

# HAAR PROJECTION NUMBERS AND FAILURE OF UNCONDITIONAL CONVERGENCE IN SOBOLEV SPACES

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ABSTRACT. For  $1 < p < \infty$  we determine the precise range of  $L_p$  Sobolev spaces for which the Haar system is an unconditional basis. We also consider the natural extensions to Triebel-Lizorkin spaces and prove upper and lower bounds for norms of projection operators depending on properties of the Haar frequency set.

## 1. INTRODUCTION

We consider the Haar system on the real line given by

$$(1) \quad \mathcal{H} = \{h_{j,\mu} : \mu \in \mathbb{Z}, j = -1, 0, 1, 2, \dots\},$$

where for  $j \in \mathbb{N} \cup \{0\}$ ,  $\mu \in \mathbb{Z}$ , the function  $h_{j,\mu}$  is defined by

$$h_{j,\mu}(x) = \mathbb{1}_{I_{j,\mu}^+}(x) - \mathbb{1}_{I_{j,\mu}^-}(x),$$

and  $h_{-1,\mu}$  is the characteristic function of the interval  $[\mu, \mu+1)$ . The intervals  $I_{j,\mu}^+ = [2^{-j}\mu, 2^{-j}(\mu + 1/2))$  and  $I_{j,\mu}^- = [2^{-j}(\mu + 1/2), 2^{-j}(\mu + 1))$  represent the dyadic children of the usual dyadic interval  $I_{j,\mu} = [2^{-j}\mu, 2^{-j}(\mu + 1))$ .

It has been shown by Marcinkiewicz [11] (based on Paley's square function result [12] for the Walsh system) that the Haar system, in contrast to the trigonometric system, represents an unconditional basis in all  $L_p([0, 1])$  if  $1 < p < \infty$ . In this paper we consider this problem in Banach spaces measuring smoothness. Triebel [20, 21, 23] showed that the Haar system represents an unconditional basis in Besov spaces  $B_{p,q}^s$  if  $1 < p, q < \infty$  and  $-1/p' < s < 1/p$ . In addition, he obtained extensions to quasi-Banach spaces. See also Ropela [14], Sickel [18], and Bourdaud [3] for related results. Note, that the endpoint case  $s = 1/p$  (and by duality the case  $s = -1/p'$ ) can be excluded by noting that all Haar functions belong to  $B_{p,q}^{1/p}$  if and

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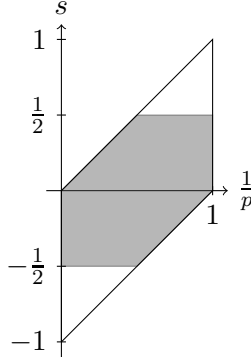


FIGURE 1. Domain for an unconditional basis in spaces  $L_p^s$

only if  $q = \infty$ . Concerning Sobolev and Triebel-Lizorkin spaces the picture is far more interesting. Triebel [23] proved that the Haar system is an unconditional basis in Sobolev (or Bessel potential) spaces  $L_p^s$ ,  $1 < p < \infty$ , if  $\max\{-1/p', -1/2\} < s < \min\{1/p, 1/2\}$ . Here the norm in  $L_p^s$  is given by  $\|f\|_{L_p^s} = \|\mathcal{D}_B^s f\|_p$  where  $\widehat{\mathcal{D}_B^s f}(\xi) = (1 + |\xi|^2)^{s/2} \widehat{f}(\xi)$ .

It has been an open question (formulated explicitly by Triebel in [23, p.95] and in [24]) whether the Haar system is an unconditional Schauder basis on  $L_p^s$  for the ranges  $1 < p < 2$ ,  $1/2 \leq s \leq 1/p$  and  $2 < p < \infty$ ,  $-1/p' \leq s \leq -1/2$ . We answer this question negatively.

It is natural to formulate the results in the class of Triebel-Lizorkin spaces  $F_{p,q}^s$  which include the  $L_p$ -Sobolev spaces  $L_p^s$ ; recall that by Littlewood-Paley theory  $L_p^s = F_{p,2}^s$  for  $1 < p < \infty$  and  $s \in \mathbb{R}$ . We emphasize that the results are already new for the special case of  $L_p^s$ -spaces.

**Theorem 1.1.** *For  $1 < p, q < \infty$ , the Haar system is an unconditional basis in  $F_{p,q}^s$  if and only if*

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}.$$

Thus the result about the Haar system in Sobolev and Triebel-Lizorkin spaces depends in a significant way on the secondary integrability parameter  $q$  while for the Besov spaces  $q$  plays no role. The “if” part of Theorem 1.1 was known and can be found in [23]. The figure above illustrates the differences of the results in Besov and Sobolev spaces.

An application of Theorem 1.1 concerns dyadic characterizations of  $F_{p,q}^s$ . For  $j \geq 1$  let  $\mathbb{1}_{j,\mu}$  be the characteristic function of the support of  $h_{j,\mu}$ . One defines the sequence space  $f_{p,q}^s$  as the space of all doubly-indexed sequences  $\{\lambda_{j,\mu}\}_{j,\mu} \subset \mathbb{C}$  for which

$$(2) \quad \|f\|_{f_{p,q}^s} = \left\| \left( \sum_{j=-1}^{\infty} 2^{jsq} \left| \sum_{\mu \in \mathbb{Z}} \lambda_{j,\mu} \mathbb{1}_{j,\mu} \right|^q \right)^{1/q} \right\|_p$$

is finite. For  $f \in \mathcal{S}(\mathbb{R})$  consider the dyadic version of the  $F_{p,q}^s$ -norm given by

$$\|f\|_{F_{p,q}^{s,\text{dyad}}} = \|\{2^j \langle f, h_{j,\mu} \rangle\}_{j,\mu}\|_{f_{p,q}^s}$$

and let  $F_{p,q}^{s,\text{dyad}}$  be the completion of  $\mathcal{S}(\mathbb{R})$  under this norm.

Triebel [23] showed that  $\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}$  is sufficient for  $F_{p,q}^{s,\text{dyad}} = F_{p,q}^s$  with equivalence of norms. He also showed that this equivalence implies that  $\mathcal{H}$  is an unconditional basis in  $F_{p,q}^s$ . Hence, Theorem 1.1 yields the necessity in the result below.

**Corollary 1.2.** *For  $1 < p, q < \infty$  we have  $F_{p,q}^{s,\text{dyad}} = F_{p,q}^s$  if and only if*

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}.$$

*Quantitative results.* We now formulate quantitative versions of Theorem 1.1. For  $j \geq 0$  define the *Haar frequency* of  $h_{j,\mu}$  to be  $2^j$ . For any subset  $E$  of the Haar system let  $\text{HF}(E)$  be the *Haar frequency set* of  $E$ , i.e.  $\text{HF}(E)$  consists of all  $2^k$  with  $k \geq 0$  for which there exists  $\mu \in \mathbb{Z}$  with  $h_{k,\mu} \in E$ . Let  $P_E$  be the orthogonal projection to the subspace spanned by  $\{h : h \in E\}$ , which is closed in  $L_2(\mathbb{R})$ . For Schwartz functions  $f$  we define

$$P_E f = \sum_{h_{j,\mu} \in E} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}.$$

For function spaces  $X$  such as  $X = F_{p,q}^s$  (or  $X = B_{p,q}^s$ ) we define growth functions depending on the cardinality of the Haar frequency set of  $E$ . First, for any  $A \subset \{2^n : n = 0, 1, \dots\}$ , set

$$(3) \quad \mathcal{G}(X, A) = \sup \{\|P_E\|_{X \rightarrow X} : \text{HF}(E) \subset A\}.$$

Define, for  $\Lambda \in \mathbb{N}$ , the upper and lower *Haar projection numbers*

$$(4) \quad \begin{aligned} \gamma^*(X; \Lambda) &= \sup \{\mathcal{G}(X, A) : \#A \leq \Lambda\}, \\ \gamma_*(X; \Lambda) &= \inf \{\mathcal{G}(X, A) : \#A \geq \Lambda\}. \end{aligned}$$

Clearly,  $\gamma_*(X; \Lambda) \leq \gamma^*(X; \Lambda)$ . If the Haar basis is an unconditional basis of  $X$  then  $\gamma^*(X; \Lambda) = O(1)$ . By the known results we have  $\gamma^*(F_{p,q}^s; \Lambda) = O(1)$  for the cases  $\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}$ . Note that for  $s \geq 1/p$  the Haar functions do not belong to  $F_{p,q}^s$ , and thus  $\gamma_*(F_{p,q}^s; \Lambda) = \infty$ . By duality,  $\gamma_*(F_{p,q}^s; \Lambda) = \infty$  for  $s \leq -1 + 1/p$ . Unlike for the scale of Besov spaces, there are intermediate ranges where the Haar system is not an unconditional basis of  $F_{p,q}^s$  but the Haar projection numbers are finite, however not uniformly bounded.

In what follows we always assume  $\Lambda > 10$ . We shall use the notation  $A \lesssim B$ , or  $B \gtrsim A$ , if  $A \leq CB$  for a positive constant depending only on  $p, q, s$ . We also use  $A \approx B$  if both  $A \lesssim B$  and  $B \lesssim A$ .

**Theorem 1.3.** (i) *For  $1 < p < q < \infty$ ,  $1/q < s < 1/p$ ,*

$$\gamma_*(F_{p,q}^s; \Lambda) \approx \gamma^*(F_{p,q}^s; \Lambda) \approx \Lambda^{s - \frac{1}{q}}.$$

(ii) For  $1 < q < p < \infty$ ,  $-1/p' < s < -1/q'$ ,

$$\gamma_*(F_{p,q}^s; \Lambda) \approx \gamma^*(F_{p,q}^s; \Lambda) \approx \Lambda^{-\frac{1}{q'}-s}.$$

Consequently the magnitude of  $\mathcal{G}(F_{p,q}^s, A)$  depends on the cardinality of  $A$  alone and we have  $\mathcal{G}(F_{p,q}^s, A) \approx (\#A)^{s-1/q}$  when  $1/q < s < 1/p$ . For the endpoint case  $s = 1/q$  or  $s = -1/q'$  we still have failure of unconditional convergence, but a new phenomenon occurs: the quantity  $\mathcal{G}(F_{p,q}^{1/q}, A)$  also depends on the density of  $\log_2(A) = \{k : 2^k \in A\}$  on intervals of length  $\approx \log_2 \#A$ . Define for any  $A$  with  $\#A \geq 2$

$$\overline{\mathcal{Z}}(A) = \max_{n \in \mathbb{Z}} \#\{k : 2^k \in A, |k - n| \leq \log_2 \#A\},$$

$$\underline{\mathcal{Z}}(A) = \min_{2^n \in A} \#\{k : 2^k \in A, |k - n| \leq \log_2 \#A\}.$$

Notice that  $1 \leq \underline{\mathcal{Z}}(A) \leq \overline{\mathcal{Z}}(A) \leq 1 + 2 \log_2 \#A$ .

**Theorem 1.4.** *Let  $A \subset \{2^n : n \geq 0\}$  such that  $\#A \geq 2$ .*

(i) For  $1 < p < q < \infty$ ,

$$\underline{\mathcal{Z}}(A)^{1-\frac{1}{q}} \lesssim \frac{\mathcal{G}(F_{p,q}^{1/q}, A)}{(\log_2 \#A)^{\frac{1}{q}}} \lesssim \overline{\mathcal{Z}}(A)^{1-\frac{1}{q}}.$$

(ii) For  $1 < q < p < \infty$ ,

$$\underline{\mathcal{Z}}(A)^{\frac{1}{q}} \lesssim \frac{\mathcal{G}(F_{p,q}^{-1/q'}, A)}{(\log_2 \#A)^{1-\frac{1}{q}}} \lesssim \overline{\mathcal{Z}}(A)^{\frac{1}{q}}.$$

We remark that  $\overline{\mathcal{Z}}(A) = O(1)$  when  $\#A \approx 2^N$  and  $\log_2(A)$  is  $N$ -separated. On the other hand, for  $A = [1, 2^N] \cap \mathbb{N}$  we have  $\underline{\mathcal{Z}}(A) \geq N$ . Hence it follows that the lower and upper Haar projection numbers for the endpoint cases have now different growth rates:

**Corollary 1.5.** *For  $\Lambda \geq 4$  we have the following equivalences.*

(i) For  $1 < p < q < \infty$ ,

$$\gamma_*(F_{p,q}^{1/q}; \Lambda) \approx (\log_2 \Lambda)^{1/q}$$

and

$$\gamma^*(F_{p,q}^{1/q}; \Lambda) \approx \log_2 \Lambda.$$

(ii) For  $1 < q < p < \infty$ ,

$$\gamma_*(F_{p,q}^{-1+1/q}; \Lambda) \approx (\log_2 \Lambda)^{1-1/q}$$

and

$$\gamma^*(F_{p,q}^{-1+1/q}; \Lambda) \approx \log_2 \Lambda.$$

The proof of the lower bounds for the lower Haar projection numbers also shows that for any infinite subset  $A$  of  $\{2^n : n \geq 0\}$  there is a subset  $E$  of the Haar system, with Haar frequency set contained in  $A$ , so that  $P_E$  does not extend to a bounded operator on  $F_{p,q}^s$ , in the  $s$ -ranges of Theorem 1.3.

