LOWER BOUNDS FOR HAAR PROJECTIONS: DETERMINISTIC EXAMPLES

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Abstract. In a previous paper by the authors the existence of Haar projections with growing norms in Sobolev-Triebel-Lizorkin spaces has been shown via a probabilistic argument. This existence was sufficient to determine the precise range of Triebel-Lizorkin spaces for which the Haar system is an unconditional basis. The aim of the present paper is to give simple deterministic examples of Haar projections that show this growth behavior in the respective range of parameters.

1. Introduction

In the recent paper [4] the authors considered the question in which range of parameters the Haar system is an unconditional basis in the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R})$, $1 < p, q < \infty$. It turned out that this is the case if and only if

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}\,. \tag{1}$$

The Haar functions (3) belong to the spaces $F^s_{p,q}$ and $B^s_{p,q}$ if $-1/p' < s < 1/p$. Moreover, by results in [6], [7], [9] they form an unconditional basis in $B^s_{p,q}$ in that range. More recently, it was shown by Triebel in [9] that in the more restrictive range (1) the Haar system is an unconditional basis also on $F^s_{p,q}$, and, as a special case when $q = 2$, in the L^p Sobolev space L^s_p. Triebel [10] asked what happens for the remaining cases corresponding to the upper and lower triangles in Figure 1.

In [4] the necessity of the condition (1) was established by showing the existence of subsets E of the Haar system $\mathcal{H}$, see (2) below, for which the corresponding projections

$$P_E f = \sum_{h_{j,k}\in E} 2^j \langle f, h_{j,k}\rangle h_{j,k}$$

are not uniformly bounded in the spaces $F^s_{p,q}$ if $1 < p < q$, $1/q \le s \le 1/p$ and $1 < q < p < \infty$, $-1/p' \le s \le -1/q'$. This shows the failure of unconditional

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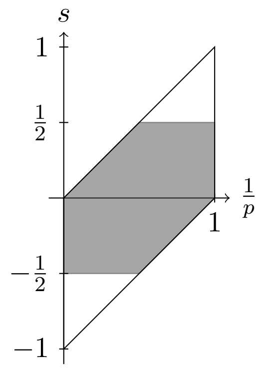

FIGURE 1. Domain for an unconditional basis in spaces L_p^s

convergence of Haar expansions in the respective spaces. The proof of the existence of such projections and sharp growth rates of their norms was based on a probabilistic argument.

The purpose of the present paper is to present a constructive, non-probabilistic argument. It turns out that the families of projections providing sharp growth rates in terms of the Haar frequency set $\mathrm{HF}(E)$ can be easily written down; however the proof of the lower bounds, with concrete testing functions is rather technical.

We consider the Haar system on the real line given by

$$\mathcal{H} = \{h_{j,\mu} \ : \ \mu \in \mathbb{Z}, j = -1, 0, 1, 2, ...\}\,, \tag{2}$$

where for $j \in \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{Z}$, the function $h_{j,\mu}$ is defined by

$$h_{j,\mu}(x) = \mathbb{1}_{I^+_{j,\mu}}(x) - \mathbb{1}_{I^-_{j,\mu}}(x)\,, \tag{3}$$

and $h_{-1,\mu}$ is the characteristic function of the interval $[\mu, \mu{+}1)$. The intervals $I^+_{j,\mu} = [2^{-j}\mu, 2^{-j}(\mu + 1/2))$ and $I^-_{j,\mu} = [2^{-j}(\mu + 1/2), 2^{-j}(\mu + 1))$ represent the dyadic children of the usual dyadic interval $I_{j,\mu} = [2^{-j}\mu, 2^{-j}(\mu{+}1))$. To formulate the main result in [4] we say that the Haar frequency of $h_{j,\mu}$ is 2^j. For a set E of Haar functions we define the Haar frequency set $\mathrm{HF}(E)$ as the set of all 2^j for which $j \in \mathbb{N} \cup \{0\}$ and 2^j is the Haar frequency of some $h \in E$.

Theorem. [4]. *(i) Let $1 < p < q < \infty$ and $1/q < s < 1/p$. Given any set $A \subset \{2^k : k \geq 0\}$ of cardinality $\geq 2^N$ there is a subset E of $\mathcal{H}$ consisting of Haar functions supported in $[0, 1]$ such that $\mathrm{HF}(E) \subset A$ and such that*

$$\|P_E\|_{F^s_{p,q} \to F^s_{p,q}} \geq c(p, q, s) 2^{N(s-1/q)}\,.$$

(ii) Let $1 < q < p < \infty$ and $-1 + 1/p < s < -1 + 1/q$. Given any set $A \subset \{2^k : k \geq 0\}$ of cardinality $\geq 2^N$ there is a subset E of $\mathcal{H}$ consisting of

Haar functions supported in $[0,1]$ *such that* $\mathrm{HF}(E)\subset A$ *and*

$$\|P_E\|_{F^s_{p,q}\to F^s_{p,q}} \ge c(p,q,s)2^{N(\frac{1}{q}-s-1)}\,.$$

There are also lower bounds in terms of powers of N for the endpoint cases $F^{-1/q'}_{p,q}$, $p>q$ and $F^{1/q}_{p,q}$, $p<q$. We note that the result of the theorem is sharp since there are the corresponding upper bounds ([4])

$$\|P_E\|_{F^s_{p,q}\to F^s_{p,q}} \le C(p,q,s)\big(\#(\mathrm{HF}(E))\big)^{s-\frac{1}{q}},$$

for $1<p<q<\infty$, $1/q<s<1/p$, and

$$\|P_E\|_{F^s_{p,q}\to F^s_{p,q}} \le C(p,q,s)\big(\#(\mathrm{HF}(E))\big)^{\frac{1}{q}-s-1},$$

for $1<q<p<\infty$, $-1/p'<s<-1/q'$. A duality argument, see [4], §2.3, shows that assertions (i), (ii) in the theorem are equivalent. It is sufficient to prove the result for N large.

