MULTILINEAR SINGULAR INTEGRAL FORMS OF CHRIST-JOURNÉ TYPE

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Abstract. We prove $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_{n+2}}(\mathbb{R}^d)$ polynomial growth estimates for the Christ-Journé multilinear singular integral forms and suitable generalizations.

Contents

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1. Introduction

1.1. **The d-commutators.** Let $0 < \epsilon < 1$ and let $\kappa \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ be a regular Calderón-Zygmund convolution kernel on $\mathbb{R}^d$, satisfying the standard size and regularity assumptions,

$$|\kappa(x)| \le C|x|^{-d}, \quad x \ne 0, \tag{1.1a}$$

$$|\kappa(x+h) - \kappa(x)| \le C\frac{|h|^\epsilon}{|x|^{d+\epsilon}}, \quad x \ne 0,\ |h| \le \frac{|x|}{2}, \tag{1.1b}$$

and the L^2 boundedness condition

$$\|\widehat{\kappa}\|_\infty \le C < \infty. \tag{1.1c}$$

Let $\|\kappa\|_{CZ(\epsilon)}$ be the smallest constant C for which the three inequalities (1.1) hold simultaneously. For convenience, in order to a priori make sense of some of the expressions in this introduction the reader may initially assume that κ is compactly supported in $\mathbb{R}^d \setminus \{0\}$.

For $a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ let $m_{x,y}a$ be the mean of a over the interval connecting x and y,

$$m_{x,y}a = \int_0^1 a(sx + (1-s)y)ds.$$

For every $y \in \mathbb{R}^d$ this is well defined for almost all $x \in \mathbb{R}^d$. Given L^∞-functions $a_1, \dots, a_n$ on $\mathbb{R}^d$ the *nth order d-commutator* associated to $a_1, \dots, a_n$, is defined by

$$\mathcal{C}[a_1, \dots, a_n]f(x) = \int \kappa(x-y)\Big(\prod_{i=1}^n m_{x,y}a_i\Big) f(y)dy.$$

One may consider $\mathcal{C}$ as an $(n+1)$-linear operator acting on $a_1, \dots, a_n, f$. Pairing with another function and renaming $a_i = f_i$, $i \le n$, $f = f_{n+1}$ one obtains the *Christ-Journé multilinear form* defined by

$$\Lambda_{\mathrm{CJ}}(f_1, \dots, f_{n+2}) = \iint \kappa(x-y)\Big(\prod_{i=1}^n m_{x,y}f_i\Big) f_{n+1}(y) f_{n+2}(x)\, dx\, dy\,. \tag{1.2}$$

In dimension $d = 1$ this operator reduces to the Calderón commutator. However the emphasis in this paper is on the behavior in dimension $d \ge 2$ where the Schwartz kernels are considerably less regular. Christ and Journé [7] showed that for a_i with $\|a_i\|_\infty \le 1$ the operator $\mathcal{C}[a_1, \dots, a_n]$ is bounded on L^p, $1 < p < \infty$, with operator norm $O(n^\alpha)$, for $\alpha > 2$. More precisely,

$$\big|\Lambda_{\mathrm{CJ}}(f_1, \dots, f_{n+2})\big| \le C_{p,\epsilon,\alpha}\|K\|_{CZ(\epsilon)}\, n^\alpha \Big(\prod_{i=1}^n \|f_i\|_\infty\Big)\|f_{n+1}\|_p \|f_{n+2}\|_{p'}, \quad \alpha > 2. \tag{1.3}$$

For related results on Calderón commutators for $d = 1$ see the discussion of previous results in §1.2 below.

The form Λ_{CJ} is not symmetric in f_i, $i = 1, \dots, n+2$, (see the discussion in §1.3 below) and it is natural to ask whether the analogous estimates hold for $f_i \in L^{p_i}$, for other choices of p_i. The problem has been proposed for example in [14] and [18], see also §1.2 for our motivation. Homogeneity considerations yield the necessary condition $\sum_{i=1}^{n+2} p_i^{-1} = 1$. In this paper we shall establish the following estimate, as a corollary of a more general result stated as Theorem 2.8 below.

Theorem 1.1. *Suppose that $d \ge 1$, $1 < p_i \le \infty$, $i = 1, \dots, n+2$, and $\sum_{i=1}^{n+2} p_i^{-1} = 1$. Let $\epsilon > 0$ and* $\min\{p_1, \dots, p_{n+2}\} \ge 1 + \delta$. *Then for Λ as in* (1.2)

$$\big|\Lambda_{\mathrm{CJ}}(f_1, \dots, f_{n+2})\big| \le C(\delta)\|\kappa\|_{CZ(\epsilon)} n^2 \log^3(2+n) \prod_{i=1}^{n+2} \|f_i\|_{p_i}\,. \tag{1.4}$$

Our main interest lies in the higher dimensional cases with $d \ge 2$. Polynomial bounds are known for $d = 1$, although the precise form of Theorem 1.1 may not have been observed before; see the discussion about previous results in §1.2.

1.2. **Background and historical remarks.**

Motivation. Our original motivation for considering estimates (1.4) for $p_i \ne \infty$ for $i \le n$ came from Bressan's problem ([4]) on incompressible mixing flows. A version of the approach chosen by Bianchini [2] leads in higher dimensions to the problem of bounding a trilinear singular integral form with even homogeneous kernels κ. One considers a smooth, time-dependent vector field $(x,t) \mapsto \vec{b}(x,t)$ which is periodic, i.e. $\vec{b}(x+k,t) = \vec{b}(x,t)$ for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, $k \in \mathbb{Z}^d$, and divergence-free, $\sum_{i=1}^d \frac{\partial b_i}{\partial x_i} = 0$. Let ϕ be the flow generated by v, i.e. we have $\frac{\partial}{\partial t}\phi_t(x) = v(\phi_t(x), t)$, $\quad \phi_0(x) = x$, so that for every t the map ϕ_t is a diffeomorphism on $\mathbb{R}^d$ satisfying $\phi(x+k,t) = k + \phi(x,t)$, for all $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$.

