SQUARE FUNCTIONS AND MAXIMAL OPERATORS ASSOCIATED WITH RADIAL FOURIER MULTIPLIERS

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Dedicated to Eli Stein

We begin with an overview on square functions for spherical and Bochner–Riesz means which were introduced by Eli Stein, and discuss their implications for radial multipliers and associated maximal functions. We then prove new endpoint estimates for these square functions, for the maximal Bochner–Riesz operator, and for more general classes of radial Fourier multipliers.

OVERVIEW

Square functions. The classical Littlewood–Paley functions on $\mathbb{R}^d$ are defined by

$$g[f] = \Big(\int_0^\infty \Big| \frac{\partial}{\partial t} P_t f \Big|^2 t \, dt \Big)^{1/2}$$

where $(P_t)_{t>0}$ is an approximation of the identity defined by the dilates of a 'nice' kernel (for example (P_t) may be the Poisson or the heat semigroup). Their significance in harmonic analysis, and many important variants and generalizations have been discussed in Stein's monographs [38], [39], [44], in the survey [45] by Stein and Wainger, and in the historical article [43].

Here we focus on L^p-bounds for two square functions introduced by Stein, for which (P_t) is replaced by a family of operators with rougher kernels or multipliers. The first is generated by the *generalized spherical means*

$$A_t^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_{|y| \le 1} (1 - |y|^2)^{\beta-1} f(x - ty) \, dy$$

defined *a priori* for Re $\beta > 0$. The definition can be extended to Re $\beta \le 0$ by analytic continuation; for $\beta = 0$ we recover the standard spherical means. In [41] Stein used (a variant of) the square function

$$\mathcal{G}_\beta f = \Big(\int_0^\infty \Big| \frac{\partial}{\partial t} A_t^\beta f \Big|^2 t \, dt \Big)^{1/2}$$

to prove L^p-estimates for the maximal function $\sup_{t>0} |A_t^{\beta-1/2+\varepsilon} f|$, in particular he established pointwise convergence for the standard spherical means when $p > \frac{d}{d-1}$ and $d \geq 3$; see also [45].

The second square function

$$G_\alpha f = \Big(\int_0^\infty \Big| \frac{\partial}{\partial t} R_t^\alpha f \Big|^2 t\, dt \Big)^{1/2},$$

generated by the Bochner–Riesz means

$$R_t^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{|\xi| \leq t} \Big(1 - \frac{|\xi|^2}{t^2} \Big)^\alpha \widehat{f}(\xi)\, e^{i\langle x, \xi \rangle}\, d\xi,$$

was introduced in Stein's 1958 paper [37] and used to control the maximal function $\sup_{t>0} |R_t^{\alpha-1/2+\varepsilon} f|$ for $f \in L^2$ in order to prove almost everywhere convergence for Bochner–Riesz means of both Fourier integrals and series (see also Chapter VII in [46]). Later, starting with the work of Carbery [3], it was recognized that sharp L^p bounds for G_α with $p > 2$ imply sharp L^p-bounds for maximal functions associated with Bochner–Riesz means and then also maximal functions associated with more general classes of radial Fourier multipliers ([4], [13]).

In [45], Stein and Wainger posed the problem of investigating the relationships between various square functions. Addressing this problem, Sunouchi [48] (in one dimension) and Kaneko and Sunouchi [23] (in higher dimensions) used Plancherel's theorem to establish among other things the uniform pointwise equivalence

$$(1) \qquad G_\alpha f(x) \approx \mathcal{G}_\beta f(x), \quad \beta = \alpha - \frac{d-2}{2}.$$

In view of this remarkable result we shall consider G_α only.

Implications for radial multipliers. We recall Stein's point of view for proving results for Fourier multipliers from Littlewood–Paley theory. Suppose the convolution operator $\mathcal{T}$ is given by $\widehat{\mathcal{T}f} = h\widehat{f}$ where h satisfies the assumptions of the Hörmander multiplier theorem. That is, if φ is a radial nontrivial C^∞ function with compact support away from the origin, and $L_\alpha^2(\mathbb{R}^d)$ is the usual Sobolev space, it is assumed that $\sup_{t>0} \|\varphi h(t\,\cdot)\|_{L_\alpha^2}$ is finite for some $\alpha > d/2$. Under this assumption $\mathcal{T}$ is bounded on L^p for $1 < p < \infty$ ([21], [39], [55]). In Chapter IV of the monograph [39], Stein approached this result by establishing the pointwise inequality

$$(2) \qquad g[\mathcal{T}f](x) \leq C \sup_{t>0} \|\varphi h(t\,\cdot)\|_{L_\alpha^2}\, g_{\lambda(\alpha)}^*[f],$$

where g is a standard Littlewood–Paley function and g_λ^* is a tangential variant of g which does not depend on the specific multiplier. As $\|g_{\lambda(\alpha)}^*[f]\|_p \lesssim \|f\|_p$ for $2 \leq p < \infty$ and $\alpha > d/2$, this proves the theorem since (under certain nondegeneracy assumptions on the generating kernel) one also has $\|g[f]\|_p \approx \|f\|_p$ for $1 < p < \infty$.

A similar point of view was later used for *radial* Fourier multipliers. Let m be a bounded function on $\mathbb{R}^+$, let $\varphi_\circ \in C_0^\infty(1, 2)$, and let T_m be defined by

$$(3) \qquad \widehat{T_m f}(\xi) = m(|\xi|)\widehat{f}(\xi).$$

The work of Carbery, Gasper and Trebels [5] yields an analogue of (2) for radial multipliers in which the g_λ^*-function is replaced with a robust version of G_α which has the same L^p boundedness properties as G_α. A variant of their argument, given by Carbery in [4], shows that one can work with G_α itself and so there is a pointwise estimate

$$(4) \qquad g[T_m f](x) \le C \sup_{t>0} \|\varphi_\circ m(t\cdot)\|_{L_\alpha^2(\mathbb{R})} G_\alpha f(x)$$

where again g is a suitable standard Littlewood–Paley function. L^p mapping properties of G_α together with (4) have been used to prove essentially sharp estimates for radial convolution operators, with multipliers in localized Sobolev spaces. However it was not evident whether (4) could also be used to capture endpoint results, for radial multipliers in the same family of spaces. We shall address this point in §1 below.

Carbery [4] also obtained a related pointwise inequality for maximal functions,

$$(5) \qquad \sup_{t>0} |T_{m(t\cdot)} f(x)| \le C\|m \circ \exp\|_{L_\alpha^2(\mathbb{R})} G_\alpha f(x),$$

which for $p \ge 2$ yields effective L^p bounds for maximal operators generated by radial Fourier multipliers from such bounds for G_α; see also Dappa and Trebels [13] for similar results. Only little is currently known about maximal operators for radial Fourier multipliers in the range $p < 2$; *cf.* Tao's work [50], [51] for examples and for partial results in two dimensions.

L^p-*bounds for* G_α. We now discuss necessary conditions and sufficient conditions on $p \in (1, \infty)$ for the validity of the inequality

$$(6) \qquad \|G_\alpha f\|_p \lesssim \|f\|_p;$$

here the notation $A \lesssim B$ is used for $A \le CB$ with an unspecified constant. By (4) it is necessary for (6) that $\alpha > 1/2$ since for $L_\alpha^2(\mathbb{R})$ to be imbedded in L^∞ we need $\alpha > 1/2$. For $1 < p < 2$ the inequality can only hold if $\alpha > \tilde{\alpha}(p) = d(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$. This is seen by writing

$$(7) \quad G_\alpha f = \left(\int_0^\infty |K_t^\alpha * f|^2 \frac{dt}{t} \right)^{1/2} \quad \text{where} \quad \widehat{K_t^\alpha}(\xi) = \alpha \frac{|\xi|^2}{t^2}\left(1 - \frac{|\xi|^2}{t^2}\right)_+^{\alpha-1}.$$

Then, for a suitable Schwartz function η, with $\widehat{\eta}$ vanishing near 0 and compactly supported in a narrow cone, and for $t \sim 1$ and large x in an open cone, we have

$$(8) \qquad K_t^\alpha * \eta(x) = c_\alpha t^d e^{it|x|}|tx|^{-\frac{d-1}{2}-\alpha} + E_t(x)$$

where E_t are lower order error terms. This leads to

$$\Big(\int_1^2 |K_t^\alpha * \eta|^2 dt \Big)^{1/2} \in L^p(\mathbb{R}^d) \quad \Longrightarrow \quad \alpha > \tilde\alpha(p) \, .$$

Note that the oscillation for large x in (8) plays no role here.

Concerning positive results for $p \le 2$, the L^2-bound for $\alpha > 1/2$ follows from Plancherel and was already observed in [37]. The case $1 < p \le 2$, $\alpha > \frac{d+1}{2}$ is covered by the Calderón–Zygmund theory for vector-valued singular integrals, and analytic interpolation yields L^p-boundedness for $1 < p < 2$, $\alpha > \tilde\alpha(p)$, see [47], [22]. There is also an endpoint result for $\alpha = \tilde\alpha(p)$, indeed one can use the arguments by Fefferman [14] for the weak type endpoint inequalities for Stein's g_λ^* function to prove that $G_{\tilde\alpha(p)}$ is of weak type (p,p) for $1 < p < 2$ (Henry Dappa, personal communication, see also Sunouchi [48] for the case $d = 1$).

The range $2 < p < \infty$ is more interesting, since now the oscillation of the kernel K_t^α plays a significant role, and, in dimensions $d \ge 2$, the problem is closely related to the Fourier restriction and Bochner–Riesz problems. A necessary condition for $p > 2$ can be obtained by duality. Inequality (6) for $p > 2$ implies that for all $b \in L^2([1,2])$ and η as above

$$(9) \qquad \Big\| \int_1^2 b(t) K_t^\alpha * \eta \, dt \Big\|_{p'} \lesssim \Big(\int_{[1,2]} |b(t)|^2 dt \Big)^{1/2} \, .$$

If we again split K_t^α as in (8), and prove suitable upper bounds for the expression involving the error terms then we see that, for $R \gg 1$,

$$\int_{|x| \ge R} \Big| \frac{\widehat b(|x|)}{|x|^{\frac{d-1}{2}+\alpha}} \Big|^{p'} dx < \infty,$$

which leads to the necessary condition $\alpha > d(\frac{1}{p'} - \frac{1}{2}) = d(\frac{1}{2} - \frac{1}{p})$.

It is conjectured that (6) holds for $2 < p < \infty$ if and only if $\alpha > \alpha(p) = \max\{d(\frac{1}{2} - \frac{1}{p}), \frac{1}{2}\}$. For $d = 1$ this can be shown in several ways, and the estimate follows from Calderón–Zygmund theory (one such proof is in [48]). The full conjecture for $d = 2$ was proved by Carbery [3], and a variable coefficient generalization of his result was later obtained in [27]. The partial result for $p > \frac{2d+2}{d-1}$ which relies on the Stein–Tomas restriction theorem is in Christ [9] and in [30]. A better range (unifying the cases $d = 2$ and $d \ge 3$) was recently obtained by the authors [25]; that is, inequality (6) holds for $\alpha > d(1/2 - 1/p)$ and $d \ge 2$ in the range $2 + 4/d < p < \infty$. This extends previous results on Bochner–Riesz means by the first author [24] and relies on Tao's bilinear adjoint restriction theorem [52]. Motivated by a still open problem of Stein [42], the authors also proved a related weighted inequality in [25], namely for $d \ge 2$, $1 \le q < \frac{d+2}{2}$,

$$\int [G_\alpha f(x)]^2 w(x) \, dx \lesssim \int |f(x)|^2 \mathfrak{W}_q w(x) \, dx, \quad \alpha > \frac{d}{2q} \, ,$$

where $\mathfrak{W}_q$ is an explicitly defined operator which is of weak type (q,q) and bounded on L^r with $q < r \le \infty$. This is an analogue of a result by Carbery and the third author in two dimensions [6] and extends a weighted inequality by Christ [9] in higher dimensions. One might expect that recent progress by Bourgain and Guth [2] on the Bochner–Riesz problem will lead to further improvements in the ranges of these results but this is currently open.