As stated above the theorem was proved in [4] by a probabilistic argument which does not identify the specific projection for which the lower bound holds. We now give an explicit and deterministic definition of such projections.

Let R be a large positive integer to be chosen later. Let $N\gg R$. Given a set of Haar frequencies $A\subset\{2^j : j\ge 1\}$ we choose a $A_N\subset A$ such that

$$2^{N-1}R^{-1}\le \#A_N\le 2^N, \tag{4a}$$

and such that $\log_2 A_N$ is R-separated; i.e., we have the property that

$$2^n\in A_N,\ 2^{\tilde n}\in A_N \implies |n-\tilde n|\ge R \text{ if } n\ne\tilde n. \tag{4b}$$

Let $E=E(N,R)$ be the collection of Haar functions $h_{j,\mu}$ with $2^j\in A_N$ and $0\le\mu\le 2^j-1$, and let P_E be the orthogonal projection to the span of E, defined initially on L^2

$$P_Ef=\sum_{2^j\in A_N}\sum_{\mu=0}^{2^j-1}2^j\langle f,h_{j,\mu}\rangle h_{j,\mu}. \tag{4c}$$

We have the following main result.

Theorem 1.1. *There is* $R=R(p,q,s)>1$ *and* $N_0=N_0(p,q,s)$ *so that for all* $N\ge N_0\gg R$ *the following lower bounds hold for the projection operators* P_E *with* $E=E(N,R)$ *as defined in* (4)*.*

(i) For $1<p<q<\infty$, $1/q<s<1/p$

$$\|P_E\|_{F^s_{p,q}\to F^s_{p,q}} \ge c(p,q,s)2^{N(s-1/q)}\,.$$

(ii) Let $1<q<p<\infty$ *and* $-1+1/p<s<-1+1/q$ *then*

$$\|P_E\|_{F^s_{p,q}\to F^s_{p,q}} \ge c(p,q,s)2^{N(\frac{1}{q}-s-1)}\,.$$

For the proof of (ii) we shall construct deterministically test functions $f_N \in F^s_{p,q}$ for which

$$\|P_E f_N\|_{F^s_{p,q}} \gtrsim \|P_E\|_{F^s_{p,q}\to F^s_{p,q}} \|f_N\|_{F^s_{p,q}}. \tag{5}$$

Here $q < p$ and $-1/p' < s < -1/q'$.

The paper is organized as follows. In §2 we will recall characterizations of the spaces $F^s_{p,q}$ in terms of local means which are convenient to work with. In §3 and §4 we give the construction of a family of test functions satisfying (5). §5 and §6 contain the core of the proof. Finally, in §7 we state some open problems.

2. Preliminaries

Let $\psi_0, \psi \in \mathcal{S}(\mathbb{R})$ such that $|\hat{\psi}_0(\xi)| > 0$ on $(-\varepsilon, \varepsilon)$ and $|\hat{\psi}(\xi)| > 0$ on $\{\xi \in \mathbb{R} : \varepsilon/4 < |\xi| < \varepsilon\}$ for some fixed $\varepsilon > 0$. We further assume vanishing moments of ψ up to order M_1 of ψ; i.e.,

$$\int \psi(x) x^n \, dx = 0 \quad \text{for} \quad n = 0, 1, ..., M_1 \, .$$

As usual we define $\psi_k := 2^k \psi(2^k \cdot)$.

Definition 2.1. Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let further $\psi_0, \psi \in \mathcal{S}(\mathbb{R})$ as above with $M_1 + 1 > s$. The Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R})$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{F^s_{p,q}} := \Big\| \Big(\sum_{k=0}^{\infty} 2^{ksq} |\psi_k * f(\cdot)|^q \Big)^{1/q} \Big\|_p$$

is finite.

The definition of the spaces $F^s_{p,q}(\mathbb{R})$, cf. [8], is usually given in terms of a compactly supported (on the Fourier side) smooth dyadic decomposition of unity. Based on vector-valued singular integral theory [1] it can be shown that the characterization given in Definition 2.1 is equivalent, see also [8, §2.4.6] and [3]. The above characterization allows for choosing ψ_0, ψ compactly supported, which is the reason for the term "local means" which in view of the localization properties of the Haar functions are useful for the purpose of this paper.