For small ε consider the truncated Bianchini semi-norm ([2]) defined by

$$B_\varepsilon[f] = \int_\varepsilon^{1/4} \int_Q \Big| f(x) - \fint_{B_r(x)} f(y)dy \Big| \, dx \, \frac{dr}{r} \, .$$

Let A be a measurable subset of $\mathbb{R}^d$ which is invariant under translation by vectors in $\mathbb{Z}^d$ (thus $A + \mathbb{Z}^d$ can be identified with a measurable subset of $\mathbb{T}^d$). Let $A^\complement = \mathbb{R}^d \setminus A$.

A calculation ([22]) shows that

$$(1.5) \quad B_\varepsilon[\mathbb{1}_{\phi_T(A)}] - B_\varepsilon[\mathbb{1}_A] = \\ V_d^{-1} \int_0^T \int_Q f(x,t) \int_{\varepsilon \le |x-y| \le 1/4} \frac{\langle x-y, \vec{b}(x,t) - \vec{b}(y,t) \rangle}{|x-y|^{d+2}} f(y,t) \, dy \, dx \, dt$$

where $Q = [0,1)^d$, $f(y,t) = \frac{1}{2}(\mathbb{1}_{\phi_t(A)} - \mathbb{1}_{\phi_t(A)^\complement})$ and V_d is the volume of the unit ball in $\mathbb{R}^d$.

This calculation leads to an alternative approach to a result by Crippa and DeLellis [12]. One has the following estimate involving general (a priori) smooth vector fields $x \mapsto v(x)$ on $\mathbb{R}^d$ satisfying $\mathrm{div}(v) = 0$. Let Dv denote its total derivative. Then for $1 < p_1, p_2, p_3 \le \infty$, $\sum_{i=1}^3 p_i^{-1} = 1$,

$$(1.6) \qquad \Big| \iint_{\varepsilon < |x-y| < N} \frac{\langle v(x) - v(y), x-y \rangle}{|x-y|^{d+2}} f(y) g(x) \, dy \, dx \Big| \lesssim \|Dv\|_{p_1} \|f\|_{p_2} \|g\|_{p_3}.$$

Here the implicit constant is independent of ε and N. One can think of (1.6) as a trilinear form acting on f, g and Dv; due to the assumption of zero divergence, the entries are not independent and one can reduce to the estimation of $d^2 - 1$ trilinear forms. In fact, (1.6) can be derived from the case $n = 1$ of Theorem 1.1, using the choices of

$$(1.7) \qquad \begin{aligned} \kappa_{ij}(x) &= \frac{x_i x_j}{|x|^{d+2}}, \quad i \ne j, \\ \kappa_i(x) &= \frac{x_i^2 - x_d^2}{|x|^{d+2}}, \quad 1 \le i < d. \end{aligned}$$

The case with f, g being characteristic functions of sets with finite measure and $Dv \in L^{p_1}$ with p_1 near 1 is of particular interest. Steve Hofmann (personal communication) has suggested that estimates such as (1.6) can also be obtained from the isotropic version of his off-diagonal $T1$ theorem [26].

Previous results. We list some previous results on the $n+2$-linear form Λ_{CJ} in (1.3), including many in dimension $d = 1$, covering the classical Calderón commutators.

(i) The first estimates of the form (1.4), for the case $d = 1$ and $n = 1$ were proved in the seminal paper by A.P. Calderón [5].

(ii) More generally, still in dimension $d = 1$, Coifman, McIntosh and Meyer [10] proved estimates of the form (1.4) for arbitrary n, with $p_1 = \cdots = p_n = \infty$ and polynomial bounds $C(n) = O(n^4)$ as $n \to \infty$. This allowed them to establish the L^2 boundedness of the Cauchy integral operator on general Lipschitz curves. See also [8] for other applications to related problems of Calderón. Christ and Journé [7] were able to improve the Coifman-McIntosh-Meyer bounds to $C(n) = O(n^{2+\varepsilon})$ (and to $O(n^{1+\varepsilon})$ for odd kernels κ).

(iii) Duong, Grafakos and Yan [14] developed a rough version of the multisingular integral theory in [21] to cover the estimates (1.4) with general exponents for $d = 1$, however their arguments yield constants $C(n)$ which are of exponential growth in n.

One should note that [14] also treats the higher Calderón commutators $\mathcal{C}[f_1, \ldots, f_n]$, with target space L^p where $p > 1/2$. For the bilinear version this had been first done by C.P. Calderón [6]. It would be interesting to obtain appropriate similar results for the d-commutators.

(iv) Muscalu [31] recently developed a new approach for proving (1.4) in dimension $d = 1$, see also [32, Theorem 4.11]. An explicit bound for the constant as $A(n, \ell)$ where ℓ is the number of indices j such that $p_j \neq \infty$ and, for fixed ℓ, $n \mapsto A(n, \ell)$ is of polynomial growth. However, by using complex interpolation (as in §15) to the case when $p_j = \infty$ for all but two j, one may remove the dependance of A on ℓ. This yields polynomial bounds for all admissible sets of exponents, as in our results.

(v) As mentioned above, crucial results for $d \geq 2$ were obtained by Christ and Journé [7] who established (1.4) for $p_1 = \cdots = p_n = \infty$ and $C(n) = O(n^{2+\varepsilon})$. Several ideas in our proof can be traced back to their work.

(vi) Hofmann [25] obtained estimates (1.4) for operators with rougher kernels κ, and an extension to weighted norm inequalities; however the induction argument in [25] only gives exponential bounds as $n \to \infty$.

(vii) For the special case that κ is an *odd and homogeneous* singular convolution kernel, estimates of the form (1.4) for $d \geq 2$ and $n = 1$ have been obtained by using the method of rotation. In [14], Duong, Grafakos and Yan use uniform results on the bilinear Hilbert transforms ([20], [37]) to obtain such estimates under the additional restriction $\min(p_1, p_2, p_3) > 3/2$, see also the survey [18].

