathbb{R}^{N+1})}$ uniformly bounded in k (here $s = G^{1/\ell_0}$ as in section 4, see (40)). We use de Leeuw's theorem to restrict to the line $\xi = -B_k(2^{N_k}+1)\lambda$ (recall that $\gamma(s) = B_k(2^{N_k}+1)s - B_k\log(2^{N_k}s) - D_k$ when $2^{-N_k} \le s \le 2^{-M_k}$). Thus we reduce matters to considering the multipliers $r_k \in M^p(\mathbb{R}^N)$ where

$$r_k(\eta,\lambda) = \int_{(u,v)\in\mathcal{R}_k} e^{i[-B_k(2^{N_k}+1)\lambda u + \eta\cdot v + \lambda\gamma(s(u,v))]}K(u,v)\,dudv$$

and we need to show that the multiplier norms $\|r_k\|_{M^p}$ become arbitrarily large with k. Next, we compare the r_k to

$$m_k(\eta,\lambda) = \int_{(u,v)\in\mathcal{R}_k} e^{i[-B_k(2^{N_k}+1)\lambda u + \eta\cdot v + \lambda\gamma(|u|)]}K(u,v)\,dudv,$$

and show that the differences $r_k - m_k$ are multipliers in $M^p(\mathbb{R}^N)$ with uniform bounds. To accomplish this we split the (u,v) integral dyadically and set

$$r_{k,j}(\eta,\lambda) = \int\limits_{\substack{2^{-2j}\le |u|^2+|v|^2\le 2^{-2j+2} \\ |v|^{1+\epsilon}\le |u|\le |v|^{1-\epsilon}}} e^{i[-B_k(2^{N_k}+1)\lambda u+\eta\cdot v+\lambda\gamma(s(u,v))]}K(u,v)\,dudv;$$

moreover we define $m_{k,j}$ analogously (replacing $\gamma(s(u,v))$ by $\gamma(|u|)$). The bound on $r_k - m_k$ follows then from

$$\|r_{k,j} - m_{k,j}\|_{L^\infty} \lesssim 2^{-\epsilon j}, \tag{48}$$

when $j \in J_k = [(1-\sigma)N_k, N_k - 100]$ since $\|r_{k,j}\|_{M^p} + \|m_{k,j}\|_{M^p} = O(1)$.

Using property i) of $s(u,v)$ in the previous section (see (40)), $s(u,v) = |u| + O(|u|^{1+\delta})$, we see that $\gamma(s) - \gamma(|u|) =$

$$B_k(2^{N_k}+1)(s-|u|) - B_k\log(s/|u|) = O(2^{N_k}B_k 2^{-j(1+\delta)}) + O(B_k 2^{-j\delta})$$

and this in turn is $O(B_k 2^{-j\delta/2})$ for $j \in J_k$ if σ in the definition of J_k is chosen small enough. Therefore the differences $r_{k,j} - m_{k,j}$ have the bound

$$|r_{k,j}(\eta,\lambda) - m_{k,j}(\eta,\lambda)| \lesssim |\lambda| B_k 2^{-j\delta/2}. \tag{49}$$

On the other hand we can use the oscillation in the u integrals for $r_{k,j}$ and $m_{k,j}$ individually to obtain a complementary bound. In fact the second derivative in u of the phase $-\lambda(B_k(2^{N_k}+1)u - \gamma(s(u))$ is

$$\lambda[\gamma'(s)s'' + \gamma''(s)(s')^2] =$$

$$\lambda B_k\big[(2^{N_k}+1-1/s)s''+(1/s^2)(s')^2\big] = \lambda B_k/u^2\big[1+O(2^{-j\delta})+O(2^{N_k}2^{-j(1+\delta)})\big].$$

However if $j \in J_k$ and $\sigma > 0$ is small, depending on $\delta > 0$, the two O terms combine to $O(2^{-j\delta/2})$ and thus we have the bound $|\lambda| B_k 2^{2j}$ from below for the second derivative of the phase. Using van der Corput's lemma and integration by parts, we see that

$$|r_{k,j}(\eta,\lambda)| \lesssim (|\lambda| B_k)^{-1/2}.$$

A simpler argument gives the same bound for $m_{k,j}$ and so

$$|r_{k,j}(\eta,\lambda) - m_{k,j}(\eta,\lambda)| \lesssim (|\lambda| B_k)^{-1/2}. \tag{50}$$

Using (49) when $|\lambda| B_k \le 2^{j\delta/4}$ and (50) when $|\lambda| B_k \ge 2^{j\delta/4}$ establishes (48).