3. Families of test functions

Let $A \subset \{2^j : j \ge 1\}$ be given and choose A_N such that

$$2^{N-1} R^{-1} \le \# A_N \le 2^N \, . \tag{6}$$

We set $\mathfrak{A}^N = \log_2 A_N$, i.e., $\mathfrak{A}^N = \{j : 2^j \in A_N\}$. Also let, for large N,

$$\begin{aligned}
\mathfrak{L}^N &= \{l : l - N \in \mathfrak{A}^N\},\\
\mathfrak{S}^N &= \{(l,\nu) : l \in \mathfrak{L}^N,\ 0 \le \nu \le 2^l - 1,\ \nu \in 2^N\mathbb{Z} + 2^{N-1}\},\\
\mathfrak{S}_l^N &= \{\nu : (l,\nu) \in \mathfrak{S}^N\}.
\end{aligned}$$

Let η be an odd C^∞ function supported in $(-2^{-4}, 2^{-4})$. Furthermore it is assumed that η has vanishing moments up to order M_0, i.e., $\int \eta(x)x^n\,dx = 0$ for $n = 0, 1, \dots, M_0$ (with M_0 some large constant), and that

$$2\int_0^{1/2} \eta(x)dx = \int_0^{1/2} \eta(x)dx - \int_{-1/2}^0 \eta(x)dx \ge 1\,. \tag{7}$$

We further define

$$\eta_{l,\nu}(x) = \eta(2^l(x - x_{l,\nu}))\,, \quad \text{where} \quad x_{l,\nu} = 2^{-l}\nu\,. \tag{8}$$

Then, clearly, $\eta_{l,\nu}$ is supported in $[x_{l,\nu} - 2^{-l-4}, x_{l,\nu} + 2^{-l-4}]$. Crucial for the subsequent analysis is the fact that for $\nu, \nu' \in \mathfrak{S}_l^N$ with $\nu \ne \nu'$ the distance of the supports of $\eta_{l,\nu}$ and $\eta_{l,\nu'}$ is at least $2^{N-l} - 2^{-l-3}$.

Let us define the family of test functions f_N by

$$f_N(x) := \sum_{l\in\mathfrak{L}^N} 2^{-ls} \sum_{\nu\in\mathfrak{S}_l^N} \eta_{l,\nu}(x). \tag{9}$$

The following $F^s_{p,q}$-norm bound for the f_N follows from [4, Prop. 4.1], which is based on a result in [2].

Proposition 3.1. *Let $1 \le q \le p < \infty$, $s \in \mathbb{R}$ and $s > -M_0$. Then*

$$\|f_N\|_{F^s_{p,q}} \lesssim_{p,q,s} (1 + 2^{-N}\#(\mathfrak{L}^N))^{1/q} \lesssim 1\,.$$

4. Lower bounds for Haar projections

Let P_E be the (family of) projections defined in (4c). By Proposition 3.1 it suffices to show

$$\|P_E f_N\|_{F^s_{p,q}} \ge c(p,q,s,R)2^{N(\frac{1}{q}-s-1)} \tag{10}$$

in the case $1 < q < p$, $-1/p' < s < -1/q'$. What remains follows by duality, cf. [4, §2.3].

Let ψ be a non-vanishing C^∞-function supported on $(-2^{-4}, 2^{-4})$ in the sense of Definition 2.1 with M_1 large enough. Setting $\psi_k = 2^k\psi(2^k\cdot)$ we have the inequality (according to Definition 2.1),

$$\Big\|\Big(\sum_{k=1}^\infty 2^{ksq}|\psi_k * g|^q\Big)^{1/q}\Big\|_q \lesssim \|g\|_{F^s_{p,q}}\,.$$

Hence, it now suffices to show (for large N)

$$\Big\|\Big(\sum_{k\in\mathfrak{A}^N} 2^{ksq}|\psi_k * P_E f_N|^q\Big)^{1/q}\Big\|_p \gtrsim 2^{N(\frac{1}{q}-s-1)}\,. \tag{11}$$

Now $\psi_k * P_E f_S$ is supported in $[-1,2]$ and, by Hölder's inequality and $p \ge q$, it is enough to verify (11) for $p = q$, i.e., we have to show

$$\Big\| \Big(\sum_{k\in\mathfrak{A}^N} 2^{ksq} \Big| \sum_{(l,\nu)\in\mathfrak{S}^N} 2^{-ls} \sum_{j\in\mathfrak{A}^N} \sum_{\mu=0}^{2^j-1} \langle \eta_{l,\nu}, h_{j,\mu}\rangle h_{j,\mu} * \psi_k \Big|^q \Big)^{1/q} \Big\|_q \gtrsim 2^{N(\frac{1}{q}-s-1)}. \tag{12}$$

Let us define

$$G^{j,N}_{k,l}(x) = \sum_{\mu=0}^{2^j-1} \sum_{\nu\in\mathfrak{S}^N_l} 2^j \langle \eta_{l,\nu}, h_{j,\mu}\rangle h_{j,\mu} * \psi_k. \tag{13}$$

Recall that for $k \in \mathfrak{A}^N$ we have $k+N \in \mathfrak{L}^N$. We shall show two inequalities. In what follows we always have $1 < q < p < \infty$, $-1/p' \le s < -1/q'$.