*Guide through the paper.* In §2 we discuss some preliminary facts about Peetre maximal functions and Triebel-Lizorkin spaces. In §3 we prove the sharp upper bounds for Haar projection operators. In §4 we provide estimates for suitable families of test functions in  $F_{p,q}^s$  for  $p > q$  and  $s \leq -1/q'$ . In §5 we determine the behavior of  $\gamma_*(F_{p,q}^s; \Lambda)$  for large  $\Lambda$ , and also the behavior of  $\gamma^*(F_{p,q}^s; \Lambda)$  if  $s < -1/q'$ . In §6 we prove refined lower bounds for the endpoint  $s = -1/q'$ ,  $p \geq q$  which yield in particular precise bounds for  $\gamma^*(F_{p,q}^{-1/q'}; \Lambda)$ . Concluding remarks are made in §7.

## 2. PRELIMINARIES

### 2.1. Littlewood-Paley decompositions and Triebel-Lizorkin spaces.

We pick functions  $\psi_0, \psi$  such that  $|\widehat{\psi}_0(\xi)| > 0$  on  $(-\varepsilon, \varepsilon)$  and  $|\widehat{\psi}(\xi)| > 0$  on  $\{\xi : \varepsilon/4 < |\xi| < \varepsilon\}$  for some fixed  $\varepsilon > 0$ . We further assume

$$(5) \quad \int \psi(x)x^n dx = 0 \text{ for } n = 0, 1, \dots, M_1$$

(if  $M_1$  is a large given integer).

Let now  $\varphi_0 \in \mathcal{S}(\mathbb{R})$  be a compactly supported function with  $\varphi_0 \equiv 1$  on  $[-4/3, 4/3]$  and  $\varphi_0 \equiv 0$  on  $\mathbb{R} \setminus [-3/2, 3/2]$ . Putting  $\varphi = \varphi_0 - \varphi_0(2\cdot)$  we obtain a smooth dyadic decomposition of unity, i.e.,  $\varphi_0(\cdot) + \sum_{k \geq 1} \varphi(2^{-k}\cdot) \equiv 1$ . In addition, we set  $\beta_0(\xi) := \varphi_0(2\xi/\varepsilon)/\psi_0(\xi)$  and  $\beta(\xi) := \varphi(2\xi/\varepsilon)/\psi(\xi)$ . Hence,  $\beta_0, \beta$  are well-defined Schwartz functions supported on  $(-3\varepsilon/4, 3\varepsilon/4)$  and  $\{\xi : \varepsilon/3 < |\xi| < 3\varepsilon/4\}$ , respectively, such that

$$(6) \quad \widehat{\psi}_0(\xi)\beta_0(\xi) + \sum_{k=1}^{\infty} \widehat{\psi}(2^{-k}\xi)\beta(2^{-k}\xi) = 1 \text{ for all } \xi \in \mathbb{R},$$

see also [7, 8]. The Triebel-Lizorkin space  $F_{p,q}^s(\mathbb{R})$  is usually defined via a smooth dyadic decomposition of unity on the Fourier side, generated for instance by  $\varphi_0$  and  $\varphi$  defined above. We define the operators  $L_k$  by  $\widehat{L_0 f}(\xi) = \varphi_0(\xi)\widehat{f}(\xi)$  and  $\widehat{L_k f}(\xi) = \varphi(2^{-k}\xi)\widehat{f}(\xi)$  and obtain the usual example for an inhomogeneous Littlewood-Paley decomposition. In particular

$$(7) \quad f = \sum_{k=0}^{\infty} L_k f$$

holds for all Schwartz functions  $f$ , with convergence in  $\mathcal{S}'(\mathbb{R})$  and all  $L_p(\mathbb{R})$ . For  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$  the Triebel-Lizorkin space  $F_{p,q}^s(\mathbb{R})$  is the collection of all tempered distributions  $f \in \mathcal{S}'(\mathbb{R})$  such that

$$(8) \quad \|f\|_{F_{p,q}^s} = \left\| \left( \sum_k 2^{ksq} |L_k f|^q \right)^{1/q} \right\|_p$$

is finite (usual modification in case  $q = \infty$ ).

Based on (6) it can be proved using vector-valued singular integrals [2], see also [22, §2.4.6.] and [15], that

$$(9) \quad \|f\|_{F_{p,q}^s} \approx \left\| \left( \sum_{k=0}^{\infty} 2^{ksq} |\psi_k * f|^q \right)^{1/q} \right\|_p$$

with  $\psi_0, \psi$  from above,  $\psi_k(x) = 2^k \psi(2^k x)$  and  $M_1 + 1 > s$ . First of all this characterization yields a useful version of (8) with operators  $\tilde{L}_k$  that reproduce the  $L_k$ . Indeed, it is easy to find compactly supported Schwartz functions  $\tilde{\varphi}_0, \tilde{\varphi}$  such that  $\tilde{\varphi}_0(\xi) = 1$  on  $\text{supp } \varphi_0$ , respectively for  $\tilde{\varphi}$ , and that  $\hat{\psi}_0 = \tilde{\varphi}_0$  and  $\hat{\psi} = \tilde{\varphi}$  are admissible for (9). With  $\tilde{L}_k$  as above we have  $\tilde{L}_k L_k = L_k$  for  $k = 0, 1, 2, \dots$ .

The above characterization (9) allows for choosing  $\psi_0, \psi$  compactly supported. Characterizations of this type are termed “local means” in Triebel [22], §2.4.6, and turn out to be convenient for the purpose of this paper.

**2.2. Peetre maximal functions.** The main tool to estimate operators in  $F_{p,q}^s$  spaces are the vector-valued maximal inequalities by Fefferman-Stein [5] and a variant due to Peetre [13]. We shall need a (variant of) an endpoint version for the Peetre maximal operators which was proved in §6.1 of [4] using an argument involving the  $\#$ -function of Fefferman-Stein [6]. Let  $\mathcal{E}(r)$  be the space of all tempered distributions whose Fourier transform is supported in  $\{\xi : |\xi| \leq r\}$ . Let

$$(10) \quad \mathfrak{M}_n^r g(x) = \sup_{|y| \leq 2^{n+2}/r} |g(x+y)|.$$

Then, the one-dimensional version of the result in [4] states that for any sequence of positive numbers  $r_k$ ,  $0 < p, q < \infty$ , and for any sequence of functions  $f_k \in \mathcal{E}(r_k)$ ,

$$(11) \quad \left\| \left( \sum_k |\mathfrak{M}_n^{r_k} f_k|^q \right)^{1/q} \right\|_p \lesssim \max\{2^{n/p}, 2^{n/q}\} \left\| \left( \sum_k |f_k|^q \right)^{1/q} \right\|_p.$$

The original result by Peetre is equivalent with the similar inequality with constant  $C_\varepsilon 2^{n\varepsilon} \max\{2^{n/p}, 2^{n/q}\}$  on the right hand side. Here we also need a vector-valued version with variable  $n$ . We formulate it for  $p \geq q$  since this is the version used here.

**Proposition 2.1.** *Let  $0 < q \leq p < \infty$ . For any sequence of positive numbers  $r_k$ , and for any doubly indexed sequence of functions  $\{f_{k,n}\}_{k,n \geq 0}$ , with  $f_{k,n} \in \mathcal{E}(r_k)$ ,*

$$(12) \quad \left\| \left( \sum_{k,n} 2^{-n} |\mathfrak{M}_n^{r_k} f_{k,n}|^q \right)^{1/q} \right\|_p \lesssim \left\| \left( \sum_{k,n} |f_{k,n}|^q \right)^{1/q} \right\|_p.$$

We omit the proof since it is a straightforward variant of the argument in §6.1 of [4].

**2.3. Duality.** We show that the statements for (i) and (ii) in Theorems 1.3 and 1.5 are equivalent, by duality.

Given an integral operator  $T$  acting on Schwartz functions let  $T'$  denote the transposed operator with the property

$$\int Tf(x)g(x)dx = \int f(y)T'g(y)dy.$$

Note that for the Haar projection operators  $P_E$  we have  $P_E = P'_E$ . Also if  $Tf = K*f$ , the operator of convolution then  $T'$  is the operator of convolution with  $K(-)$ .

Assume that  $1 < p, q < \infty$  and let  $s \in \mathbb{R}$  such that  $P_E : F_{p',q'}^{-s} \rightarrow F_{p',q'}^{-s}$  is bounded with operator norm  $A$ . Then we need to show that

$$(13) \quad \left\| \left( \sum_k 2^{ksq} |L_k P_E f|^q \right)^{1/q} \right\|_p \lesssim A \left\| \left( \sum_k 2^{ksq} |L_k f|^q \right)^{1/q} \right\|_p,$$

with implicit constant depending only on  $p, q, s$  and the choice of the Schwartz functions defining  $L_k$ .

To see (13) we may assume that the  $k$ -sum on the left hand side is extended over a finite subset  $\mathfrak{K}$  of  $\mathbb{N} \cup \{0\}$ . Then there is  $G = \{G_k\} \in L_{p'}(\ell_{q'})$  with  $\|G\|_{L_{p'}(\ell_{q'})} \leq 1$  so that the left hand side of (13) is finite and equal to

$$\begin{aligned} & \int \sum_{k=0}^{\infty} 2^{ks} L_k P_E f(x) G_k(x) dx \\ &= \sum_{k=0}^{\infty} 2^{ks} \sum_{j=0}^{\infty} \int L_k P_E L_j \tilde{L}_j f(x) G_k(x) dx \\ &= \int \sum_{j=0}^{\infty} \tilde{L}_j f(y) L'_j P'_E [\sum_k 2^{ks} L'_k G_k](y) dy. \end{aligned}$$

Using Hölder's inequality, we estimate the last displayed expression by

$$\begin{aligned} & \left\| \left( \sum_j 2^{jsq} |\tilde{L}_j f|^q \right)^{1/q} \right\|_p \left\| \left( \sum_{j=0}^{\infty} 2^{-jsq} |L'_j P'_E [\sum_k 2^{ks} L'_k G_k]|^{q'} \right)^{1/q'} \right\|_{p'} \\ & \lesssim \|f\|_{F_{p,q}^s} \|P_E [\sum_k 2^{ks} L'_k G_k]\|_{F_{p',q'}^{-s}} \lesssim \|f\|_{F_{p,q}^s} A \left\| \sum_k 2^{ks} L'_k G_k \right\|_{F_{p',q'}^{-s}} \end{aligned}$$

by assumption. Finally

$$\begin{aligned} \left\| \sum_k 2^{ks} L'_k G_k \right\|_{F_{p',q'}^{-s}} &= \left\| \left( \sum_j 2^{-jsq} \left| \sum_{k=j-2}^{j+2} 2^{ks} L_j L'_k G_k \right|^{q'} \right)^{1/q'} \right\|_{p'} \\ &\leq C_{p'} \left\| \left( \sum_k |G_k|^{q'} \right)^{1/q'} \right\|_{p'} \lesssim 1, \end{aligned}$$

and (13) is proved.

## 3. UPPER BOUNDS FOR HAAR PROJECTIONS

For the upper bounds asserted in Theorem 1.3 it suffices to consider the projection numbers  $\gamma^*(F_{p,q}^s; \Lambda)$  and  $\gamma_*(F_{p,q}^s; \Lambda)$  for the choice of  $\Lambda = 2^N$ , for large  $N$ . The following theorem gives a refined version of these upper bounds.

For a subset  $E$  of the Haar system let

$$(14) \quad Z_N(E) = \max_{k \in \mathbb{N}} \#\{n : 2^n \in \text{HF}(E), |n - k| \leq N\}$$

Clearly  $1 \leq Z_N(E) \leq 2N + 1$  for all  $E \subset \mathcal{H}$ .

**Theorem 3.1.** *Let  $1 < q < p < \infty$ ,  $N \geq 2$ . There is  $N_0(p, q, s)$  such that for  $N \geq N_0(p, q, s)$  the following holds for subsets  $E$  of the Haar system with  $\#\text{HF}(E) \leq 2^N$  (with implicit constants depending on  $p, q, s$ ).*

(i) *If  $-1/p' < s < -1/q'$  then*

$$\|P_E\|_{F_{p,q}^s \rightarrow F_{p,q}^s} \lesssim 2^{N(-s - \frac{1}{q'})}.$$

(ii) *For the case  $s = -1/q'$  we have*

$$\|P_E\|_{F_{p,q}^{-1/q'} \rightarrow F_{p,q}^{-1/q'}} \lesssim N^{1-1/q} Z_N(E)^{1/q}.$$

The remainder of this section is devoted to the proof of Theorem 3.1.