We note that one can modify the argument in [14] to remove this restriction, and also to obtain a version for $n \geq 2$. Indeed let $\kappa_\Omega(x) = |x|^{-d}\Omega(x/|x|)$ with $\Omega \in L^1(S^{d-1})$ and $\Omega(\theta) = -\Omega(-\theta)$. Let

$$\mathcal{C}_\Omega[f_1, \ldots, f_n]f_{n+1}(x) = \int \kappa_\Omega(x - y) f_{n+1}(y) \prod_{i=1}^{n} \int_0^1 f_i((1 - s_i)x + s_i y) ds_i \, dy;$$

then

$$\mathcal{C}_\Omega[f_1, \ldots, f_n]f_{n+1}(x) = \frac{1}{2} \int_{S^{d-1}} \Omega(\theta)\, \mathcal{C}_\theta[f_1, \ldots, f_n, f_{n+1}](x) \, d\theta \tag{1.8}$$

where

$$\mathcal{C}_\theta[f_1, \ldots, f_{n+1}](x) = p.v. \int_{-\infty}^{\infty} f_{n+1}(x - s\theta) \Big(\prod_{i=1}^{n} \int_0^1 f_i(x - us\theta) du \Big) \frac{ds}{s}$$

Now if $e_1 = (1, 0, \ldots, 0)$ and R_θ is a rotation with $R_\theta e_1 = \theta$ we have

$$\mathcal{C}_\theta[f_1, \ldots, f_{n+1}](x) = \mathcal{C}_{e_1}[f_1 \circ R_\theta, \ldots, f_{n+1} \circ R_\theta](R_\theta^{-1} x)$$

and thus the operator norms of $\mathcal{C}_\theta$ are independent of θ. One notices that

$$\mathcal{C}_{e_1}[f_1, \ldots, f_{n+1}](x_1, x') = p.v. \int_{-\infty}^{\infty} \frac{1}{x_1 - y_1} f_{n+1}(y_1, x') \prod_{i=1}^{n} \Big(\int_0^1 f_i((1 - u)x_1 + uy_1, x') du \Big) dy_1 ,$$

the Calderón commutator acting in the first variable. The one-dimensional results for the commutators in [5], [14] can now be applied to show that for $\sum_{i=1}^{n+2} p_i^{-1} = 1$, $p_i > 1$,

$$\Big| \int \mathcal{C}_\Omega[f_1, \ldots, f_n] f_{n+1}(x) f_{n+2}(x) dx \Big| \lesssim C(p_1, \ldots, p_{n+2}) \|\Omega\|_{L^1(S^{d-1})} \prod_{i=1}^{n+2} \|f_i\|_{L^{p_i}}.$$

Note that the assumption κ odd is crucial in formula (1.8) and thus the argument does not seem to be applicable to the d-commutators associated with the convolution kernels in (1.7).

(viii) When $n = 1$ it is known that the Christ-Journé commutator $\mathcal{C}[a]$ (with $a \in L^\infty$) is of weak type $(1,1)$. This has been shown by Grafakos and Honzík [19] in two dimensions and by one of the authors [34] in all dimensions. It is an open problem whether the higher degree d-commutators ($n \geq 2$) are of weak type $(1,1)$ in dimension $d \geq 2$.

1.3. **Towards a more general result.** In order to prove Theorem 1.1 it suffices to prove estimate (1.4) for the cases where two of the exponents, say p_i, p_j, $1 \leq i < j \leq n+2$ belong to $(1,\infty)$ and the other n exponents are equal to ∞. Equivalently, if ϖ is a permutation of $\{1,\dots,n+2\}$ and

$$\Lambda_{\mathrm{CJ}}^{\varpi}(f_1,\dots,f_{n+2}) = \Lambda_{\mathrm{CJ}}(f_{\varpi(1)},\dots,f_{\varpi(n+2)})$$

one has to show, for $1 < p < \infty$, the inequalities

$$\Lambda_{\mathrm{CJ}}^{\varpi}[f_1,\dots,f_{n+2}]\big| \leq C_{\delta,p} n^2 (\log n)^3 \|\kappa\|_{CZ(\delta)} \Big(\prod_{i=1}^{n} \|f_i\|_\infty\Big) \|f_{n+1}\|_p \|f_{n+2}\|_{p'}\,, \tag{1.9}$$

uniformly in ϖ.

Formally the operator $\Lambda_{\mathrm{CJ}}^{\varpi}$ takes the form

$$\Lambda_{\mathrm{CJ}}^{\varpi}(f_1,\dots,f_{n+2}) = \iiint K^{\varpi}(\alpha, x-y) f_{n+2}(x) f_{n+1}(y) \prod_{i=1}^{n} f_i(x - \alpha_i(x-y))\, d\alpha\, dx\, dy. \tag{1.10}$$

The case $\varpi = id$ in (1.9) is covered already by the original result of Christ and Journé. Thus by the symmetry in $\{1,\dots,n\}$ and essential symmetry in $\{n+1, n+2\}$ (with a change of variable $\alpha_j \mapsto (1-\alpha_j)$) two cases remain of particular interest:

- If ϖ^i is the permutation that interchanges i and $n+1$ and leaves all $k \notin \{i, n+1\}$ fixed then the kernel K^{ϖ^i} is given by
$$K^{\varpi^i}(\alpha, v) = \begin{cases} |\alpha_i|^{d-n-1} \kappa(\alpha_i v) & \text{if } \alpha_i \geq 1,\ 0 \leq \alpha_j \leq \alpha_i,\ j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$
- If $1 \leq i, j \leq n$, $i \neq j$ and ϖ^{ij} is the permutation with $\varpi^{ij}(i) = n+1$, $\varpi^{ij}(j) = n+2$ and $\varpi^{ij}(k) = k$ for $k \notin \{i, j, n+1, n+2\}$ then the kernel $K^{\varpi^{ij}}$ is given by
$$\begin{aligned} K^{\varpi^{ij}}(\alpha, v) &= |\alpha_i - \alpha_j|^{d-n-1} \kappa((\alpha_i - \alpha_j)(x-y)) \\ &\qquad \text{either if } \alpha_i \leq 0,\ \alpha_j \geq 1,\ \alpha_i \leq \alpha_k \leq \alpha_j \text{ for } k \neq i, j; \\ &\qquad \text{or if } \alpha_j \leq 0,\ \alpha_i \geq 1,\ \alpha_j \leq \alpha_k \leq \alpha_i \text{ for } k \neq i, j; \\ K^{\varpi^{ij}}(\alpha, v) &= 0 \text{ otherwise.} \end{aligned}$$

Once (1.9) is proved for $\varpi = id$, $\varpi = \varpi^i$, $\varpi = \varpi^{ij}$, the general result follows by complex interpolation for multilinear operators, see [1, Theorem 4.4.1].