By the equivalence (1) one can interpret the boundedness of G_α as a regularity result for spherical means and then for solutions of the wave equation. By a somewhat finer analysis in conjunction with the use of the Fefferman–Stein #-function [17] the authors obtained an $L^p(L^2)$ endpoint result, local in time, in fact not just for the wave equation, but also for other dispersive equations. Namely if $\gamma > 0$, $d \ge 2$, $2 + 4/d < p < \infty$ then

$$(10) \qquad \left\| \left(\int_{-1}^{1} \left| e^{it(-\Delta)^{\gamma/2}} f \right|^2 dt \right)^{1/2} \right\|_p \lesssim \|f\|_{B^p_{s,p}}, \quad \frac{s}{\gamma} = d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}.$$

Here $B^p_{s,p}$ is the Besov space which strictly contains the Sobolev space L^p_s for $p > 2$.

Concerning endpoint estimates, many such results for Bochner–Riesz multipliers and variants had been previously known (*cf.* [10], [11], [12], [33], [34], [49]). For the Bochner–Riesz means R^λ_t with the critical exponent $\lambda(p) = d(1/2 - 1/p) - 1/2$, Tao [50] showed that if for some $p_1 > 2d/(d-1)$ the L^{p_1} boundedness holds for all $\lambda > \lambda(p_1)$, then one also has a bound in the limiting case, for $p_1 < p < \infty$, namely $R^{\lambda(p)}_t$ maps $L^{p,1}$ to L^p, and $L^{p'} \to L^{p',\infty}$. In contrast no positive result for $G_{d/2-d/p}$ seems to have been known, even for the version with dilations restricted to $(1/2, 2)$. It should be emphasized that, despite the pointwise equivalence of the two square functions in (1), the sharp regularity result (10) does not imply a corresponding endpoint bound for $G_{d/2-d/p}$ (in fact the latter is not bounded on L^p). In this paper we will prove a sharp result for $G_{d/2-d/p}$ in the restricted open range of the Stein–Tomas adjoint restriction theorem, and obtain related results for maximal operators and Fourier multipliers.

1. ENDPOINT RESULTS

Theorem 1.1. *Let* $d \ge 2$, $\frac{2(d+1)}{d-1} < p < \infty$ *and* $\alpha = d(\frac{1}{2} - \frac{1}{p})$. *Then*

$$(11) \qquad \qquad \|G_\alpha f\|_p \le C\|f\|_{L^{p,2}}.$$

Here $L^{p,q}$ denotes the Lorentz space. We note that the $L^p \to L^p$ boundedness fails; moreover $L^{p,2}$ cannot be replaced by a larger space $L^{p,\nu}$ for $\nu > 2$. This can be shown by the argument in (9) namely, if $b \in L^2([1,2])$ then the function $\hat{b}(|\cdot|)(1 + |\cdot|)^{-\frac{d}{p'} + \frac{1}{2}}$ belongs to $L^{p',2}$ but not necessarily to $L^{p',r}$ for $r < 2$. The space $L^{p,2}$ has occured earlier in endpoint results related to other square functions, see [31], [35], [53].

The pointwise bound (5) and Theorem 1.1 yield a new bound for maximal functions, in particular for multipliers in the Sobolev space $L^2_{d/2-d/p}$ which

are compactly supported away from the origin. This Sobolev condition is too restrictive to give any endpoint bound for the maximal Bochner–Riesz operator. However such a result can be deduced from a related result on maximal functions

$$M_m f(x) = \sup_{t>0} |\mathcal{F}^{-1}[m(t|\cdot|)\widehat{f}](x)|$$

with m compactly supported away from the origin. Our assumptions involve the Besov space $B^2_{\alpha,q}$ (which is L^2_α when $q = 2$) and thus the following result seems to be beyond the scope of a square function estimate when $q \neq 2$.

Theorem 1.2. *Let* $d \geq 2$, $\frac{2(d+1)}{d-1} < p < \infty$, $\alpha = d(\frac{1}{2} - \frac{1}{p})$ *and* $p' \leq q \leq \infty$. *Assume that* m *is supported in* $(1/2, 2)$ *and that* m *belongs to the Besov space* $B^2_{\alpha,q}$. *Then*

$$\|M_m f\|_{L^p} \leq C\|m\|_{B^2_{\alpha,q}}\|f\|_{L^{p,q'}} .$$

We apply this to the Bochner–Riesz maximal operator R^λ_* defined by

$$R^\lambda_* f(x) = \sup_{t>0} |R^\lambda_t f(x)|.$$

Split $(1 - t^2)^\lambda_+ = u_\lambda(t) + m_\lambda(t)$ where m_λ is supported in $(1/2, 2)$ and $u_\lambda \in C_0^\infty(\mathbb{R})$. Then the maximal function $M_{u_\lambda} f$ is pointwise controlled dominated by the Hardy–Littlewood maximal function and thus bounded on L^p for all $p > 1$. The function m_λ belongs to the Besov space $B^2_{\lambda+1/2,\infty}$ and Theorem 1.2 with $q = \infty$ yields a maximal version of (the dual of) Christ's endpoint estimate in [11].

Corollary 1.3. *Let* $d \geq 2$, $\frac{2(d+1)}{d-1} < p < \infty$, *and* $\lambda = d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$. *Then*

$$\|R^\lambda_* f\|_p \leq C\|f\|_{L^{p,1}} .$$

We now consider operators T_m with radial Fourier multipliers, as defined in (3), which do not necessarily decay at ∞. The pointwise bounds (4), Theorem 1.1 and duality yield optimal $L^p \to L^{p,2}$ estimates in the range $1 < p < \frac{2(d+1)}{d+3}$, for Hörmander type multipliers with localized L^2_α conditions in the critical case $\alpha = d(\frac{1}{p} - \frac{1}{2})$. This demonstrates the effectiveness of Stein's point of view in (2) and (4).

The following more general theorem is again beyond the scope of a square function estimate. We use dilation invariant assumptions involving localizations of Besov spaces $B^2_{\alpha,q}$. We note that in [33] it had been left open whether one could use endpoint Sobolev space or Besov spaces with $q > 1$ in (12) below.

Theorem 1.4. *Let* $d \geq 2$, $1 < p < \frac{2(d+1)}{d+3}$, $\alpha = d(\frac{1}{p} - \frac{1}{2})$ *and* $p \leq q \leq \infty$. *Let* $\varphi_\circ$ *be a nontrivial* C_0^∞ *function supported in* $(1, 2)$. *Assume*

$$(12) \qquad \sup_{t>0} \|\varphi_\circ m(t \cdot)\|_{B^2_{\alpha,q}} < \infty .$$

Then T_m *maps* L^p *to* $L^{p,q}$ *and* $L^{p',q'}$ *to* $L^{p'}$.

It is not hard to see that the assumption (12) is independent of the choice of the particular cutoff $\varphi_\circ$. The result is sharp as T_m does not map L^p to $L^{p,r}$ for $r < q$. This can be seen by considering some test multipliers of Bochner–Riesz type. Indeed, let Φ_1 be a radial C^∞ function, with $\Phi_1(x) = 1$ for $2^{-1/2} \le |x| \le 2^{1/2}$ and supported in $\{1/2 < |x| < 2\}$ and similarly let χ be a radial C^∞ function compactly supported away from the origin and so that $\chi(\xi) = 1$ in a neighborhood of the unit sphere. Set (now with $p < 2$)

$$m(\xi) = \chi(\xi) \sum_{j=1}^{\infty} c_j \int (1 - |\xi - \eta|^2)_+^{d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}} 2^{jd} \widehat{\Phi}_1(2^j \eta) \, d\eta \,.$$

We first remark that if we write $m(\xi) = m_\circ(|\xi|)$, then $m_\circ \in B^2_{\alpha,q}(\mathbb{R})$ if and only if $m \in B^2_{\alpha,q}(\mathbb{R}^d)$ (here we use that $m_\circ$ is compactly supported away from the origin). Now considering the explicit formula for the kernel of Bochner–Riesz means (*cf.* (41) below) it is easy to see that $m \in B^2_{d/p-d/2,q}(\mathbb{R}^d)$ if and only if $\{c_j\}_{j=1}^{\infty}$ belongs to ℓ^q; moreover the necessary condition $\mathcal{F}^{-1}[m] \in L^{p,q}$ is satisfied if and only if $\{c_j\}$ belongs to ℓ^q. These considerations show the sharpness of Theorem 1.4 and also the sharpness of Theorem 1.2.

For the operator T_m acting on the subspace L^p_{rad}, consisting of radial L^p functions, the estimate corresponding to Theorem 1.4 has been known to be true in the optimal range $1 < p < \frac{2d}{d+1}$. In fact Garrigós and the third author [18] obtained an actual characterization of classes of Hankel multipliers which yields, for $p \le q \le \infty$,

$$\|T_m\|_{L^p_{rad} \to L^{p,q}} \approx \sup_{t>0} \left\| \mathcal{F}^{-1}\big[\phi(|\cdot|) m(t|\cdot|)\big] \right\|_{L^{p,q}(\mathbb{R}^d)} \text{ if } 1 < p < \frac{2d}{d+1}.$$

This easily implies the $L^p_{rad} \to L^{p,q}$ boundedness under assumption (12), see [18]. Similarly, if in Theorem 1.4 we replace the range $(1, \frac{2d+2}{d+3})$ with the smaller p-range $(1, \frac{2d-2}{d+1})$ (applicable only in dimension $d \ge 4$) the result follows from the characterization of radial L^p Fourier multipliers acting on general L^p functions in a recent article by Heo, Nazarov and the third author [19]. There it is proved that for $p \le q \le \infty$,

$$(13) \quad \|T_m\|_{L^p \to L^{p,q}} \approx \sup_{t>0} \left\| \mathcal{F}^{-1}\big[\phi(|\cdot|) m(t|\cdot|)\big] \right\|_{L^{p,q}(\mathbb{R}^d)} \text{ if } 1 < p < \frac{2d-2}{d+1} \,.$$

The remainder of this paper is devoted to the proofs of the above theorems. They are mostly based on ideas in [19]. It remains an interesting open problem to extend the range of (13), in particular to prove such a result for some $p > 1$ in dimensions two and three. Moreover it would be interesting to prove the above theorems beyond the Stein–Tomas range.

2. Convolution with spherical measures

In this section we prove an inequality for convolutions with spherical measures acting on functions with a large amount of cancellation. It can be

used to obtain results such as Theorems 1.4 for radial multipliers which are compactly supported away from the origin.

To formulate this inequality let η be a Schwartz function on $\mathbb{R}^d$ and let ψ be a radial C^∞ function with compact support in $\{x : |x| \leq 1\}$ and such that

$$\widehat{\psi}(\xi) = u(|\xi|)$$

vanishes of order $10d$ at the origin. For $j \geq 1$ let $I_j = [2^j, 2^{j+1}]$ and denote by σ_r the surface measure on the sphere of radius r which is centered at the origin. Thus the norm of σ_r as a measure is $O(r^{d-1})$. We recall the Bessel function formula

$$(14) \qquad \widehat{\sigma}_r(\xi) = r^{d-1}\mathcal{J}(r|\xi|) \text{ with } \mathcal{J}(s) = c(d)s^{-\frac{d-2}{2}}J_{\frac{d-2}{2}}(s),$$

which implies $|\widehat{\sigma}_r(\xi)| \lesssim r^{d-1}(1+r|\xi|)^{-\frac{d-1}{2}}$. In view of the assumed cancellation of ψ, we have

$$(15) \qquad \|\widehat{\psi * \sigma_r}\|_\infty = O(r^{(d-1)/2}).$$

In what follows let ν be a probability measure on $[1, 2]$. We will need to work with functions with values in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, \frac{dr}{r})$ and write

$$\|F\|_{L^p(L^1(\mathcal{H}))} = \Big\| \int_1^2 \Big(\int_0^\infty |F_t(r,\cdot)|^2 \frac{dr}{r} \Big)^{1/2} d\nu(t) \Big\|_p.$$

Proposition 2.1. *Let* $1 \leq p < \frac{2(d+1)}{d+3}$. *Then*

$$\Big\| \sum_{j\geq 1} \int_1^2 \int_{I_j} \psi * \sigma_{rt} * \eta * F_{t,j}(r, \cdot)\, dr\, d\nu(t) \Big\|_p \lesssim \Big(\sum_{j\geq 1} 2^{jd} \|F_j\|_{L^p(L^1(\mathcal{H}))}^p \Big)^{1/p}.$$

The measure ν is used here to unify the proofs of Theorems 1.2 and 1.4. For our applications we are only interested in two such measures. For Theorems 1.1 and 1.4 we take for ν the Dirac measure at $t = 1$ (and consequently in this case we can set $\sigma_{rt} = \sigma_r$ and eliminate all t-integrals in the proofs below). For the application to Theorem 1.2 we take for ν the Lebesgue measure on $[1, 2]$.