It therefore suffices to reach a contradiction under the assumption that the $\{m_k\}$ are Fourier multipliers of $L^p(\mathbb{R}^N)$, with uniform bounds in k. We may drop the restriction $|v|^{1+\epsilon} \le |u| \le |v|^{1-\epsilon}$ in the integration defining the m_k since K is integrable outside this region (the errors being uniform M^p multipliers; they are Fourier transforms of finite Borel measures) and denoting the resulting multipliers by $\widetilde{m}_k$, we use de Leeuw's theorem to see that for every $\Sigma \in \mathbb{S}^{N-2}$, the functions $m_{k,\Sigma}(\rho,\lambda) = \chi_{(0,\infty)}(\rho)\,\widetilde{m}_k(\rho\Sigma,\lambda)$ are multipliers in $M^p(\mathbb{R}^2)$ with bounds uniformly in k. We choose $\Sigma \in \mathbb{S}^{N-2}$ so that (45) is satisfied.

For $\rho > 0$, we have $m_{k,\Sigma}(\rho,\lambda) =$

$$\int_{10^{10}2^{-N_k}\le|u|,|v|\le 2^{-(1-\sigma)N_k}} e^{i[-\lambda B_k(2^{N_k}+1)u+\lambda\gamma(|u|)+\rho\Sigma\cdot v]}K(u,v)\,dudv$$

and in this region, $\gamma(|u|) = B_k(2^{N_k}+1)|u| - B_k\log(2^{N_k}|u|) - D_k$ and so

$$e^{i\lambda[B_k\log 2^{N_k}+D_k]}m_{k,\Sigma}(\rho,\lambda) = \int_{|u|\in I_k} e^{i\lambda[B_k(2^{N_k}+1)(|u|-u)-B_k\log|u|]}\int_{|v|\in I_k}K(u,v)e^{i\rho\Sigma\cdot v}dvdu$$

where $I_k := [10^{10}2^{-N_k}, 2^{-(1-\sigma)N_k}]$. The factor $e^{i\lambda[B_k\log 2^{N_k}+D_k]}$ gives rise to a fixed translation and so does not affect the assumption that the $m_{k,\Sigma}(\rho,\lambda)$ have uniform bounds in $M^p(\mathbb{R}^2)$ and therefore may be removed.

Next, up to an error which is in $M^p(\mathbb{R}^2)$, we may replace the region of integration $|v| \in I_k$ in $m_{k,\Sigma}(\rho,\lambda)$ with $v \in \mathbb{R}^{N-1}$ since K is integrable in the complementary region (again the errors are Fourier transforms of finite Borel measures). Splitting the u integration where $u > 0$ and $u < 0$ and making the changes of variables $u \to -u$ in the second part, matters are reduced to examining the function $(\rho,\lambda) \mapsto m^1_{k,\Sigma}(\rho,\lambda) + m^2_{k,\Sigma}(\rho,\lambda)$ where

$$m^1_{k,\Sigma}(\rho,\lambda) = \int_{u\in I_k} e^{-i\lambda B_k\log u}\int_{v\in\mathbb{R}^{N-1}}K(u,v)e^{i\rho\Sigma\cdot v}dvdu$$

and

$$m^2_{k,\Sigma}(\rho,\lambda) = \int_{u\in I_k} e^{i\lambda B_k[2(2^{N_k}+1)u-\log u]}\int_{v\in\mathbb{R}^{N-1}}K(-u,v)e^{i\rho\Sigma\cdot v}dvdu.$$

Here we see an important implication of the construction of γ; the linear part of the phase has cancelled in $m^1_{k,\Sigma}$ and this will allow us pass from $m^1_{k,\Sigma}$ to $e^{iB_k\lambda\log(\rho)}$ whereas the linear part of the phase in $m^2_{k,\Sigma}$ produces large enough oscillation to keep it well-behaved as a multiplier.

