EXTENSIONS OF THE STEIN-TOMAS THEOREM

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ABSTRACT. We prove an endpoint version of the Stein-Tomas restriction theorem, for a general class of measures, and with a strengthened Lorentz space estimate. A similar improvement is obtained for Stein's estimate on oscillatory integrals of Carleson-Sjölin-Hörmander type and some spectral projection operators on compact manifolds, and for classes of oscillatory integral operators with one-sided fold singularities.

1. INTRODUCTION AND STATEMENT OF RESULTS

Fourier restriction. Our first result concerns an endpoint version of the L^2 Stein-Tomas Fourier restriction theorem ([29], [30], [27]), in the following general setup as formulated by Mockenhaupt [21], and also by Mitsis [20].

Let $0 < a < d$, $0 < b \le a/2$, and consider a probability measure μ on $\mathbb{R}^d$. We assume that, for positive finite constants $A \ge 1$, $B \ge 1$, μ satisfies

$$(1.1) \qquad \sup_{\mathrm{rad}(\mathcal{B}) \le 1} \frac{\mu(\mathcal{B})}{\mathrm{rad}(\mathcal{B})^a} \le A$$

where the supremum is taken over all balls $\mathcal{B}$ with radius ≤ 1

$$(1.2) \qquad \sup_{|\xi| \ge 1} |\xi|^b |\widehat{\mu}(\xi)| \le B.$$

The number $\inf\{a : (1.1) \text{ holds for some } A < \infty\}$ is often referred to as the 'dimension' of μ and the number $\inf\{2b : (1.2) \text{ holds for some } B < \infty\}$ is the 'Fourier dimension' of μ.

The Stein-Tomas theorem (originally for surface measure on the sphere) is concerned with $L^p(\mathbb{R}^d, dx) \to L^2(d\mu)$ estimates for the Fourier transform. Stein, in the 1960's, proved that such estimates hold for some $p > 1$ if (1.2) holds for some $b > 0$. Tomas [29] improved Stein's estimate and obtained an almost sharp range. His proof was used in [21], [20], to show that, given (1.1) and (1.2),

$$(1.3) \qquad \mathcal{F} : L^p(dx) \to L^2(d\mu), \qquad 1 \le p < p_\circ(a, b) := \frac{2(d - a + b)}{2(d - a) + b}.$$

For surface measure on hypersurfaces with nonvanishing curvature one has $a = 2b = d - 1$, which gives the familiar parameter $p_\circ = \frac{2(d+1)}{d+3}$. The article [21] was primarily concerned with measures on Salem sets, i.e. singular measures supported on a-dimensional subsets of the real line which satisfy (1.2) for $b < a/2$ (with the parameter B depending on b).

Stein (*cf.* [30]) proved an endpoint $L^p \to L^2(d\mu)$ estimate for the surface measure on a sphere, using interpolation with an analytic family of kernels. As shown by Greenleaf [11] this approach can also be used when μ is surface measure on an imbedded submanifold of $\mathbb{R}^d$, in order to get the endpoint bound for $p = p_\circ(a, b)$. However it is not clear how to extend the analytic interpolation argument (and neither the alternative interpolation argument in [10], [22]) to the general class of measures satisfying (1.1), (1.2). Here we establish the endpoint version of (1.3) and further strengthen it by replacing $L^{p_\circ}$ with the larger and generally optimal Lorentz space $L^{p_\circ, 2}$.

Theorem 1.1. *Let* $0 < a < d$, $0 < b \leq a/2$, *and let* μ *be a probability measure satisfying* (1.1), (1.2) *with constants* $A \geq 1$, $B \geq 1$. *Let* $p_\circ = \frac{2(d-a+b)}{2(d-a)+b}$. *Then*

$$(1.4) \qquad \int |\widehat{f}|^2 d\mu \leq C^2 A^{\frac{b}{d-a+b}} B^{\frac{d-a}{d-a+b}} \|f\|^2_{L^{p_\circ, 2}(\mathbb{R}^d)}.$$

If a, b *are chosen from a compact interval* $I \subset (0, \infty)$ *then the constant* C *depends only on* d *and* I.

The proof of Theorem 1.1 is given in §2.

Remarks. (i) By interpolation with the trivial $L^1 \to L^\infty$ bound we see that (1.4) implies

$$(1.5) \qquad \left(\int |\widehat{f}|^q d\mu \right)^{1/q} \leq C^{\frac{2}{q}} A^{\frac{b}{(d-a+b)q}} B^{\frac{d-a}{(d-a+b)q}} \|f\|_{L^p(\mathbb{R}^d)}$$

for $1 \leq p \leq p_\circ(a, b)$, $q = \frac{b}{d-a+b}p'$. The dependence of the constant on A and B for $p < p_\circ(a, b)$ has been relevant in the work by Łaba and Pramanik [19].

By real interpolation Theorem 1.1 also implies $\mathcal{F} : L^{p,s}(dx) \to L^q(d\mu)$ for $1 < p < p_\circ(a, b)$, $q \leq \frac{b}{d-a+b}p'$, $s \leq q$. Here $p' = \frac{p}{p-1}$, the conjugate exponent. In some instances (e.g. [31], [16], [28]) the $L^p \to L^q$ result for the critical $q = \frac{b}{d-a+b}p'$ is known even for some $p > p_\circ(a, b)$ and in such cases the Lorentz improvement of Theorem 1.1 for $q = 2$ is of course trivial by interpolation.

(ii) Our estimates follow from off-diagonal $L^p \to L^q$ bounds for the convolution operator with kernel $\widehat{\mu}$. These are known for the surface measure on spheres, in particular for this example the restricted weak type estimate in Proposition 2.1 below is a special case of S. Gutiérrez' result [13] on Bochner-Riesz operators with negative index. Related off-diagonal estimates are also featured in [1] where complex interpolation is used (and which contains also several earlier references), and, more recently, in the article [18] by Keel and

Tao where real interpolation for bilinear operators is used to obtain endpoint $L_t^2(L_x^{p,2})$ Strichartz estimates (with the Lorentz norms in the slices).

Sharpness of the Lorentz exponent. We consider the case of surface measure on the sphere and show that for this example the Lorentz exponent in Theorem 1.1 is optimal. Indeed we show that $\mathcal{F}$ does not map $L^{p,s} \to L^q$ for $q = \frac{d-1}{d+1}p'$ and $s > q$. This is seen by a superposition of standard Knapp examples at different scales. Namely let

$$g(\xi', \xi_d) = \sum_{k=1}^{N} 2^{k(d-1)/q} \eta_1(2^k|\xi'|)\eta_0(2^{2k-5}(|\xi_d - 1|))$$

where η_1, η_0 are suitable bump functions on $(\frac{3}{4}, \frac{5}{4})$ and $(-1, 1)$, respectively. Then $(\int_{S^{d-1}} |g|^q d\sigma)^{1/q} \approx N^{1/q}$. Also $f = \mathcal{F}^{-1}[g]$ is bounded and the measure of the set $\{x : |f(x)| > 2^{-j}\}$ is bounded by $C2^{k_j(d+1)}$ where $(d+1-\frac{d-1}{q})k_j \in [j-c, j+c]$. Hence if $k_j < N$ the measure of this set is bounded by $C'2^{j\frac{(d+1)q}{(d+1)q-(d-1)}} = C'2^{jp}$ and if $k_j \geq N$ it is $\lesssim 2^{Np}$. Thus the $L^{p,s}$ norm of f is $O(N^{1/s})$ and if the Fourier restriction operator maps $L^{p,s}$ to $L^q(d\mu)$ then $s \leq q$.