*Two preliminary estimates.* We state two lemmata which will be used frequently when estimating the Haar projection operators  $P_E$ . In what follows let  $\psi_k$  be as in §2.1.

**Lemma 3.2.** *Let  $k \leq j$ . Then, with  $y_{j,\mu} := 2^{-j}(\mu + \frac{1}{2})$ , the support of  $h_{j,\mu} * \psi_k$  is contained in  $[y_{j,\mu} - 2^{-k}, y_{j,\mu} + 2^{-k}]$ . Moreover,*

$$\|h_{j,\mu} * \psi_k\|_\infty \lesssim 2^{2k-2j}.$$

*Proof.* The support property is immediate due to the support property of  $\psi_k$ . Since  $\int h_{j,\mu}(y) dy = 0$  we have

$$h_{j,\mu} * \psi_k(x) = \int (\psi_k(x - y) - \psi_k(x - y_{j,\mu})) h_{j,\mu}(y) dy$$

and using  $\psi_k' = O(2^{2k})$ , we get

$$|h_{j,\mu}(x)| \lesssim \int_{y_{j,\mu}-2^{-j}}^{y_{j,\mu}+2^{-j}} 2^{2k} |y - y_{j,\mu}| dy \lesssim 2^{2k-2j}. \quad \square$$

**Lemma 3.3.** *Let  $0 < p \leq \infty$ . (i) Suppose that  $k \geq j$ , and let  $x \in \mathbb{R}$  such that*

$$\min \{|x - 2^{-j}\mu|, |x - 2^{-j}(\mu + \frac{1}{2})|, |x - 2^{-j}(\mu + 1)|\} \geq 2^{-k}.$$

*Then  $h_{j,\mu} * \psi_k(x) = 0$ .*

(ii)  *$\|h_{j,\mu} * \psi_k\|_p \lesssim 2^{-k/p}$  for  $k \geq j$ .*



*Proof.* (i) follows by the support and cancellation properties of  $\psi_k$  and fact that  $h_{j,\mu}$  is constant on  $I_{j,\mu}^+$ ,  $I_{j,\mu}^-$ , and  $I_{j,\mu}^0$ . Since  $\|\psi_k\|_1 \lesssim 1$  we have (ii) for  $p = \infty$ . From (i) we then get (ii) for all  $p$ .  $\square$

*Basic reductions.* We use the Peetre type maximal operators  $\mathfrak{M}_n^r$  defined in (10); it will be convenient to use the notation  $\mathcal{M}_n^k = \mathfrak{M}_n^{2^k}$  so that

$$(15) \quad \mathcal{M}_n^k g(x) = \sup_{|y| \leq 2^{-k+n+2}} |g(x+y)|.$$

In the remainder of the chapter we assume  $1 < q < \infty$ , and  $E$  will denote a subset of  $\mathcal{H}$  satisfying

$$(16) \quad \#(\text{HF}(E)) < 2^{N+1}.$$

Let  $\psi_k$  be as in §4. Theorem 3.1 follows from

$$\left\| \left( \sum_{k \in \mathbb{N}} 2^{ksq} |\psi_k * P_E f|^q \right)^{1/q} \right\|_p \lesssim \max\{2^{N(\frac{1}{q}-s-1)}, N^{1-1/q} Z^{1/q}\} \|f\|_{F_{p,q}^s},$$

with  $Z := \mathcal{Z}_N(E)$ . This in turn follows from

$$(17) \quad \left\| \left( \sum_{k \in \mathbb{N}} 2^{ksq} |\psi_k * P_E [\sum_l 2^{-ls} \psi_l * f_l]|^q \right)^{1/q} \right\|_p \\ \lesssim \max\{2^{N(\frac{1}{q}-s-1)}, N^{1-1/q} Z^{1/q}\} \left\| \left( \sum_{l=0}^{\infty} |f_l|^q \right)^{1/q} \right\|_p$$

for all  $\{f_l\}$  with  $f_l \in \mathcal{E}(2^l)$ .

Given a set  $E$  of Haar functions, we set  $E_j = \{\mu : h_{j,\mu} \in E\}$ . We link  $j = k + m$  and  $l = k + m + n$  and define, for  $m, n \in \mathbb{Z}$ ,  $k = 0, 1, \dots$ ,

$$(18) \quad T_{m,n}^k f = \sum_{\mu \in E_{k+m}} 2^{k+m} \langle \psi_{k+m+n} * f, h_{k+m,\mu} \rangle \psi_k * h_{k+m,\mu}, \\ \text{if } k+m \in \text{HF}(E), k+m+n \geq 0,$$

and  $T_{m,n}^k = 0$  if  $k+m \notin \text{HF}(E)$  or  $k+m+n < 0$ . Then

$$(19) \quad 2^{ks} \psi_k * P_E [\sum_l 2^{-ls} \psi_l * f_l] = \sum_m \sum_n 2^{-s(m+n)} T_{m,n}^k f_{k+m+n}.$$

In preparation for the proof of (17) we first state estimates of  $T_{m,n}^k f$  in terms of the Peetre type maximal operators  $\mathcal{M}_n^k$  (15), or in some cases just the Hardy-Littlewood maximal operator  $M$ .

**Lemma 3.4.** *Let  $k \geq 0$ ,  $k+m \geq 0$ . The following estimates hold for continuous  $f$ .*

(i) For  $m \geq 0$  and  $n \geq 0$ ,

$$|T_{m,n}^k f(x)| \lesssim 2^{-m-n} \mathcal{M}_{n+m}^{k+n+m} f(x).$$

(ii) For  $m \geq 0$  and  $n \leq 0$ ,

$$|T_{m,n}^k f(x)| \lesssim \begin{cases} 2^{n-m} \mathcal{M}_{m+n}^{k+m+n} f(x) & \text{if } n \geq -m \\ 2^{n-m} Mf(x) & \text{if } n < -m. \end{cases}$$

(iii) For  $m \leq 0$  and  $n \geq 0$ ,

$$|T_{m,n}^k f(x)| \lesssim 2^{-n} \mathcal{M}_n^{k+n+m} f(x).$$

(iv) For  $m \leq 0$  and  $n \leq 0$ ,

$$|T_{m,n}^k f(x)| \lesssim Mf(x).$$

*Proof.* Let  $m \geq 0$ ,  $n \geq 0$ . For  $x \in \text{supp}(\psi_k * h_{k+m,\mu})$  we have (with  $\tilde{\psi} := \psi(\cdot)$ )

$$\begin{aligned} 2^{k+m} |\langle \psi_{k+m+n} * f, h_{k+m,\mu} \rangle| &= 2^{k+m} |\langle f, \tilde{\psi}_{k+m+n} * h_{k+m,\mu} \rangle| \\ &\lesssim 2^{k+m} 2^{-k-m-n} \sup_{y: |x-y| \leq 2^{2-k}} |f(y)| \\ &\lesssim 2^{-n} \mathcal{M}_{m+n}^{k+m+n} f(x). \end{aligned}$$

Now  $\psi_k * h_{k+m,\mu} = O(2^{-2m})$  and for fixed  $x$  the  $\mu$ -sum in (18) contributes  $O(2^m)$  terms. This yields (i).

Let  $m \geq 0$ ,  $-m \leq n \leq 0$ . We now have  $\|\tilde{\psi}_{k+m+n} * h_{k+m,\mu}\|_\infty \lesssim 2^{2n}$ , by Lemma 3.2 and therefore

$$\begin{aligned} 2^{k+m} |\langle \psi_{k+m+n} * f, h_{k+m,\mu} \rangle| &= 2^{k+m} |\langle f, \tilde{\psi}_{k+m+n} * h_{k+m,\mu} \rangle| \\ &\lesssim 2^{k+m} 2^{2n} 2^{-k-m-n} \sup_{y: |x-y| \leq 2^{2-k}} |f(y)| \\ &\lesssim 2^n \mathcal{M}_{m+n}^{k+m+n} f(x). \end{aligned}$$

As in the previous case  $\psi_k * h_{k+m,\mu} = O(2^{-2m})$  and there are  $O(2^m)$   $\mu$ -terms that contribute. This leads to (ii) in the case when  $m+n \geq 0$ . If  $m \geq 0$  and  $n \leq -m$  then we have instead

$$\begin{aligned} 2^{k+m} |\langle f, \tilde{\psi}_{k+m+n} * h_{k+m,\mu} \rangle| \\ \lesssim 2^{k+m} 2^{2n} \int_{|x-y| \leq 2^{-k-m-n+2}} |f(y)| dy \lesssim 2^n Mf(x), \end{aligned}$$

which gives the second estimate in (ii).

Next assume  $m \leq 0$ . Now we use that  $h_{k+m,\mu} * \tilde{\psi}_{k+m+n}$  is supported in the union of three intervals of length  $2^{-k-m-n}$  centered at the endpoints and the middle point of  $\text{supp}(h_{k+m,\mu})$ . Thus, for  $x \in \text{supp}(\psi_k * h_{k+m,\mu})$ ,

$$\begin{aligned} 2^{k+m} |\langle f, \tilde{\psi}_{k+m+n} * h_{k+m,\mu} \rangle| \\ \lesssim 2^{-n} \sup_{|x-y| \leq 2^{-k-m+2}} |f(x-y)| \lesssim 2^{-n} \mathcal{M}_n^{k+m+n} f(x) \end{aligned}$$

and (iii) follows since every  $x$  is contained in  $\text{supp}(\psi_k * h_{k+m,\mu})$  for at most three choices of  $\mu$ .

When  $m \leq 0$ ,  $n \leq 0$  we can estimate instead

$$2^{k+m} |\langle \psi_{k+m+n} * f, h_{k+m,\mu} \rangle| \lesssim Mf(x),$$

for  $x \in \text{supp}(\psi_k * h_{k+m,\mu})$  and obtain (iv).  $\square$

By the Peetre type inequality (11) we see that for  $m \geq 0$  the  $L_\rho \rightarrow L_\rho$  operator norm of  $T_{m,n}^k$  when acting on functions in  $\mathcal{E}(C2^{k+m+n})$ , is  $O(2^{-(m+n)/\rho'})$  if  $n \geq 0$  and  $O(2^{2n}2^{-(m+n)/\rho'})$  when  $n \leq 0$ . It will be useful to observe an improvement in  $m$  which we will apply for  $\rho = p$  and  $\rho = q$ .

**Lemma 3.5.** *Let  $m, k \geq 0$ . Then for  $1 \leq \rho \leq \infty$  and  $f \in L_\rho$ ,*

$$\|T_{m,n}^k f\|_\rho \lesssim \begin{cases} 2^{-n/\rho'} 2^{-m} \|f\|_\rho, & n \geq 0, \\ 2^{n-m} \|f\|_\rho, & n \leq 0, \end{cases}$$

where the implicit constant depends only on  $\rho$ .

*Proof.* We have  $\|\psi_k * h_{k+m,\mu}\|_\infty = O(2^{-2m})$ , by the cancellation property of  $h_{k+m,\mu}$ . We decompose  $\mathbb{R}$  into dyadic intervals of length  $2^{-k}$ , labeled  $J_{k,\nu}$  for  $\nu \in \mathbb{Z}$ . We say that  $\mu \sim \nu$  if  $\mu \in E_{k+m}$  and  $I_{k+m,\mu}$  intersects  $J_{k,\nu}$  or one of its neighbors.

We first examine the case  $n \geq 0$ . Now  $\psi_{k+n+m} * h_{k+m,\mu}$  is supported on a set  $V_\mu^{k+m,n}$  of measure  $O(2^{-k-m-n})$ , namely the union of three intervals of length  $2^{-k-m-n+1}$  centered at the two endpoints and the midpoint of the interval  $I_{k+m,\mu}$ .