Thus we want to study multilinear forms of the type

$$\Lambda[K](b_1,\dots,b_{n+2}) = \iiint K(\alpha, x-y) b_{n+2}(x) b_{n+1}(y) \prod_{i=1}^{n} b_i(x - \alpha_i(x-y))\, d\alpha\, dx\, dy, \tag{1.11}$$

where $x \in \mathbb{R}^d$, $\alpha \in \mathbb{R}^n$, and $K(\alpha, x)$ is a Calderón-Zygmund kernel in the x variable which depends on a parameter $\alpha \in \mathbb{R}^n$. We will impose some regularity conditions on the α variable. The basic example, corresponding to the Christ-Journé multilinear forms, is

$$K(\alpha, x) = \mathbb{1}_{[0,1]^n}(\alpha) \kappa(x)$$

where κ is a regular Calderón convolution kernel satisfying the conditions (1.1).

Our goal is to

- To introduce a reasonably general class $\mathcal{K}_\varepsilon$ of kernels $K(\alpha, x)$, for which linear forms of type (1.11) are closed under adjoints. If ϖ is a permutation of $\{1, \dots, n+2\}$, then the multilinear form $\Lambda[K](b_{\varpi(1)}, \dots, b_{\varpi(n+2)})$ should be written as $\Lambda[K^\varpi](b_1, \dots, b_{n+2})$ for a suitable K^ϖ, with appropriate bounds on K^ϖ in the class $\mathcal{K}_\varepsilon$.
- To prove estimates for this same class of kernels that cover the estimates for the d-commutators in Theorem 1.1.

Roughly the class of admissible kernels consists of those K for which the norm $\|\cdot\|_{\mathcal{K}_\varepsilon}$ defined in (2.3), (2.4) below is finite; see §2 for further discusion of the spaces of distributions on which this definition is made. The extension to the class $\mathcal{K}_\varepsilon$ allows us to substantially extend the class of allowable convolution kernels κ in the definition of the d-commutators, see Example 2.2 below.

Let $p_1, \dots, p_{n+2} \in (1, \infty]$ with $\sum_{j=1}^{n+2} p_j^{-1} = 1$, and let $p_0 = \min_{1 \le j \le n+2} p_j$. For $b_j \in L^{p_j}(\mathbb{R}^d)$ we shall prove the inequality

$$|\Lambda[K](b_1, \dots, b_{n+2})| \le C_{p_0,d,\varepsilon} \|K\|_{\mathcal{K}_\varepsilon} n^2 \log^3(2+n) \prod_{i=1}^{n+2} \|b_i\|_{p_i}. \tag{1.12}$$

The expression on the left hand side makes a priori sense at least for K supported in a compact subset of $\mathbb{R}^N \times (\mathbb{R}^d \setminus \{0\})$ (and this restriction does not enter in the estimate). The kernels in $\mathcal{K}_\varepsilon$ can be thought of sums of dilates of functions in a weighted Besov space; this will be made precise in §3. These weighted Besov spaces are closely related to Besov spaces of forms on $\mathbb{RP}^{n+d}$. This motivated some of the considerations in §3 and §4.

A key point of the $\mathcal{K}_\varepsilon$ norms is that they depend on n in a natural way so that the term $n^2 \log^3(2+n)$ in (1.12) does not become trivial. We shall derive a stronger version in the next section in Theorem 2.10 below in which dependence on the $\mathcal{K}_\varepsilon$ occurs in a very weak (logarithmic) way. In fact one can define an endpoint space $\mathfrak{K}_0$ which contains the union of the spaces $\mathcal{K}_\varepsilon$, so that the inequality

$$|\Lambda[K](b_1, \dots, b_{n+2})| \le C_{p_0,d,\varepsilon} \|K\|_{\mathfrak{K}_0} n^2 \log^3 \big(2 + n \frac{\|K\|_{\mathcal{K}_\varepsilon}}{\|K\|_{\mathfrak{K}_0}}\big) \prod_{i=1}^{n+2} \|b_i\|_{p_i}. \tag{1.13}$$

holds. A crucial point about the classes $\mathcal{K}_\varepsilon$ is that if K belongs to K_ε then all K^ϖ in (1.10) belong to some $\mathcal{K}_{\epsilon'}$ class with polynomial bound in n. One can then see that if inequality (1.13) holds for $(p_1, \dots, p_{n+2}) = (\infty, \dots, \infty, p_0, p_0')$ then the same is true for the kernels K^ϖ. This invariance under adjoints will be discussed in §4.

The strategy of proving (1.13) for $p_1 = \dots = p_n = \infty$ then follows Christ and Journé [7], with the main inequalities outlined in §5. The subsequent sections contain the details of the proofs.

Selected Notation

- We use the notation $A \lesssim B$ to denote $A \le CB$, where C is a constant independent of any relevant parameters. C is allowed to depend on d and ε, but *not on* n.
- For two nonnegative numbers a, b we occasionally write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$
- The Euclidean ball in $\mathbb{R}^d$ of radius r and with center x is denoted by $B^d(x, r)$.
- For a function g on $\mathbb{R}^d$ we define dilation operators which leave the $L^1(\mathbb{R}^d)$ norm invariant by
$$g^{(t)}(x) := t^d g(tx).$$

- For a function ς on $\mathbb{R}^n \times \mathbb{R}^d$ we define dilation operators in the x-variable by
$$\varsigma^{(t)}(\alpha, x) := t^d \varsigma(\alpha, tx).$$
- For a kernel K on $\mathbb{R}^d \times \mathbb{R}^d$ we define dilated versions by
$$\mathrm{Dil}_t K(x,y) := t^d K(tx, ty)\,.$$
- Given Banach spaces E_1, E_2 we denote by $\mathcal{L}(E_1, E_2)$ the Banach space of bounded linear operators from E_1 to E_2.
- We denote by $C_0^\infty(\mathbb{R}^d)$ the space of compactly supported C^∞ functions. The subspace $C_{0,0}^\infty(\mathbb{R}^d)$ consists of all $f \in C_0^\infty(\mathbb{R}^d)$ with $\int f(x)dx = 0$.
- Let $\mathcal{V}$ be an index set, and for each $\nu \in \mathbb{Z}$, let $\{\varSigma_N^\nu\}$ be a sequence of operators in $\mathcal{L}(E_1, E_2)$. We say that $\varSigma_N^\nu$ converges in the *strong operator topology* to $\varSigma^\nu \in \mathcal{L}(E_1, E_2)$, with *equiconvergence with respect to* $\mathcal{V}$, if for every $f \in E_1$ and every $\varepsilon > 0$ there exists a positive integer $N(\varepsilon, f)$ such that $\|\varSigma_N^\nu f - \varSigma^\nu f\|_{E_2} < \varepsilon$ for all $N > N(\varepsilon, f)$, $\nu \in \mathcal{V}$.