Operators of Carleson-Sjölin-Hörmander type. We consider oscillatory integral operators T_λ given by

$$(1.6) \qquad T_\lambda f(x) = \int \zeta(x, y) e^{i\lambda\varphi(x,y)} f(y)\, dy;$$

here $\lambda > 1$, $\zeta \in C_c^\infty(\Omega_L \times \Omega_R)$ where Ω_L is an open set in $\mathbb{R}^d$ and Ω_R is an open set in $\mathbb{R}^{d-1}$. The phase is real-valued and smooth on $\Omega := \Omega_L \times \Omega_R$ and the following conditions are assumed.

First, the mixed Hessian φ''_{xy} has maximal rank

$$(1.7) \qquad \operatorname{rank}\varphi''_{xy} = d - 1$$

on Ω. This implies that for every $x \in \Omega_L$ the variety

$$(1.8) \qquad \Sigma_x := \{\varphi'_x(x, y) : y \in \Omega_R\}$$

is an immersed hypersurface in $(\mathbb{R}^d)^*$. The second hypothesis is then that for every $x \in \Omega_L$ the hypersurface Σ_x has nonvanishing Gaussian curvature everywhere. Analytically this means that for any unit vector $u = (u_1, \dots, u_d)$ we have the condition

$$(1.9) \qquad u^{\mathrm{t}}\varphi_{xy} = 0 \implies \det\left(\nabla_{yy}^2(u^{\mathrm{t}}\varphi_x)\right) \neq 0,$$

for all points in Ω.

In [16] Hörmander raised the question whether conditions (1.7), (1.9) imply

$$(1.10) \qquad \|T_\lambda\|_{L^p(\mathbb{R}^{d-1}) \to L^q(\mathbb{R}^d)} \lesssim \lambda^{-d/q}, \quad q = \frac{d+1}{d-1}p',$$

for $1 < p < \frac{2d}{d-1}$. As he pointed out a limiting argument yields the analogous estimate for the adjoint of the Fourier restriction operator; the relevant phase function is $\varphi(x,y) = \langle x, \Gamma(y) \rangle$ where Γ parametrizes a hypersurface with nonvanishing curvature. The optimal result in two dimensions was proved in [16] following earlier results by Fefferman and Stein [9] and by Carleson and Sjölin [8]. Bourgain [4] showed that in dimension $d \geq 3$ there are classes of phase-functions satisfying (1.7), (1.9) for which (1.10) fails for any $p > 2$. Earlier, Stein [27] had established (1.10) in the range $1 \leq p \leq 2$. Here we are concerned with a Lorentz space strengthening of (1.10) for the endpoint $p = 2$ of Stein's result, with L^q replaced by $L^{q,2}$.

Following [22] we slightly generalize the setup of Stein's theorem and relax the curvature assumptions on the manifolds Σ_x in (1.8), namely, we assume that for every point on Σ_x at least κ principal curvatures do not vanish. This is equivalent to

$$(1.11) \qquad u^{\mathrm{t}} \varphi_{xy} = 0 \implies \operatorname{rank} \left(\nabla^2_{yy}(u^{\mathrm{t}} \varphi_x) \right) \geq \kappa,$$

for all points in Ω, and all unit vectors u. The case $\kappa = d-1$ corresponds to the setup described above and the case $\kappa = d-2$ occurs in problems with conical structures.

Theorem 1.2. *Let T_λ be as in (1.6), with φ satisfying (1.7), (1.11). Let $q_\circ = 2 + 4\kappa^{-1}$. Then*

$$\|T_\lambda\|_{L^2(\mathbb{R}^{d-1}) \to L^{q_\circ,2}(\mathbb{R}^d)} \lesssim \lambda^{-d/q_\circ}.$$

We give the proof of Theorem 1.2 in §3.

Spectral projection operators on compact manifolds. As an application we mention a slight improvement of the $L^2 \to L^q$ endpoint bounds for spectral projection operators associated to the Laplace-Beltrami operator on general compact Riemannian manifolds, due to Sogge [25]. See also [24] for a result covering first order pseudo-differential operators and then some higher order differential operators.

Following the latter paper, and [26], we consider a classical elliptic pseudo-differential operator of first order on a d-dimensional compact manifold M which is self-adjoint with respect to some given density. We denote by $p(x,\xi)$ the principal symbol, which is homogeneous of degree one with respect to ξ, and only vanishes for $\xi = 0$. Our hypothesis is that the co-spheres

$$\Sigma_x = \{\xi : p(x,\xi) = 1\}$$

are convex, with non-vanishing Gaussian curvature everywhere (this property is referred to as "strict convexity" in [24]). Of course the main example is given by $P = \sqrt{-\Delta}$ where Δ is the Laplace-Beltrami operator on M. Consider the finite dimensional space of eigenfunctions of P whose eigenvalues belong to $[\lambda, \lambda + 1]$, for $\lambda \gg 1$, and the self-adjoint projection to this finite-dimensional subspace. We denote this projection operator $\chi_\lambda(P)$ (where χ_λ is the characteristic function of $[\lambda, \lambda + 1]$). By the results in [25], [24] the

$L^2(M) \to L^q(M)$ operator norm of $\chi_\lambda(P)$ is $O(\lambda^{d(1/2-1/q)-1/2})$ in the sharp range $q_\circ := \frac{2d+2}{d-1} \leq q \leq \infty$; in particular one has the bound $O(\lambda^{1/q_\circ})$ for $q = q_\circ$. The argument in [24] relies on the small time parametrix construction for solutions of the wave equation in [14], and so does the treatment in ch.5 of [26]. In the latter the $L^2 \to L^q$ estimates for $\chi_\lambda(P)$ are directly reduced to $L^2 \to L^q$ inequalities for oscillatory integral operators of Carleson-Sjölin-Hörmander type. Thus using this approach Theorem 1.2 can be used to derive the following endpoint result.