Thus

$$\begin{aligned} & \|T_{m,n}^k f\|_\rho \\ & \lesssim \left( \sum_\nu \int_{J_{k,\nu}} \left[ \sum_{\mu:\mu\sim\nu} |\psi_k * h_{k+m,\mu}(x)| 2^{k+m} |\langle h_{k+m,\mu} * \psi_{k+n+m}, f \rangle| \right]^\rho dx \right)^{1/\rho} \\ & \lesssim 2^{-k/\rho} 2^{-2m} 2^{k+m} \left( \sum_\nu \left[ \sum_{\mu:\mu\sim\nu} \int_{V_\mu^{k+m,n}} |f(y)| dy \right]^\rho \right)^{1/\rho} \\ & \lesssim 2^{k/\rho'} 2^{-m} \left( \sum_\nu \sum_{\mu:\mu\sim\nu} \int_{V_\mu^{k+m,n}} |f(y)|^\rho dy 2^{-(k+m+n)\rho/\rho'} 2^{m\rho/\rho'} \right)^{1/\rho} \end{aligned}$$

since the measure of  $\text{meas}(V_\mu^{k+m,n}) = O(2^{-k-m-n})$  and since for each  $\nu$  there are at most  $O(2^m)$  integers  $\mu$  with  $\mu \sim \nu$ . Now each  $y$  is contained in a bounded number of the sets  $V_\mu^{k+m,n}$  and for each  $\mu$  the number of  $\nu$  with  $\mu \sim \nu$  is also bounded. Hence the expression on the last displayed line is dominated by a constant times

$$2^{-m} 2^{-n/\rho'} \left( \sum_\nu \sum_{\mu:\mu\sim\nu} \int_{V_\mu^{k+m,n}} |f(y)|^\rho dy \right)^{1/\rho} \lesssim 2^{-m} 2^{-n/\rho'} \|f\|_\rho.$$

which proves the assertion for  $n \geq 0$ .

For the case  $n \leq 0$  we have  $\|\psi_{k+m+n} * h_{k+m,\mu}\|_\infty = O(2^{2n})$ . The function is supported in an interval  $I_{k+m,\mu}^n$  of length  $C2^{-k-m-n}$ , centered at an  $x_{k+m,\mu}$  in the support of  $h_{k+m,\mu}$ . We estimate

$$\begin{aligned} & \left( \sum_\nu \int_{J_{k,\nu}} \left[ \sum_{\mu:\mu\sim\nu} |\psi_k * h_{k+m,\mu}(x)| 2^{k+m} |\langle h_{k+m,\mu} * \psi_{k+m+n}, f \rangle| \right]^\rho dx \right)^{1/\rho} \\ & \lesssim 2^{-k/\rho} 2^{-2m} 2^{k+m} 2^{2n} \left( \sum_\nu \left[ \sum_{\mu:\mu\sim\nu} \int_{I_{k+m,\mu}^n} |f(y)| dy \right]^\rho \right)^{1/\rho} \\ & \lesssim 2^{k/\rho'} 2^{-m} 2^{2n} \left( \sum_\nu 2^{m\rho/\rho'} \sum_{\mu:\mu\sim\nu} 2^{-(k+m+n)\rho/\rho'} \int_{I_{k+m,\mu}^n} |f(y)|^\rho dy \right)^{1/\rho}. \end{aligned}$$

We now interchange summations and integration and observe that for each  $\mu$  there are only  $O(1)$  values of  $\nu$  with  $\mu \sim \nu$ . This leads to

$$\|T_{m,n}^k f\|_\rho \lesssim 2^{-m} 2^{2n-n/\rho'} \left( \int |f(y)|^\rho \#\{\mu : y \in I_{k+m,\mu}^n\} dy \right)^{1/\rho}$$

and since for each  $y$  there are at most  $O(2^{-n})$  values of  $\mu$  with  $y \in I_{k+m,\mu}^n$  the asserted inequality for  $n \leq 0$  follows.  $\square$

In what follows we use operators  $U_k$  defined by  $\widehat{U_k f}(\xi) = \Phi(2^{-k}\xi)\widehat{f}(\xi)$  where  $\Phi \in C_c^\infty(\mathbb{R})$  supported in  $(-4, 4)$  satisfying  $\Phi(\xi) = 1$  for  $|\xi| \leq 2$ . Notice that  $U_k f_k = f_k$  for  $f_k \in \mathcal{E}(2^k)$ . In order to facilitate interpolation we shall replace  $f_{k+m+n}$  on the right hand side by  $U_{k+m+n} g_{k+m+n}$  where  $\vec{g} = \{g_l\}_{l=0}^\infty$  is an arbitrary function in  $L_p(\ell_q)$ .

The main inequalities needed to prove Theorem 3.1 for the case  $s < -1/q'$  are stated in the following proposition (which also provides useful information for the case  $s = -1/q'$ ). Recall that  $\#\text{HF}(E) \leq 2^{N+1}$ .

**Proposition 3.6.** *Let  $1 < q < p < \infty$ , and  $\varepsilon > 0$ .*

(i) *For  $m \geq 0$  and  $n \geq 0$ ,*

$$\begin{aligned} & \left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\ & \lesssim_\varepsilon \min\{2^{-\frac{n}{q'}} 2^{-m(1-\frac{1}{q}+\frac{1}{p}+\varepsilon)}, 2^{N(\frac{1}{q}-\frac{1}{p})} 2^{-\frac{n}{p'}} 2^{-m}\} \|\vec{g}\|_{L_p(\ell_q)}. \end{aligned}$$

(ii) *For  $m \geq 0$  and  $-m \leq n \leq 0$ ,*

$$\begin{aligned} & \left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\ & \lesssim 2^{n-m} \min\{2^{(m+n)(\frac{1}{q}-\frac{1}{p}+\varepsilon)}, 2^{N(\frac{1}{q}-\frac{1}{p})}\} \|\vec{g}\|_{L_p(\ell_q)}. \end{aligned}$$

(iii) For  $m \geq 0$  and  $n \leq -m \leq 0$ ,

$$\left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+n+m}|^q \right)^{1/q} \right\|_p \lesssim 2^{n-m} \|\vec{g}\|_{L_p(\ell_q)}.$$

(iv) For  $m \leq 0$  and  $n \geq 0$ ,

$$\begin{aligned} \left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\ \lesssim \min\{2^{-n(1-\frac{1}{q})}, 2^{N(\frac{1}{q}-\frac{1}{p})} 2^{-n(1-\frac{1}{p})}\} \|\vec{g}\|_{L_p(\ell_q)}. \end{aligned}$$

(v) For  $m \leq 0$  and  $n \leq 0$ ,

$$\left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \lesssim \|\vec{g}\|_{L_p(\ell_q)}.$$

*Proof.* We prove (i) by interpolation. We first observe that, by Lemma 3.5,

$$\left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_q \lesssim 2^{-m} 2^{-n/q'} \|\vec{g}\|_{L_q(\ell_q)}.$$

By Lemma 3.4, (i), and (11),

$$\begin{aligned} \left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_r \\ \lesssim 2^{-m/q'} 2^{-n/q'} \left\| \left( \sum_k |U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_r \\ \lesssim 2^{-m/q'} 2^{-n/q'} \|\vec{g}\|_{L_r(\ell_q)} \end{aligned}$$

which we choose for  $r \gg p > q$  large. Interpolation yields

$$(20) \quad \left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \lesssim_\varepsilon 2^{-m(1-\frac{1}{q}+\frac{1}{p}-\varepsilon)} 2^{-n/q'} \|\vec{g}\|_{L_p(\ell_q)}$$

with  $\varepsilon = \frac{1-q/p}{r-q}$ . Letting  $r \rightarrow \infty$  we obtain the first bound stated in (i). We also have by Hölder's inequality ( $q < p$ ) and  $\#\text{HF}(E) \leq 2^N$

$$\begin{aligned} \left\| \left( \sum_k |T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\ \lesssim 2^{N(\frac{1}{q}-\frac{1}{p})} \left( \sum_k \|T_{m,n}^k U_{k+m+n} g_{k+m+n}\|_p^p \right)^{1/p} \\ \lesssim 2^{N(\frac{1}{q}-\frac{1}{p})} 2^{-m} 2^{-n/p'} \|\vec{g}\|_{L_p(\ell_p)} \\ (21) \quad \lesssim 2^{N(\frac{1}{q}-\frac{1}{p})} 2^{-m} 2^{-n/p'} \|\vec{g}\|_{L_p(\ell_q)}. \end{aligned}$$

Here, for the second inequality we have used Lemma 3.5, with  $\rho = p$  and for the third the embedding  $\ell_q \subset \ell_p$ . This concludes the proof of (i).

Inequalities (ii), (iii), (iv) follow more directly from the corresponding statements in Lemma 3.4, combined with an application of (11).  $\square$

For the case  $s = -1/q'$  we also need

**Proposition 3.7.** *Let  $1 < q \leq p < \infty$  and  $Z = Z_N(E)$  with  $E$  as in (16).*

(i) *For  $m \geq 0$ ,*

$$\begin{aligned} \left\| \left( \sum_k \sum_{0 \leq n \leq N} |2^{(m+n)/q'} T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\ \lesssim Z^{1/q} 2^{-m(\frac{1}{p}-\varepsilon)} \|\vec{g}\|_{L_p(\ell_q)}. \end{aligned}$$

(ii) *For  $m \leq 0$ ,*

$$\left\| \left( \sum_k \sum_{0 \leq n \leq N} |2^{n/q'} T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \lesssim Z^{1/q} \|\vec{g}\|_{L_p(\ell_q)}.$$

*Proof.* For the proof of (i) we interpolate between

$$(22) \quad \left\| \left( \sum_k \sum_{0 \leq n \leq N} |2^{(m+n)/q'} T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_q \lesssim Z^{1/q} 2^{-m/q} \|\vec{g}\|_{L_q(\ell_q)}$$

and

$$(23) \quad \left\| \left( \sum_k \sum_{0 \leq n \leq N} |2^{(m+n)/q'} T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \lesssim Z^{1/q} \|\vec{g}\|_{L_p(\ell_q)}$$

which we use for large  $p$ .

Recall  $T_{m,n}^k = 0$  if  $2^{k+m} \notin \text{HF}(E)$ . To see (22) we interchange summation and integration and use Lemma 3.5 for  $\rho = q$  to estimate the left hand side by

$$\begin{aligned} & \left( \sum_{k: 2^{k+m} \in \text{HF}(E)} \sum_{0 \leq n \leq N} [2^{-m/q} \|g_{k+m+n}\|_q]^q \right)^{1/q} \\ & \lesssim 2^{-m/q} \left( \sum_l \|g_l\|_q^q \#\{n : 0 \leq n \leq N, 2^{l-n} \in \text{HF}(E)\} \right)^{1/q} \\ & \lesssim 2^{-m/q} Z^{1/q} \|\vec{g}\|_{L_q(\ell_q)}. \end{aligned}$$

To see (23) we use Proposition 2.1. By Lemma 3.4, (i), we have that the left hand side of (23) is dominated by

$$\begin{aligned}
 & \left\| \left( \sum_{k:2^{k+m} \in \text{HF}(E)} \sum_{0 \leq n \leq N} |2^{-(m+n)/q} \mathcal{M}_{n+m}^{k+n+m} U_{k+n+m} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\
 & \lesssim \left\| \left( \sum_{k:2^{k+m} \in \text{HF}(E)} \sum_{0 \leq n \leq N} |U_{k+n+m} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\
 & \lesssim \left\| \left( \sum_{k:2^{k+m} \in \text{HF}(E)} \sum_{0 \leq n \leq N} |g_{k+m+n}|^q \right)^{1/q} \right\|_p \\
 & \lesssim \left\| \left( \sum_l |g_l|^q \#\{n : 0 \leq n \leq N, 2^{l-n} \in \text{HF}(E)\} \right)^{1/q} \right\|_p
 \end{aligned}$$

and the last expression is  $\lesssim Z^{1/q} \|\vec{g}\|_{L_p(\ell_q)}$ . This concludes the proof of (i).