We first give a proof for the L^p bound of each term in the j-sum, which uses standard arguments ([14], [15]).

Lemma 2.2. *Let* $1 \leq p \leq \frac{2(d+1)}{d+3}$. *Then*

$$\Big\| \int_1^2 \int_{I_j} \psi * \sigma_{rt} * F_t(r, \cdot)\, dr\, d\nu(t) \Big\|_2 \lesssim 2^{jd/2} \|F\|_{L^p(L^1(\mathcal{H}))}.$$

Proof. We use Plancherel's theorem and then the Stein–Tomas restriction theorem [54]. With $\mathcal{J}$ as in (14) so that $\widehat{\sigma}_r(\xi) = r^{d-1}\mathcal{J}(r|\xi|)$ and $\widehat{\psi}(\xi) =$

$u(|\xi|)$, we get from the restriction theorem

$$\left\| \int_1^2 \int_{I_j} \psi * \sigma_{rt} * F_t(r, \cdot) \, dr \, d\nu(t) \right\|_2^2$$

$$= c \int |u(\rho)|^2 \int_{S^{d-1}} \left| \int_1^2 \int_{I_j} (rt)^{d-1} \mathcal{J}(rt\rho) \widehat{F}_t(r, \rho\xi') \, dr \, d\nu(t) \right|^2 d\sigma(\xi') \, \rho^{d-1} d\rho$$

$$\lesssim \int |u(\rho)|^2 \rho^{\frac{2d}{p}-d-1} \left\| \int_1^2 \int_{I_j} (rt)^{d-1} \mathcal{J}(rt\rho) F_t(r, \cdot) \, dr \, d\nu(t) \right\|_p^2 d\rho$$

$$\lesssim \left\| \int_1^2 \left(\int |u(\rho)|^2 \rho^{\frac{2d}{p}-d-1} \left| \int_{I_j} r^{d-1} \mathcal{J}(rt\rho) F_t(r, \cdot) \, dr \right|^2 d\rho \right)^{1/2} d\nu(t) \right\|_p^2.$$

In the last step we have used Minkowski's integral inequality. We claim that, for fixed $x \in \mathbb{R}^d$ and $t \in [1, 2]$,
(16)
$$\int |u(\rho)|^2 \rho^{\frac{2d}{p}-d-1} \left| \int_{I_j} r^{d-1} \mathcal{J}(rt\rho) F_t(r, x) \, dr \right|^2 d\rho \lesssim \int_{I_j} \left| F_t(r, x) \right|^2 r^{d-1} dr \, ,$$

with the implicit constant uniform in x, t, and the lemma follows by substituting this in the previous display.

To see (16) we first notice that for a radial $H(w) = H_\circ(|w|)$ we have

$$\int H_\circ(r) r^{d-1} \mathcal{J}(r|\xi|) \, dr = c_d \widehat{H}(\xi).$$

Thus, if we take $H^{x,t}(w) = \chi_{I_j}(|w|) F_t(|w|, x)$, the left-hand side of (16) is a constant multiple of

$$\int |\widehat{\psi}(\xi)|^2 |\xi|^{\frac{2d}{p}-2d} |\widehat{H^{x,t}}(t\xi)|^2 d\xi$$

$$\lesssim \int |\widehat{H^{x,t}}(\xi)|^2 d\xi = c \int |H^{x,t}(w)|^2 dw = c' \int_{I_j} |F_t(r, x)|^2 r^{d-1} \, dr \, ,$$

and we are done. In the inequality we used that $\widehat{\psi}$ vanishes of high order at the origin. $\qquad\square$

If we fix j and assume that $F_{Q,t}(r, \cdot)$ is supported for all r in a cube Q of sidelength 2^j then the expression $\int_1^2 \int_{I_j} \psi * \sigma_{rt} * F_{Q,t}(r, \cdot) dr \, d\nu(t)$ is supported in a similar slightly larger cube. From this it quickly follows that

$$\left\| \int_1^2 \int_{I_j} \psi * \sigma_{rt} * F_{Q,t}(r, \cdot) \, dr \, d\nu(t) \right\|_p$$

$$\lesssim 2^{jd/p} \left\| \int_1^2 \left(\int_0^\infty |F_{Q,t}(r, \cdot)|^2 \frac{dr}{r} \right)^{1/2} d\nu(t) \right\|_p.$$

This estimate is however insufficient to prove Proposition 2.1 for $p > 1$. We shall also need the following orthogonality lemma.

Lemma 2.3. *Let $J_1, J_2 \subset (0, \infty)$ be intervals and let E_1, E_2 be compact sets in $\mathbb{R}^d$ with $dist(E_1, E_2) \geq M \geq 1$. Suppose that for every $r \in J_i$, the function $x \mapsto f_i(r, x)$ is supported in E_i. Then, for $t_1, t_2 \in [1, 2]$,*

$$\Big| \int_{J_1} \int_{J_2} \big\langle \psi * \sigma_{r_1 t_1} * f_1(r_1, \cdot), \psi * \sigma_{r_2 t_2} * f_2(r_2, \cdot) \big\rangle dr_1 dr_2 \Big|$$

$$\lesssim M^{-\frac{d-1}{2}} \prod_{i=1}^{2} \Big[\int_1^2 \int \Big(\int_{J_i} |f_i(r, y)|^2 r^{d-1} dr \Big)^{1/2} dy \Big].$$

Proof. We follow [19] and apply Parseval's identity and polar coordinates in ξ. Then,

$$\big\langle \psi * \sigma_{r_1 t_1} * f_1(r_1, \cdot), \psi * \sigma_{r_2 t_2} * f_2(r_2, \cdot) \big\rangle$$

$$= c \int |\widehat{\psi}(\xi)|^2 \widehat{\sigma}_{r_1 t_1}(\xi) \overline{\widehat{\sigma}_{r_2 t_2}(\xi)} \iint f_1(r_1, y_1) \overline{f_2(r_2, y_2)} e^{i\langle \xi, y_2 - y_1 \rangle} \, dy_1 dy_2 d\xi$$

$$= c' \int |u(\rho)|^2 (r_1 t_1)^{d-1} \mathcal{J}(r_1 t_1 \rho)(r_2 t_2)^{d-1} \mathcal{J}(r_2 t_2 \rho) \times$$

$$\iint f_1(r_1, y_1) \overline{f_2(r_2, y_2)} \mathcal{J}(\rho | y_1 - y_2 |) \, dy_1 \, dy_2 \, \rho^{d-1} d\rho,$$

so that the left-hand side of the desired inequality is equal to a constant multiple of

$$(17) \quad \int \iint |u(\rho)|^2 \int_{J_1} (r_1 t_1)^{d-1} \mathcal{J}(r_1 t_1 \rho) f_1(r_1, y_1) \, dr_1$$

$$\times \int_{J_2} (r_2 t_2)^{d-1} \mathcal{J}(r_2 t_2 \rho) \overline{f_2(r_2, y_2)} \, dr_2 \ \mathcal{J}(\rho | y_1 - y_2 |) \, dy_1 \, dy_2 \, \rho^{d-1} d\rho.$$

Now define two radial kernels by $H_i^{y_i}(w) = f_i(|w|, y_i) \chi_{J_i}(|w|)$ so that the expression (17) can be written as a constant times

$$(18) \quad \int \iint |\widehat{\psi}(\xi)|^2 \widehat{H}_1^{y_1}(t_1 \xi) \overline{\widehat{H}_2^{y_2}(t_2 \xi)} \, \mathcal{J}(|\xi| | y_1 - y_2 |) \, dy_1 dy_2 \, d\xi.$$

Then, using the decay for Bessel functions and the M-separation assumption,

$$|\mathcal{J}(|\xi| | y_1 - y_2 |)| \lesssim (1 + \rho M)^{-\frac{d-1}{2}}, \quad y_i \in E_i, \ i = 1, 2.$$

By the Cauchy–Schwarz inequality, the left-hand side of the desired inequality is thus bounded by

$$\prod_{i=1,2} \Big[\int_{y_i \in \mathbb{R}^d} \Big(\int \frac{|\widehat{\psi}(\xi)|^2}{(1 + |\xi| M)^{\frac{d-1}{2}}} |\widehat{H_i^{y_i}}(t_i \xi)|^2 d\xi \Big)^{1/2} dy \Big]$$

$$\lesssim M^{-\frac{d-1}{2}} \prod_{i=1,2} \Big[\int_{y \in \mathbb{R}^d} \|\widehat{H_i^{y_i}}\|_2 dy \Big],$$

and by Plancherel's theorem this is

$$\lesssim M^{-\frac{d-1}{2}} \prod_{i=1,2} \Big[\int \Big(\int_{w_i \in \mathbb{R}^d} |f_i(|w|, y)|^2 \chi_{J_i}(|w|) dw \Big)^{1/2} dy \Big]$$

$$\lesssim M^{-\frac{d-1}{2}} \prod_{i=1,2} \Big[\int \Big(\int_{J_i} |f_i(r, y)|^2 r^{d-1} dr \Big)^{1/2} dy \Big],$$

and so we are done. □

Proof of Proposition 2.1. The case $p = 1$ is trivial and we assume $p > 1$ in what follows. For $z \in \mathbb{Z}^d$ consider the cube q_z of all x with $z_i \leq x_i < z_i + 1$ for $i = 1, \ldots, d$. Let

$$\gamma_{j,z}(f) = \sup_{x \in q_z} \int_1^2 \Big(\int_0^\infty \Big| \int \eta(x - y) F_{j,t}(r, y) \, dy \Big|^2 \frac{dr}{r} \Big)^{1/2} d\nu(t),$$

and since η is a Schwartz function it is straightforward to verify that, for every j,

$$(19) \qquad \Big(\sum_{z \in \mathbb{Z}^d} |\gamma_{j,z}(f)|^p \Big)^{1/p} \lesssim \Big\| \int_1^2 \Big(\int_0^\infty |F_{j,t}(r, \cdot)|^2 \frac{dr}{r} \Big)^{1/2} d\nu(t) \Big\|_p,$$

with the implicit constant independent of j. If $\gamma_{j,z}(f) \neq 0$ we set

$$b_{j,z,t}(r, x) = [\gamma_{j,z}(f)]^{-1} \chi_{q_z}(x) \int \eta(x - y) F_{j,t}(r, y) \, dy$$

and if $\gamma_{j,z}(f) = 0$ we set $b_{j,z,t} = 0$. Then

$$(20) \qquad \sum_{z \in \mathbb{Z}^d} \sup_{x \in q_z} \int_1^2 \Big(\int_0^\infty |b_{j,z,t}(r, x)|^2 \frac{dr}{r} \Big)^{1/2} d\nu(t) \leq 1.$$

Let

$$V_{j,z}(x) = \int_1^2 \int_{I_j} \psi * \sigma_{rt} * b_{j,z,t}(r, x) \, dr \, d\nu(t).$$

In view of (19) it suffices to show that for arbitrary functions $z \mapsto \gamma_{j,z}$ on $\mathbb{Z}^d$ we have, for $1 < p < \frac{2(d+1)}{d+3}$,

$$(21) \qquad \Big\| \sum_{j \geq 1} \sum_{z \in \mathbb{Z}^d} \gamma_{j,z} V_{j,z} \Big\|_p \lesssim \Big(\sum_{j \geq 1} \sum_{z \in \mathbb{Z}^d} |\gamma_{j,z}|^p 2^{jd} \Big)^{1/p}$$

where the implicit constant is independent of the specific choices of the $b_{j,z,t}$ (satisfying (20) with $b_{j,z,t}$ supported in q_z). Let μ_d denote the measure on $\mathbb{N} \times \mathbb{Z}^d$ given by

$$\mu_d(E) = \sum_{j \geq 1} 2^{jd} \#\{z \in \mathbb{Z}^d : (j, z) \in E\}.$$

Then (21) expresses the $L^p(\mathbb{Z}^d \times \mathbb{N}, \mu_d) \to L^p(\mathbb{R}^d)$ boundedness of an operator $\mathcal{T}$. In the open p-range it suffices by real interpolation to show that $\mathcal{T}$

maps $L^{p,1}(\mathbb{Z}^d \times \mathbb{N}, \mu_d)$ to $L^{p,\infty}(\mathbb{R}^d)$. This amounts to checking the restricted weak-type inequality

$$(22) \qquad \operatorname{meas}\big(\big\{ x : \big| \sum_{j \geq 1} \sum_{z \in \mathcal{E}_j} V_{j,z} \big| > \lambda \big\}\big) \lesssim \lambda^{-p} \sum_{j \geq 1} 2^{jd} \#(\mathcal{E}_j)$$

where $\mathcal{E}_j$ are finite subsets of $\mathbb{Z}^d$. Now for each (j, z) the term $V_{j,z}$ is supported on a ball of radius $C2^{j+1}$ and therefore the entire sum is supported on a set of measure $\lesssim \sum_{j \geq 1} 2^{jd} \#(\mathcal{E}_j)$. Thus the estimate (22) holds for $\lambda \leq 10$. Assume now that $\lambda > 10$.