Corollary 1.3. *For $\lambda \geq 1$, $q_\circ = \frac{2d+2}{d-1}$, the operators $\lambda^{-1/q_\circ}\chi_\lambda(P)$ map $L^2(M)$ to $L^{q_\circ,2}(M)$ and $L^{q'_\circ,2}(M)$ to $L^2(M)$, with operator norms uniform in λ.*

Operators with one-sided fold singularities. One can also prove Lorentz-space improvements of the endpoint $L^2(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ results for oscillatory integral operator with one-sided fold singularities, obtained by Greenleaf and the second author in [12]. Here one considers the operator defined by

$$(1.12) \qquad \mathcal{T}_\lambda f(x) = \int \zeta(x,y)e^{i\lambda\Phi(x,y)}f(y)\,dy,$$

where $\zeta \in C_c^\infty(\Omega_1 \times \Omega_2)$ and now $\Omega_1, \Omega_2 \subset \mathbb{R}^d$. The phase Φ is smooth and real-valued in $\Omega_1 \times \Omega_2$ and T_λ now acts on functions of d variables. We assume that the map

$$\pi_L : (x,y) \mapsto (x, \Phi_x(x,y))$$

has only fold singularities in $\Omega_1 \times \Omega_2$, i.e.

$$(1.13) \qquad \begin{aligned} &\operatorname{rank} \Phi''_{xy} \geq d-1 \\ \det(\Phi_{xy})(x,y) = 0,\ &\Phi_{xy}b = 0,\ b \neq 0 \quad \Longrightarrow \quad \langle b, \nabla_y \rangle(\det \Phi_{xy}) \neq 0. \end{aligned}$$

For an integer κ, $0 \leq \kappa \leq d-1$ we say that *Hypothesis* (π_L, κ) is satisfied if the $(d-1)$-dimensional immersed hypersurfaces

$$\mathcal{L}_x = \{\Phi_x(x,y) : \det \Phi_{xy} = 0\}$$

have at least κ nonvanishing principal curvatures at every point. Notice that the case $\kappa = 0$ is included (and contains no particular assumption).

Theorem 1.4. *Let $\mathcal{T}_\lambda$ be as in (1.12), with Φ satisfying (1.13) and also Hypothesis (π_L, κ). Let $q_1 = \frac{2\kappa+4}{\kappa+1}$. Then*

$$\|\mathcal{T}_\lambda\|_{L^2(\mathbb{R}^d) \to L^{q_1,2}(\mathbb{R}^d)} \lesssim \lambda^{-d/q_1}.$$

The $L^2 \to L^{q_1}$ bounds are in [12]. Given the preparations in that work the proof of Theorem 1.4 is very similar to the proof of Theorem 1.2. We sketch the argument in §4.

Remarks. (i) In two dimensions, under the stronger hypothesis of *two-sided* fold singularities, together with the appropriate curvature assumptions, such estimates can be derived from the sharp $L^p \to L^q$ results for

$q = 3p'/2$ and $q > 5/2$, obtained by Bennett and the second author in [2]. The above mentioned example by Bourgain suggests that higher dimensional analogues of those estimates will not hold in the full generality of our setup here.

(ii) From Theorem 1.4 one can obtain an $L^2_{\mathrm{comp}} \to L^{q,2}_{\mathrm{loc}}$ improvement for Fourier integral operators with fold singularities in [12], using arguments in that paper. In particular this covers the $L^2 \to L^{3,2}$ and $L^{3/2,2} \to L^2$ estimates for translation invariant averages over curves in $\mathbb{R}^3$, with non-vanishing curvature and torsion. The corresponding Lebesgue space estimates had been already obtained by Oberlin [23] and his paper was the starting point for the variable results in [12]. The version of Theorem 1.4 for one-sided folds, in its adjoint formulation, also implies the optimal $L^{3/2,2}_{\mathrm{comp}}(\mathbb{R}^3) \to L^2_{\mathrm{loc}}(\mathbb{R}^3)$ estimate for the restricted X-ray transform associated to *well-curved* line complexes in $\mathbb{R}^3$, see [12] for further discussion.

(iii) As observed in Appendix I of [2] the Lorentz-space improvement of the abovementioned result by Oberlin can also be obtained by interpolation from better $L^p \to L^q$ bounds for $p > 2$ for oscillatory integrals. It is presently unknown whether such better results hold for just one-sided folds and suitable curvature assumptions, even in $\mathbb{R}^3$.

(iv) Theorem 1.4 can be used to slightly improve estimates for eigenfunctions of the Laplace-Beltrami operator Δ on a compact manifold M, when restricted to hypersurfaces, see Burq, Gérard, Tzvetkov [5] and Hu [17]. Assume that e_λ is an eigenfunction for Δ satisfying $\Delta e_\lambda = -\lambda^2 e_\lambda$. It is proved in [17] that for any hypersurface $S \subset M$, the quotient $\|e_\lambda\|_{L^q(S)}/\|e_\lambda\|_{L^2(M)}$ is $O(\lambda^{(d-1)(1/2-1/q)})$ for $\frac{2d}{d-1} \le q \le \infty$. Note that $(d-1)(1/2-1/q) = 1/q$ for the endpoint $q = 2d/(d-1)$. Hu's result is based on an application of Theorem 2.2 in [12]. The improved estimate

$$\|e_\lambda\|_{L^{q,2}(S)} \lesssim (1+\lambda)^{1/q}\|e_\lambda\|_{L^2(M)}, \quad q = \frac{2d}{d-1}.$$

is obtained by applying instead Theorem 1.4 (in $d-1$ dimensions, with $\kappa = d-2$) in her argument.

2. PROOF OF THEOREM 1.1

The theorem is a consequence of the following convolution inequality for the Fourier transform of μ (*cf.* [13] for the case of surface measure on the sphere).

Proposition 2.1. *Let μ satisfy* (1.1), (1.2) *and define* $Tf = f * \widehat{\mu}$. *Let*

$$(2.1) \qquad \rho = \frac{(d-a+2b)(d-a+b)}{(d-a)^2 + 3b(d-a) + b^2}, \qquad \sigma = \frac{d-a+2b}{b}.$$

Then T is of restricted weak type (ρ, σ) and of restricted weak type (σ', ρ'), both with operator norm $O(A^{\frac{b}{d-a+b}} B^{\frac{d-a}{d-a+b}})$. Moreover, if $\rho < p < \sigma'$ and

$\frac{1}{p} - \frac{1}{q} = \frac{d-a}{d-a+b}$, *then for any* $s \in (0, \infty]$,

(2.2) $$\|f * \widehat{\mu}\|_{L^{q,s}} \leq \mathcal{C}(p,s) A^{\frac{b}{d-a+b}} B^{\frac{d-a}{d-a+b}} \|f\|_{L^{p,s}}.$$

In particular (2.2) *holds for* $p = p_\circ(a,b)$, $q = (p_\circ(a,b))'$.