  Given bounded operators $T_j^\nu \in \mathcal{L}(E_1, E_2)$, $j \in \mathbb{Z}$, we say that $\sum_j T_j^\nu$ converges in the strong operator topology, with equiconvergence with respect to $\mathcal{V}$, if the sequence of partial sums $\varSigma_N = \sum_{j=-N}^N T_j^\nu$ converges in the strong operator topology with equiconvergence with respect to $\mathcal{V}$.
- Given bounded k-linear operators L, L_N, defined on a k-tuple $(A_1, \dots, A_k)$ of normed spaces with values in a normed space B, we say that L_N converges to L in the strong operator topology (as $N \to \infty$) if $\|L_N(a_1, \dots, a_k) - L(a_1, \dots, a_k)\|_B \to 0$ for all $(a_1, \dots, a_k) \in A_1 \times \dots \times A_k$. When $B = \mathbb{C}$ or $\mathbb{R}$ then there is no difference between strong and weak operator topologies, and we omit the word strong.
- The spaces $L\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^d)$ are defined in §2.1.
- The operators P_k, Q_k, $\mathcal{Q}_k$ and $\overline{Q}_k[u]$ are introduced in §6 (although Q_k is already used in earlier sections). The class $\mathfrak{U}$ is defined in Definition 6.2.
- The semi-norms $\|\cdot\|_{\mathcal{K}_{\varepsilon,i}}$, $i = 1,2,3,4,5$ and the spaces $\mathcal{K}_\varepsilon$ are defined in §2.1. The related spaces $\mathfrak{K}_\varepsilon$ are defined in §2.2.
- The semi-norms $\|\cdot\|_{\mathcal{B}_{\varepsilon,i}}$, $i = 1,2,3,4$, and the spaces $\mathcal{B}_\varepsilon$ are defined in §2.2.
- The Schur classes Int^1, Int^∞, $\mathrm{Int}^1_\varepsilon$, $\mathrm{Int}^\infty_\varepsilon$ and the regularity classes $\mathrm{Reg}^1_{\varepsilon,\mathrm{lt}}$, $\mathrm{Reg}^\infty_{\varepsilon,\mathrm{lt}}$, $\mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}$, $\mathrm{Reg}^\infty_{\varepsilon,\mathrm{rt}}$ are defined in §8.1.1.
- The singular integral classes SI, $\mathrm{SI}^1_\varepsilon$, $\mathrm{SI}^\infty_\varepsilon$ and annular integrability classes Ann^1, Ann^∞, $\mathrm{Ann}_{\mathrm{av}}$ are defined in §8.1.2.
- The Carleson condition for operators and norm $\|\cdot\|_{\mathrm{Carl}}$ is given in Definition 8.14. The atomic boundedness condition, with norm $\|\cdot\|_{\mathrm{At}}$ is given in Definition 8.15.
- The Op_ϵ, Op_0 norms are defined in §8.3.
- The notion of a Carleson function and the norm $\|\cdot\|_{carl}$ is given in definition 11.2.

2. Statements of the main results

2.1. **The classes $\mathcal{K}_\varepsilon$.** We first introduce certain classes of tempered distributions on $\mathbb{R}^n \times \mathbb{R}^d$ which satisfy integrability properties in the first (α-)variable and contain all kernels allowable in (1.11). For each $N \in \mathbb{N}_0$ consider the space $M\mathcal{S}'_N(\mathbb{R}^n \times \mathbb{R}^d)$ defined as normed spaces of tempered distributions K on $\mathbb{R}^n \times \mathbb{R}^d$ for which there is $C > 0$ so that for all $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^d)$

$$|\langle K, f\rangle| \le C \sup_{\alpha \in \mathbb{R}^n, x \in \mathbb{R}^d} \sum_{|\gamma| \le N} (1+|x|)^N |\partial_x^\gamma f(\alpha, x)|. \tag{2.1}$$

Here $\langle K, f\rangle$ denotes the pairing between distributions and test functions and the minimal C in (2.1) is the norm in $M\mathcal{S}'_N(\mathbb{R}^n \times \mathbb{R}^d)$. The space $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ is the space of tempered distributions K on $\mathbb{R}^n \times \mathbb{R}^d$ for which (2.1) holds *for some* $N \in \mathbb{N}$. Note that $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ can be seen as an inductive limit of the normed spaces $M\mathcal{S}'_N(\mathbb{R}^n \times \mathbb{R}^d)$, and this gives $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ the structure of a locally convex topological vector space. A net $\{f_\imath\}_{i\in\mathfrak{I}}$ is Cauchy in this topology if there exists an N so that all $f_\imath$ belong to $M\mathcal{S}'_N(\mathbb{R}^n \times \mathbb{R}^d)$ for some fixed N and so that $f_\imath$ is Cauchy in the norm topology of $M\mathcal{S}'_N(\mathbb{R}^n \times \mathbb{R}^d)$. It is easy to see the normed spaces $M\mathcal{S}'_N(\mathbb{R}^n \times \mathbb{R}^d)$ are complete and thus $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ is complete. Let $M(\mathbb{R}^n)$ be the space of bounded Borel measures on $\mathbb{R}^n$. $K \in M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ gives rise to a continuous linear operator $\beta_K : \mathcal{S}(\mathbb{R}^d) \to M(\mathbb{R}^n)$ defined by

$$\langle \beta_K(\phi_2), \phi_1\rangle := \langle K, \phi_1 \otimes \phi_2\rangle \text{ for } \phi_1 \in \mathcal{S}(\mathbb{R}^n),\ \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

Let $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ be the subspace of $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ consisting of those K for which $\beta_K(\phi_2) \in L^1(\mathbb{R}^n)$, for all $\phi_2 \in \mathcal{S}(\mathbb{R}^d)$. $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ is a closed subspace of $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ and inherits its complete locally convex topology.