First we change variables $(\tilde{u},\tilde{v}) = (\frac{u}{\rho}, uv)$ and (after replacing $(\tilde{u},\tilde{v})$ by (u,v)) we see that

$$m^1_{k,\Sigma}(\rho,\lambda) = e^{iB_k\lambda\log(\rho)}\int_{u\in I_{k,\rho}} e^{-i\lambda B_k\log u}\widehat{K_+}(u\Sigma)\frac{du}{u} \tag{51}$$

and

$$m^2_{k,\Sigma}(\rho,\lambda) = e^{iB_k\lambda\log(\rho)}\int_{u\in I_{k,\rho}} e^{i\lambda B_k[2(2^{N_k}+1)u-\log u]}\widehat{K_-}(u\Sigma)\frac{du}{u} \tag{52}$$

where $I_{k,\rho} = [10^{10}2^{-N_k}\rho, 2^{-(1-\sigma)N_k}\rho]$ (recall (46)).

Next, we restrict (λ, ρ) to the region $|\lambda| \lesssim B_k^{-1}$ and $2^{(1-\sigma/2)N_k} \leq \rho \leq 2^{(1-\sigma/4)N_k}$ by introducing

$$\chi(\rho, \lambda) = \varphi(B_k\lambda)[\phi(2^{-N_k(1-\sigma/2)}\rho) - \varphi(2^{-N_k(1-\sigma/4)}\rho)]$$

where $\varphi \in C_0^\infty(\mathbb{R})$ is $\equiv 1$ in a neighborhood of 0 and setting

$$\widetilde{m}^1_{k,\Sigma}(\rho, \lambda) = \chi(\rho, \lambda)\, m^1_{k,\Sigma}(\rho, \Sigma) \quad \text{and} \quad \widetilde{m}^2_{k,\Sigma}(\rho, \lambda) = \chi(\rho, \lambda)\, m^2_{k,\Sigma}(\rho, \Sigma).$$

We will see that $\widetilde{m}^2_{k,\Sigma}$ are classical Marcinkiewicz multipliers uniformly in k and Σ and that the $M^p(\mathbb{R}^2)$ norms of $\widetilde{m}^1_{k,\Sigma}$ become large with k by reducing to showing that the same is true for the multipliers $b_k(\rho, \lambda) = \chi(\rho, \lambda)e^{iB_k\lambda\log(\rho)}$. But first we need the following lemma.

Lemma 5.1. *Suppose K is a smooth homogeneous Calderón-Zygmund kernel on $\mathbb{R}^N$ satisfying the addition cancellation condition $\widehat{K_\pm}(0) = 0$ as described in* (47)*. Then*

(1) $\widehat{K_\pm}(u\Sigma), \partial_u\widehat{K_\pm}(u\Sigma) = O(u^{-M})$ *as $u \to \infty$ for any M, uniformly for $\Sigma \in \mathbb{S}^{N-2}$;*
(2) $\widehat{K_\pm}(u\Sigma) = O(u\log(1/u))$ *as $u \to 0^+$ and* $\partial_u\widehat{K_\pm}(u\Sigma) = O(\log(1/u))$ *as $u \to 0^+$, uniformly for $\Sigma \in \mathbb{S}^{N-2}$.*

Proof. Part (1) follows since $K_\pm$ is smooth and integrable and that the same is true for any derivative of $K_\pm$.

For part (2) we have by (47),

$$\widehat{K_\pm}(u\Sigma) = \int_{\mathbb{R}^{N-1}} K(\pm 1, v)[e^{iu\Sigma\cdot v} - 1]\, dv$$

and so

$$|\widehat{K_\pm}(u\Sigma)| \lesssim \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|v|)^N} \min(1, u|v|)\, dv$$

which is $O(u\log(u))$ for small $u > 0$. The derivative estimate follows in a similar way except one has to be mindful of the fact that the derivative of the integrand defining $\widehat{K_\pm}$ (with respect to u) is not absolutely integrable. The details involve integrations by parts and are left to the reader.

□

We now extend the integration region $u \in I_{k,\rho}$ in (51) and (52) to $0 < u < \infty$, the error in both instances being a classical Marcinkiewicz multiplier, uniformly in k. To see this we write the sum of errors as

$$(53) \quad e^{i\lambda B_k\log(\rho)} \int_{u\notin I_{k,\rho}} e^{-i\lambda B_k\log(u)} \Big[\widehat{K_+}(u\Sigma) + e^{i\lambda 2B_k(2^{N_k}+1)u}\widehat{K_-}(u\Sigma)\Big]\, \frac{du}{u}$$

and observe that Lemma 5.1 immediately shows that the integral is $O\Big((\rho 2^{-N_k})^{1-\epsilon} + (\rho 2^{-N_k(1-\sigma)})^{-M}\Big)$ for any $\epsilon, M > 0$. Now consider the partial derivative with respect to λ of (53); the above bound on the integral shows that the part of the $\partial/\partial\lambda$ derivative which falls on the factor $e^{i\lambda B_k \log(\rho)}$ is