For the proof of (ii) we use Lemma 3.4, (iii) and again Proposition 2.1 to see that

$$\begin{aligned}
 & \left\| \left( \sum_k \sum_{0 \leq n \leq N} |2^{n/q'} T_{m,n}^k U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\
 & \lesssim \left\| \left( \sum_{k:2^{k+m} \in \text{HF}(E)} \sum_{0 \leq n \leq N} |2^{-n/q} \mathcal{M}_n^{k+m+n} U_{k+m+n} g_{k+m+n}|^q \right)^{1/q} \right\|_p \\
 & \lesssim \left\| \left( \sum_{k:2^{k+m} \in \text{HF}(E)} \sum_{0 \leq n \leq N} |g_{k+m+n}|^q \right)^{1/q} \right\|_p \\
 & \lesssim Z^{1/q} \|\vec{g}\|_{L_p(\ell_q)}. \quad \square
 \end{aligned}$$

*Proof of Theorem 3.1.* By the triangle inequality in  $L_p(\ell_q)$  we have

$$\begin{aligned}
 & \left\| \left( \sum_{k \in \mathbb{N}} 2^{ksq} |\psi_k * P_E[\sum_{l=0}^{\infty} 2^{-ls} \psi_l * f_l]|^q \right)^{1/q} \right\|_p \\
 (24) \quad & \leq \sum_{m \geq 0} 2^{-sm} [I_m + II_m + III_m + IV_m] + \sum_{m < 0} 2^{-sm} [V_m + VI_m] \\
 & \quad + \sum_{m \geq 0} \sum_{n < -m} 2^{-s(n+m)} VII_{m,n} + \sum_{m < 0} \sum_{n < 0} 2^{-s(n+m)} VIII_{m,n},
 \end{aligned}$$

where for  $m \geq 0$ ,

$$(25a) \quad I_m = \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \left| \sum_{n > \max\{N-m, 0\}} 2^{-sn} T_{m,n}^k f_{k+m+n} \right|^q \right)^{1/q} \right\|_p,$$

$$(25b) \quad II_m = \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \left| \sum_{0 \leq n \leq \max\{N-m, 0\}} 2^{-sn} T_{m,n}^k f_{k+m+n} \right|^q \right)^{1/q} \right\|_p,$$

$$(25c) \quad III_m = \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \left| \sum_{\substack{n \leq 0 \\ m+n \geq N}} 2^{-sn} T_{m,n}^k f_{k+m+n} \right|^q \right)^{1/q} \right\|_p,$$

$$(25d) \quad IV_m = \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \left| \sum_{\substack{n \leq 0 \\ 0 \leq m+n \leq N}} 2^{-sn} T_{m,n}^k f_{k+m+n} \right|^q \right)^{1/q} \right\|_p,$$

and, for  $m \leq 0$ ,

$$(25e) \quad V_m = \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \left| \sum_{n > N} 2^{-sn} T_{m,n}^k f_{k+m+n} \right|^q \right)^{1/q} \right\|_p,$$

$$(25f) \quad VI_m = \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \left| \sum_{0 \leq n \leq N} 2^{-sn} T_{m,n}^k f_{k+m+n} \right|^q \right)^{1/q} \right\|_p.$$

Moreover

$$(25g) \quad VII_{m,n} = \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} |T_{m,n}^k f_{k+m+n}|^q \right)^{1/q} \right\|_p, \quad n < \min\{-m, 0\}.$$

When  $-1/p' < s < -1/q'$  we estimate the terms  $I_m, \dots, VI_m$  by another use of the triangle inequality in  $L_p(\ell_q)$ , with respect to the  $n$  summation. When  $s = -1/q'$  we still do this for the terms  $I_m, V_m$  but argue differently for the terms involving the restriction  $0 \leq n \leq N$ . In what follows we shall need to distinguish the cases  $s < -1/q'$  and  $s = -1/q'$  in various estimates and therefore write  $I(s), II(s), \dots$  for the expressions  $I, II, \dots$ , resp.

By Proposition 3.6, (i) we have

$$\begin{aligned} \sum_{m \geq 0} 2^{-sm} I_m(s) &\lesssim \|f\|_{L_p(\ell_q)} \times \\ &2^{N(\frac{1}{q}-\frac{1}{p})} \left[ \sum_{m \geq N} 2^{-(1+s)m} \sum_{n \geq 0} 2^{-n(s+\frac{1}{p})} + \sum_{0 \leq m < N} 2^{-(1+s)m} \sum_{n \geq N-m} 2^{-n(s+\frac{1}{p})} \right] \end{aligned}$$

and the constant for  $q \leq p < \infty$  is easily seen to be  $O(2^{N(-s-1/q')})$ .



Next we have  $II_m(s) = 0$  for  $m > N$ . For the terms with  $0 \leq m \leq N$ , we get again by Proposition 3.6, (i),

$$\begin{aligned} & \sum_{0 \leq m \leq N} 2^{-sm} II_m(s) \\ & \lesssim \sum_{0 \leq m \leq N} 2^{-m(s+1)} 2^{m(\frac{1}{q} - \frac{1}{p} - \varepsilon)} \sum_{0 \leq n \leq N-m} 2^{-n(s + \frac{1}{q'})} \|\vec{f}\|_{L_p(\ell_q)} \\ & \lesssim \max\{|s + 1/q'| 2^{-N(s + \frac{1}{q'})}, N\} \|\vec{f}\|_{L_p(\ell_q)} \end{aligned}$$

which contributes the desired bound for  $-1/p' < s < -1/q'$ . For  $s = -1/q'$  we use Hölder's inequality in the  $n$ -sum followed by Proposition 3.7 to get

$$\begin{aligned} & \sum_{0 \leq m \leq N} 2^{m/q'} II_m(-1/q') \\ & \lesssim \sum_{0 \leq m \leq N} 2^{m/q'} N^{1/q'} \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \sum_{\substack{0 \leq n \leq \\ \max\{N-m, 0\}}} |2^{n/q'} T_{m,n}^k f_{k+m+n}|^q \right)^{1/q} \right\|_p \\ & \lesssim N^{1/q'} \sum_{m \geq 0} 2^{-m(\frac{1}{p} - \varepsilon)} Z^{1/q} \|\vec{f}\|_{L_p(\ell_q)} \lesssim N^{1/q'} Z^{1/q} \|\vec{f}\|_{L_p(\ell_q)}. \end{aligned}$$

For  $-1/p' < s \leq -1/q'$  we have by Proposition 3.6, (ii),

$$\begin{aligned} \sum_{m \geq 0} 2^{-ms} III_m(s) & \leq \sum_{m \geq N} \sum_{N-m \leq n \leq 0} 2^{-(m+n)s} 2^{n-m} 2^{N(\frac{1}{q} - \frac{1}{p})} \|\vec{f}\|_{L_p(\ell_q)} \\ & \lesssim 2^{-N/p} 2^{-N(s+1-1/q)} \|\vec{f}\|_{L_p(\ell_q)}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \sum_{m \geq 0} 2^{-ms} IV_m(s) \\ & \lesssim \sum_{m \geq 0} \sum_{\substack{-m \leq n \leq \\ \min\{N-m, 0\}}} 2^{-(m+n)s} 2^{n-m} 2^{(m+n)(\frac{1}{q} - \frac{1}{p} + \varepsilon)} \|\vec{f}\|_{L_p(\ell_q)} \\ & + \sum_{m \geq N} \sum_{N-m \leq n \leq 0} 2^{-(m+n)s} 2^{n-m} 2^{N(\frac{1}{q} - \frac{1}{p})} \|\vec{f}\|_{L_p(\ell_q)} \end{aligned}$$

with the implicit constant depending on  $\varepsilon > 0$ . Since  $p < \infty$  we may choose  $0 < \varepsilon < 1/p$ . We evaluate various geometric series and obtain for  $s \leq -1/q'$

$$\sum_{m \geq 0} 2^{-ms} IV_m(s) \lesssim 2^{-N(\frac{1}{p} - \varepsilon)} 2^{-N(s+1-\frac{1}{q})} \|\vec{f}\|_{L_p(\ell_q)}.$$

Next we consider the terms with  $m \leq 0$ . We use the second estimate in Proposition 3.6, (iv), and  $s > -1/p'$  to obtain

$$\begin{aligned} \sum_{m \leq 0} 2^{-sm} V_m(s) &\lesssim 2^{N(\frac{1}{q} - \frac{1}{p})} \sum_{m \leq 0} 2^{-sm} \sum_{n \geq N} 2^{-n(s+1-\frac{1}{p})} \|\vec{f}\|_{L_p(\ell_q)} \\ &\lesssim 2^{N(-s-1+\frac{1}{q})} \|\vec{f}\|_{L_p(\ell_q)}. \end{aligned}$$

For the terms  $VI_m(s)$  we need to separately treat the case  $s < -1/q'$  and  $s = -1/q'$ . For  $s < -1/q'$  we use the first estimate in Proposition 3.6, (iv) and get

$$\begin{aligned} \sum_{m \leq 0} 2^{-sm} VI_m(s) &\lesssim \sum_{m \leq 0} 2^{-sm} \sum_{0 \leq n \leq N} 2^{n(-s-1+\frac{1}{q})} \|\vec{f}\|_{L_p(\ell_q)} \\ &\lesssim 2^{N(-s-1+\frac{1}{q})} \|\vec{f}\|_{L_p(\ell_q)}. \end{aligned}$$

For  $s = -1/q'$  we argue as for the  $II_m(-1/q')$  terms above and use Hölder's inequality in the  $n$  sum followed by Proposition 3.7 (ii) to get

$$\begin{aligned} &\sum_{m \leq 0} 2^{m/q'} VI_m(-1/q') \\ &\lesssim \sum_{m \leq 0} 2^{m/q'} N^{1/q'} \left\| \left( \sum_{\substack{k \geq 0 \\ k+m \in \text{HF}(E)}} \sum_{0 \leq n \leq N} |2^{n/q'} T_{m,n}^k f_{k+m+n}|^q \right)^{1/q} \right\|_p \\ &\lesssim N^{1/q'} \sum_{m \leq 0} 2^{m/q'} Z^{1/q} \|\vec{f}\|_{L_p(\ell_q)} \lesssim N^{1/q'} Z^{1/q} \|\vec{f}\|_{L_p(\ell_q)}. \end{aligned}$$

Finally, the inequalities

$$\sum_{m \geq 0} \sum_{n < -m} 2^{-(m+n)s} V II_m(s) \lesssim \|\vec{f}\|_{L_p(\ell_q)}$$

and

$$\sum_{m \leq 0} \sum_{n \leq 0} 2^{-(m+n)s} V II_m(s) \lesssim \|\vec{f}\|_{L_p(\ell_q)}$$

follow immediately from Proposition 3.6, (iii), (v), resp.  $\square$

#### 4. BOUNDS FOR FAMILIES OF TEST FUNCTIONS

It will be convenient to use characterizations of function spaces by compactly supported localizations (i.e. the local means in [22]), see §2.1. In what follows let  $M_0, M_1$  be positive integers, and we shall always assume that  $-M_0 < s < M_1$ .

Let  $\psi_0, \psi$  be  $C^\infty$  functions supported in  $(-1/2, 1/2)$  so that  $\widehat{\psi}_0(\xi) \neq 0$  for  $|\xi| \leq 1$  and so that  $\psi(\xi) \neq 0$  for  $1/4 \leq |\xi| \leq 1$ , moreover  $\widehat{\psi}$  vanishes of order  $M_1$  at 0. Thus the cancellation condition (5) holds. Let  $\psi_k = 2^k \psi(2^k \cdot)$  for  $k = 1, 2, \dots$ . We shall use the characterization of  $F_{p,q}^s$  using the  $\psi_k$ , see (9). Now we will define some test functions which will be used to establish the

lower bounds in Theorem 1.3. In what follows we fix an integer  $m \geq 0$ ; all implicit constants will be independent of  $m$ .

Let  $\eta$  be a  $C^\infty$  function supported in  $(-1/2, 1/2)$  such that  $\int \eta(x)x^n dx = 0$  for  $n = 1, \dots, M_0$ . Let, for  $l \geq m$ ,  $\mathcal{P}_l^m$  be a set of  $2^{m-l}$ -separated points in  $[0, 1]$ . That is  $\mathcal{P}_l^m = \{x_{l,1}, \dots, x_{l,N(l)}\}$  with  $N(l) \leq 2^{l-m}$  and  $x_{l,\nu} < x_{l,\nu+1}$  with  $x_{l,\nu+1} - x_{l,\nu} \geq 2^{m-l}$ . Define

$$(26) \quad \eta_{l,\nu} = \eta(2^l(x - x_{l,\nu})).$$

Let  $\mathfrak{L}^m$  be a finite set of nonnegative integers  $\geq m$  and assume  $\#\mathfrak{L}^m \geq 2^m$ . Let

$$(27a) \quad \mathfrak{S}^m = \{(l, \nu) : l \in \mathfrak{L}^m, x_{l,\nu} \in \mathcal{P}_l^m\};$$

and

$$(27b) \quad \mathfrak{S}_l^m = \{\nu : (l, \nu) \in \mathfrak{S}^m\}$$

For any indexed sequence  $\{a_{l,\nu}\}$  satisfying  $\sup_{l,\nu} |a_{l,\nu}| \leq 1$ , we define for  $l \in \mathfrak{L}^m$ ,

$$(28) \quad g_m(x) = \sum_{l \in \mathfrak{L}^m} 2^{-ls} \sum_{\nu \in \mathfrak{S}_l^m} a_{l,\nu} \eta_{l,\nu}(x).$$

If the families  $\mathfrak{L}^m$ ,  $m = 1, 2, \dots$ , are disjoint, we set

$$g := \sum_{m \geq 1} \beta_m g_m.$$

The proof of the following proposition is a modification of the proof of a corresponding result by Christ and one of the authors ([4]).