Proof of Theorem 1.1. Taking Proposition (2.1) for granted the conclusion follows from (2.2) for $s = 2$, $p = p_\circ$. Indeed, by Tomas' T^*T argument, with $\widetilde{f} := f(-\cdot)$,

(2.3) $$\int |\widehat{f}(\xi)|^2 d\mu = \int \overline{f}(x)\, \widetilde{f} * \widehat{\mu}(-x)\, dx \leq \|f\|_{L^{p_\circ,2}} \|\widetilde{f} * \widehat{\mu}\|_{L^{p_\circ',2}}$$
$$\lesssim A^{\frac{b}{d-a+b}} B^{\frac{d-a}{d-a+b}} \|f\|^2_{L^{p_\circ,2}}.$$

$\square$

Remark 2.2 (*Bourgain's interpolation argument*). In the proof of Proposition 2.1 we use a trick introduced by Bourgain [3] in his proof of an endpoint bound for the spherical maximal function, see §6.2 in [6] for an abstract analogue. In this version we are given pairs of spaces $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$, and operators T_j that map A_i to B_i and we assume that $\|T_j\|_{A_0 \to B_0} \leq M_0 2^{-j\beta_0}$ and $\|T_j\|_{A_1 \to B_1} \leq M_1 2^{j\beta_1}$ for some $\beta_0 > 0$, $\beta_1 > 0$. Let $\vartheta = \frac{\beta_0}{\beta_0 + \beta_1}$. Then the result is that $T = \sum T_j$ maps the Lions-Peetre interpolation space $\overline{A}_{\vartheta,1}$ to $\overline{B}_{\vartheta,\infty}$ with operator norm $C(\beta_0, \beta_1) M_0^{1-\vartheta} M_1^{\vartheta}$. In applications we are mostly dealing with Lebesgue or Lorentz spaces and the result then involves a restricted weak type estimate, as in [3].

Proof of Proposition 2.1. We prove the $L^{\rho,1} \to L^{\sigma,\infty}$ inequality. We use the Tomas approach in [29], [21] and dyadically decompose $\widehat{\mu}$. Let χ_0 be smooth and supported in $\{x : |x| < 1\}$ and let $\chi_0(x) = 1$ for $|x| \leq 1/2$. For $j \geq 1$ let $\chi_j(x) = \chi_0(2^{-j}x) - \chi_0(2^{1-j}x)$, so $\sum_{j=0}^{\infty} \chi_j(x) \equiv 1$. Let $\mu_j = \mu * \mathcal{F}^{-1}[\chi_j]$. Since μ is a probability measure it is clear that $\|\mu_0\|_\infty + \|\widehat{\mu}_0\|_\infty \lesssim 1$. As $A, B \geq 1$ it is easily verified (for details *cf.* [21]) that for $j \geq 0$ assumption (1.2) implies

(2.4) $$\|\widehat{\mu}_j\|_\infty \lesssim B 2^{-jb}$$

and that assumption (1.1) implies

(2.5) $$\|\mu_j\|_\infty \lesssim A 2^{j(d-a)}.$$

Therefore, if we define $T_j f = f * \widehat{\mu}_j$, we have $\|T_j\|_{L^1 \to L^\infty} \lesssim B 2^{-jb}$ and $\|T_j\|_{L^2 \to L^2} \lesssim A 2^{j(d-a)}$.

Now let

(2.6) $$\theta = \frac{d-a}{d-a+b}$$

so that $(1-\theta)(d-a) + \theta(-b) = 0$. We calculate that for $p_\circ = \frac{2(d-a+b)}{2(d-a)+b}$ we have $(1-\theta)(\frac{1}{2}, \frac{1}{2}) + \theta(1, \frac{1}{\infty}) = (\frac{1}{p_\circ}, 1 - \frac{1}{p_\circ})$. Now the two inequalities for

T_j allow us to apply Bourgain's interpolation trick; the result is that the operator $T = \sum T_j$ is of restricted weak type $(p_\circ, p'_\circ)$, with operator norm $\leq C A^{1-\theta} B^\theta$, and if I is a compact subinterval of $(0, \infty)$ then for $a, b \in I$ the constants $C(a, b)$ depend only on I. Thus we have proved

$$\|f * \widehat{\mu}\|_{L^{p'_\circ, \infty}} \lesssim A^{\frac{b}{d-a+b}} B^{\frac{d-a}{d-a+b}} \|f\|_{L^{p_\circ, 1}}.$$

By applying Tomas' argument we get

$$(2.7) \quad \int |\widehat{f}(\xi)|^2 d\mu = \int \overline{f}(x) \widetilde{f} * \widehat{\mu}(-x) dx$$

$$\lesssim \|f\|_{L^{p_\circ, 1}} \|\widetilde{f} * \widehat{\mu}\|_{L^{p'_\circ, \infty}} \lesssim A^{\frac{b}{d-a+b}} B^{\frac{d-a}{d-a+b}} \|f\|^2_{L^{p_\circ, 1}}$$

which is weaker than (2.3).

We use (2.7) to bound $\|f * \widehat{\mu}_j\|_2$. By Plancherel's theorem and (2.5),

$$\|f * \widehat{\mu}_j\|_2 = \left(\int |\widehat{f}(\xi)|^2 |\mu_j(\xi)|^2 d\xi \right)^{1/2}$$

$$\lesssim A^{1/2} 2^{j\frac{d-a}{2}} \left(\int |\widehat{f}(\xi)|^2 |\mu_j(\xi)| d\xi \right)^{1/2}$$

$$\lesssim A^{1/2} 2^{j\frac{d-a}{2}} \left(\int |\mathcal{F}^{-1}[\chi_j](\xi)| \int |\widehat{f}(\eta + \xi)|^2 d\mu(\eta) \, d\xi \right)^{1/2}.$$

By (2.7), this is

$$\lesssim A^{1/2} 2^{j\frac{d-a}{2}} \left(\int \frac{2^{jd}}{(1 + 2^j |\xi|)^{d+1}} A^{1-\theta} B^\theta \|f e^{\langle 2\pi i \cdot, \xi \rangle}\|^2_{L^{p_\circ, 1}} \, d\xi \right)^{1/2}$$

and hence we obtain

$$(2.8) \qquad \|f * \widehat{\mu}_j\|_2 \lesssim A^{1-\theta/2} B^{\theta/2} 2^{j\frac{d-a}{2}} \|f\|_{L^{p_\circ, 1}}.$$

We interpolate this estimate with the $L^1 \to L^\infty$ bound $O(B 2^{-jb})$. Let

$$(2.9) \qquad \gamma = \frac{d-a}{d-a+2b}.$$

so that $(1 - \gamma)\frac{d-a}{2} + \gamma(-b) = 0$. A calculation shows that if ρ, σ are in (2.1), then $(1 - \gamma)(\frac{1}{p_\circ}, \frac{1}{2}) + \gamma(1, \frac{1}{\infty}) = (\frac{1}{\rho}, \frac{1}{\sigma})$. Thus, again by Bourgain's interpolation trick, the operator of convolution with $\widehat{\mu}$ is of restricted weak type (ρ, σ), with operator norm $\lesssim (A^{1-\frac{\theta}{2}} B^{\frac{\theta}{2}})^{1-\gamma} B^\gamma$. One calculates from (2.6), (2.9) that $(1-\gamma)(1 - \frac{\theta}{2}) = \frac{b}{d-a+b} = 1 - \theta$ and $(1-\gamma)\frac{\theta}{2} + \gamma = \frac{d-a}{d-a+b} = \theta$ which yields

$$\|f * \widehat{\mu}\|_{L^{\sigma, \infty}} \lesssim A^{1-\theta} B^\theta \|f\|_{L^{\rho, 1}},$$

as claimed. The corresponding $L^{\sigma', 1} \to L^{\rho', \infty}$ inequality follows by duality. Finally, inequality (2.2) for $\rho < p < \sigma'$ follows by real interpolation between the $L^{\rho, 1} \to L^{\sigma, \infty}$ and the $L^{\sigma', 1} \to L^{\rho', \infty}$ inequality. To obtain the $L^{p_\circ, s} \to L^{p'_\circ, s}$ inequality note that $(1/p_\circ, 1/p'_\circ)$ is the midpoint of the interval with endpoints $(1/\rho, 1/\sigma)$ and $(1/\sigma', 1/\rho')$. $\qquad\square$