$$O\Big(B_k \log(\rho)\big[(\rho 2^{-N_k})^{1-\epsilon} + (\rho 2^{-N_k(1-\sigma)})^{-M}\big]\Big)$$

which is $\lesssim 1/|\lambda|$ in the above restricted region for ρ and λ. The remaining estimate for $\partial/\partial\lambda$ as well as the other derivative estimates for $\partial/\partial\rho$ and $\partial^2/\partial\rho\partial\lambda$ are easier to obtain and therefore our underlying assumption that the $m_{k,\Sigma}$ are uniform $L^p(\mathbb{R}^2)$ multipliers implies that $\mathfrak{m}^1_{k,\Sigma} + \mathfrak{m}^2_{k,\Sigma}$ where

$$\mathfrak{m}^1_{k,\Sigma}(\rho,\lambda) = \chi(\rho,\lambda) e^{i\lambda B_k \log(\rho)} \int_0^\infty e^{-i\lambda B_k \log(u)} \widehat{K_+}(u\Sigma)\, \frac{u}{u}$$

and

$$\mathfrak{m}^2_{k,\Sigma}(\rho,\lambda) = \chi(\rho,\lambda) e^{i\lambda B_k \log(\rho)} \int_0^\infty e^{-i\lambda B_k (2(2^{N_k}+1)u + \log(u))} \widehat{K_-}(u\Sigma)\, \frac{u}{u}$$

are uniform $L^p(\mathbb{R}^2)$ Fourier multipliers.

Lemma 5.2. *The function $\mathfrak{m}^2_{k,\Sigma}(\rho,\lambda)$ is a classical Marcinkiewicz multiplier, uniformly in k and Σ.*

Proof. Once again, the most difficult derivative estimate of $\mathfrak{m}^2_{k,\Sigma}$ to obtain is part of the $\partial/\partial\lambda$ derivative which differentiates the exponential $e^{i\lambda B_k \log(\rho)}$:

$$iB_k \log(\rho) e^{i\lambda B_k \log(\rho)} \int_0^\infty e^{i\lambda[2B_k(2^{N_k}+1)u - B_k \log(u)]} \widehat{K_-}(u\Sigma)\, \frac{du}{u}.$$

We split the integral into three parts around $u = 2^{-N_k}$;

$$\int_{u<c2^{-N_k}} + \int_{c2^{-N_k}<u<C2^{-N_k}} + \int_{C2^{-N_k}<u} \cdots \quad \frac{du}{u}$$

where $c > 0$ and $C > 0$ are small and large absolute constants, respectively. Recall that $\lambda B_k \lesssim 1$ and $\log(p) \lesssim N_k$ on the support of χ. Using Lemma 5.1 and integrating by parts for the first and third integrals, one easily obtains the bound $O(N_k^2[2^{N_k} B_k \lambda]^{-1})$ for these integrals (the bound for the second integral follows by a simple size estimate for the integrand) and hence we see that the above expression is $O(\log(\rho) N_k^2 2^{-N_k} \lambda^{-1})$ which in turn is $O(\lambda^{-1})$ on the support of χ. The other derivative estimates for $\mathfrak{m}_{k,\Sigma}$ are easier; we omit the details.

□

We are left to show that the functions

$$\mathfrak{m}^1_{k,\Sigma} = \chi(\rho,\lambda) e^{i\lambda B_k \log(\rho)} \int_0^\infty e^{-i\lambda B_k \log(u)} \widehat{K_+}(u\Sigma) du/u$$

$$:= \chi(\rho,\lambda)e^{i\lambda B_k \log(\rho)} I(\lambda B_k)$$

do not have uniform $M^p(\mathbb{R}^2)$ bounds.