We decompose $\mathbb{R}^d$ into dyadic 'half open' cubes of sidelength 2^j and let $\mathfrak{Q}_j$ be the collection of these 2^j-cubes. For each $Q \in \mathfrak{Q}_j$ let Q^* be the cube with same center as Q but sidelength 2^{j+5}. Note that for $z \in Q$ the term $V_{j,z}$ is supported in Q^*. Letting

$$\mathfrak{Q}_j(\lambda) := \{ Q \in \mathfrak{Q}_j \,:\, \#(\mathcal{E}_j \cap Q) > \lambda^p \}$$

and

$$\Omega = \bigcup_j \bigcup_{Q \in \mathfrak{Q}_j(\lambda)} Q^*,$$

we have the favorable estimate

$$\operatorname{meas}(\Omega) \leq 2^5 \sum_{j \geq 1} \sum_{Q \in \mathfrak{Q}_j(\lambda)} |Q| \leq 2^5 \sum_{j \geq 1} 2^{jd} \sum_{Q \in \mathfrak{Q}_j(\lambda)} \frac{\#(\mathcal{E}_j \cap Q)}{\lambda^p}$$
$$\lesssim \lambda^{-p} \sum_{j \geq 1} 2^{jd} \#(\mathcal{E}_j) \,.$$

Thus the remaining estimates need only involve the 'good' part of $\mathcal{E}_j$;

$$\mathcal{E}_j^\lambda = \bigcup_{Q \in \mathfrak{Q}_j \setminus \mathfrak{Q}_j(\lambda)} Q \cap \mathcal{E}_j.$$

Note that every subset of diameter $C2^j$, with $C > 1$, contains $\lesssim C^d \lambda^p$ points in $\mathcal{E}_j^\lambda$. Letting

$$V_j = \sum_{z \in \mathcal{E}_j^\lambda} V_{j,z} \,,$$

it remains to show that

$$\operatorname{meas}\big(\big\{ x : \big| \sum_{j \geq 1} V_j(x) \big| > \lambda \big\}\big) \lesssim \lambda^{-p} \sum_{j \geq 1} 2^{jd} \#(\mathcal{E}_j) \,.$$

This will follow from

$$(23) \qquad \Big\| \sum_{j \geq 1} V_j \Big\|_2^2 \leq C \lambda^{\frac{2p}{d+1}} \log \lambda \sum_{j \geq 1} 2^{jd} \#(\mathcal{E}_j)$$

and Tshebyshev's inequality since, for $p < \frac{2(d+1)}{d+3}$ and $\lambda > 1$,

$$\lambda^{\frac{2p}{d+1} - 2} \log \lambda \leq C_p \lambda^{-p}.$$

Proof of (23). Setting $N(\lambda) = 10\log_2\lambda$, we treat the sums over $j \leq N(\lambda)$ and $j > N(\lambda)$ separately. Using the Cauchy–Schwarz inequality for the first sum,

$$(24) \quad \Big\|\sum_{j\geq 1} V_j\Big\|_2^2 \lesssim \log(\lambda) \sum_{j\leq N(\lambda)} \|V_j\|_2^2 + \sum_{j>N(\lambda)} \|V_j\|_2^2$$
$$+ \sum_{j>N(\lambda)} \sum_{N(\lambda)<k<j-10} |\langle V_j, V_k\rangle|.$$

Since the expression $\sum_{z\in\mathcal{E}_j^\lambda \cap Q} V_{j,z}$ is supported in Q^* it follows easily from Lemma 2.2 (applied with the endpoint exponent $\frac{2(d+1)}{d+3}$) that

$$\|V_j\|_2^2 \lesssim \sum_{Q\in\mathfrak{Q}_j} \Big\| \int_1^2 \Big(\int_{I_j} 2^{jd}\Big| \sum_{z\in Q\cap\mathcal{E}_j^\lambda} b_{j,z,t}(r,\cdot)\Big|^2 \frac{dr}{r}\Big)^{1/2} d\nu(t)\Big\|_{\frac{2(d+1)}{d+3}}^2.$$

Since $Q\cap\mathcal{E}_j^\lambda$ contains no more than λ^p points we have by (20)

$$\Big\| \int_1^2 \Big(\int_{I_j} 2^{jd}\Big| \sum_{z\in Q\cap\mathcal{E}_j^\lambda} b_{j,z,t}(r,\cdot)\Big|^2 \frac{dr}{r}\Big)^{1/2} d\nu(t)\Big\|_{\frac{2(d+1)}{d+3}}^2$$
$$\lesssim 2^{jd}\big(\#(\mathcal{E}_j^\lambda \cap Q)\big)^{\frac{d+3}{d+1}} \lesssim 2^{jd}\#(\mathcal{E}_j\cap Q)\lambda^{p\frac{2}{d+1}}$$

and thus

$$(25) \quad \sum_{j=1}^\infty \|V_j\|_2^2 \lesssim \lambda^{p\frac{2}{d+1}} \sum_j 2^{jd}\#\mathcal{E}_j.$$

Thus we get the asserted bound (23) for the sum of the first two terms on the right-hand side of (24).

It remains to estimate the mixed terms $\langle V_j, V_k\rangle$ for $N(\lambda) < k < j - 10$. For fixed j,k we let $I_{j,k}^n = [2^k n, 2^k(n+1)] \cap I_j$ with $n \in \mathbb{Z}$, $n \approx 2^{j-k}$. Then with

$$V_{j,z,t}^{k,n} := \psi * \int_{I_{j,k}^n} \sigma_{rt} * b_{j,z,t}(r,\cdot)\, dr$$
$$V_{k,z'} := \psi * \int_1^2 \int_{I_k} \sigma_{rs} * b_{k,z',s}(r,\cdot)\, dr\, d\nu(s)$$

we can write

$$\langle V_j, V_k\rangle = \int_1^2 \sum_n \sum_{z\in\mathcal{E}_j^\lambda} \sum_{z'\in\mathfrak{Z}_k(n,z,t)} \langle V_{j,z,t}^{k,n}, V_{k,z'}\rangle\, d\nu(t);$$

here, in view of the support properties, we were able to restrict the z' summation to the set

$$\mathfrak{Z}_k(n,z,t) := \{z' \in \mathcal{E}_k^\lambda : \big||z'-z| - nt2^k\big| \leq C2^k\},$$

with C a suitable constant. Observe that for $z' \in \mathfrak{Z}_k(n, z, t)$, with $k \leq j - 10$, we have $|z - z'| \approx 2^j$ since $nt2^k \in I_j$.

By Lemma 2.3 (applied with the parameter $M \approx 2^j$) we have for fixed z, z', t,

$$(26) \quad |\langle V_{j,z,t}^{k,n}, V_{k,z'}\rangle|| \lesssim 2^{-j\frac{d-1}{2}} \int_{|y-z|\leq C} h_{j,k,t}^{z,n}(y)\, dy \int_{|y'-z'|\leq C} h_k^{z'}(y')\, dy'$$

with

$$h_{j,k,t}^{z,n}(y) = \Big(\int_{I_{j,k}^n} |b_{j,z,t}(r,y)|^2 r^{d-1} dr \Big)^{1/2},$$

$$h_k^{z'}(y) = \int_1^2 \Big(\int_{I_k} |b_{k,z',s}(r,y)|^2 r^{d-1} dr \Big)^{1/2} d\nu(s).$$

By our normalization assumption (20),

$$(27) \quad \int_1^2 \Big(\sum_n |h_{j,k,t}^{z,n}(y)|^2 \Big)^{1/2} d\nu(t) \lesssim 2^{\frac{jd}{2}} \qquad \text{and} \qquad h_k^{z'}(y') \lesssim 2^{\frac{kd}{2}}$$

and, by the Cauchy–Schwarz inequality, we also have

$$(28) \quad \int_1^2 \sum_n |h_{j,k,t}^{z,n}(y)| d\nu(t) \lesssim 2^{\frac{jd}{2}} 2^{\frac{j-k}{2}}.$$

Altogether, using (26) and (27),

$$|\langle V_j, V_k\rangle|$$
$$\lesssim 2^{-j\frac{d-1}{2}} \sum_{z \in \mathcal{E}_j^\lambda} \sum_n \int_1^2 \int_{|y-z|\leq C} h_{j,k,t}^{z,n}(y)\, dy\, 2^{kd/2} \#(\mathfrak{Z}_k(n, z, t))\, d\nu(t).$$

Recall that for every cube Q of sidelength 2^k the set $\mathfrak{Z}_k(n, z, t) \cap Q$ contains at most λ^p points. Moreover, for each z, n, t there are no more than $O(2^{(j-k)(d-1)})$ dyadic cubes of sidelength 2^k which intersect $\mathfrak{Z}_k(z, n, t)$. Thus

$$\#(\mathfrak{Z}_k(n, z, t)) \lesssim \lambda^p 2^{(j-k)(d-1)}.$$

This and (28) yield, for $k \leq j - 10$,

$$|\langle V_j, V_k\rangle| \lesssim 2^{-j\frac{d-1}{2}} \sum_{z \in \mathcal{E}_j^\lambda} \int_{|y-z|\leq C} \int_1^2 \sum_n h_{j,k,t}^{z,n}(y) d\nu(t)\, dy\, 2^{kd/2} \lambda^p 2^{(j-k)(d-1)}$$

$$\lesssim 2^{-j\frac{d-1}{2}} \#(\mathcal{E}_j^\lambda) 2^{\frac{jd}{2}} 2^{\frac{j-k}{2}} 2^{\frac{kd}{2}} \lambda^p 2^{(j-k)(d-1)} \lesssim \lambda^{-p} 2^{-k\frac{d-1}{2}} 2^{jd} \#(\mathcal{E}_j^\lambda).$$

By summing a geometric series, we see that

$$\sum_{j > N(\lambda)} \sum_{N(\lambda) < k < j-10} |\langle V_j, V_k\rangle| \lesssim \lambda^p 2^{-N(\lambda)\frac{d-1}{2}} \sum_{j \geq 1} 2^{jd} \#\mathcal{E}_j,$$

and by the choice of $N(\lambda) = 10 \log_2 \lambda$, we have $\lambda^p 2^{-N(\lambda)\frac{d-1}{2}} \lesssim \lambda^{p-5} \lesssim 1$. This gives the desired estimate (indeed a better estimate) for the third term on the right-hand side of (24) and finishes the proof of (23). $\qquad\square$

Lorentz space estimates. We will use the following interpolation lemma in which we allow any $d > 0$; this is the only place where d does not necessarily denote the dimension.

Lemma 2.4. *Let $1 \le p_0 < p_1$, $d > 0$, and, for $j \in \mathbb{N}$, let S_j be an operator acting on functions on a measure space $(\mathcal{M}, \mu)$ with values in a Banach space $\mathcal{B}$. Suppose that the inequality*

$$(29) \qquad \Big\| \sum_{j \ge 1} S_j g_j \Big\|_{p_i} \le M_i \Big(\sum_{j \ge 1} 2^{jd} \|g_j\|_{L^{p_i}(\mathcal{B})}^{p_i} \Big)^{1/p_i}$$

holds for $i = 0, 1$. Then for $p_0 < p < p_1$, $\frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}$, and $p \le q \le \infty$,

$$\Big\| \sum_{j \ge 1} 2^{-jd/p} S_j f_j \Big\|_{L^{p,q}} \le C_{p,q} M_0^{1-\theta} M_1^\theta \Big\| \Big(\sum_{j \ge 1} |f_j|_{\mathcal{B}}^q \Big)^{1/q} \Big\|_{L^p}$$

with $q = \infty$ interpreted as usual by taking a supremum.