3. Proof of Theorem 1.2

We may use a partition of unity and a compactness argument to reduce to the situation that the amplitude ζ has support in $\{(x, y) : |x| < \varepsilon^2, |y| < \varepsilon^2\}$, for small $\varepsilon > 0$. After changes of variable in x and in y we may assume that

$$(3.1) \qquad \qquad \varphi_{x'y}(0,0) = I_{d-1}$$

$$(3.2) \qquad \qquad \varphi_{x'yy}(0,0) = 0$$

$$(3.3) \qquad \qquad \varphi_{x_dy}(0,0) = 0$$

$$(3.4) \qquad \qquad \text{rank } \varphi_{x_dyy}(0,0) \geq \kappa.$$

The conclusion of the Theorem is equivalent with the case $s = 2$ of

$$\|T_\lambda T_\lambda^*\|_{L^{q_\circ',s}(\mathbb{R}^d) \to L^{q_\circ,s}(\mathbb{R}^d)} \lesssim \lambda^{-2d/q_\circ}.$$

We split the operator $T_\lambda T_\lambda^*$. Let $\eta_0 \in C_0^\infty(\mathbb{R})$ so that $\eta_0(s) = 1$ for $|s| \leq 1/2$ and η_0 supported in $(-1, 1)$. For $j \geq 1$, let $\eta_j = \eta_0(2^{-j}\cdot) - \eta_0(2^{-j+1}\cdot)$. We set

$$(3.5) \quad \begin{aligned} b_j(w, z, y) &= \zeta(w, y)\overline{\zeta(z, y)}\eta_j(\lambda(w_d - z_d))\eta_0(\varepsilon^{-1}\lambda 2^{-j}|w' - z'|) \\ \widetilde{b}_j(w, z, y) &= \zeta(w, y)\overline{\zeta(z, y)}\eta_j(\lambda(w_d - z_d))(1 - \eta_0(\varepsilon^{-1}\lambda 2^{-j}|w' - z'|)) \end{aligned}$$

and let $\mathcal{S}_j^\lambda$ be the operators with integral kernel

$$S_j^\lambda(w, z) = \int b_j(w, z, y)e^{i\lambda(\varphi(w,y) - \varphi(z,y))}\,dy;$$

also let $\widetilde{S}_j^\lambda(w, z)$ be similarly defined with $\widetilde{b}_j$ in place of b_j. Then

$$(3.6) \qquad \qquad T_\lambda T_\lambda^* = \sum_{j \geq 0} \mathcal{S}_j^\lambda + \sum_{j \geq 0} \widetilde{\mathcal{S}}_j^\lambda.$$

Note that b_j is supported where $|w' - z'| \ll |w_d - z_d|$ and $|w_d - z_d| \approx 2^j \lambda^{-1}$.

For integration by parts arguments we analyze

$$\varphi_y'(w, y) - \varphi_y'(z, y) =$$
$$\int_0^1 (w' - z')^{\mathrm{t}}\varphi_{x'y}(z + s(w - z))\,ds + \int_0^1 (w_d - z_d)\varphi_{x_dy}(z + s(w - z))\,ds$$

and by (3.1), (3.3), $\varphi_{x'y} = I_{d-1} + O(\varepsilon^2)$, $\varphi_{x_dy} = O(\varepsilon^2)$. On the support of $\widetilde{b}_j$ we have $|w' - z'| \geq c\varepsilon|w_d - z_d|$ and thus

$$|\varphi_y'(w, y) - \varphi_y'(z, y)| \geq |w' - z'| - C\varepsilon^2|w_d - z_d|$$
$$\geq \varepsilon|w - z| \text{ for } (w, z, y) \in \text{supp}\widetilde{b}_j.$$

Integration by parts with respect to y yields

$$\left| \sum_{j \geq 0} \widetilde{S}_j^\lambda(w, z) \right| \leq C_{\varepsilon, N}(1 + \lambda|w - z|)^{-N}.$$

From Schur's Lemma and subsequent interpolation with a trivial L^∞ bound we get for $2 < q < \infty$, $0 < s \le \infty$

$$(3.7) \qquad \Big\| \sum_{j\ge 0} \widetilde{\mathcal{S}}_j^\lambda \Big\|_{L^{q',s}\to L^{q,s}} \lesssim \lambda^{-2d/q};$$

moreover, by the support properties of b_0 and Schur's lemma

$$(3.8) \qquad \big\| \mathcal{S}_0^\lambda \big\|_{L^{q',s}\to L^{q,s}} \lesssim \lambda^{-2d/q}.$$

We shall use these inequalities for $s = 2$.

The main task is to show that

$$(3.9) \qquad \Big\| \sum_{j>0} \mathcal{S}_j^\lambda \Big\|_{L^{q_\circ',s}\to L^{q_\circ,s}} \lesssim \lambda^{-2d/q_\circ}, \quad q_\circ = 2 + 4\kappa^{-1}.$$

The crucial step in the proof of (3.9) is to establish part (ii) in

Proposition 3.1. *(i) For $j > 0$,*

$$(3.10) \qquad \big\| \mathcal{S}_j^\lambda \big\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim 2^{-j\kappa/2}.$$

(ii) For $q_\circ = 2 + 4\kappa^{-1}$, $j > 0$

$$(3.11) \quad \big\| \mathcal{S}_j^\lambda \big\|_{L^{q_\circ',1}(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} + \big\| \mathcal{S}_j^\lambda \big\|_{L^2(\mathbb{R}^d)\to L^{q_\circ,\infty}(\mathbb{R}^d)} \lesssim 2^{j/2}\lambda^{-d(\frac{1}{q_\circ}+\frac{1}{2})}.$$

Proof that Proposition 3.1 implies (3.9). We interpolate (3.10) with the two inequalities in (3.11). Let $\rho = \frac{2(\kappa+1)(\kappa+2)}{\kappa^2+6\kappa+4}$, $\sigma = \frac{2\kappa+2}{\kappa}$ (which coincide with the definition in (2.1) for the parameters $(a,b) = (d-1, \kappa/2)$). Then $(\frac{1}{\rho}, \frac{1}{\sigma}) = (\frac{1-\gamma}{p_\circ}+\gamma, \frac{1-\gamma}{2})$ with $p_\circ = q_\circ'$ and $\gamma = (1+\kappa)^{-1}$ as in (2.9). We argue as in the proof of Proposition 2.1, and obtain, by real interpolation and Bourgain's trick, that

$$(3.12) \qquad \Big\| \sum_{j>0} \mathcal{S}_j^\lambda \Big\|_{L^{\rho,1}\to L^{\sigma,\infty}} + \Big\| \sum_{j>0} \mathcal{S}_j^\lambda \Big\|_{L^{\sigma',1}\to L^{\rho',\infty}} \lesssim \lambda^{-d(1-\frac{1}{\rho}+\frac{1}{\sigma})}.$$