**Proposition 4.1.** *Let  $s > -M_0$ .*

(i) *If  $1 \leq p, q < \infty$  then*

$$\|g_m\|_{F_{p,q}^s} \lesssim_{p,q,s} \left\| \left( \sum_{l \in \mathfrak{L}^m} \left| \sum_{\nu \in \mathfrak{S}_l^m} a_{l,\nu} \mathbb{1}_{l,\nu} \right|^q \right)^{1/q} \right\|_p$$

and

$$\|g\|_{F_{p,q}^s} \lesssim_{p,q,s} \left\| \left( \sum_m |\beta_m|^q \sum_{l \in \mathfrak{L}^m} \left| \sum_{\nu \in \mathfrak{S}_l^m} a_{l,\nu} \mathbb{1}_{l,\nu} \right|^q \right)^{1/q} \right\|_p.$$

(ii) *If  $1 \leq q \leq p < \infty$  then there exists  $C_{p,q,s}$  such that*

$$(29) \quad \|g_m\|_{F_{p,q}^s} \leq C_{p,q,s} (2^{-m} \#(\mathfrak{L}^m))^{1/q}$$

and

$$(30) \quad \|g\|_{F_{p,q}^s} \leq C_{p,q,s} \left( \sum_{m \geq 1} |\beta_m|^q 2^{-m} \#(\mathfrak{L}^m) \right)^{1/q}.$$

*Proof.* The functions  $\{\eta_{l,\nu}\}_{l,\nu}$  represent a family of “smooth atoms” in the sense of Frazier/Jawerth [8, Thm. 4.1 and §12] which immediately implies the relations in (i). Here we need the pairwise disjointness of the sets  $\mathfrak{L}^m$ .

We continue with proving

$$(31) \quad \left\| \left( \sum_{l \in \mathfrak{L}^m} \left| \sum_{\nu \in \mathfrak{S}_l^m} \mathbb{1}_{I_{l,\nu}} \right|^q \right)^{1/q} \right\|_p \lesssim_{p,q} (2^{-m} \#(\mathfrak{L}^m))^{1/q}.$$

Then (i) together with (31) and  $\sup_{l,\nu} |a_{l,\nu}| \leq 1$  gives (29).

Indeed, let  $G_l(x) = \sum_{\nu \in \mathfrak{S}_l^m} \mathbb{1}_{I_{l,\nu}}(x)$  and  $G(x) = \left( \sum_{l \in \mathfrak{L}^m} G_l(x)^q \right)^{1/q}$ . In order to control the  $L_p$ -norm of  $G$  we use the dyadic version of the Fefferman-Stein interpolation theorem for  $L_q$  and  $BMO$ . Note that the proof of [6, Thm. 5] gives the dyadic version of  $\#$ -function estimate and thus one can work with the  $BMO_{\text{dyad}}$  norm in [6, Cor. 2]. Consequently it suffices to show that the norms of  $G$  in  $L_q$  and  $BMO_{\text{dyad}}$  are bounded by  $C(2^{-m} \#(\mathfrak{L}^m))^{1/q}$ . This is immediate for the  $L_q$  norm. For the  $BMO_{\text{dyad}}$  norm we have to show that

$$(32) \quad \sup_J \inf_c \frac{1}{|J|} \int_J |G(y) - c| dy \lesssim (2^{-m} \#(\mathfrak{L}^m))^{1/q},$$

where the sup is taken over all dyadic intervals and the inf is taken over all complex numbers.

For a fixed dyadic interval  $J$  with midpoint  $x_J$  we define

$$c_{J,l} = \begin{cases} \sum_{\nu \in \mathfrak{S}_l^m} \mathbb{1}_{I_{l,\nu}}(x_J) & \text{if } 2^{-l} \geq |J| \\ 0 & \text{if } 2^{-l} < |J| \end{cases}$$

and

$$c_J = \left( \sum_{l \in \mathfrak{L}^m} c_{J,l}^q \right)^{1/q}.$$

Fix  $J$ . Then

$$\begin{aligned} \frac{1}{|J|} \int_J |G(y) - c_J| dy &\leq \frac{1}{|J|} \int_J \left| \left( \sum_{l \in \mathfrak{L}^m} G_l(y)^q \right)^{1/q} - \left( \sum_{l \in \mathfrak{L}^m} c_{J,l}^q \right)^{1/q} \right| dy \\ &\leq \frac{1}{|J|} \int_J \left( \sum_{l \in \mathfrak{L}^m} |G_l(y) - c_{J,l}|^q \right)^{1/q} dy \\ &\leq \left( \sum_{l \in \mathfrak{L}^m} \frac{1}{|J|} \int_J |G_l(y) - c_{J,l}|^q dy \right)^{1/q}. \end{aligned}$$

Here we have used the triangle inequality in  $\ell_q$  and Hölder’s inequality on the interval  $J$ . Note that

$$G_l(y) = c_{J,l} \text{ if } y \in J \text{ and } 2^{-l} \geq |J|.$$

Since  $c_{J,l} = 0$  if  $2^{-l} < |J|$  we get from the previous estimate

$$(33) \quad \frac{1}{|J|} \int_J |G(y) - c_J| dy \leq \left( \sum_{\substack{l \in \mathfrak{L}^m \\ 2^{-l} < |J|}} \frac{1}{|J|} \int_J |G_l(y)|^q dy \right)^{1/q}.$$

Now by the definition of  $G_l$  we have

$$\int_J |G_l(y)|^q dy \leq \begin{cases} 2^{-l} & \text{if } 2^{-m}|J| \leq 2^{-l} < |J| \\ 2^{-m}|J| & \text{if } 2^{-l} < 2^{-m}|J| \end{cases}$$

and thus

$$\begin{aligned} \sum_{\substack{l \in \mathfrak{L}^m \\ 2^{-l} < |J|}} \frac{1}{|J|} \int_J |G_l(y)|^q dy &\leq \sum_{l: 2^{-m}|J| \leq 2^{-l} < |J|} (2^l|J|)^{-1} + \sum_{l \in \mathfrak{L}^m} 2^{-m} \\ &\lesssim (1 + 2^{-m} \#(\mathfrak{L}^m)). \end{aligned}$$

Since we assume that  $\#(\mathfrak{L}^m) \geq 2^m$  this finishes the proof of (32).

Finally, (30) is a consequence of the second relation in (i), the triangle inequality in  $L_{p/q}$  and (31). In fact, the second relation in (i) can be rewritten to

$$\|g\|_{F_{p,q}^s} \lesssim \left\| \sum_{m \geq 1} |\beta_m|^q \sum_{l \in \mathfrak{L}^m} \left| \sum_{\nu \in \mathfrak{S}_l^m} a_{l,\nu} \mathbb{1}_{l,\nu} \right|^q \right\|_{p/q}^{1/q}$$

Since  $p/q \geq 1$  we obtain

$$\|g\|_{F_{p,q}^s} \lesssim \left( \sum_m |\beta_m|^q \left\| \sum_{l \in \mathfrak{L}^m} \left| \sum_{\nu \in \mathfrak{S}_l^m} \mathbb{1}_{l,\nu} \right|^q \right\|_{p/q} \right)^{1/q}$$

and (31) finishes the proof.  $\square$

## 5. LOWER BOUNDS FOR HAAR PROJECTION NUMBERS

In this section we require that  $\psi$  is supported on  $(-2^{-4}, 2^{-4})$  and that  $\int \psi(x)x^M dx = 0$  for  $M = 0, 1, \dots, M_0$  for some large integer  $M_0$ , and let  $\psi_k = 2^k \psi(2^k \cdot)$ . Let  $\Psi(x) = \int_{-\infty}^x \psi(t) dt$ , the primitive which is also supported in  $(-2^{-4}, 2^{-4})$ . For  $h_{0,0} = \chi_{[0,1/2)} - \chi_{[1/2,1)}$  we have

$$\psi * h_{0,0}(x) = \Psi(x) + \Psi(x-1) - 2\Psi(x - \frac{1}{2})$$

and therefore  $\psi * h_{0,0}(x) = -2\Psi(x - \frac{1}{2})$  for  $x \in [1/4, 3/4]$ . Thus there is  $c_0 > 0$  and a subinterval  $J \subset [1/4, 3/4]$  so that

$$|\psi * h_{0,0}(x)| \geq c_0, \text{ for } x \in J.$$

For  $k = 0, 1, 2, \dots$  and  $\mu \in \mathbb{Z}$  let  $J_{k,\mu} = 2^{-k}\mu + 2^{-k}J$  which is a subinterval of the middle half of  $I_{k,\mu}$ , of length  $\gtrsim 2^{-k}$ , and we have

$$(34) \quad |\psi_k * h_{k,\mu}(x)| \geq c_0 \text{ for } x \in J_{k,\mu}.$$

We now prepare for the definition of a suitable family of test functions. Let  $\eta$  be an odd  $C^\infty$  function supported in  $(-2^{-5}, 2^{-5})$  so that  $\int \eta(x)x^M dx = 0$  for  $M = 0, 1, \dots, M_0$  and so that

$$(35) \quad 2 \int_0^{1/2} \eta(x) dx = \int_0^{1/2} \eta(x) dx - \int_{-1/2}^0 \eta(x) dx \geq 1.$$

We now pick an arbitrary set  $A$  of Haar frequencies and  $N$  so that

$$(36) \quad \Lambda < \#A + 1, \text{ and } 2^N \leq \#A < 2^{N+1},$$

and fix  $N$  for the remainder of the section. Define, for  $n = N, N-1, \dots, 1$

$$(37) \quad \eta_{k,n,\mu}(y) = \eta(2^{k+n}(x - 2^{-k}\mu - 2^{-k-1}))$$

Let  $r_k$  denote the Rademacher function on  $[0, 1]$ . For  $t \in [0, 1]$  and  $2^k \in A$  let

$$(38a) \quad \Upsilon_k(y) = \sum_{n=0}^N \alpha_n \Upsilon_{k,n}(y)$$

with

$$(38b) \quad \Upsilon_{k,n}(y) = 2^{n(-s+1/q)} \sum_{\mu=0}^{2^k-1} \eta_{k,n,\mu}(y).$$

Let

$$(39a) \quad f_{n,t}(y) = 2^{-N/q} \sum_{2^k \in A} r_k(t) 2^{-ks} \Upsilon_{k,n}(y)$$

and

$$(39b) \quad f_t(y) = \sum_{n=1}^N \alpha_n f_{n,t}.$$

**Lemma 5.1.** *The following estimates hold uniformly in  $t \in [0, 1]$ .*

(i) For  $n = 1, \dots, N$ ,

$$\|f_{n,t}\|_{F_{p,q}^s} \leq C_{p,q,s}.$$

(ii) Suppose that  $\log_2 A$  is  $N$ -separated (i.e.  $2^j \in A$ ,  $2^{j'} \in A$ ,  $j \neq j'$  implies  $|j - j'| \geq N$ ). Then

$$\|f_t\|_{F_{p,q}^s} \leq C_{p,q,s} \left( \sum_{n=1}^N |\alpha_n|^q \right)^{1/q}.$$

*Proof.* Let  $\mathfrak{L}^n = \{l : 2^{l-n} \in A\}$ . Then

$$f_{n,t}(y) = 2^{(n-N)/q} \sum_{l \in \mathfrak{L}^n} r_{l-n}(t) 2^{-ls} \sum_{\mu} \eta(2^l(x - 2^{n-l}\mu - 2^{n-l-1}))$$

and (i) follows from Proposition 4.1 since  $2^{-n}\#(\mathfrak{L}^n) \lesssim 2^{N-n}$ . Since the sets  $\mathfrak{L}^n$ ,  $n = 1, \dots, N$ , are essentially disjoint (ii) follows as well.  $\square$

For  $t \in [0, 1]$  let

$$(40) \quad T_t g(x) = \sum_{2^j \in A} r_j(t) \sum_{\mu=0}^{2^j-1} 2^j \langle h_{j,\mu}, g \rangle h_{j,\mu}(x).$$

We seek to derive a lower bound for  $\|T_{t_1} f_{t_2}\|_{F_{p,q}^s}$  for most  $t_1, t_2$ . This is accomplished by

**Proposition 5.2.** *Let  $-1 < s \leq -1/q'$ . Let  $f_{N,t}$  as in (39a) (with  $n = N$ ) and  $f_t$  as in (39b). Then there is  $c > 0$  such that the following relations*

(i)

$$\left( \int_0^1 \int_0^1 \|T_{t_1} f_{N,t_2}\|_{F_{p,q}^s}^q dt_1 dt_2 \right)^{1/q} \geq c 2^{N(-s-1/q')}$$

and (ii)

$$\left( \int_0^1 \int_0^1 \|T_{t_1} f_{t_2}\|_{F_{p,q}^s}^q dt_1 dt_2 \right)^{1/q} \geq c \left| \sum_{n=0}^N \alpha_n 2^{n(-s-\frac{1}{q'})} \right|$$

hold true.