Proof. Let μ^d denote the measure on $\mathbb{N} \times \mathcal{M}$ given by

$$\mu^d(E) = \sum_{j \ge 1} 2^{jd} \int_{x:(j,x)\in E} d\mu \,.$$

By real interpolation of the assumptions (29) we have

$$(30) \qquad \Big\| \sum_{j \ge 1} S_j g_j \Big\|_{L^{p,q}} \le C_{p,q} M_0^{1-\theta} M_1^\theta \big\| \{g_j\} \big\|_{L^{p,q}(\mu^d,\mathcal{B})} \,.$$

We may apply this with $g_j = 2^{-jd/p} f_j$ and then our assertion follows from the inequality

$$(31) \qquad \big\| \{2^{-jd/p} f_j\} \big\|_{L^{p,q}(\mu^d,\mathcal{B})} \le \big\| \{f_j\} \big\|_{L^p(\ell^q(\mathcal{B}))} \,.$$

The case for $p = q$ is immediate. We also have

$$\mu^d\big(\big\{(j,x) : 2^{-\frac{jd}{p}} |f_j(x)|_{\mathcal{B}} > \lambda\big\}\big) \le \mu^d\big(\big\{(j,x) : 2^{-\frac{jd}{p}} \sup_k |f_k(x)|_{\mathcal{B}} > \lambda\big\}\big)$$

$$= \int \sum_{\substack{j : 2^{jd} < \\ \sup_k |f_k(x)|_{\mathcal{B}}^p \lambda^{-p}}} 2^{jd} \, dx \le \lambda^{-p} \int \sup_k |f_k(x)|_{\mathcal{B}}^p \, dx,$$

which yields (31) for $q = \infty$. By complex interpolation (with fixed p) we obtain (31) for $p \le q \le \infty$. $\qquad\square$

As an immediate consequence of Lemma 2.4 we obtain a Lorentz space version of Proposition 2.1 which is the main ingredient in the proof of Theorem 1.2.

Corollary 2.5. *Let* $1 < p < \frac{2(d+1)}{d+3}$ *and* $p \leq q \leq \infty$. *Then*

$$\Big\| \sum_{j \geq 1} 2^{-\frac{jd}{p}} \int_1^2 \int_{I_j} \psi * \eta * \sigma_{rt} * F_{j,t}(r,\cdot) \, dr \, d\nu(t) \Big\|_{L^{p,q}}$$

$$\lesssim \Big\| \int_1^2 \Big(\sum_{j \geq 1} |F_{j,t}|_{\mathcal{H}}^q \Big)^{1/q} d\nu(t) \Big\|_p .$$

A preparatory result. For the proof of Theorems 1.1 and 1.4 we shall need a more technical variant of the corollary which is compatible with atomic decompositions. In what follows we let ν be Dirac measure at $t = 1$ so that the integrals in t disappear. Let $\ell \geq 1$ and for $\mathfrak{z} \in \mathbb{Z}^d$ let

$$R_{\mathfrak{z}}^\ell = \{x : 2^\ell \mathfrak{z}_i \leq x_i < 2^\ell(\mathfrak{z}_i + 1), \ i = 1, \dots, d\};$$

these sets form a grid of disjoint cubes with sidelength 2^ℓ covering $\mathbb{R}^d$. In the following proposition we use the conclusion of Proposition 2.1 as our hypothesis.

Proposition 2.6. *Suppose that, for some* $p_1 \in (1, 2)$,

$$\Big\| \sum_{j \geq \ell+2} \int_{I_j} \psi * \sigma_r * \eta * F_j(r, \cdot) \, dr \Big\|_{p_1} \lesssim \Big(\sum_{j \geq 1} 2^{jd} \|F_j\|_{L^{p_1}(\mathcal{H})}^{p_1} \Big)^{1/p_1}.$$

Let $b_{j,\mathfrak{z}} \in L^2(\mathcal{H})$ *with* $\|b_{j,\mathfrak{z}}\|_{L^2(\mathcal{H})} \leq 1$, *let* $\beta_j(\mathfrak{z}) \in \mathbb{C}$ *and define*

$$S_j \beta_j(x) = \sum_{\mathfrak{z}} \beta_j(\mathfrak{z}) \Big(\psi * \eta * \int_{I_j} \sigma_r * \big(\chi_{R_{\mathfrak{z}}^\ell} b_{j,\mathfrak{z}}(r, \cdot) \big) \, dr \Big)$$

Then, for $1 < p < p_1$ *and* $p \leq q \leq \infty$,

$$\Big\| \sum_{j \geq \ell+2} 2^{-jd/p} S_j \beta_j \Big\|_{L^{p,q}} \leq C_p 2^{\ell d(1/p - 1/2) - \varepsilon(p))} \Big(\sum_{\mathfrak{z} \in \mathbb{Z}^d} \Big(\sum_{j \geq 1} |\beta_j(\mathfrak{z})|^q \Big)^{p/q} \Big)^{1/p},$$

where $\varepsilon(p) = \frac{(d-1)p_1'}{2}(\frac{1}{p} - \frac{1}{p_1})$.

Proof. We argue as in [20], Prop. 3.1. First note that

$$(32) \qquad \Big\| \sum_{j \geq \ell+2} S_j \beta_j \Big\|_{p_1} \lesssim 2^{\ell d(1/p_1 - 1/2)} \Big(\sum_{j \geq 1} 2^{jd} \sum_{\mathfrak{z}} |\beta_j(\mathfrak{z})|^{p_1} \Big)^{1/p_1}.$$

Indeed, by hypothesis the left-hand side is dominated by a constant times

$$\Big(\sum_{j \geq 1} 2^{jd} \Big\| \sum_{\mathfrak{z}} \beta_j(\mathfrak{z}) \chi_{R_{\mathfrak{z}}^\ell} b_{j,\mathfrak{z}} \Big\|_{L^{p_1}(\mathcal{H})}^{p_1} \Big)^{1/p_1}$$

$$\leq \Big(\sum_{j \geq 1} 2^{jd} \sum_{\mathfrak{z}} |\beta_j(\mathfrak{z})|^{p_1} \big\| \chi_{R_{\mathfrak{z}}^\ell} b_{j,\mathfrak{z}} \big\|_{L^{p_1}(\mathcal{H})}^{p_1} \Big)^{1/p_1}$$

and after using Hölder's inequality on each $R_{\mathfrak{z}}^\ell$ and the L^2 normalization of $b_{j,\mathfrak{z}}$ we obtain (32).

There is a better L^1 bound. Note that for $r \approx 2^j$ the term $\psi * \sigma_r * b_{j,\mathfrak{z}}(r, \cdot)$ is supported on an annulus with radius $\approx 2^j$ and width 2^ℓ. We use the Cauchy–Schwarz inequality on this annulus and then (15) and estimate

$$\Big\| \sum_{j \geq \ell+2} \psi * \eta * \sum_{\mathfrak{z}} \beta_j(\mathfrak{z}) \int_{I_j} \sigma_r * (b_{j,\mathfrak{z}}(r, \cdot) \chi_{R_{\mathfrak{z}}^\ell}) \, dr \Big\|_1$$

$$\lesssim \sum_{j \geq \ell+2} \sum_{\mathfrak{z}} |\beta_j(\mathfrak{z})| \int_{I_j} \|\psi * \sigma_r * (b_{j,\mathfrak{z}}(r, \cdot) \chi_{R_{\mathfrak{z}}^\ell})\|_1 \, dr$$

$$\lesssim \sum_{j \geq \ell+2} \sum_{\mathfrak{z}} |\beta_j(\mathfrak{z})| \int_{I_j} (2^\ell 2^{j(d-1)})^{1/2} \|\psi * \sigma_r * (b_{j,\mathfrak{z}}(r, \cdot) \chi_{R_{\mathfrak{z}}^\ell})\|_2 \, dr$$

$$\lesssim 2^{\ell/2} \sum_{j \geq 1} 2^{j(d-1)} \sum_{\mathfrak{z}} |\beta_j(\mathfrak{z})| \int_{I_j} \|b_{j,\mathfrak{z}}(r, \cdot)\|_2 \, dr ,$$

and by Cauchy–Schwarz on I_j and the normalization assumption on $b_{j,\mathfrak{z}}$ we get

(33) $$\Big\| \sum_{j \geq \ell+2} S_j \beta_j \Big\|_1 \lesssim 2^{\ell/2} \sum_{j \geq 1} 2^{jd} \sum_{\mathfrak{z}} |\beta_j(\mathfrak{z})| .$$

Now Lemma 2.4 is used to interpolate (32) and (33) and the assertion follows. $\square$

3. Proof of Theorem 1.2

We start with a simple fact on Besov spaces, namely if ζ is a C^∞ function supported on a compact subinterval of $(0, \infty)$ then

(34) $$\|\zeta(|\cdot|)g(|\cdot|)\|_{B_{\alpha,q}^2(\mathbb{R}^d)} \lesssim \|g\|_{B_{\alpha,q}^2(\mathbb{R})} , \quad \alpha > 0.$$

To see this note that the corresponding inequality with Sobolev spaces L_α^2, $\alpha = 0, 1, 2, \ldots$ is true by direct computation, and then (34) follows by real interpolation.

Next if $\mathcal{F}^{-1}[m(|\cdot|)](x) = \kappa(|x|)$ we can use polar coordinates to see that

(35) $$\|m(|\cdot|)\|_{B_{\alpha,q}^2(\mathbb{R}^d)} \approx \Big(\sum_{j=0}^{\infty} \Big[\int_{I_j} |\kappa(r)|^2 r^{2\alpha+d-1} dr \Big]^{q/2} \Big)^{1/q} ;$$

here, as in §2, $I_j = [2^j, 2^{j+1}]$ for $j \geq 1$, and $I_0 = (0, 2]$.

We shall first prove a dual version of a bound for a maximal operator where the dilations are restricted to $[1, 2]$.

Proposition 3.1. *Let $d \geq 2$, $1 < p < \frac{2(d+1)}{d+3}$, $\alpha = d(\frac{1}{p} - \frac{1}{2})$, $p \leq q \leq \infty$. Then, for $m \in B_{\alpha,q}^2$ with support in $(1/2, 2)$,*

(36) $$\Big\| \int_1^2 T_{m(t\cdot)} f_t \, dt \Big\|_{L^{p,q}} \lesssim \|m\|_{B_{\alpha,q}^2} \Big\| \int_1^2 |f_t| \, dt \Big\|_p.$$

Proof. Let ϕ be a radial C^∞-function so that $\widehat{\phi}$ is supported in $\{1/8 \leq |\xi| \leq 8\}$ and equal to one in $\{1/4 \leq |\xi| \leq 4\}$. Then $T_{m(t\cdot)}f_t = T_{m(t\cdot)}(\phi * f_t)$ for $1 \leq t \leq 2$. Also

$$T_{m(t\cdot)}f = \int_0^\infty \kappa(r) t^{1-d} \sigma_{rt} * \phi * f \, dr, \quad 1 \leq t \leq 2,$$

where κ is bounded and smooth, and the right-hand side of (35) is finite with $\alpha = d/p - d/2$. We may split $\phi = \psi * \eta$ where $\psi \in C_c^\infty$ with $\widehat{\psi}$ vanishing of high order at the origin. It then suffices to show that

$$(37) \quad \Big\| \int_1^2 \int_2^\infty \kappa(r) t^{1-d} \psi * \sigma_{rt} * f_t \, dr \, dt \Big\|_{L^{p,q}}$$

$$\lesssim \Big(\sum_{j=1}^\infty \Big(\int_{I_j} |\kappa(r)|^2 r^{2d/p} \frac{dr}{r} \Big)^{q/2} \Big)^{1/q} \Big\| \int_1^2 |f_t| \, dt \Big\|_p$$

This estimate follows by applying Corollary 2.5. Take ν to be Lebesgue measure on $[1,2]$, use the tensor product

$$F_{j,t}(r,x) = 2^{jd/p} \chi_{I_j}(r) \kappa(r) \, t^{1-d} f_t(x)$$

and observe that $\|F_j\|_{L^p(L^1(\mathcal{H}))}$ can be estimated by the right-hand side of (37). $\qquad\square$

We also need a standard 'orthogonality' estimate, in Lorentz spaces.

Lemma 3.2. *Let $\{\beta_k\}_{k\in\mathbb{Z}}$ a family of L^1-functions, satisfying*
   (i) $\sup_k \|\beta_k\|_{L^1(\mathbb{R}^d)} < \infty$,
   (ii) $\sup_\xi \sum_{k\in\mathbb{Z}} |\widehat{\beta}_k(\xi)| < \infty$.
Then

$$(38) \quad \Big\| \sum_k \beta_k * f_k \Big\|_{L^{p,q}} \lesssim \Big(\sum_k \|f_k\|_{L^{p,q}}^p \Big)^{1/p}, \quad 1 < p < 2, \ p \leq q \leq \infty,$$

and

$$(39) \quad \Big(\sum_k \|\beta_k * f\|_{L^{p,q}}^p \Big)^{1/p} \lesssim \|f\|_{L^{p,q}}, \quad 2 < p < \infty, \ 1 \leq q \leq p.$$

Here the functions $\{f_k\}$ are allowed to have values in a Hilbert space $\mathcal{H}$ (and f may have values in $\mathcal{H}'$).