Now $1 - \rho^{-1} + \sigma^{-1} = 2q_\circ^{-1}$ and we may interpolate the two inequalities in (3.12) to deduce the assertion (3.9). $\qquad\square$

Proof of Proposition 3.1. Note that by (3.4), (3.2) the determinant of a symmetric $\kappa \times \kappa$ minor of $\varphi_{x_d yy}$ is nonzero near the origin. This means that for $|w'-z'| \le \varepsilon|w_d - z_d| \approx 2^j\lambda^{-1}$, the corresponding minor of

$$\lambda\big[\varphi_{y'y'}(w,y) - \varphi_{y'y'}(z,y)\big] =$$
$$2^j \frac{w_d - z_d}{2^j\lambda^{-1}} \int_0^1 \varphi_{x_d y'y'}(z + s(w - z))ds + O(\varepsilon 2^j)$$

has determinant $\approx 2^j$. Now inequality (3.10) follows easily by a stationary phase argument with respect to the relevant κ coordinates.

For part (ii) we need only prove the $L^2 \to L^{q_\circ,\infty}$ inequality since the $L^{p_\circ,1} \to L^2$ inequality follows by taking adjoints and replacing φ by $-\varphi$.

We first notice that $\mathcal{S}_j^\lambda$ is identically zero if $2^j \gtrsim \varepsilon\lambda$ and in all other cases $\mathcal{S}_j^\lambda$ is essentially local on balls of diameter $\approx 2^j \lambda^{-1}$. This means if $r \geq 2^j/\lambda$ and f is supported in the ball $B(a,r)$ centered at $a \in \mathbb{R}^d$ then $\mathcal{S}_j^\lambda f$ is supported in $B(a, Cr)$. Therefore it suffices to prove the inequality for functions f supported in $B(a, 2^j\lambda^{-1})$. We set

$$\text{(3.13)} \qquad \mu = 2^j, \qquad \delta = 2^j\lambda^{-1},$$

and change variables $w = a + \delta u$ and $z = a + \delta v$.

Then

$$\mathcal{S}_j^\lambda f(a + \delta v) = \delta^d \mathcal{R}_\mu[f(a + \delta \cdot)]$$

where

$$\text{(3.14)} \qquad \mathcal{R}_\mu g(u) \equiv \mathcal{R}_\mu^{a,\delta} g(u) = \iint e^{-i\mu\Psi(u,v,y)} \beta(u,v,y) dy\, g(v) dv,$$

with

$$\text{(3.15)} \qquad \Psi(u,v,y) \equiv \Psi(u,v,y;a,\delta) = \frac{1}{\delta}\Big[\varphi(a + \delta u, y) - \varphi(a + \delta v, y)\Big]$$

$$= \int_0^1 \langle u - v, \nabla_x \varphi(a + \delta(v + s(u - v)), y)\rangle\, ds$$

and

$$\beta(u,v,y) = b_j(a + \delta u, a + \delta v, y)$$

with $\delta = 2^j\lambda^{-1}$. By rescaling

$$\|\mathcal{S}_j^\lambda\|_{L^2 \to L^{q_\circ,\infty}} \lesssim \delta^{\frac{d}{2} + \frac{d}{q_\circ}} \sup_{|a| \lesssim \varepsilon^2} \|\mathcal{R}_\mu^{a,\delta}\|_{L^2 \to L^{q_\circ,\infty}}, \qquad \delta = 2^j\lambda^{-1} = \mu\lambda^{-1}$$

and thus we just need to show that for $\mu \geq 1$

$$\|\mathcal{R}_\mu\|_{L^2 \to L^{q_\circ,\infty}} \lesssim \mu^{-\frac{d-1}{2} - \frac{d}{q_\circ}}.$$

This of course follows from

$$\text{(3.16)} \qquad \|\mathcal{R}_\mu \mathcal{R}_\mu^*\|_{L^{q_\circ',1} \to L^{q_\circ,\infty}} \lesssim \mu^{1-d-2d/q_\circ}.$$

We proceed to show (3.16) using an analogue of Tomas' interpolation argument. The Schwartz kernel of $\mathcal{R}_\mu \mathcal{R}_\mu^*$ is given by

$$\text{(3.17)} \qquad K_\mu(u, \widetilde{u}) = \iiint e^{i\mu(\Psi(u,v,y+h) - \Psi(\widetilde{u},v,y))} \gamma(u, \widetilde{u}; v, y, h)\, dv\, dy\, dh$$

where $\gamma(u, \widetilde{u}; v, y, h) = \beta(u, v, y + h)\overline{\beta(\widetilde{u}, v, y)}$.

We now reduce the number of frequency variables by a straightforward stationary phase arguments. Let

$$\theta(u, \widetilde{u}, v', v_d, y, h) = \Psi(u, v, y + h) - \Psi(\widetilde{u}, v, y)$$

$$= \frac{1}{\delta}\Big[\varphi(a + \delta u, y + h) - \varphi(a + \delta v, y + h) - \varphi(a + \delta\widetilde{u}, y) + \varphi(a + \delta v, y)\Big].$$

Then the partial Hessian of θ with respect to the (v', h)-variables is given by

$$\begin{pmatrix} -\delta\varphi_{x'x'}(a+\delta v, y+h) & -\varphi_{x'y}(a+\delta v, y+h) \\ -\varphi_{yx'}(a+\delta v, y+h) & \langle u-v, \int_0^1 \varphi_{xyy}(a+\delta(v+s(u-v))y)\,ds\rangle \end{pmatrix}.$$

It is clearly nondegenerate on the support of our cutoff functions (with small ε). We observe that

$$\theta_{v'} = 0 \quad \Longleftrightarrow \quad \varphi_{x'}(a+\delta v, y+h) = \varphi_{x'}(a+\delta v, y)$$

$$\theta_h = 0 \quad \Longleftrightarrow \quad \langle u-v, \int_0^1 \varphi_{xy}(a+\delta(v+s(u-v))y)\,ds\rangle = 0$$

and these equations are solved by $h = 0$ and $v' = \mathfrak{v}'(u, v_d, y)$ for some smooth $\mathfrak{v}'$. We now observe that when θ is evaluated at $h = 0$, the result is independent of v, in fact

$$\theta(u, \widetilde{u}, \mathfrak{v}'(u, v_d, y), v_d, y, 0) = \frac{1}{\delta}\left[\varphi(a+\delta u, y) - \varphi(a+\delta\widetilde{u}, y)\right] = \Psi(u, \tilde{u}, y).$$

The method of stationary phase (applied in the (v', h)-variables) gives

$$(3.18) \qquad K_\mu(u, \widetilde{u}) = \mu^{1-d} \int e^{i\mu\Psi(u, \widetilde{u}, y)}\alpha(u, \widetilde{u}, y)dy$$

for suitable smooth amplitudes α depending smoothly on the parameters a and δ.