Proposition 4.1. *There is $c_1 > 0$ such that*

$$\Big(\sum_{k\in\mathfrak{A}^N} 2^{ksq} \big\| 2^{-(k+N)s} G^{k,N}_{k,k+N} \big\|_q^q \Big)^{1/q} \ge c_1 R^{-1/q} 2^{N(\frac{1}{q}-1-s)}. \tag{14}$$

Proposition 4.2. *There is $\varepsilon = \varepsilon(s,q) > 0$ such that*

$$\Big(\sum_{k\in\mathfrak{A}^N} 2^{ksq} \Big\| \sum_{\substack{(j,l)\in\mathfrak{A}^N\times\mathfrak{L}^N \\ (j,l)\neq(k,k+N)}} 2^{-ls} G^{j,N}_{k,l} \Big\|_q^q \Big)^{1/q} \le C_2 (1 + 2^{-R\varepsilon} 2^{N(\frac{1}{q}-1-s)}). \tag{15}$$

If we choose R large enough (depending on p, q, s) then the two propositions imply Theorem 1.1.

5. Proof of Proposition 4.1

Let $\Psi(x) = \int_{-\infty}^x \psi(t)dt$, supported also in $(-2^{-4}, 2^{-4})$. Note that

$$\psi * h_{0,0}(x) = \Psi(x) + \Psi(x-1) - 2\Psi(x - \tfrac{1}{2})$$

and hence $\psi * h_{0,0}(x) = -2\Psi(x-\frac{1}{2})$ if $x \in [1/4, 3/4]$. Thus there is $c > 0$ and an interval $J \subset [1/4, 3/4]$ so that

$$|\psi * h_{0,0}(x)| \ge c \text{ for } x \in J\,.$$

For $k = 0, 1, 2, \dots$ and $\mu \in \mathbb{Z}$ let $J_{k,\mu} = 2^{-k}\mu + 2^{-k}J$, a subinterval of the middle half of $I_{k,\mu}$ of length $\gtrsim 2^{-k}$. We then get the estimate

$$|\psi_k * h_{k,\mu}(x)| \ge c_0 \text{ for } x \in J_{k,\mu}. \tag{16}$$

The left-hand side of (14) is

$$2^{-Ns} \Big(\sum_{k\in\mathfrak{A}^N} \Big\| \sum_{\mu=0}^{2^k-1} \sum_{\nu\in\mathfrak{S}^N_{k+N}} 2^k \langle h_{k,\mu}, \eta_{k+N,\nu}\rangle h_{k,\mu} * \psi_k \Big\|_q^q \Big)^{1/q}.$$

Now $h_{k,\mu}$ is supported in $I_{k,\mu} = [2^{-k}\mu, 2^{-k}(\mu+1)]$. If $\nu \in \mathfrak{S}^N_{k+N}$ then $\nu = 2^N m + 2^{N-1}$ for some integer m and in this case $\eta_{k+N,\nu}$ is supported in $[2^{-k}m+2^{-k-1}-2^{-k-N-4}, 2^{-k}m+2^{-k-1}+2^{-k-N-4}]$. For fixed μ this interval intersects $I_{k,\mu}$ only if $m=\mu$, and thus for the scalar products $\langle h_{k,\mu}, \eta_{k+N,\nu}\rangle$ (with $\nu \in \mathfrak{S}^N_{k+N}$) we only get a contribution for $\nu = \nu_N(\mu) := 2^N\mu + 2^{N-1}$. We calculate

$$\begin{aligned}\langle h_{k,\mu}\eta_{k+N,\nu_N(\mu)}\rangle &= \int h_{0,0}(2^k x-\mu)\eta(2^{k+N}(x-2^{-k}\mu-2^{-k-1}))dx\\ &= 2^{-k}\int \eta(2^N(y-1/2))h_{0,0}(y)\,dy\\ &= 2^{-k}\int_{-1/2}^0 \eta(2^N y) - 2^{-k}\int_0^{1/2}\eta(2^N y)\,dy\\ &= -2^{-N-k+1}\int_0^{1/2}\eta(u)\,du\,,\end{aligned}$$

where we have used that $\mathrm{supp}(\eta(2^N\cdot))$ is contained in $(-2^{-N-4}, 2^{-N-4})$. By (7) we get

$$\big|\langle h_{k,\mu}, \eta_{k+N,\nu_N(\mu)}\rangle\big| \ge 2^{-k-N}.$$

Recall that $J_{k,\mu'}$ is contained in the middle half of $I_{k,\mu'}$. Now

$$\mathrm{supp}(\psi_k * h_{k,\mu}) \subset [2^{-k}\mu - 2^{-k-4}, 2^{-k}(\mu+1)+2^{-k-4}]$$

and thus, given μ,μ', the support of $\psi_k * h_{k,\mu}$ can intersect $J_{k,\mu'}$ only if $\mu=\mu'$. Hence, if we set $\Omega_k = \cup_{\mu=1}^{2^{k-1}} J_{k,\mu}$ we have, using also (16),

$$\begin{aligned}\big\|G^{k,N}_{k,k+N}\big\|_q^q &\ge \int_{\Omega_k} |G^{k,N}_{k,k+N}(x)|^q dx\\ &\ge \sum_{\mu=1}^{2^{k-1}} \big|2^k\langle h_{k,\mu}, \eta_{k+N,\nu_N(\mu)}\rangle\big|^q \int_{J_{k,\mu}} |\psi_k * h_{k,\mu}(x)|^q dx\\ &\ge c2^{-Nq}\,,\end{aligned}$$

where $c>0$ does neither depend on R nor N. Since $\mathrm{card}(\mathfrak{A}^N) \gtrsim 2^{N-1}/R$ we obtain the lower bound (14) after summing in k.