We now define the Banach space $\mathcal{K}_\varepsilon$ used in (1.12). For $K \in L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ and $\eta \in \mathcal{S}(\mathbb{R}^d)$ it makes sense to write $K(\alpha, \cdot) * \eta$ for the convolution of K and η in the x-variable. This yields an L^1 function in the α variable, which depends smoothly on x. For $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^d)$, let

$$K^{(t)}(\alpha, x) := t^d K(\alpha, tx)$$

and we extend this to $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ by continuity in the usual way. Fix $\eta \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$\inf_{\theta\in S^{d-1}} \sup_{\tau>0} |\widehat{\eta}(\tau\theta)| > 0, \tag{2.2}$$

where $\widehat{\eta}$ denotes the Fourier transform of η.

Definition 2.1. Let η be as in (2.2), and $0 < \varepsilon \le 1$.

(i) Define five semi-norms by

$$\|K\|_{\mathcal{K}^\eta_{\varepsilon,1}} := \sup_{\substack{1\le i\le n\\ t>0}} \int (1 + |\alpha_i|)^\varepsilon \|\eta * K^{(t)}(\alpha, \cdot)\|_{L^2(\mathbb{R}^d)}\, d\alpha, \tag{2.3a}$$

$$\|K\|_{\mathcal{K}^\eta_{\varepsilon,2}} := \sup_{\substack{1\le i\le n\\ t>0\\ 0<h\le 1}} h^{-\varepsilon} \int \|\eta * [K^{(t)}(\alpha + he_i, \cdot) - K^{(t)}(\alpha, \cdot)]\|_{L^2(\mathbb{R}^d)}\, d\alpha, \tag{2.3b}$$

$$\|K\|_{\mathcal{K}_{\varepsilon,3}} := \sup_{\substack{1\le i\le n\\ R>0}} \iint_{R\le|x|\le 2R} (1 + |\alpha_i|)^\varepsilon |K(\alpha, x)|\, dx\, d\alpha, \tag{2.3c}$$

$$\|K\|_{\mathcal{K}_{\varepsilon,4}} := \sup_{\substack{1\le i\le n\\ R>0\\ 0<h\le 1}} h^{-\varepsilon} \iint_{R\le|x|\le 2R} |K(\alpha + he_i, x) - K(\alpha, x)|\, dx\, d\alpha, \tag{2.3d}$$

$$\|K\|_{\mathcal{K}_{\varepsilon,5}} := \sup_{\substack{R>2\\ y\in\mathbb{R}^d}} R^\varepsilon \iint_{|x|\ge R|y|} |K(\alpha, x - y) - K(\alpha, x)|\, dx\, d\alpha\,. \tag{2.3e}$$

(ii) The space $\mathcal{K}_\varepsilon$ is the subspace of $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ consisting of those K for which the norm

$$\|K\|_{\mathcal{K}_\varepsilon} := \|K\|_{\mathcal{K}^\eta_{\varepsilon,1}} + \|K\|_{\mathcal{K}^\eta_{\varepsilon,2}} + \|K\|_{\mathcal{K}_{\varepsilon,3}} + \|K\|_{\mathcal{K}_{\varepsilon,4}} + \|K\|_{\mathcal{K}_{\varepsilon,5}} \tag{2.4}$$

is finite.

The definition of $\|\cdot\|_{\mathcal{K}_\varepsilon}$ depends on a choice of $\eta \in \mathcal{S}(\mathbb{R}^d)$ satisfying (2.2). However, the equivalence class of the norm does not depend on the choice, and the constants in the equivalences of different choices of η will not depend on n. This is made explicit in Lemma 3.1 below.

Example 2.2. Let $\epsilon \in (0,1)$ and let $\kappa \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ be a convolution kernel in $\mathbb{R}^d$ satisfying

$$\|\widehat{\kappa}\|_\infty \le C \tag{2.5}$$

and

$$\sup_{R\ge 2} R^\epsilon \sup_{y\in\mathbb{R}^d} \int_{|x|\ge R|y|} |\kappa(x-y)-\kappa(x)|dx \le C. \tag{2.6}$$

Let

$$K(x,\alpha) = \chi_{[0,1]^n}(\alpha)\kappa(x)\,.$$

Then $K \in \mathcal{K}_\delta(\mathbb{R}^n \times \mathbb{R}^d)$ for $\delta < \epsilon$ and

$$\|K\|_{\mathcal{K}_\delta} \lesssim_{\delta,\epsilon} C. \tag{2.7}$$

The details of (2.7) are left to the reader.

We state a preliminary version of our boundedness result (see Theorem 2.8 below for a more definitive version).

Theorem 2.3. *Let $\varepsilon > 0$, $\delta > 0$ and η as in* (2.2)*.*

(i) There is a constant $C = C(d,\delta,\varepsilon,\eta)$ such that the following statement holds a priori for all kernels in $\mathcal{K}_\varepsilon$ which also belong to $L^1(\mathbb{R}^n \times \mathbb{R}^d)$. The multilinear form

$$\Lambda[K](b_1,\dots,b_{n+2}) = \iiint K(\alpha, x-y) b_{n+2}(x) b_{n+1}(y) \prod_{i=1}^n b_i(x-\alpha_i(x-y))\,d\alpha\,dx\,dy,$$

satisfies

$$|\Lambda[K](b_1,\dots,b_{n+2})| \le Cn^2\log^3(1+n)\|K\|_{\mathcal{K}_\varepsilon} \prod_{i=1}^{n+2} \|b_i\|_{p_i}\,, \tag{2.8}$$

for all $b_i \in L^{p_i}(\mathbb{R}^d)$, $1+\delta < p_i < \infty$, $\sum_{i=1}^{n+2} p_i^{-1} = 1$.

(ii) The multilinear form $(K, b_1,\dots,b_{n+2}) \mapsto \Lambda[K](b_1,\dots,b_{n+2})$ extends to a bounded multilinear form on $\mathcal{K}_\varepsilon \times L^{p_1} \times \dots \times L^{p_{n+2}}$ satisfying (2.8) *for all $K \in \mathcal{K}_\varepsilon$.*

The proof of Theorem 2.3 we will heavily rely on a decomposition theorem for the class $\mathcal{K}_\varepsilon$, to which we now turn. This decomposition will specify further part (ii) of the theorem, i.e. describe how to extend the result from part (i) to all kernels in $\mathcal{K}_\varepsilon$.