We now decompose the kernel in a way analogous to (3.6). Split coordinates in $\mathbb{R}^d$ as $u = (u', u_d)$ and let, for $\ell = 0, 1, 2, \ldots$ (and η_ℓ as in (3.5))

$$\alpha_\ell(u, \widetilde{u}, y) = \alpha(u, \widetilde{u}, y)\eta_\ell(\mu(u_d - \widetilde{u}_d))\eta_0(\varepsilon^{-1}\mu 2^{-\ell}|u' - \widetilde{u}'|)$$

$$\widetilde{\alpha}_\ell(u, \widetilde{u}, y) = \alpha(u, \widetilde{u}, y)\eta_\ell(\mu(u_d - \widetilde{u}_d))(1 - \eta_0(\varepsilon^{-1}\mu 2^{-\ell}|u' - \widetilde{u}'|)).$$

Let $\mathcal{V}_\ell^\mu$ denote the operators with integral kernel

$$V_\ell^\mu = \mu^{1-d} \int e^{i\mu\Psi(u, \widetilde{u}, y)}\alpha_\ell(u, \widetilde{u}, y)\,dy$$

and let $\widetilde{\mathcal{V}}_\ell^\mu$ and the kernel $\widetilde{V}_\ell^\mu$ be analogously defined with $\widetilde{\alpha}_\ell$ in place of α_ℓ. Then

$$(3.19) \qquad \mathcal{R}_\mu\mathcal{R}_\mu^* = \sum_{\ell\geq 0}\widetilde{\mathcal{V}}_\ell^\mu + \sum_{\ell\geq 0}\mathcal{V}_\ell^\mu.$$

The straightforward argument used for (3.7) and (3.8) now yields for $2 < q < \infty$, $0 < s \leq \infty$,

$$(3.20) \qquad \left\|\mu^{d-1}\mathcal{V}_0^\mu\right\|_{L^{q',s}\to L^{q,s}} + \left\|\mu^{d-1}\sum_{\ell\geq 0}\widetilde{\mathcal{V}}_\ell^\mu\right\|_{L^{q',s}\to L^{q,s}} \lesssim \mu^{-2d/q}.$$

We need to prove the appropriate $L^1 \to L^\infty$ and $L^2 \to L^2$ bounds for $\mathcal{V}_\ell^\mu$ which are

$$(3.21) \qquad \left\| \mu^{d-1} \mathcal{V}_\ell^\mu \right\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \lesssim 2^{-\ell\kappa/2} \,,$$

$$(3.22) \qquad \left\| \mu^{d-1} \mathcal{V}_\ell^\mu \right\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \lesssim 2^\ell \mu^{-d} \,.$$

Then, by Bourgain's interpolation trick, (3.21) and (3.22) imply (3.16).

It only remains to prove (3.21) and (3.22). The inequality (3.10) (written for (ℓ, μ, α_ℓ) in place of (j, λ, γ_j)) immediately yields (3.21).

For the L^2 bound we observe that only the cases $2^\ell \lesssim \varepsilon\mu$ are relevant and that $\mathcal{V}_\ell^\mu$ is essentially local on balls of diameter $\approx 2^\ell \mu^{-1}$. Therefore it suffices to prove the inequality for f supported in the ball $B(b, r)$ with $r = 2^\ell \mu^{-1}$. We rescale and set $u = b + 2^\ell \mu^{-1}\omega$ and $\widetilde{u} = b + 2^\ell \mu^{-1}\widetilde{\omega}$. Then

$$(3.23) \qquad \mathcal{V}_\ell^\mu f(b + 2^\ell \mu^{-1}\omega) = (2^\ell \mu^{-1})^d \mathcal{W}^\ell[f(b + 2^\ell \mu^{-1}\cdot)](\omega)$$

where

$$(3.24) \qquad \mathcal{W}^\ell h(\omega) = \iint e^{-i2^\ell \Theta(\omega, \widetilde{\omega}, y)} \sigma(\omega, \widetilde{\omega}, y) dy \, h(\widetilde{\omega}) d\widetilde{\omega},$$

with

$$\Theta(\omega, \widetilde{\omega}, y) = \frac{\mu}{2^\ell \delta} \left[\varphi(a + \delta b + \tfrac{2^\ell \delta}{\mu}\omega, y) - \varphi(a + \delta b + \tfrac{2^\ell \delta}{\mu}\widetilde{\omega}, y) \right]$$

$$= \left\langle \omega - \widetilde{\omega}, \int_0^1 \varphi_x(a + \delta b + 2^\ell \lambda^{-1}(\widetilde{\omega} + s(\omega - \widetilde{\omega})), y) ds \right\rangle .$$

Recall that $\mu = 2^j$ and $\delta = 2^j \lambda^{-1}$ so that $2^\ell \delta \mu^{-1} = 2^\ell \lambda^{-1} \leq \varepsilon$. The phase Θ and therefore the operator $\mathcal{W}^\ell$ depend on the points a, b but the estimates will be uniform.

We now show that

$$(3.25) \qquad \|\mathcal{W}^\ell\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \lesssim 2^{-\ell(d-1)}$$

which, by (3.23), implies that $\|\mu^{d-1} \mathcal{V}_\ell^\mu\|_{L^2} \lesssim (2^\ell \mu^{-1})^d 2^{-\ell(d-1)}$. This is (3.22).

Finally, (3.25) follows from a standard T^*T estimate for oscillatory integral operators associated to a canonical graph, here with $d - 1$ frequency variables y and space variables ω', $\widetilde{\omega}'$, with frozen ω_d, $\widetilde{\omega}_d$. For this result we refer to Lemma 2.3 in [12] which is built on an argument in [15]. The required estimate follows after noting that

$$\begin{pmatrix} \Theta_{\omega'\widetilde{\omega}'} & \Theta_{\omega'y} \\ \Theta_{y\widetilde{\omega}'} & \Theta_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \varphi_{x'y} \\ \varphi_{yx'} & \langle \omega - \widetilde{\omega}, \varphi_{xyy} \rangle \end{pmatrix} \Big|_{(0,0)} + O(\varepsilon)$$

has determinant bounded away from 0. This is immediate from (3.1) (provided that ε is chosen small). $\qquad \square$