6. Proof of Proposition 4.2

We first collect several standard and elementary facts about the Haar coefficients.

Lemma 6.1. *(i) If* $\mathrm{supp}(\eta_{l,\nu})$ *is contained either in* $I^+_{j,\mu}$, *or in* $I^-_{j,\mu}$, *or in* $I^{\complement}_{j,\mu}$ *then* $\langle \eta_{l,\nu}, h_{j,\mu}\rangle = 0$.

(ii)

$$|\langle \eta_{l,\nu}, h_{j,\mu}\rangle| \lesssim \begin{cases} 2^{-l} & \text{if } l \ge j,\\ 2^{l-2j} & \text{if } l \le j.\end{cases}$$

Lemma 6.2. *(i) Suppose that $k \ge j$. If the distance of x to the three points $2^{-j}\mu$, $2^{-j}(\mu+\frac12)$, $2^{-j}(\mu+1)$ is at least 2^{-k} then $h_{j,\mu} * \psi_k(x) = 0$.*

(ii) For $k \ge j$ we have $\|h_{j,\mu} * \psi_k\|_q \lesssim 2^{-k/q}$.

Lemma 6.3. *Let $k \le j$. Let $y_{j,\mu} := 2^{-j}(\mu+\frac12)$, the midpoint of the interval $I_{k,\mu}$. Then the support of $h_{j,\mu} * \psi_k$ is contained in $[y_{j,\mu} - 2^{-k}, y_{j,\mu} + 2^{-k}]$. Also,*

$$\|h_{j,\mu} * \psi_k\|_\infty \lesssim 2^{2k-2j}.$$

The proofs of Lemmata 6.1, 6.2 and 6.3 are straightforward and can be looked up e.g. in [4].

We have the following estimates when $j \le k$.

Lemma 6.4. *Let $l \ge N$. For $1 \le q \le \infty$,*

(17a) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{j-l} 2^{(j-k)/q}, \qquad k \ge j, \quad l \ge j+N,$$

(17b) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{j-l} 2^{(l-N-k)/q}, \qquad k \ge j, \quad j \le l \le j+N,$$

(17c) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{l-j} 2^{(l-k-N)/q}, \qquad k \ge j, \quad l \le j.$$

Proof. Let $l \ge j+N$. By Lemma 6.2, (i), the function $G^{j,N}_{k,l}$ is supported on the union of $O(2^j)$ intervals of length 2^{-k}, i.e. on a set of measure $O(2^{j-k})$. By Lemma 6.1 we have, for fixed μ, that $\langle \eta_{l,\nu}, h_{j,\mu}\rangle \ne 0$ only for a finite number of indices ν, and we always have $2^j |\langle \eta_{l,\nu}, h_{j,\mu}\rangle| \lesssim 2^{j-l}$. Thus (17a) follows.

Now let $j \le l \le j+N$. Since the sets $\mathrm{supp}(\eta_{l,\nu})$ with $\nu \in \mathfrak{S}^N_l$ are 2^{N-2-l} separated, and $2^{N-l} \ge 2^{-j}$, we see from Lemma 6.2 that $G^{j,N}_{k,l}$ is supported on the union of $O(2^{l-N})$ intervals of length 2^{-k}, i.e. on a set of measure $O(2^{l-k-N})$. As in the previous case $2^j|\langle \eta_{l,\nu}, h_{j,\mu}\rangle| \lesssim 2^{j-l}$, and (17b) follows.

Let $l \le j$. As in the previous case $G^{j,N}_{k,l}$ is supported on a set of measure $O(2^{l-k-N})$. By Lemma 6.1, (ii), we have now $2^j|\langle \eta_{l,\nu}, h_{j,\mu}\rangle| \lesssim 2^{l-j}$, and (17c) follows. $\square$

For $k \le j$ we have

Lemma 6.5. *Let $l \ge N$. For $1 \le q \le \infty$,*

(18a) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{k-l}, \qquad k \le j \le l-N,$$

(18b) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{k-j-N}, \qquad k \le l-N \le j \le l,$$

(18c) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{2k-j-l} 2^{(l-k-N)/q}, \qquad l-N \le k \le j \le l,$$

(18d) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{2k-2j} 2^{(l-k-N)/q}, \qquad l-N \le k \le l \le j,$$

(18e) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{l+k-2j-N}, \qquad k \le l-N \le l \le j,$$

(18f) $$\|G^{j,N}_{k,l}\|_q \lesssim 2^{3k+l-4j} 2^{-N/q}, \qquad l \le k \le j.$$

Proof. Let, for $\rho \in \mathbb{Z}$, $I^*_{k,\rho} = \cup_{i=-1,0,1} I_{k,\rho+i}$ the triple interval. Then for $j \geq k$ the function $h_{j,\mu} * \psi_k$ is supported in at most five of the intervals $I^*_{k,\rho}$.

For the case (18a) we have $2^{-l+N} \leq 2^{-j} \leq 2^{-k}$. By Lemma 6.1 we have, for each μ, $\sum_\nu 2^j |\langle \eta_{l,\nu}, h_{j,\mu}\rangle| \lesssim 2^{j-l}$. By Lemma 6.3 $|h_{j,\mu} * \psi_k(x)| \lesssim 2^{2k-2j}$ and for fixed x there are at most $O(2^{j-k})$ terms with $h_{j,\mu} * \psi_k(x) \neq 0$. Hence $|G^{j,N}_{k,l}(x)| \lesssim 2^{k-l}$ and (18a) follows.