Using the decay properties of $\widehat{K_+}$ we see that $I'(t) = O(1)$. Furthermore our nondegeneracy condition on K (45) implies $I(0) \neq 0$ and so $|I(t)| \gtrsim 1$ for t in a small neighborhood of 0. Therefore choosing the support of ϕ in the definition of χ small enough, $\lambda \to \phi(B_k\lambda)[I(B_k\lambda)]^{-1}$ is a Fourier multiplier on L^p, with bounds uniform in k. Therefore, our initial assumption that the singular integral operator $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^{N+1})$ for some $p < 2$ implies that $b_k(\rho,\lambda) = \chi(\rho,\lambda)e^{iB_k\lambda\log(\rho)}$ are Fourier multipliers on $L^p(\mathbb{R}^2)$, with bounds uniform in k. However the proof of Proposition 1.2 in [10] shows that

$$\|b_k\|_{M^p(\mathbb{R}^2)} \gtrsim N_k^{(1/p-1/2)/2}. \tag{54}$$

This is verified by testing the multiplier operator associated to the b_k's on functions of the form f_k where

$$\widehat{f}_k(\rho,\lambda) = \sum_{\ell\in L_k} \beta(\rho - e^{\ell^2})\widehat{a}(B_k\lambda);$$

here $L_k = \{\ell : (1-\sigma/2)N_k \leq \ell^2 \leq (1-\sigma/4)N_k\}$, a is an appropriate Schwartz function, and β is a C^∞ function supported in $\{\rho : |\rho| \leq 1\}$ with $\beta(\rho) = 1$ if $|\rho| \leq 1/2$. By Littlewood-Paley theory,

$$\|f_k\|_{L^q} \approx N_k^{1/2}, \quad 1 < q < \infty,$$

but on the other hand, $N_k^{1/2p} \lesssim \|b_k\|_{M^p}\|f_k\|_{L^p}$ which implies (54) (for more details, see [10]).

As $\|b_k\|_{M^p}$ is not uniformly bounded it follows that $\mathcal{H}$ fails to be bounded on any $L^p(\mathbb{R}^{N+1})$ for $p \neq 2$, completing the proof of Theorem 1.3. □

References

[1] J. Bennett, *Oscillatory singular integrals with variable flat phases, and related operators*, Phd thesis, University of Edinburgh, 1999.

[2] A. Carbery, M. Christ, J. Vance, S. Wainger and D. Watson, *Operators associated to flat plane curves: L^p estimates via dilation methods*, Duke Math. J. **59** (1989), 675-700.

[3] A. Carbery, J. Vance, S. Wainger, and D. Watson, *The Hilbert transform and maximal function along flat curves, dilations, and differential equations*, Amer. J. Math. **116** (1994), 1203–1239.

[4] H. Carlsson, M. Christ, A. Córdoba, J. Douandikoetxea, J.L. Rubio de Francia, J. Vance, S. Wainger, and D. Weinberg, *L^p estimates for maximal functions and Hilbert transforms along flat convex curves in $\mathbf{R}^2$*, Bull. Amer. Math. Soc. **14** (1986), 263-267.

[5] J. Duoandikoetxea, and J.L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541-561.

[6] M. Jodeit, *A note on Fourier multipliers*, Proc. Amer. Math. Soc. **27** (1971), 423-424.

[7] W. Kim, S. Wainger, J. Wright, and S. Ziesler, *Singular integrals and maximal functions associated to surfaces of revolution*, Bull. London Math. Soc. **28** (1996), 291-296.

[8] A. Nagel, J. Vance, S. Wainger, and D. Weinberg, *Hilbert transforms for convex curves*, Duke Math. J. **50** (1983), 735-744.

[9] H. Schulz, *Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms*, Indiana Univ. Math. J. **40** (1991), 1267-1275.

[10] A. Seeger, S. Wainger, J. Wright, and Sarah Ziesler, *Classes of singular integrals along curves and surfaces*, Trans. Amer. Math. Soc. **351** (1999), 3757-3769.

[11] E.M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Princeton University Press, Princeton, NJ, 1993.

[12] E.M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239-1295.

[13] J.O. Strömberg, personal communication.

[14] S. Wainger, J. Wright, and S. Ziesler, *Singular integrals associated to hypersurfaces: L^2 theory*, J. Funct. Anal. **167** (1999), 148-182.

A. Seeger, Department of Mathematics, University of Wisconsin, Madison, Wisconsin, 53706

E-mail address: `seeger@math.wisc.edu`

S. Wainger, Department of Mathematics, University of Wisconsin, Madison, Wisconsin, 53706

E-mail address: `wainger@math.wisc.edu`

J. Wright, School of Mathematics, University of Edinburgh, JCMB, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland

E-mail address: `j.r.wright@ed.ac.uk`

S. Ziesler, Department of Mathematics and Computer Science, Dominican University, River Forest, Illinois 60305

E-mail address: `snziesler@comcast.net`