4. Proof of Theorem 1.4

The proof is quite analogous to the proof of Theorem 1.2, and therefore we will give only a sketch. As discussed in [12] one can assume after suitable changes of variables in the x and the y coordinates that, with the y-variables split as $y = (y', y_d)$,

$$(4.1) \qquad \begin{pmatrix} \Phi_{x'y'} & \Phi_{x'y_d} \\ \Phi_{x_dy'} & \Phi_{x_dy_d} \end{pmatrix} \Big|_{(0,0)} = \begin{pmatrix} I_{d-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(4.2) \qquad \Phi_{x_dy_dy_d}(0,0) \neq 0 \,,$$

$$(4.3) \qquad \Phi_{x_dy'y_d}(0,0) = 0 \,.$$

(4.2) reflects the fold condition on π_L. Moreover,

$$(4.4) \qquad \operatorname{rank}(\Phi_{x_dy'y'}) \geq \kappa$$

which expresses the curvature condition. Note that by (4.2), (4.3) and (4.4),

$$(4.5) \qquad \operatorname{rank}(\Phi_{x_dyy}) \geq \kappa + 1.$$

We shall argue as in the proof of Theorem 1.2 and show that

$$\|\mathcal{T}_\lambda \mathcal{T}_\lambda^*\|_{L^{q_1',s}(\mathbb{R}^d) \to L^{q_1,s}(\mathbb{R}^d)} \leq \lambda^{-2d/q_1}.$$

We proceed splitting the operator $\mathcal{T}_\lambda \mathcal{T}_\lambda^*$ as in (3.6) with the only difference that φ is replaced with Φ and now the y integrations are over small open sets in $\mathbb{R}^d$. The proof of the estimate analogous to (3.7) is exactly the same, and then again the main task is to establish that

$$(4.6) \qquad \Big\|\sum_{j>0} \mathcal{S}_j^\lambda\Big\|_{L^{q_1',s} \to L^{q_1,s}} \lesssim \lambda^{-2d/q_1}, \qquad q_1 = \frac{2\kappa+4}{\kappa+1} \,.$$

The following estimates are analogous to Proposition 3.1:

$$(4.7) \qquad \big\|\mathcal{S}_j^\lambda\big\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \lesssim 2^{-j(\kappa+1)/2},$$

and

$$(4.8) \qquad \big\|\mathcal{S}_j^\lambda\big\|_{L^{q_1',1}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} + \big\|\mathcal{S}_j^\lambda\big\|_{L^2(\mathbb{R}^d) \to L^{q_1,\infty}(\mathbb{R}^d)} \lesssim 2^{j/2} \lambda^{-(d/q_1+d/2)} \,.$$

Given (4.7) and (4.8) Bourgain's interpolation argument shows that

$$(4.9) \qquad \Big\|\sum_{j>0} \mathcal{S}_j^\lambda\Big\|_{L^{\rho_1,1} \to L^{\sigma_1,\infty}} + \Big\|\sum_{j>0} \mathcal{S}_j^\lambda\Big\|_{L^{\sigma_1',1} \to L^{\rho_1',\infty}} \lesssim \lambda^{-d(1-\frac{1}{\rho_1}+\frac{1}{\sigma_1})}$$

where ρ_1, σ_1 are as in (2.1), with $a = d-1$, $b = (\kappa+1)/2$. Then $(\frac{1}{\rho_1}, \frac{1}{\sigma_1}) = (\frac{1-\gamma}{q_1'} + \gamma, \frac{1-\gamma}{2})$ with $\gamma = (2+\kappa)^{-1}$. Since $1 - \rho_1^{-1} + \sigma_1^{-1} = 2q_1^{-1}$ we get (4.6).

It remains to establish the estimates (4.7) and (4.8). Again, (4.7) follows using the method of stationary phase and the better bound is due to the condition (4.5). The estimate (4.8) is proved analogously to (3.11) above. The phase functions Ψ, θ, Θ as well as the operators $\mathcal{R}_\mu$, $\mathcal{V}_\ell^\mu$, $\widetilde{\mathcal{V}}_\ell^\mu$ and then

$\mathcal{W}^\ell$ are defined as in the proof of Proposition 3.1, with the exception that all y integrations are over a small open set in $\mathbb{R}^d$ (instead $\mathbb{R}^{d-1}$ above). We need to show the analogue of (3.16), namely that the $L^{q'_1,1} \to L^{q_1,\infty}$ operator norm of $\mathcal{R}_\mu \mathcal{R}_\mu^*$ is $O(\mu^{1-d-2d/q_1})$. As above this follows from the analogues of (3.21), (3.22) which read now

$$(4.10) \qquad \left\| \mu^{d-1} \mathcal{V}_\ell^\mu \right\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \lesssim 2^{-\ell(\kappa+1)/2} \, ,$$

$$(4.11) \qquad \left\| \mu^{d-1} \mathcal{V}_\ell^\mu \right\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \lesssim 2^{\ell/2} \mu^{-d} \, .$$

The stationary phase argument using (4.5) implies (4.10).

Turning to (4.11) the previous rescaling argument reduces matters to a better estimate for the $\mathcal{W}^\ell$, namely the bound

$$(4.12) \qquad \|\mathcal{W}^\ell\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \lesssim 2^{-\ell(d-\frac{1}{2})} \, .$$

The gain of a factor of $2^{-\ell/2}$ compared to (3.25) (already crucial in [12]) comes from the fact that we now have d frequency variables y and that we can use the fold condition for π_L. We freeze $\omega_d, \widetilde{\omega}_d$ and observe that in the domain of Θ

$$(4.13) \qquad |\omega' - \widetilde{\omega}'| \ll |\omega_d - \widetilde{\omega}_d| \approx 1.$$

As before we check the standard condition for a phase function parametrizing a canonical graph for the phase $(\omega', \widetilde{\omega}', y) \mapsto \Theta(\omega, \widetilde{\omega}, y)$. That is, the determinant of

$$\begin{pmatrix} \Theta_{\omega'\widetilde{\omega}'} & \Theta_{\omega'y'} & \Theta_{\omega'y_d} \\ \Theta_{y'\widetilde{\omega}'} & \Theta_{y'y'} & \Theta_{y'y_d} \\ \Theta_{y_d\widetilde{\omega}'} & \Theta_{y_dy'} & \Theta_{y_dy_d} \end{pmatrix} = \begin{pmatrix} 0 & \Phi_{x'y'} & \Phi_{x'y_d} \\ \Phi_{y'x'} & \langle\omega - \widetilde{\omega}, \Phi_{xy'y'}\rangle & \langle\omega - \widetilde{\omega}, \Phi_{xy'y_d}\rangle \\ \Phi_{y_dx'} & \langle\omega - \widetilde{\omega}, \Phi_{xy_dy'}\rangle & \langle\omega - \widetilde{\omega}, \Phi_{xy_dy_d}\rangle \end{pmatrix} \Bigg|_{(0,0)}$$
$$+ O(\varepsilon)$$

is bounded away from 0. It is easily seen from (4.1), (4.3) that this determinant is equal to $\langle\omega - \widetilde{\omega}, \Phi_{xy_dy_d}\rangle + O(\varepsilon)$ and by (4.13) this is equal to $(\omega_d - \widetilde{\omega}_d)\Phi_{x_dy_dy_d} + O(\varepsilon)$. Thus the canonical graph condition is satisfied by the fold condition (4.2) and (4.12) follows from Lemma 2.3 in [12]. $\square$

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