Now consider the case (18b), $2^{-l} \leq 2^{-j} \leq 2^{-l+N} \leq 2^{-k}$. Let $M_k(x)$ be the number of indices μ for which there exists a ν with $\langle \eta_{l,\nu}, h_{j,\mu}\rangle \neq 0$ and for which $h_{j,\mu} * \psi_k(x) \neq 0$. Since the supports of the $\eta_{l,\nu}$ are 2^{N-2-l} separated, and $2^{N-l} \geq 2^{-j}$ we have $M_k(x) \lesssim 2^{l-N-k}$. The upper bounds for $\sum_\nu 2^j |\langle \eta_{l,\nu}, h_{j,\mu}\rangle|$ and for $|h_{j,\mu} * \psi_k(x)|$ are as in the previous case. Hence $|G^{j,N}_{k,l}(x)| \lesssim 2^{j-l} 2^{2k-2j} 2^{l-N-k} = 2^{k-j-N}$ and (18b) follows.

Next consider the case (18c), $2^{-l} \leq 2^{-j} \leq 2^{-k} \leq 2^{-l+N}$. As in the previous case, $|h_{j,\mu} * \psi_k(x)| \lesssim 2^{2k-2j}$, and $2^j |\langle \eta_{l,\nu}, h_{j,\mu}\rangle| \lesssim 2^{j-l}$. Also since the supports of the $\eta_{l,\nu}$ are 2^{-l+N-2}-separated and $2^{-l+N} \geq 2^{-k} \geq 2^{-j} \geq 2^{-l}$ there are, for every x only $O(1)$ indices ν, and $O(1)$ indices μ so that $\langle \eta_{l,\nu}, h_{j,\mu}\rangle \neq 0$ and $h_{j,\mu} * \psi_k(x) \neq 0$. Hence $\|Gf\|_\infty \lesssim 2^{2k-2j} 2^{j-l} = 2^{2k-j-l}$. Finally, again, because $2^{-l+N} \geq 2^{-k}$ the support of $G^{j,N}_{k,l}$ is contained in a union of $O(2^{l-N})$ intervals of length $O(2^{-k})$ and thus in a set of measure $O(2^{l-N-k})$. Now (18c) follows.

Consider the case (18d), $2^{-j} \leq 2^{-l} \leq 2^{-k} \leq 2^{-l+N}$. Since $2^{-l+N} \geq 2^{-k}$ there are, for any x, only $O(1)$ indices ν such there exists a μ with $\langle \eta_{l,\nu}, h_{j,\mu}\rangle \neq 0$ and $h_{j,\mu} * \psi_k(x) \neq 0$; moreover the set of x for which this can happen is a union of $O(2^{l-N})$ intervals of length $O(2^{-k})$ and thus of measure $O(2^{l-N-k})$. By Lemma 6.3, $\|h_{j,\mu} * \Psi_k\|_\infty \lesssim 2^{2k-2j}$, and by Lemma 6.1 we have, for fixed ν, $\sum_\mu 2^j |\langle \eta_{l,\nu}, h_{j,\mu}\rangle| \lesssim 2^{j-l} 2^j 2^{l-2j} \lesssim 1$ and thus $\|G^{j,N}_{k,l}\|_\infty \lesssim 2^{2k-2j}$. Together with the support property of $G^{j,N}_{k,l}$ this shows (18d).

Next consider the case (18e), $2^{-j} \leq 2^{-l} \leq 2^{N-l} \leq 2^{-k}$. For each x we have $\psi_k * h_{j,\mu}(x) \neq 0$ only for those μ with $|2^{-j}\mu - x| \leq 2 \cdot 2^{-k}$. We can have $\langle \eta_{l,\nu}, h_{j,\mu}\rangle \neq 0$ for some of such μ only when $|2^{-l}\nu - x| \leq 2 \cdot 2^{-k}$ and because of the 2^{-l+N-2}-separateness of the sets $\mathrm{supp}(\eta_{l,\nu})$ there are at most 2^{l-N-k} indices ν with this property. For each such ν there are at most $O(2^{j-l})$ indices μ such that $\langle \eta_{l,\nu}, h_{j,\mu}\rangle \neq 0$. We use the bounds $2^j \langle \eta_{l,\nu}, h_{j,\mu}\rangle = O(2^{l-j})$ and $\psi_k * h_{j,\mu}(x) = O(2^{2k-2j})$ to see that $|G^{j,N}_{k,l}(x)| \lesssim 2^{l-N-k} 2^{j-l} 2^{l-j} 2^{2k-2j}$; hence $\|G^{j,N}_{k,l}\|_\infty \lesssim 2^{l+k-2j-N}$ which gives (18e).