*Proof.* Note that (i) is a special case of (ii) (with the choice  $\alpha_N = 1$ ,  $\alpha_n = 0$  for  $n \leq N$ ). The left hand side in (ii) is equivalent with

$$\left( \int_0^1 \int_0^1 \left\| \left( \sum_{k=0}^{\infty} 2^{ksq} |\psi_k * T_{t_1} f_{t_2}|^q \right)^{1/q} \right\|_p^q dt_1 dt_2 \right)^{1/q}.$$

Since  $\psi_k * T_{t_1} f_{t_2}$  is supported in  $[-1, 2]$ , we can use Hölder's inequality (with  $p \geq q$ ) to see that this expression is bounded below by a positive constant times

$$(41) \quad \begin{aligned} & \left( \int_0^1 \int_0^1 \left\| \left( \sum_{2^k \in A} 2^{ksq} |\psi_k * (T_{t_1} f_{t_2})|^q \right)^{1/q} \right\|_q^q dt_1 dt_2 \right)^{1/q} \\ & = \left( \sum_{2^k \in A} 2^{ksq} \left\| \left( \int_0^1 \int_0^1 |\psi_k * T_{t_1} f_{t_2}(x)|^q dt_1 dt_2 \right)^{1/q} \right\|_q^q \right)^{1/q}. \end{aligned}$$

For fixed  $x$  we have

$$\psi_k * T_{t_1} f_{t_2}(x) = 2^{-N/q} \sum_{2^j \in A} \sum_{2^l \in A} r_j(t_1) r_l(t_2) 2^{-ls} \sum_{\mu=0}^{2^j-1} 2^j \langle \Upsilon_l, h_{j,\mu} \rangle \psi_k * h_{j,\mu}(x)$$

and, by Khinchine's inequality,

$$\begin{aligned} & \left( \int_0^1 \int_0^1 |\psi_k * (T_{t_1} f_{t_2})(x)|^q dt_1 dt_2 \right)^{1/q} \\ & \geq c(q) 2^{-N/q} \left( \sum_{2^j \in A} \sum_{2^l \in A} \left| 2^{-ls} \sum_{\mu=0}^{2^j-1} 2^j \langle \Upsilon_l, h_{j,\mu} \rangle \psi_k * h_{j,\mu}(x) \right|^2 \right)^{1/2} \end{aligned}$$

Hence, for  $2^k \in A$ , picking up only the terms with  $j = k$  and  $l = k$ ,

$$\begin{aligned} & \left( \int_0^1 \int_0^1 \left| \psi_k * (T_{t_1} f_{t_2})(x) \right|^q dt_1 dt_2 \right)^{1/q} \\ & \gtrsim 2^{-N/q} \left| 2^{-ks} \sum_{\mu=0}^{2^k-1} 2^k \langle \Upsilon_k, h_{k,\mu} \rangle \psi_k * h_{k,\mu}(x) \right|. \end{aligned}$$

Observe that the supports of  $h_{k,\mu}$  and  $\eta_{k,m,\mu'}$  are disjoint when  $\mu \neq \mu'$ . Thus

$$\begin{aligned} 2^k \langle \Upsilon_k, h_{k,\mu} \rangle &= \sum_{n=0}^N \alpha_n 2^{n(-s+1/q)} \sum_{\mu'=0}^{2^k-1} 2^k \langle \eta_{k,n,\mu'}, h_{k,\mu} \rangle \\ &= \sum_{n=0}^N \alpha_n 2^{n(-s+1/q)} 2^k \langle \eta_{k,n,\mu}, h_{k,\mu} \rangle. \end{aligned}$$

Furthermore

$$\begin{aligned} & 2^k \langle \eta_{k,n,\mu}, h_{k,\mu} \rangle \\ &= 2^k \int \eta(2^{k+n}(x - 2^{-k}\mu - 2^{-k-1})) h_{0,0}(2^k x - \mu) dx \\ &= \int \eta(2^n(y - \frac{1}{2})) h_{0,0}(y) dy \\ &= \int_{-1/2}^0 \eta(2^n y) dy - \int_0^{1/2} \eta(2^n y) dy \\ &= -2^{-n+1} \int_0^{1/2} \eta(u) du \end{aligned}$$

where in the last line we have used that  $\eta$  is odd and supported in  $(-2^{-4}, 2^{-4})$ .

Next we observe that

$$\psi_k * h_{k,\tilde{\mu}}(x) = 0, \text{ for } x \in J_{k,\mu}, \mu \neq \tilde{\mu}.$$

So from the above we get, for  $x \in J_{k,\mu}$

$$\begin{aligned} & 2^k \sum_{\tilde{\mu}=0}^{2^k-1} \langle \Upsilon_k, h_{k,\tilde{\mu}} \rangle \psi_k * h_{k,\tilde{\mu}}(x) \\ &= 2^k \langle \Upsilon_k, h_{k,\mu} \rangle \psi_k * h_{k,\mu}(x) \\ &= -2 \psi_k * h_{k,\mu}(x) \int_0^1 \eta(u) du \sum_{n=0}^N \alpha_n 2^{n(\frac{1}{q}-1-s)}. \end{aligned}$$



Finally we can prove the lower bound for the expression (41) and obtain

$$\begin{aligned}
 & \left( \sum_{2^k \in A} 2^{ksq} \left\| \left( \int_0^1 \int_0^1 |\psi_k * (T_{t_1} f_{t_2})(x)|^q dt_1 dt_2 \right)^{1/q} \right\|_q \right)^{1/q} \\
 & \gtrsim \left( \sum_{2^k \in A} 2^{ksq} \sum_{\mu=0}^{2^k-1} \int_{J_{k,\mu}} \left[ 2^{-N/q} 2^{-ks} 2^k |\langle \Upsilon_k, h_{k,\mu} \rangle| |\psi_k * h_{k,\mu}(x)| \right]^q dx \right)^{1/q} \\
 & \gtrsim \left| \int_0^{1/2} \eta(x) dx \sum_{n=0}^N \alpha_n 2^{n(-s-\frac{1}{q'})} \right| \left( \sum_{2^k \in A} 2^{-N} \sum_{\mu=0}^{2^k-1} \int_{J_{k,\mu}} |\psi_k * h_{k,\mu}(x)|^q dx \right)^{1/q} \\
 & \gtrsim \left| \sum_{n=0}^N \alpha_n 2^{n(-s-\frac{1}{q'})} \right|.
 \end{aligned}$$

Here we have used (34), (35), and (36), and the condition  $s \leq -1/q'$ .  $\square$

*Growth of  $\gamma_*(F_{p,q}^s, \Lambda)$ ,  $s < -1/q'$ .* Take  $A$  as in (36). Let  $f_{t,N}$  be as in (39a) with  $n = N$ , so that  $\|f_{t,N}\|_{F_{p,q}^s} \lesssim 1$ . By Proposition 5.2 there exist  $t_1, t_2$  in  $[0, 1]$  so that

$$\|T_{t_1} f_{N,t_2}\|_{F_{p,q}^s} \gtrsim 2^{N(-s-\frac{1}{q'})}$$

Hence

$$\|T_{t_1}\|_{F_{p,q}^s \rightarrow F_{p,q}^s} \geq c_{p,q,s} 2^{N(-s-\frac{1}{q'})}$$

Now let

$$(42) \quad E^\pm := \{h_{j,\mu} : 2^j \in A, r_j(t_1) = \pm 1, \mu = 0, \dots, 2^j - 1\}.$$

Then

$$T_{t_1} = P_{E^+} - P_{E^-}$$

and thus at least one of  $P_{E^+}$  or  $P_{E^-}$  has operator norm bounded below by  $c_{p,q,s} 2^{N(-s-\frac{1}{q'})}$ . Since  $\text{HF}(E^\pm) \subset A$  we get

$$\mathcal{G}(F_{p,q}^s, A) \gtrsim 2^{N(-s-\frac{1}{q'})}, \quad s \leq -1/q'$$

and the asserted lower bound for  $\mathcal{G}(F_{p,q}^s; \Lambda)$  follows in the range  $s < -1/q'$ . By Theorem 3.1 we also have

$$\mathcal{G}(F_{p,q}^{-1/q'}, A) \leq c(p, q, s) \Lambda^{-s-1/q'}.$$

Thus  $\gamma^*(F_{p,q}^{-s}; \Lambda) \approx \Lambda^{-s-1/q'}$  for large  $\Lambda$ .

*Remark.* The above arguments already give a lower bound  $c(\log \Lambda)^{1/q'}$  in the endpoint case, for the lower Haar projection numbers  $\gamma_*(F_{p,q}^{-1/q'}, \Lambda)$ . Let  $A'$  be an  $2N$  separated subset with  $\#(A') \geq (2N)^{-1} \#A$ . Let  $f_t$  as in (39) with  $A$  replaced by  $A'$  and with the choices  $s = -1/q'$  and  $\alpha_n = 1$ ,  $n = 1, \dots, N$ . Then  $\|f_t\|_{F_{p,q}^{-1/q'}} \lesssim N^{1/q'}$ . By Proposition 5.2 there exist  $t_1, t_2$  in  $[0, 1]$  so that  $\|T_{t_1} f_{t_2}\|_{F_{p,q}^{-1/q'}} \gtrsim N$ . Hence  $\|T_{t_1}\|_{F_{p,q}^{-1/q'} \rightarrow F_{p,q}^{-1/q'}} \geq c_{p,q} N^{1/q'}$ .

Now let  $E^\pm$  be as in (42). Then  $\max_\pm \|P_{E^\pm}\|_{F_{p,q}^{-1/q'} \rightarrow F_{p,q}^{-1/q'}} \geq \frac{c_{p,q}}{2} N^{1/q'}$ . Thus  $\mathcal{G}(F_{p,q}^{-1/q'}, A) \geq \mathcal{G}(F_{p,q}^{-1/q'}, A') \gtrsim N^{1/q'}$  and hence  $\gamma_*(F_{p,q}^{-1/q'}; \Lambda) \gtrsim (\log \Lambda)^{1/q'}$ .

## 6. LOWER BOUNDS FOR THE ENDPOINT CASE

In this section we prove the lower bounds in Theorem 1.4. The following result provides a slightly sharper bound where a min is replaced by an average.

**Theorem 6.1.** *Assume  $\#A \geq 4^N$  and that  $4^N$  disjoint intervals  $I_\kappa$ ,  $\kappa = 1, \dots, 4^N$  are given such that the length of  $I_\kappa$  is  $N$ , and such that  $I_\kappa \cap \log_2 A \neq \emptyset$ . Let*

$$(43) \quad Z = \frac{1}{4^N} \sum_{\kappa=1}^{4^N} \#(I_\kappa \cap \log_2 A).$$

Then, for  $q \leq p < \infty$ ,

$$\mathcal{G}(F_{p,q}^{-1/q'}; A) \geq c(p, q) N^{1-1/q} Z^{1/q}.$$

*Proof that Theorem 6.1 implies Theorem 1.4.* The upper bounds follow easily from Theorem 3.1. For the lower bounds let  $A \subset \{2^n : n \geq 1\}$  be of large cardinality and let  $N$  be such that  $8^{N-1} \leq \#A \leq 8^N$ . Let  $\underline{Z}(A) = Z$ . Then we can find  $M_N$  disjoint intervals

$$I_i = (n_i - 3N, n_i + 3N)$$

with midpoints  $n_i \in \log_2(A)$ ,  $i = 1, \dots, M_N$  so that  $M_N \geq 8^{N-1}/N$  and so that each  $I_i$  contains at least  $Z$  points in  $\log_2(A)$ . Each  $I_i$  contains a subinterval  $\tilde{I}_i$  of length  $N$  which contains at least  $Z/6$  points. This means that the hypothesis of Theorem 6.1 is satisfied, and we get  $\mathcal{G}(F_{p,q}^{-1/q'}; A) \gtrsim c(p, q) N^{1/q'} (Z/6)^{1/q}$ . Part b) of Theorem 1.4 follows since  $\#\log_2(A) \approx N$ . Part a) follows by duality.  $\square$

**Proof of Theorem 6.1.** Let  $b_\kappa$  be the largest integer in  $I_\kappa$  and

$$(44) \quad \mathfrak{L} = \{b_\kappa + N : \kappa = 1, \dots, 4^N\},$$

$$(45a) \quad \mathfrak{A}(\kappa) = \{j \in I_\kappa : 2^j \in A\},$$

$$(45b) \quad \mathcal{E}(\kappa) = \{(j, \mu) : j \in \mathfrak{A}(\kappa), \mu \in 2^{j-b_\kappa+N+2}\mathbb{Z}, 1 \leq \mu < 2^j\},$$

$$(45c) \quad \mathcal{E} = \bigcup_{\kappa=1}^{4^N} \mathcal{E}(\kappa).$$

Let further  $\eta$  be as in (35) and

$$(46) \quad H_l(x) = \sum_{1 \leq \sigma \leq N} 2^{-\sigma} \sum_{\substack{\rho \in \mathbb{N}: \\ 0 < 2^{2N+2-l}\rho < 1}} \eta(2^{l-\sigma}(x - 2^{2N+2-l}\rho)).$$

Define, for  $t \in [0, 1]$

$$(47) \quad f_t(x) = \sum_{l \in \mathfrak{L}} r_l(t) 2^{l/q'} H_l(x).$$

**Lemma 6.2.** *We have*

$$\|f_t\|_{F_{p,q}^{-1/q'}} \leq C(p, q) N^{1/q}.$$

uniformly in  $t$ .