2.2. **Decomposition of kernels in $\mathcal{K}_\varepsilon$.** In the following definition $e_1,\dots,e_n$ will denote the standard basis of $\mathbb{R}^n$.

Definition 2.4. For $n, d \in \mathbb{N}$ and $0 \le \varepsilon \le 1$ we define four (semi-)norms

$$\|\varsigma\|_{\mathcal{B}_{\varepsilon,1}} := \max_{1\le i\le n} \iint (1+|\alpha_i|)^{\varepsilon} |\varsigma(\alpha, v)|\, d\alpha\, dv, \tag{2.9a}$$

$$\|\varsigma\|_{\mathcal{B}_{\varepsilon,2}} := \sup_{\substack{0<h\le 1\\ 1\le i\le n}} h^{-\varepsilon} \iint |\varsigma(\alpha + he_i, v) - \varsigma(\alpha, v)|\, d\alpha\, dv, \tag{2.9b}$$

$$\|\varsigma\|_{\mathcal{B}_{\varepsilon,3}} := \sup_{0<|h|\le 1} |h|^{-\varepsilon} \iint |\varsigma(\alpha, v+h) - \varsigma(\alpha, v)|\, d\alpha\, dv, \tag{2.9c}$$

$$\|\varsigma\|_{\mathcal{B}_{\varepsilon,4}} := \iint (1+|v|)^{\varepsilon} |\varsigma(\alpha, v)|\, d\alpha\, dv. \tag{2.9d}$$

Let $\mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ be the space of those $\varsigma \in L^1(\mathbb{R}^n \times \mathbb{R}^d)$ such that the norm

$$\|\varsigma\|_{\mathcal{B}_\varepsilon} := \|\varsigma\|_{\mathcal{B}_{\varepsilon,1}} + \|\varsigma\|_{\mathcal{B}_{\varepsilon,2}} + \|\varsigma\|_{\mathcal{B}_{\varepsilon,3}} + \|\varsigma\|_{\mathcal{B}_{\varepsilon,4}} \tag{2.10}$$

is finite.

For $0 < \varepsilon < 1$ the space $\mathcal{B}_\epsilon$ is a type of Besov space, hence the notation. See also §4.5 below. Recall the notation $\varsigma^{(t)}(\alpha, x) := t^d \varsigma(\alpha, tx)$.

Definition 2.5. (i) Let $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\int \phi(x)dx = 1$, let Q_j denote the operator of convolution with $2^{jd}\phi(2^j\cdot) - 2^{(j-1)d}\phi(2^{j-1}\cdot)$. When acting on $K \in LS'(\mathbb{R}^n \times \mathbb{R}^d)$, we define Q_jK by taking the convolution in $\mathbb{R}^d$.

(ii) Set

$$\varsigma_j[K] := (Q_jK)^{(2^{-j})}. \tag{2.11}$$

(iii) For $K \in LS'(\mathbb{R}^n \times \mathbb{R}^d)$ let

$$\|K\|_{\mathfrak{K}_0} = \sup_{j\in\mathbb{Z}} \|\varsigma_j[K]\|_{L^1(\mathbb{R}^n\times\mathbb{R}^d)}\,. \tag{2.12}$$

(iv) Let $\mathfrak{K}_\varepsilon$ be the space of all $K \in LS'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|K\|_{\mathfrak{K}_\varepsilon} := \sup_{j\in\mathbb{Z}} \|\varsigma_j[K]\|_{\mathcal{B}_\varepsilon(\mathbb{R}^n\times\mathbb{R}^d)}$$

is finite.

The relation between the spaces $\mathcal{K}_\varepsilon$ and $\mathfrak{K}_\varepsilon$ is given in the following theorem.

Theorem 2.6. *(i) A distribution $K \in L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ belongs to $\bigcup_{0<\varepsilon<1} \mathcal{K}_\varepsilon$ if and only if there exists an $\varepsilon > 0$ and a bounded set $\{\varsigma_j : j \in \mathbb{Z}\} \subset \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ satisfying*

$$\int \varsigma_j(\alpha, v)\, dv = 0$$

for all j, α and

$$K = \sum_{j\in\mathbb{Z}} \varsigma_j^{(2^j)},$$

holds with convergence in the topology on $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ (and thus also in the sense of distributions).

(ii) Let $K \in \mathcal{K}_\varepsilon$. Then for $\delta < \varepsilon$,

$$\|K\|_{\mathfrak{K}_\delta} \le C_{\delta,\varepsilon,d} \|K\|_{\mathcal{K}_\varepsilon}\,.$$

(iii) Let $K \in \mathfrak{K}_\epsilon$. Then for $\delta < \epsilon/2$

$$\|K\|_{\mathcal{K}_\delta} \le C_{\delta,\varepsilon,d} \|K\|_{\mathfrak{K}_\epsilon}\,.$$

2.3. **Boundedness of multilinear forms.** For any $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ and for $b_i \in L^{p_i}(\mathbb{R}^d)$ with $\sum_{i=1}^{n+2} p_i^{-1} = 1$ the multilinear form

$$\Lambda[\varsigma](b_1, \dots, b_{n+2}) = \iiint \varsigma(\alpha, x-y) b_{n+2}(x) b_{n+1}(y) \prod_{i=1}^{n} b_i(x - \alpha_1(x-y))\, dx\, dy\, d\alpha$$

is well defined; more precisely we have

Lemma 2.7. *Let $\varsigma \in L^1(\mathbb{R}^n \times \mathbb{R}^d)$. Suppose for $1 \le l \le n+2$, $b_i \in L^{p_i}(\mathbb{R}^d)$ with $\sum_{i=1}^{n+2} p_i^{-1} = 1$. Then, for all $j \in \mathbb{Z}$,*

$$\left|\Lambda[\varsigma^{(2^j)}](b_1, \dots, b_{n+2})\right| \le \|\varsigma\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)} \prod_{i=1}^{n+2} \|b_i\|_{p_i}.$$

Proof. This follows easily by Hölder's inequality. □

Theorem 2.6 suggests to define the form $\Lambda[K]$, for $K \in \mathcal{K}_\varepsilon$, as the limit of partial sums

$$\sum_{j=-N}^{N} \Lambda[\varsigma_j^{(2^j)}](b_1, \dots, b_{n+2}) \tag{2.13}$$

as $N \to \infty$.