Finally, for (18f), $2^{-j} \leq 2^{-k} \leq 2^{-l}$, we use $l \geq N$. Then by the separation of the sets $\mathrm{supp}(\eta_{l,\nu})$ we see that $G^{j,N}_{k,l}$ is supported on the union of $O(2^{l-N})$ intervals I_ν of length $O(2^{-l})$ (containing $2^{-l}\nu$ with $\nu \in \mathfrak{S}^N_l$). Thus $G^{j,N}_{k,l}$ is supported on a set of measure 2^{-N}. For $x \in I_\nu$ there are at most $O(2^{k-j})$ indices μ with $\psi_k * h_{j,\mu}(x) \neq 0$. For any such μ we have again

$2^j\langle\eta_{l,\nu},h_{j,\mu}\rangle = O(2^{l-j})$ and $\psi_k * h_{j,\mu}(x) = O(2^{2k-2j})$. Thus we have the bound $|G^{j,N}_{k,l}(x)| \lesssim 2^{k-j}2^{l-j}2^{2k-2j}$; hence $\|G^{j,N}_{k,l}\|_\infty \lesssim 2^{3k+l-4j}$ and by the estimate for the support of $G^{j,N}_{k,l}$ we obtain (18f). $\square$

Now let $\mathcal{P}$ be the set of pairs $(m,n)\in\mathbb{Z}\times\mathbb{Z}$ such that at least one of the inequalities $|m|\ge R$, $|n|\ge R$ is satisfied. Change variables to write $j=k+m$, and $l=k+N+n$. We estimate the left hand side of (15) as

(19)
$$\Big(\sum_{k\in\mathfrak{A}^N}\Big\|\sum_{\substack{(m,n):\\(k+m,k+N+n)\in\mathfrak{A}^N\times\mathfrak{L}^N\\(m,n)\neq(0,0)}} 2^{-s(n+N)}G^{k+m,N}_{k,k+N+n}\Big\|_q^q\Big)^{1/q} \lesssim \sum_{(m,n)\in\mathcal{P}}\mathcal{V}_{m,n}$$

where

$$\mathcal{V}_{m,n} = 2^{-s(n+N)}\Big(\sum_{\substack{k\in\mathfrak{A}^N:\\(k+m,k+N+n)\in\mathfrak{A}^N\times\mathfrak{L}^N}}\|G^{k+m,N}_{k,k+N+n}\|_q^q\Big)^{1/q}.$$

We may rewrite the inequalities in Lemma 6.4 and Lemma 6.5 and get estimates in terms of m,n. Using $\#\mathfrak{A}^N=O(2^N)$ this leads to the following inequalities for $m\le 0$.

(20a) $\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-1-s)}2^{-n(1+s)+m(1+\frac1q)}$, $m\le 0$, $n\ge m$,

(20b) $\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-s-1)}2^{n(\frac1q-1-s)+m}$, $m\le 0$, $m-N\le n\le m$,

(20c) $\mathcal{V}_{m,n}\lesssim 2^{(N+n)(\frac1q-s)}2^{N+n-m}$, $m\le 0$, $n\le m-N$.

For $m\ge 0$ we get from Lemma 6.5

(21a) $\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-s-1)}2^{-n(1+s)}$, $0\le m\le n$,

(21b) $\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-s-1)}2^{-m-sn}$, $0\le n\le m\le n+N$,

(21c) $\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-s-1)}2^{n(\frac1q-1-s)-m}$, $n\le 0\le m\le n+N$,

(21d) $\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-s)}2^{-2m+n(\frac1q-s)}$, $n\le 0\le n+N\le m$,

(21e) $\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-s)}2^{-2m}2^{n(1-s)}$, $0\le n\le m-N$,

(21f) $\mathcal{V}_{m,n}\lesssim 2^{(N+n)(1-s)-4m}$, $n+N\le 0\le m$.

Now use the assumption $-1<s<\frac1q-1$, and conclude by summing in m, n for the various parts. First consider the case $m\le 0$. By (20a),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\m\le 0,\,n\ge m}}\mathcal{V}_{m,n}\lesssim 2^{N(\frac1q-1-s)}\Big(\sum_{m\le -R}\sum_{n\ge m}+\sum_{-R\le m\le 0}\sum_{n\ge R}\Big)2^{m(1+\frac1q)}2^{-n(1+s)}$$

(22a)
$$\lesssim\big(2^{-R(\frac1q-s)}+2^{-R(1+s)}\big)\,2^{N(\frac1q-1-s)}.$$

By (20b),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ m\le 0,\, m-N\le n\le m}} \mathcal{V}_{m,n} \lesssim \Big(\sum_{m\le -R}\sum_{n\le m} + \sum_{-R\le m\le 0}\sum_{n\le -R}\Big) 2^{m+(n+N)(\frac1q-s-1)}$$

$$\lesssim 2^{-R(\frac1q-s-1)}\, 2^{N(\frac1q-1-s)}\,. \tag{22b}$$

By (20c),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ n+N\le m\le 0}} \mathcal{V}_{m,n} \lesssim \sum_{m\le 0}\sum_{n\le m-N} 2^{(n+N)(\frac1q-s+1)-m} \lesssim_{q,s} 1\,. \tag{22c}$$

We now turn to the terms with $m \ge 0$. Observe that the conditions $(m,n)\in\mathcal{P}$, $0\le m\le n$ imply $n\ge R$, and by (21a) we get