*Proof.* For  $\sigma = 1, \dots, N$  let

$$\mathfrak{L}(\sigma) = \{b_\kappa + N - \sigma : \kappa = 1, \dots, 4^N\}.$$

Thus the  $\mathfrak{L}(\sigma)$  are disjoint sets, of cardinality  $2^{2N}$  each. Let  $\{r_j\}_{j=1}^\infty$  be the system of Rademacher functions and define, for  $t \in [0, 1]$ ,

$$\begin{aligned} g_{\sigma,t} &= \sum_{\kappa=1}^{2^{2N}} 2^{(b_\kappa + N - \sigma)/q'} \sum_{\substack{\rho \in \mathbb{N}: \\ 0 < 2^{2N+2-l} \rho < 1}} r_{l+\sigma}(t) \eta(2^{b_\kappa + N - \sigma}(x - 2^{N+2-b_\kappa} \rho)) \\ &= \sum_{l \in \mathfrak{L}(\sigma)} 2^{l/q'} \sum_{\substack{\rho \in \mathbb{N}: \\ 0 < 2^{2N+2-l} \rho < 1}} r_{l+\sigma}(t) \eta(2^l(x - 2^{2N+2-l-\sigma} \rho)) \end{aligned}$$

so that

$$f_t = \sum_{\sigma=1}^N 2^{-\sigma/q} g_{\sigma,t}.$$

We apply Proposition 4.1 with the parameter  $N$  replaced by  $2N$  and  $m = 2N - \sigma$ . Clearly, the points  $2^{l-\sigma} 2^{2N+2-l} \rho$  are then  $2^{m-l}$  separated. By inequality (30) with  $\beta_{2N-\sigma} = 2^{-\sigma/q}$ ,

$$\|f_t\|_{F_{p,q}^{-1/q'}} \lesssim \left( \sum_{\sigma=1}^N (2^{-\sigma/q})^q 2^{\sigma-2N} \#(\mathfrak{L}(\sigma)) \right)^{1/q} \lesssim N^{1/q}. \quad \square$$

Define for  $t \in [0, 1]$

$$\mathcal{T}_t f(x) = \sum_{(j,\mu) \in \mathcal{E}} r_j(t) 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}(x).$$

**Proposition 6.3.** *Let  $q < p < \infty$ . Then there is  $c(p, q) > 0$  such that for large  $N$*

$$(48) \quad \left( \int_0^1 \int_0^1 \|\mathcal{T}_{t_1} f_{t_2}\|_{F_{p,q}^{-1/q'}}^q dt_2 dt_1 \right)^{1/q} \geq c(p, q) N Z^{1/q}.$$

*Proof.* By (9) and Hölder's inequality it suffices to show

$$(49) \quad \left( \int_0^1 \int_0^1 \left\| \left( \sum_{\kappa} \sum_{k \in \mathfrak{A}(\kappa)} 2^{kq/q'} |\psi_k * \mathcal{T}_{t_1} f_{t_2}|^q \right)^{1/q} \right\|_q^q dt_1 dt_2 \right)^{1/q} \gtrsim N Z^{1/q}.$$

If we interchange variables and apply Khinchine's inequality then (49) follows from

$$(50) \quad \left( \sum_{\kappa} \sum_{k \in \mathfrak{A}(\kappa)} 2^{kq/q'} \left\| \left( \sum_j \sum_{l \in \mathfrak{L}} 2^j \langle 2^{l/q'} H_l, h_{j,\mu} \rangle \psi_k * h_{j,\mu} \right)^2 \right\|_q^{1/2} \right)^{1/q} \gtrsim NZ^{1/q}.$$

We drop all terms with  $(j, l) \neq (k, b_\kappa + N)$  and see that the left hand side of (50) is bounded below by

$$(51) \quad \left( \sum_{\kappa} \sum_{k \in \mathfrak{A}(\kappa)} 2^{kq/q'} \left\| \sum_{(k,\mu) \in \mathcal{E}(\kappa)} 2^{(b_\kappa + N)/q'} 2^k \langle H_{b_\kappa + N}, h_{k,\mu} \rangle \psi_k * h_{k,\mu} \right\|_q^q \right)^{1/q}.$$

Let  $J_{k,\mu}$  be as in (34). With

$$\zeta_{\kappa,\sigma,\rho}(x) = \eta(2^{b_\kappa + N - \sigma}(x - 2^{N+2-b_\kappa}\rho))$$

we have

$$\langle H_{b_\kappa + N}, h_{k,\mu} \rangle = \sum_{1 \leq \sigma \leq N} 2^{-\sigma} \sum_{\substack{\rho: \\ 0 < 2^{N+2-b_\kappa}\rho < 1}} \langle \zeta_{\kappa,\sigma,\rho}, h_{k,\mu} \rangle.$$

Recall that by (45b) we only consider  $\mu$  of the form  $\mu = \mu_n := 2^{k-b_M+N+2}\eta$  for  $n \in \mathbb{N}$ . For those  $\mu$ ,

$$\begin{aligned} & 2^k \langle \zeta_{\kappa,\sigma,\rho}, h_{k,\mu_n} \rangle \\ &= 2^k \int \eta(2^{b_\kappa + N - \sigma}(x - 2^{N+2-b_\kappa}\rho)) h_{0,0}(2^k x - \mu_n) dx \\ &= \int \eta(2^{b_\kappa + N - \sigma - k}u + 2^{b_\kappa + N - \sigma - k}\mu_n - 2^{2N+2-\sigma}\rho) h_{0,0}(u) du \\ &= \int \eta(2^{b_\kappa + N - \sigma - k}u + 2^{2N+2-\sigma}(n - \rho)) h_{0,0}(u) du. \end{aligned}$$

For  $k \in \mathfrak{A}(\kappa)$  we have  $2^{b_\kappa + N - \sigma - k} \leq \frac{1}{4}2^{2N - \sigma + 2}$  and since  $\eta$  is supported in  $(2^{-5}, 2^5)$  we see that

$$2^{k-\sigma} \langle \zeta_{\kappa,\sigma,\rho}, h_{k,\mu_n} \rangle = \begin{cases} 2^{k-b_\kappa-N} \int_0^{1/2} \eta(u) du & \text{if } n = \rho, \\ 0 & \text{if } n \neq \rho. \end{cases}$$

Hence,

$$(52) \quad 2^k \langle H_{b_\kappa + N}, h_{k,\mu_n} \rangle = N 2^{k-b_\kappa-N} \int_0^{1/2} \eta(u) du.$$

The intervals  $J_{k,\mu_n}$  are disjoint. Hence the expression (51) is bounded below by

$$c \left( \sum_{\kappa} \sum_{k \in \mathfrak{A}(\kappa)} 2^{kq/q'} \times \sum_{\substack{n: \\ 0 < 2^{k-b_\kappa+N+2} n < 2^k}} \int_{J_{k,\mu_n}} \left| 2^{(b_\kappa+N)/q'} N 2^{k-b_\kappa-N} \int_0^{1/2} \eta(u) du \right|^q dx \right)^{1/q}.$$

The measure of  $\cup_{n: 0 < 2^{k-b_\kappa+N+2} n < 2^k} J_{k,\mu_n}$  is  $\approx 2^{b_\kappa-N-k}$ . Hence, the last expression is bounded below by

$$\begin{aligned} & c' \left( \sum_{\kappa} \sum_{k \in \mathfrak{A}(\kappa)} 2^{kq/q'} 2^{b_\kappa-N-k} [2^{(b_\kappa+N)/q'} N 2^{k-b_\kappa-N}]^q \right)^{1/q} \\ & \gtrsim \left( \sum_{\kappa} \sum_{k \in \mathfrak{A}(\kappa)} 2^{-2N} N^q \right)^{1/q} \gtrsim N Z^{1/q}. \end{aligned}$$

This proves (50) and completes the proof of the proposition.  $\square$

*Proof of Theorem 6.1, conclusion.* By Lemma 6.2 and Proposition 6.3 there exist  $t_1, t_2$  in  $[0, 1]$  such that  $\|f_t\|_{F_{p,q}^{-1/q'}} \lesssim N^{1/q}$  and  $\|T_{t_1} f_{t_2}\|_{F_{p,q}^{-1/q'}} \gtrsim N Z^{1/q}$ . Hence

$$\|T_{t_1}\|_{F_{p,q}^{-1/q'} \rightarrow F_{p,q}^{-1/q'}} \geq c_{p,q} N^{1-1/q} Z^{1/q}.$$

As in the previous section, if  $E^\pm$  is as in (42). Then

$$\max_{\pm} \|P_{E^\pm}\|_{F_{p,q}^{-1/q'} \rightarrow F_{p,q}^{-1/q'}} \geq \frac{c_{p,q}}{2} N^{1-1/q} Z^{1/q}.$$

Thus  $\mathcal{G}(F_{p,q}^{-1/q'}, A) \gtrsim N^{1/q'} Z^{1/q}$ .  $\square$

## 7. CONCLUDING REMARKS

7.1. It is possible to disprove the unconditional basis property also in the case  $q/(q+1) \leq p \leq 1$  and  $s \geq 1/q$  via complex interpolation, see Figure 2 above. Indeed, if  $E$  is a subset of the Haar system the (quasi-)norm of the corresponding projection operator  $P_E$  interpolates as follows

$$(53) \quad \|P_E\|_{F_{p,q}^s \rightarrow F_{p,q}^s} \lesssim \|P_E\|_{F_{p_0,q}^{s_0} \rightarrow F_{p_0,q}^{s_0}}^{1-\theta} \cdot \|P_E\|_{F_{p_1,q}^{s_1} \rightarrow F_{p_1,q}^{s_1}}^\theta$$

with  $1/p = (1-\theta)/p_0 + \theta/p_1$ ,  $s = (1-\theta)s_0 + \theta s_1$ . Since  $P_E : F_{p_0,q}^{s_0} \rightarrow F_{p_0,q}^{s_0}$  is  $O(1)$  we obtain the relation

$$\|P_E\|_{F_{p_1,q}^{s_1} \rightarrow F_{p_1,q}^{s_1}} \gtrsim \|P_E\|_{F_{p,q}^s \rightarrow F_{p,q}^s}^{1/\theta}.$$

Choosing  $(1/p, s)$  as in Figure 2 below we obtain large quasi-norms in the given quasi-Banach region. For the endpoint case the argument has to be modified by interpolating along the  $s = 1/q$  line. Putting in (53) the upper bounds from Theorem 3.1 and the lower bounds from Theorem 6.1 we obtain

“large” projections norms. These observations show that the shaded region displayed in Figure 2 is the correct one for Hardy-Sobolev spaces  $F_{p,2}^s$  on the real line, see also [23], §2.2.3, Page 82.

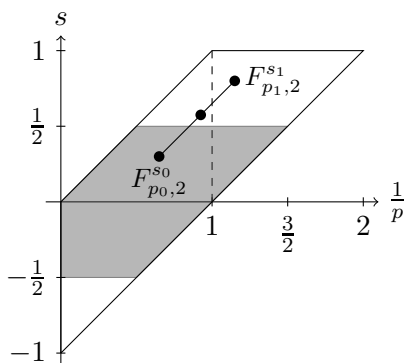


FIGURE 2. The Haar basis in Hardy-Sobolev spaces, complex interpolation

7.2. Concerning the spaces  $F_{p,q}^s(\mathbb{R}^n)$  we expect a similar picture as in Figure 1. However, for the quasi-Banach situation there will be a  $n$ -dependence, see [23], §2.3.2, Page 94.

7.3. The corresponding problem for the Faber-Schauder basis, i.e., the family of hat functions that are integrals of associated Haar functions (*cf.* [23, Chapt. 3]), can be derived from the results in this paper. Due to the shift of regularity of the basis functions there will be corresponding shifts in the parameter domain (shaded region), *cf.* Figures 1, 2 together with [23], §3.1.2, Page 127.

7.4. The proofs in this paper of the existence of projection operators with large norm are probabilistic. It is also possible to explicitly construct subsets of the Haar system for which the corresponding projections have large operator norms. This is done in the subsequent paper [17].

7.5. It will be shown in a forthcoming paper [9] that there are suitable enumerations of  $\mathcal{H}$  which form a Schauder basis of  $L_p^s$ , for  $-1/p' < s < 1/p$ . This result has also extensions to  $F_{p,q}^s$  spaces.

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