Proof. By duality (38) and (39) are equivalent. To see (38) we define $\mathfrak{m}_d$ to be the product measure on $\mathbb{R}^d \times \mathbb{Z}$ of Lebesgue measure on $\mathbb{R}^d$ and counting measure on $\mathbb{Z}$. Define an operator P acting on functions $(x,k) \mapsto f_k(x)$, letting $F = \{f_k\}$, by $PF = \sum_k \beta_k * f_k$. By assumption (i) P maps the space $L^1(\mathbb{R}^d \times \mathbb{Z}, \mathfrak{m}_d)$ to $L^1(\mathbb{R}^d)$ and by the almost orthogonality assumption (ii) it maps $L^2(\mathbb{R}^d \times \mathbb{Z}, \mathfrak{m}_d)$ to $L^2(\mathbb{R}^d)$. Hence by real interpolation P maps $L^{p,q}(\mathbb{R}^d \times \mathbb{Z}, \mathfrak{m}_d)$ to $L^{p,q}(\mathbb{R}^d)$ for all $1 < p < 2$ and $q > 0$. Let

$$E_{k,m}(F) = \{x : |f_k(x)|_{\mathcal{H}} > 2^m\}.$$

If $p \le q$ we have, by the triangle inequality in $\ell^{q/p}$,

$$\|F\|_{L^{p,q}(\mathfrak{m}_d;\mathcal{H})} \approx \Big(\sum_m 2^{mq}\Big|\sum_k \mathrm{meas}(E_{k,m}(F))\Big|^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

$$\le \Big(\sum_k \Big(\sum_m 2^{mq}\Big|\mathrm{meas}(E_{k,m}(F))\Big|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} \approx \big(\sum_k \|f_k\|^p_{L^{p,q}(\mathcal{H})}\big)^{1/p},$$

where for $q = \infty$ we make the usual modification. This proves (38). $\qquad\square$

Proof of Theorem 1.2, conclusion. Now let $\frac{2(d+1)}{d-1} < p < \infty$ and $p \le q \le \infty$. Let ϕ be as above and define L_k by $\widehat{L_k f}(\xi) = \widehat{\phi}(2^{-k}\xi)\widehat{f}(\xi)$. We may then estimate

$$\|M_m f\|_p \le \Big(\sum_{k \in \mathbb{Z}} \big\| \sup_{1 \le t \le 2} |T_{m(2^k t\cdot)} L_k f| \big\|_p^p\Big)^{1/p}.$$

For every $k \in \mathbb{Z}$,

$$\big\| \sup_{1 \le t \le 2} |T_{m(2^k t\cdot)} L_k f| \big\|_p \le C\|m\|_{B^2_{d/2 - d/p, q}} \|L_k f\|_{L^{p,q'}};$$

this follows for $k = 0$ by duality from Proposition 3.1, and then for general k by scaling. By Lemma 3.2

$$\Big(\sum_{k \in \mathbb{Z}} \|L_k f\|^p_{L^{p,q'}}\Big)^{1/p} \lesssim \|f\|_{L^{p,q'}}$$

and combining the estimates we are done. $\qquad\square$

4. Proofs of Theorems 1.1 and 1.4

Many endpoint bounds for convolution operators on Lebesgue spaces can be obtained by interpolation involving a Hardy space estimate and an L^2 estimate; this idea goes back to [40], [17]. In some instances it has been advantageous to use Hardy space or BMO methods such as atomic decompositions or the Fefferman–Stein #-maximal function directly on L^p to prove theorems which cannot immediately be obtained by interpolation (see for example endpoint questions treated in [32], [46], [25], [19], [29]). We formulate such a result suitable for application in the proofs of Theorems 1.1 and 1.4. In order to give a unified treatment we need to consider vector-valued operators.

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. We consider translation invariant operators mapping $L^2(\mathcal{H}_1)$ to $L^2(\mathcal{H}_2)$, with convolution kernels having values in the space $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. On the Fourier transform side, the operators are given by $\widehat{Tf}(\xi) = M(\xi)\widehat{f}(\xi)$ where $\widehat{f}(\xi) \in \mathcal{H}_1$, $\widehat{Tf}(\xi) \in \mathcal{H}_2$, with $\sup_\xi |M(\xi)|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} < \infty$. If S is an $L^2(\mathbb{R}^d)$ convolution operator with scalar kernel (and multiplier) and $\mathcal{H}$ is a Hilbert space then S extends to a bounded operator on $L^2(\mathbb{R}^d, \mathcal{H})$, denoted temporarily by $S \otimes Id_{\mathcal{H}}$. If T is as before with $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$-valued kernel then

$(S \otimes Id_{\mathcal{H}_2})T = T(S \otimes Id_{\mathcal{H}_1})$. With a slight abuse of notation we shall continue to write S for either $S \otimes Id_{\mathcal{H}_2}$ and $S \otimes Id_{\mathcal{H}_1}$.

We need to formulate a hypothesis which will be used for convolution operators with multipliers compactly supported away from the origin.

Hypothesis 4.1. Let $1 < p < 2$, $p \le q \le \infty$, $\varepsilon > 0$ and $A > 0$. We say that the kernel $\mathcal{K}$ satisfies $\mathrm{Hyp}(p, q, \varepsilon, A)$ if for every $\ell \ge 0$ one can split the kernel into a short and long range contribution

$$\mathcal{K} = \mathcal{K}_\ell^{\mathrm{sh}} + \mathcal{K}_\ell^{\mathrm{lg}}$$

so that the following properties hold:

(i) $\mathcal{K}_\ell^{\mathrm{sh}}$ is supported in $\{x : |x| \le 2^{\ell+10}\}$.

(ii) $\sup_{\xi \in \mathbb{R}^d} |\mathcal{F}[\mathcal{K}_\ell^{\mathrm{sh}}](\xi)|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \le A$.

(iii) For every family of L^2 functions $\{a_{\mathfrak{z}}\}_{\mathfrak{z} \in \mathbb{Z}^d}$, with $\mathrm{supp}\,(a_{\mathfrak{z}}) \in R_{\mathfrak{z}}^\ell$ and $\sup_{\mathfrak{z}} \|a_{\mathfrak{z}}\|_{L^2(\mathcal{H}_1)} \le 1$, and for $\gamma \in \ell^p(\mathbb{Z}^d)$ the inequality

$$\Big\| \sum_{\mathfrak{z}} \mathcal{K}_\ell^{\mathrm{lg}} * (\gamma(\mathfrak{z})a_{\mathfrak{z}}) \Big\|_{L^{p,q}} \le A 2^{\ell(d(\frac{1}{p} - \frac{1}{2}) - \varepsilon)} \Big(\sum_{\mathfrak{z}} |\gamma(\mathfrak{z})|^p \Big)^{1/p}$$

holds.

Theorem 4.2. *Given $p \in (1, 2)$, $p \le q \le \infty$, $\varepsilon > 0$ and $A > 0$ suppose that $\mathcal{K}^k$, $k \in \mathbb{Z}$ are $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$-valued kernels satisfying hypothesis $\mathrm{Hyp}(p, q, \varepsilon, A)$. Define the convolution operator T_k by*

$$T_k f(x) = \int 2^{kd} \mathcal{K}^k(2^k(x - y)) f(y) dy$$

Let η be a scalar Schwartz function with $\widehat{\eta}$ supported in $\{\xi : 1/4 \le |\xi| \le 4\}$ and let $\eta_k = 2^{kd}\eta(2^k \cdot)$. Then the operator $f \mapsto \sum_{k \in \mathbb{Z}} \eta_k * T_k f$, initially defined on $\mathcal{H}_1$ valued Schwartz functions with compact Fourier support away from the origin, extends to an operator acting on all $f \in L^p(\mathcal{H}_1)$ so that the inequality

$$\Big\| \sum_k \eta_k * T_k f \Big\|_{L^{p,q}(\mathcal{H}_2)} \le C_p A \|f\|_{L^p(\mathcal{H}_1)}$$

holds.

The proof of Theorem 4.2 is by now quite standard, but for completeness we include it in Appendix A below. Given Theorem 4.2 we now show how it can be used to deduce Theorems 1.1 and 1.4 from the results in §2.

Remark 4.3. We actually prove a slightly more general result: Assuming that the estimate of Proposition 2.1 holds for some exponent $p_1 \in (1, \frac{2d}{d+1})$ then the conclusion of Theorem 1.4 holds for $1 < p < p_1$ and the conclusion of Theorem 1.1 holds for $p_1' < p < \infty$. A similar remark also applies to Theorem 1.2.

Proof of Theorem 1.1. With p_1 as in Remark 4.3, by duality and changes of variables $t = 2^k s$ it is enough to show that, for $1 < p < p_1$ and $\alpha = d(\frac{1}{p} - \frac{1}{2})$,

(40)
$$\Big\| \sum_{k \in \mathbb{Z}} \int_1^2 \mathcal{F}^{-1} \Big[\frac{|\xi|^2}{2^{2k} s^2} \Big(1 - \frac{|\xi|^2}{2^{2k} s^2} \Big)_+^{\alpha - 1} \widehat{f_s} \Big] \frac{ds}{s} \Big\|_{L^{p,2}} \lesssim \Big\| \Big(\int_1^2 |f_s|^2 \frac{ds}{s} \Big)^{1/2} \Big\|_p .$$

Let ϕ be such that $\widehat{\phi}$ is supported in $\{1/4 \le |\xi| \le 4\}$ with $\widehat{\phi}(\xi) = 1$ in $\{1/3 \le |\xi| \le 3\}$. Let

(41)
$$\mathcal{J}_\alpha(\rho) = \rho^{-\frac{d-2}{2} - \alpha} J_{\frac{d-2}{2} + \alpha}(\rho)$$

so that $\mathcal{F}[\mathcal{J}_\alpha(t | \cdot |)](\xi) = c_\alpha t^{-d}(1 - |\xi|^2/t^2)_+^{\alpha - 1}$ (see Chapter VII of [46]). In particular $\mathcal{J}_0 = \mathcal{J}$ as in (14). Let $\phi_k = 2^{kd}\phi(2^k \cdot)$. Then (40) follows from

(42) $\quad \Big\| \sum_{k \in \mathbb{Z}} \phi_k * \int_1^2 \int \mathcal{J}_\alpha(s|y|) f_s(\cdot - y) dy \frac{ds}{s} \Big\|_{L^{p,2}} \lesssim \Big\| \Big(\int_1^2 |f_s|^2 \frac{ds}{s} \Big)^{1/2} \Big\|_p .$

The reduction of (40) to (42) involves incorporating irrelevant powers of s in the definition of f_s and an application of standard estimates for vector-valued singular integrals ([39]) to handle the contribution of $(1 - |\xi|^2)_+^{\alpha - 1}$ away from the unit sphere. We omit the details.

We now split $\phi = \eta * \psi * \psi$ where $\widehat{\eta}$ has the same support as $\widehat{\phi}$ and ψ is a radial C_0^∞ function supported in $\{x : |x| \le 1/10\}$, furthermore $\widehat{\psi}$ vanishes to order $10d$ at the origin. If $\mathcal{H}_1 = L^2([1,2], dr)$ then we wish to apply Theorem 4.2 with the $\mathcal{H}_1'$ valued kernel $\mathcal{K}^k \equiv \mathcal{K}$ (independent of k) defined by

(43)
$$\langle \mathcal{K}(x), v \rangle = \int_1^2 v(s) \int_0^\infty \mathcal{J}_\alpha(sr) \psi * \sigma_r(x) dr \, ds .$$

We define the corresponding short range kernel $\mathcal{K}_\ell^{\mathrm{sh}}$ by letting the r-integral in (43) extend over $[0, 2^{\ell+2}]$ and the long range kernel $\mathcal{K}_\ell^{\mathrm{lg}}$ by letting the r-integral extend over $(2^{\ell+2}, \infty)$.