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ 0\le m\le n-N}} \mathcal{V}_{m,n} \lesssim 2^{N(\frac1q-1-s)} \sum_{n\ge R} 2^{-n(1+s)} n$$

$$\lesssim R 2^{-R(1+s)} 2^{N(\frac1q-1-s)}\,. \tag{23a}$$

By (21b),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ 0\le n\le m\le n+N}} \mathcal{V}_{m,n} \lesssim 2^{N(\frac1q-1-s)}\Big(\sum_{m\ge R}\sum_{0\le n\le m} + \sum_{n\ge R}\sum_{m\ge n}\Big) 2^{-sn} 2^{-m}$$

$$\lesssim 2^{-R(1+s)} 2^{N(\frac1q-1-s)}\,. \tag{23b}$$

By (21c),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ n\le 0\le m\le n+N}} \mathcal{V}_{m,n} \lesssim \Big(\sum_{-N\le n\le 0}\sum_{m\ge R} + \sum_{-N\le n\le -R}\sum_{0\le m\le N+n}\Big) 2^{(N+n)(\frac1q-1-s)} 2^{-m}$$

$$\lesssim \big(2^{-R} + 2^{-R(\frac1q-1-s)}\big) 2^{N(\frac1q-1-s)}\,. \tag{23c}$$

By (21d),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ n\le 0\le n+N\le m}} \mathcal{V}_{m,n} \lesssim \sum_{-N\le n\le 0}\sum_{m\ge n+N} 2^{-2m+(n+N)(\frac1q-s)}$$

$$\lesssim \sum_{-N\le n\le 0} 2^{(n+N)(\frac1q-s-2)} \lesssim 1\,. \tag{23d}$$

By (21e),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ 0\le n\le m-N}} \mathcal{V}_{m,n} \lesssim 2^{N(\frac1q-s)} \sum_{n\ge 0} 2^{n(1-s)} \sum_{m\ge n+N} 2^{-2m}$$

$$\lesssim 1\,. \tag{23e}$$

By (21f),

$$\sum_{\substack{(m,n)\in\mathcal{P}\\ n+N\le 0\le m}} \mathcal{V}_{m,n} \lesssim \sum_{n\le -N} 2^{(n+N)(1-s)} \sum_{m\ge 0} 2^{-4m} \lesssim 1\,. \tag{23f}$$

We combine (19) with the various estimates in (22) and (23) to obtain (15) with a positive $\varepsilon < \min\{s+1, \frac{1}{q}-s-1\}$.

7. Concluding remarks

7.1. *Endpoint cases.* For the endpoint cases $p<q$, $s=1/q$, and $q<p$, $s=-1/q'$ it has been shown in [4] that for any $A\subset\{2^j : j\in\mathbb{N}\}$ with $\#A\approx 2^N$ there is a set E of Haar functions supported in $[0,1]$ such that $\mathrm{HF}(E)\subset A$ and such that

$$\begin{aligned} &\|P_E\|_{F^{1/q}_{p,q}\to F^{1/q}_{p,q}} \gtrsim N^{1/q}, \quad 1<p<q<\infty\\ &\|P_E\|_{F^{-1/q'}_{p,q}\to F^{-1/q'}_{p,q}} \gtrsim N^{1-1/q}, \quad 1<q<p<\infty \end{aligned}$$

If A is N-separated then these bounds are sharp, as they are matched with corresponding upper bounds. In all cases the upper bounds are $O(N)$ and in some cases the lower bounds may be $\approx N$. The proofs of these results in [4] rely on probabilistic arguments. A combination with the ideas in this paper (using in particular the R-separation in the frequency sets in (4b)) also yields lower bounds for explicit examples of projections. The details are somewhat lengthy (cf. §6) and we shall not pursue this here.

7.2. *Some open problems.*

7.2.1. *Test functions for the case $p<q$.* Our proofs reduce the case $p<q$ to the case $q<p$ by duality. It would be interesting to identify suitable test functions in the case $p<q$ and get a proof which establishes directly the lower bounds in the range $1/q\le s<1/p$.

7.2.2. *Multipliers for Haar expansions.* Let $m\in\ell^\infty(\mathbb{N}\times\mathbb{Z})$. Consider the operator defined on L^2 by

$$T_m f = \sum_{j,\mu} m(j,\mu) 2^j \langle f, h_{j,\mu}\rangle h_{j,\mu}\,.$$

When $\max\{-1/p',-1/q'\}<s<\min\{1/p,1/q\}$ the operator T_m is bounded on $F^s_{p,q}$ with operator norm $\lesssim \|m\|_\infty$ since the Haar system is an unconditional basis in this case. What is the right condition for boundedness on $F^s_{p,q}$ when either $p<q$, $1/q\le s<1/p$, or $q<p$, $-1/p'<s\le -1/q'$?

7.2.3. *Quasi-greedy bases.* Compared to unconditionality there is a weaker property of a basis in a Banach space, called "quasi-greedy", see [5, §1.4], which is highly relevant for non-linear approximation. It is known that "unconditionality" implies "quasi-greedy" but not the other way around, see [5, §1.1]. Hence it is a natural question (asked by V. Temlyakov) whether the Haar basis $\mathcal{H}$ is quasi-greedy in $F^s_{p,q}$ if $1 < p < q$, $1/q \le s < 1/p$ and $1 < q < p < \infty$, $-1/p' < s \le -1/q'$. Note that this is already open in the case $q = 2$, corresponding to L^p Sobolev spaces L^s_p.

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