Clearly the support condition (i) in Hypothesis 4.1 holds. Note that $d/p - d/2 > 1/2$ for $p < 2d/(d+1)$. Thus to check condition (ii) of Hypothesis 4.1 it suffices to verify that

$$\sup_{\xi \in \mathbb{R}^d} \Big(\int_1^2 \Big| \int_0^{2^{\ell+2}} \mathcal{J}_\alpha(rs) \widehat{\psi}(\xi)^2 \widehat{\sigma_r}(\xi) dr \Big|^2 ds \Big)^{1/2} \le A_\alpha , \qquad \alpha > 1/2 .$$

Writing $\widehat{\psi}(\xi) = u(|\xi|)$, this reduces to

(44) $\quad \sup_{\rho > 0} |u(\rho)|^2 \Big(\int_1^2 \Big| \int_0^{2^{\ell+2}} \mathcal{J}_\alpha(rs) \mathcal{J}(r\rho) r^{d-1} dr \Big|^2 ds \Big)^{1/2} \lesssim A_\alpha .$

We may take the r-integral over $[1, 2^{\ell+2}]$ since the estimate for the contribution for $r \in [0, 1]$ is immediate. We use the standard asymptotic expansions

for the modified Bessel-function $\mathcal{J}_\alpha$,

$$(45) \quad \mathcal{J}_\alpha(u) = u^{-\frac{d-1}{2}-\alpha} \Big[\sum_{n=0}^{1} u^{-n} \big(c_{n,\alpha}^+ e^{iu} + c_{n,\alpha}^- e^{-iu} \big) + O(|u|^{-2}) \Big], \quad u \geq 1$$

and also the analogous expansion for $\mathcal{J} = \mathcal{J}_0$. If we consider only the leading terms in both asymptotic expansions we are led to bound

$$\sup_{\rho > 0} \frac{|u(\rho)|^2}{\rho^{\frac{d-1}{2}}} \Big(\int_1^2 \Big| \int_1^{2^{\ell+2}} e^{ir(\pm s \pm \rho)} r^{-\alpha} dr \Big|^2 ds \Big)^{1/2} \lesssim A_\alpha, \quad \alpha > 1/2,$$

which follows from Plancherel's theorem on $\mathbb{R}$. The other terms with lower order or nonoscillatory error terms are similar or more straightforward. Note that we also use $|u(\rho)| \leq \rho^{10d}$ for $\rho \in (0,1)$. This establishes condition (ii) in Hypothesis 4.1.

Finally we verify condition (iii). Let $\{a_{\mathfrak{z}}\}_{\mathfrak{z} \in \mathbb{Z}^d}$ be $L^2(\mathcal{H}_1)$ functions with $\sup_{\mathfrak{z}} \|a_{\mathfrak{z}}\|_{L^2(\mathcal{H}_1)} \leq 1$, supported on 2^ℓ-cubes with disjoint interiors. We then need to show that

$$(46) \quad \Big\| \sum_{j \geq \ell+2} \int_{I_j} \psi * \psi * \sigma_r * \sum_{\mathfrak{z}} \gamma(\mathfrak{z}) \int_1^2 \mathcal{J}_\alpha(sr) a_{\mathfrak{z}}(s, \cdot) ds \Big\|_{L^{p,2}}$$

$$\lesssim A 2^{\ell(d(\frac{1}{p} - \frac{1}{2}) - \varepsilon)} \Big(\sum_{\mathfrak{z}} |\gamma(\mathfrak{z})|^p \Big)^{1/p}.$$

Setting

$$c_{j,\mathfrak{z}} = \Big(\int_{I_j} \int_{R_{\mathfrak{z}}} \Big| \int_1^2 \mathcal{J}_\alpha(sr) a_{\mathfrak{z}}(s, x) ds \Big|^2 \frac{dr}{r} dx \Big)^{1/2}$$

we may apply Proposition 2.6 for $q = 2$ with

$$\beta_j(\mathfrak{z}) = 2^{jd/p} \gamma(\mathfrak{z}) c_{j,\mathfrak{z}}, \quad \text{and} \quad b_{j,\mathfrak{z}}(r,x) = \chi_{I_j}(r) c_{j,\mathfrak{z}}^{-1} \int_1^2 \mathcal{J}_\alpha(sr) a_{\mathfrak{z}}(s,x) ds$$

if $c_{j,\mathfrak{z}} \neq 0$ and $b_{j,\mathfrak{z}} = 0$ if $c_{j,\mathfrak{z}} = 0$. We can then dominate the left-hand side of (46) by a constant times

$$2^{\ell(\frac{d}{p} - \frac{d}{2} - \varepsilon(p))} \Big(\sum_{\mathfrak{z}} \Big(\sum_j |\beta_j(\mathfrak{z})|^2 \Big)^{p/2} \Big)^{1/p}$$

with $\varepsilon(p) > 0$ for $p < p_1$. We are only left to show that for fixed $\mathfrak{z}$

$$\Big(\sum_j |\beta_j(\mathfrak{z})|^2 \Big)^{1/2} \lesssim |\gamma(\mathfrak{z})|$$

where the implicit constant is uniform in $\mathfrak{z}$. This estimate follows from

$$(47) \quad \sum_{j \geq \ell+2} 2^{2jd/p} \int_{I_j} \Big| \int_1^2 \mathcal{J}_\alpha(sr) a_{\mathfrak{z}}(s,x) ds \Big|^2 \frac{dr}{r} \lesssim \int_1^2 |a_{\mathfrak{z}}(s,x)|^2 ds$$

and integration over $x \in R_{\mathfrak{z}}$. To see (47) we use again the asymptotics (45). The estimate for the oscillatory terms (with $n = 0, 1$) becomes

$$\sum_{j \geq \ell+2} 2^{2jd/p} \int_{I_j} r^{-2\alpha-2n-d} \Big| \int_1^2 e^{\pm isr} s^{-\frac{d-1}{2}-\alpha-n} a_{\mathfrak{z}}(s,x) ds \Big|^2 dr$$

$$\lesssim \int_1^2 |a_{\mathfrak{z}}(s,x)|^2 ds ,$$

and since $\alpha = d/p - d/2$ it suffices to show

$$\int \Big| \int_1^2 e^{\pm isr} v(s) a_{\mathfrak{z}}(s,x) dt \Big|^2 dr \lesssim \int_1^2 |a_{\mathfrak{z}}(s,x)|^2 ds$$

with $\sup_s |v(s)| \leq C$. But this is an immediate consequence of Plancherel's theorem. Lastly, if in (47) we put the error term $O((sr)^{-\alpha-\frac{d-1}{2}-2})$ for $\mathcal{J}_\alpha(sr)$ the resulting expression can be easily estimated by

$$\int_{2^\ell}^\infty r^{-3} dr \Big[\int_1^2 |a_{\mathfrak{z}}(s,x)| ds \Big]^2 \lesssim \int_1^2 |a_{\mathfrak{z}}(s,x)|^2 ds .$$

This concludes the proof of (47), and thus the proof of Theorem 1.1. $\square$

Proof of Theorem 1.4. We apply Theorem 4.2 with $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}$. It is easy to see that it suffices to show that, for $\alpha = d(1/p - 1/2)$,

$$\Big\| \sum_{k \in \mathbb{Z}} \mathcal{F}^{-1} \big[m_k(2^{-k}|\cdot|) \widehat{\eta}(2^{-k}\cdot) \widehat{f} \big] \Big\|_{L^{p,q}} \lesssim \sup_k \|m_k\|_{B^2_{\alpha,q}} \|f\|_p,$$

where m_k are functions in $B^2_{\alpha,q}(\mathbb{R})$ supported in $(1/2, 2)$ and η is a radial Schwartz function with $\widehat{\eta}$ supported in the annulus $\{1/4 < |\xi| < 4\}$. Now write $\mathcal{F}^{-1}[m_k(|\cdot|)](x) = \kappa_k(|x|)$. Using polar coordinates and (34) we see that

$$(48) \quad \Big(\sum_{j \geq 1} \Big[\int_{I_j} |\kappa_k(r)|^2 r^{2d/p} \frac{dr}{r} \Big]^{q/2} \Big)^{1/q} \lesssim \|m_k\|_{B^2_{\alpha,q}}, \quad \alpha = d(1/p - 1/2),$$

and of course $\sup_{0 < r \leq 1} |\kappa_k(r)| < \infty$. With ψ as in the proof of Theorem 1.1 it suffices to show that the kernels

$$K_\ell^k = K_\ell^{k,\mathrm{sh}} + K_\ell^{k,\mathrm{lg}} = \Big[\int_0^{2^{\ell+2}} + \int_{2^{\ell+2}}^\infty \Big] \kappa_k(r) \, \psi * \psi * \sigma_r \, dr$$

satisfy the assumptions of Hypothesis 4.1, uniformly in k. Note that by (15)

$$\Big| \mathcal{F} \Big[\int_{I_j} \kappa_k(r) \, \psi * \psi * \sigma_r \, dr \Big](\xi) \Big| \lesssim \int_{I_j} |\kappa_k(r)| r^{\frac{d-1}{2}} dr$$

$$\lesssim 2^{-j(\frac{d}{p}-\frac{d+1}{2})} \Big(\int_{I_j} |\kappa_k(r)|^2 r^{2d/p} \frac{dr}{r} \Big)^{1/2}$$

and since $p < \frac{2d}{d+1}$ we may sum in j to deduce that $\sup_k \|\widehat{K}_\ell^{k,\mathrm{sh}}\|_\infty < \infty$.

We turn to the kernels $K_\ell^{k,\mathrm{lg}}$ and again show using Proposition 2.6 that they suffice condition (iii) in Hypothesis 4.1. Define

$$\beta_{k,j}(\mathfrak{z}) = \gamma(\mathfrak{z})\Big(\int_{I_j} |\kappa_k(r)|^2 r^{2d/p} \frac{dr}{r} \Big)^{1/2}$$

and

$$b_{k,j,\mathfrak{z}}(r,x) = 2^{-1}[\beta_{k,j}(\mathfrak{z})]^{-1} 2^{jd/p} \chi_{I_j}(r) \kappa_k(r) a_{\mathfrak{z}}(x)$$

if $\beta_{k,j}(\mathfrak{z}) \neq 0$ (and $b_{k,j,\mathfrak{z}} = 0$ otherwise). Then

$$\|b_{k,j,\mathfrak{z}}\|_{L^2(\mathcal{H})} = \Big(\int \int_0^\infty |b_{j,z}(r,x)|^2 \frac{dr}{r} dx \Big)^{1/2} \leq 1.$$

Now

$$K_\ell^{k,\mathrm{lg}} * \sum_{\mathfrak{z}} \gamma(\mathfrak{z}) a_{\mathfrak{z}} = \sum_{\mathfrak{z}} \sum_{j \geq \ell+2} \beta_{k,j}(\mathfrak{z}) \int_{I_j} \psi * \psi * \sigma_r * b_{k,j,\mathfrak{z}}(r,\cdot) dr$$

and by Proposition 2.6 we have

$$\Big\| K_\ell^{k,\mathrm{lg}} * \sum_{\mathfrak{z}} a_{\mathfrak{z}} \Big\|_p \lesssim 2^{\ell(\frac{d}{p} - \frac{d}{2} - \varepsilon(p))} \Big(\sum_{\mathfrak{z}} \Big(\sum_j |\beta_{k,j}(\mathfrak{z})|^q \Big)^{p/q} \Big)^{1/p},$$

with $\varepsilon(p) > 0$ for $p < p_1$. Finally, by (48)

$$\Big(\sum_{\mathfrak{z}} \Big(\sum_j |\beta_{k,j}(\mathfrak{z})|^q \Big)^{p/q} \Big)^{1/p} \lesssim \Big(\sum_{\mathfrak{z}} |\gamma(\mathfrak{z})|^p \Big)^{1/p} \|m_k\|_{B^2_{d(1/p-1/2),q}},$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

APPENDIX A. PROOF OF THEOREM 4.2

By normalization we may assume that Hypothesis $\mathrm{Hyp}(p,q,\varepsilon,A)$ holds with $A = 1$. We use atomic decompositions in L^p which are constructed from square functions, based on the ideas by Chang and Fefferman [8]. A convenient and useful form is given by an ℓ^2-valued version of Peetre's maximal square function (*cf.* [28], [55]),

$$\mathfrak{S}f(x) = \Big(\sum_k \sup_{|y| \leq 100d\cdot 2^{-k}} |\mathcal{L}_k f(x+y)|_{\mathcal{H}_1}^2 \Big)^{1/2},$$

where $\mathcal{L}_k f = \phi_k * f$, with $\phi_k = 2^{kd}\phi(2^k\cdot)$, and ϕ is a radial Schwartz function with $\widehat{\phi}$ supported in $\{\xi : 1/5 < |\xi| < 5\}$. Then

$$\|\mathfrak{S}f\|_p \leq C_p \|f\|_{L^p(\mathcal{H}_1)}, \quad 1 < p < \infty.$$

We closely follow the argument in [20]. Choose ϕ by splitting the function η in the statement of Theorem 4.2 as

$$\eta = \psi * \phi$$

where ψ is a radial C_0^∞-function with support in $\{x : |x| < 1/4\}$ whose Fourier transform vanishes to order $10d$ at the origin. We set $\psi_k = 2^{kd}\psi(2^k\cdot)$, then $\eta_k = \psi_k * \phi_k$ and we have

$$\sum_k \eta_k T_k f = \sum_k \psi_k T_k \mathcal{L}_k f \,.$$

For $k \in \mathbb{Z}$, we tile $\mathbb{R}^d$ by the dyadic cubes of sidelength 2^{-k} and write $L(Q) = -k$ if the sidelength of a dyadic cube Q is 2^{-k}. For each $n \in \mathbb{Z}$, let

$$\Omega_n = \{x : \mathfrak{S}f(x) > 2^n\}.$$

Let $\mathcal{Q}^n_{-k}$ be the set of all dyadic cubes of sidelength 2^{-k} which have the property that $|Q \cap \Omega_n| \geq |Q|/2$ but $|Q \cap \Omega_{n+1}| < |Q|/2$. Let

$$\Omega_n^* = \{x : M\chi_{\Omega_n}(x) > 100^{-d}\}$$

with M the Hardy–Littlewood maximal operator. The set Ω_n^* is open, contains Ω_n and satisfies $|\Omega_n^*| \lesssim |\Omega_n|$. Let $\mathcal{W}_n$ be the set of all dyadic cubes W for which the 50-fold dilate of W is contained in Ω_n^* and W is maximal with respect to this property. The collection $\{W\}$ forms a Whitney-type decomposition of Ω_n^*. The interiors of the Whitney cubes are disjoint. For each $W \in \mathcal{W}_n$ we denote by W^* the tenfold dilate of W; the dilates $\{W^* : W \in \mathcal{W}_n\}$ have still bounded overlap.

Note that each $Q \in \mathcal{Q}^n_{-k}$ is contained in a unique $W \in \mathcal{W}_n$. For $W \in \mathcal{W}_n$, set

$$a_{k,W,n} = \sum_{\substack{Q \in \mathcal{Q}^n_{-k} \\ Q \subset W}} (\mathcal{L}_k f)\chi_Q \,,$$

and for any dyadic cube W define

$$a_{k,W} = \sum_{n:W \in \mathcal{W}_n} a_{k,W,n}.$$

The functions $a_{k,W,n}$ can be considered as 'atoms', but without the usual normalization. For fixed n one has

(49) $$\sum_{W \in \mathcal{W}_n} \sum_k \|a_{k,W,n}\|^2_{L^2(\mathcal{H}_1)} \lesssim 2^{2n}\mathrm{meas}(\Omega_n).$$

Indeed (arguing as in [8]) the left-hand side is equal to

$$\sum_{Q \in \mathcal{W}_n} \sum_k \sum_{Q \in \mathcal{Q}^n_{-k}} \int_Q |\mathcal{L}_k f(x)|^2_{\mathcal{H}_1} \, dx$$

$$\leq 2 \sum_{Q \in \mathcal{W}_n} \sum_k \sum_{Q \in \mathcal{Q}^n_{-k}} \int_{Q \cap (\Omega_n \setminus \Omega_{n+1})} \sup_{|y| \leq 2^{-k}\sqrt{d}} |\mathcal{L}_k f(x+y)|^2_{\mathcal{H}_1} \, dx$$

$$\leq 2 \int_{\Omega_n \setminus \Omega_{n+1}} \mathfrak{S}f(x)^2 \leq 2\,\mathrm{meas}(\Omega_n)2^{2(n+1)} \,.$$

Let $T_{k,\ell}^{\mathrm{lg}}$, $T_{k,\ell}^{\mathrm{sh}}$, be the convolution operator with kernels $2^{kd}\mathcal{K}_\ell^{k,\mathrm{lg}}(2^k\cdot)$ and $2^{kd}\mathcal{K}_\ell^{k,\mathrm{sh}}(2^k\cdot)$, respectively. The desired estimate will follow once we establish the short range inequality

$$(50) \qquad \Big\| \sum_k \sum_{\ell \geq 0} \sum_{\substack{W \in \cup_n \mathcal{W}_n \\ L(W)=-k+\ell}} \psi_k * T_{k,\ell}^{\mathrm{sh}} a_{k,W} \Big\|_{L^p(\mathcal{H}_2)} \lesssim \|\mathfrak{S}f\|_p.$$

and for fixed $\ell \geq 0$ the long range inequality

$$(51) \qquad \Big\| \sum_k \sum_{\substack{W \in \cup_n \mathcal{W}_n \\ L(W)=-k+\ell}} \psi_k * T_{k,\ell}^{\mathrm{lg}} a_{k,W} \Big\|_{L^{p,q}(\mathcal{H}_2)} \lesssim 2^{-\ell\varepsilon} \|\mathfrak{S}f\|_p.$$

Proof of (50). We prove that for $1 < r < 2$ and for fixed $n \in \mathbb{Z}$

$$(52) \qquad \Big\| \sum_k \sum_{\ell \geq 0} \sum_{\substack{W \in \mathcal{W}_n \\ L(W)=-k+\ell}} \psi_k * T_{k,\ell}^{\mathrm{sh}} a_{k,W,n} \Big\|_{L^r(\mathcal{H}_2)}^r \leq C_r \, 2^{nr} \mathrm{meas}(\Omega_n).$$

By 'real interpolation' (cf. Lemma 2.2 in [19]) it follows that the stronger estimate

$$\Big\| \sum_n \sum_k \sum_{\ell \geq 0} \sum_{\substack{W \in \mathcal{W}_n \\ L(W)=-k+\ell}} \psi_k * T_{k,\ell}^{\mathrm{sh}} a_{k,W,n} \Big\|_{L^p(\mathcal{H}_2)}^p \lesssim \sum_n 2^{np} \mathrm{meas}(\Omega_n).$$

holds and this implies (50) since $\sum_n 2^{np}\mathrm{meas}(\Omega_n) \lesssim \|\mathfrak{S}f\|_p^p$.

Since the expression inside the norm in (52) is supported in Ω_n^* we see that the left-hand side of (52) is dominated by

$$(53) \qquad \mathrm{meas}(\Omega_n^*)^{1-r/2} \Big\| \sum_k \sum_{\ell \geq 0} \sum_{\substack{W \in \mathcal{W}_n \\ L(W)=-k+\ell}} \psi_k * T_{k,\ell}^{\mathrm{sh}} a_{k,W,n} \Big\|_{L^2(\mathcal{H}_2)}^r.$$

The convolution operators with kernel ψ_k are almost orthogonal and thus we can dominate the left-hand side of (53) by a constant times

$$(54) \quad \mathrm{meas}(\Omega_n^*)^{1-r/2} \Big(\sum_k \Big\| \sum_{\ell \geq 0} \sum_{\substack{W \in \mathcal{W}_n \\ L(W)=-k+\ell}} T_{k,\ell}^{\mathrm{sh}} a_{k,W,n} \Big\|_{L^2(\mathcal{H}_2)}^2 \Big)^{r/2}.$$

Now, for each W with $L(W) = -k + \ell$, the function $T_{k,\ell}^{\mathrm{sh}} a_{k,W,n}$ is supported in the expanded cube W^*. The cubes W^* with $W \in \Omega_j$ have bounded overlap, and therefore the expression (54) is

$$(55) \qquad \lesssim \mathrm{meas}(\Omega_n^*)^{1-r/2} \Big(\sum_k \sum_{\ell \geq 0} \sum_{\substack{W \in \mathcal{W}_n \\ L(W)=-k+\ell}} \big\| T_{k,\ell}^{\mathrm{sh}} a_{k,W,n} \big\|_{L^2(\mathcal{H}_2)}^2 \Big)^{r/2}.$$

Now we have for fixed W

$$\big\| T_{k,\ell}^{\mathrm{sh}} a_{k,W,n} \big\|_{L^2(\mathcal{H}_2)} \lesssim \|a_{k,W,n}\|_{L^2(\mathcal{H}_1)}.$$

By (49) we have

$$\sum_k \sum_{\ell \geq 0} \sum_{\substack{W \in \mathcal{W}_n \\ L(W) = -k+\ell}} \|a_{k,W,n}\|_2^2 \lesssim \sum_{W \in \mathcal{W}_n} \sum_k \|a_{k,W,n}\|_2^2 \lesssim 2^{2n} \mathrm{meas}(\Omega_n).$$

Since $\mathrm{meas}(\Omega_n^*) \lesssim \mathrm{meas}(\Omega_n)$ it follows that the right-hand side of (55) is dominated by a constant times $\mathrm{meas}(\Omega_n)2^{nr}$ which then yields (52) and finishes the proof of the short range estimate.

Proof of (51). We use the first estimate in Lemma 3.2, with $\beta_k = \psi_k$, and the $\mathcal{H}_2$ valued functions $F_k = \sum_{\substack{W: \\ L(W) = -k+\ell}} T_{k,\ell}^{\mathrm{lg}} a_{k,W}$. We then see that (51) follows from

$$(56) \qquad \sum_k \Big\| \sum_{\substack{W: \\ L(W) = -k+\ell}} T_{k,\ell}^{\mathrm{lg}} a_{k,W} \Big\|_{L^{p,q}(\mathcal{H}_2)}^p \lesssim 2^{-\ell \varepsilon p} \sum_n \mathrm{meas}(\Omega_n) 2^{np}.$$

By rescaling and assumption (iii) in Definition 4.1 we have for every k

$$\Big\| T_{k,\ell}^{\mathrm{lg}} \big[\sum_{\substack{W: \\ L(W) = -k+\ell}} a_{k,W} \big] \Big\|_{L^{p,q}(\mathcal{H}_2)}$$

$$\lesssim 2^{-\ell\varepsilon} 2^{(\ell-k)d(\frac{1}{p}-\frac{1}{2})} \Big(\sum_{\substack{W: \\ L(W) = -k+\ell}} \|a_{k,W}\|_{L^2(\mathcal{H}_1)}^p \Big)^{1/p}.$$

Thus in order to finish the proof we need the inequality

$$(57) \qquad \sum_k \sum_{\substack{W: \\ L(W) = -k+\ell}} 2^{(\ell-k)d(1-\frac{p}{2})} \|a_{k,W}\|_{L^2(\mathcal{H}_1)}^p \lesssim \sum_n 2^{np} \mathrm{meas}(\Omega_n).$$

For fixed k and fixed W, the functions $a_{k,W,n}$, $n \in \mathbb{Z}$ live on disjoint sets (since the dyadic cubes of sidelength 2^{-k} are disjoint and each such cube is in exactly one family $\mathcal{Q}_{-k}^n$). Therefore

$$\|a_{k,W}\|_{L^2(\mathcal{H}_1)} \lesssim \Big(\sum_n \|a_{k,W,n}\|_{L^2(\mathcal{H}_1)}^2 \Big)^{1/2}$$

and thus we can bound the left-hand side of (57) by

$$\sum_n \sum_{k \in \mathbb{Z}} \sum_{\substack{W \in \mathcal{W}_n: \\ L(W) = -k+\ell}} 2^{(\ell-k)d(1-\frac{p}{2})} \|a_{k,W,n}\|_{L^2(\mathcal{H}_1)}^p$$

$$\leq \sum_n \Big(\sum_{k \in \mathbb{Z}} \sum_{\substack{W \in \mathcal{W}_n: \\ L(W) = -k+\ell}} \mathrm{meas}(W) \Big)^{1-p/2} \Big(\sum_k \sum_{\substack{W \in \mathcal{W}_n: \\ L(W) = -k+\ell}} \|a_{k,W,n}\|_{L^2(\mathcal{H}_1)}^2 \Big)^{p/2}$$

$$\leq \sum_n \mathrm{meas}(\Omega_n^*)^{1-p/2} \Big(\sum_k \sum_{W \in \mathcal{W}_n} \|a_{k,W,n}\|_{L^2(\mathcal{H}_1)}^2 \Big)^{p/2};$$

here we used the disjointness of Whitney cubes in $\mathcal{W}_n$. By (49) the last displayed expression is bounded by

$$C \sum_n \text{meas}(\Omega_n^*)^{1-p/2}(2^{2n}\text{meas}(\Omega_n))^{p/2} \lesssim \sum_n 2^{np}\text{meas}(\Omega_n)$$

which gives (57). $\qquad\square$

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