Our main boundedness result (a sharper version of Theorem 2.3) is

Theorem 2.8. *Let $0 < \delta < 1$, let $p_1, \dots, p_{n+2} \in [1+\delta, \infty]$ with $\sum_{l=1}^{n+2} p_l^{-1} = 1$.*

(i) Let $\mathfrak{I}$ be a finite subset of $\mathbb{Z}$ and let $\{\varsigma_j : j \in \mathfrak{I}\}$ be a subset of $\mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ so that for every $j \in \mathfrak{I}$, $\int \varsigma_j(\alpha, x)\, dx = 0$ for almost all $\alpha \in \mathbb{R}^n$. Let

$$K_{\mathfrak{I}} = \sum_{j \in \mathfrak{I}} \varsigma_j^{(2^j)}.$$

Then for $b_l \in L^{p_l}(\mathbb{R}^d)$ we have

$$|\Lambda[K_{\mathfrak{I}}](b_1, \dots, b_{n+2})| \le C_{\epsilon,d,\delta} n^2 \big(\sup_{j \in \mathbb{Z}} \|\varsigma_j\|_{L^1(\mathbb{R}^{n+d})}\big) \log^3\Big(2 + n \frac{\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{\mathcal{B}_\epsilon}}{\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{L^1}}\Big) \prod_{l=1}^{n+2} \|b_l\|_{p_l}$$

where the constant $C_{\epsilon,d,\delta}$ is independent of n and $\mathfrak{I}$.

(ii) Let $K \in \mathcal{K}^\varepsilon$ so that $K = \sum_{j\in\mathbb{Z}} \varsigma_j^{(2^j)}$ in $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ with $\int \varsigma_j(\alpha, x) dx = 0$ for almost all $\alpha \in \mathbb{R}^n$. Let $\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} < \infty$, $b_1 \in L^{p_1}$, …, $b_{n+2} \in L^{p_{n+2}}$. Then $\sum_{j=-\infty}^{\infty} \Lambda[\varsigma_j^{(2^j)}]$ converges in the operator topology of $(n+2)$-linear functionals to a limit $\Lambda[K]$ satisfying

$$|\Lambda[K](b_1, \dots, b_{n+2})| \le C_{p_0,\epsilon,d} n^2 \|K\|_{\mathfrak{K}_0} \log^3\Big(2 + n \frac{\|K\|_{\mathfrak{K}_\varepsilon}}{\|K\|_{\mathfrak{K}_0}}\Big) \prod_{l=1}^{n+2} \|b_l\|_{p_l}.$$

We now turn to the multilinear forms defined by adjoint operators. More generally, given a permutation ϖ on $\{1, \dots, n+2\}$ we define the multilinear form $\Lambda^\varpi[\varsigma]$ by

$$\Lambda^\varpi[\varsigma](b_1, \dots, b_{n+2}) = \Lambda[\varsigma](b_{\varpi(1)}, \dots, b_{\varpi(n+2)}). \tag{2.14}$$

We have the following crucial result which will be proved in §4. It shows that operators of the form (2.13), and their limits as $N \to \infty$, are closed under adjoints.

Theorem 2.9. *Let $\epsilon > 0$. There exists $\epsilon' \geq c(\epsilon)$ (independent of n) such that for any permutation ϖ of $\{1, \dots, n+2\}$ there exists a bounded linear transformation $\ell_\varpi : \mathcal{B}_\epsilon(\mathbb{R}^n \times \mathbb{R}^d) \to \mathcal{B}_{\epsilon'}(\mathbb{R}^n \times \mathbb{R}^d)$ with*

$$(\ell_\varpi \varsigma)^{(t)} = \ell_\varpi(\varsigma^{(t)}), \quad t > 0,$$

and

$$\Lambda^\varpi[\varsigma] = \Lambda[\ell_\varpi \varsigma]\,,$$

such that

$$\|\ell_\varpi \varsigma\|_{\mathcal{B}_{\epsilon'}} \lesssim n^2 \|\varsigma\|_{\mathcal{B}_\epsilon}$$

and

$$\|\ell_\varpi \varsigma\|_{L^1} = \|\varsigma\|_{L^1}.$$

Furthermore, if $\int \varsigma(\alpha, v)\, dv = 0$ a.e. then also $\int \ell_\varpi \varsigma(\alpha, v)\, dv = 0$ a.e.

In light of Theorem 2.9, the result in Theorem 2.8 is closed under taking adjoints, and therefore follows from the following result and complex interpolation (see §15).

Theorem 2.10. *Let $\delta > 0$, $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$, $p \in [1+\delta, 2]$, and let $p' = p/(p-1)$. For $b_{n+1} \in L^p(\mathbb{R}^d)$, $b_{n+2} \in L^{p'}(\mathbb{R}^d)$ we have*

$$|\Lambda[K](b_1, \dots, b_{n+2})| \leq C_{\epsilon,d,\delta} n^2 \sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{L^1} \log^3\big(2+n \frac{\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{\mathcal{B}_\epsilon}}{\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{L^1}}\big) \Big(\prod_{l=1}^n \|b_l\|_\infty\Big) \|b_{n+1}\|_p \|b_{n+2}\|_{p'}.$$

The structure of the proof of Theorem 2.10 will be discussed in §5, and the details of the proof will be given in subsequent sections.

2.4. **Remarks on Besov spaces.**

2.4.1. *Equivalent norms.* In Definition 2.4 we chose a particular form of the norm $\|\cdot\|_{\mathcal{B}_\epsilon}$ which is well suited for our goal to prove estimates with polynomial growth in n. There are other equivalent norms which could be used, for instance, one might replace the expression

$$\sup_{\substack{0<h\leq 1\\ 1\leq i\leq n}} h^{-\epsilon} \iint |\varsigma(\alpha + he_i, v) - \varsigma(\alpha, v)|\, d\alpha\, dv$$

with

$$\sup_{0<|h|\leq 1} |h|^{-\epsilon} \iint |\varsigma(\alpha + h, v) - \varsigma(\alpha, v)|\, d\alpha\, dv$$

and one ends up with a comparable norm. These two choices differ by a factor which is polynomial in n. Fortunately, the result in Theorem 2.8 only involves $\|\varsigma_j\|_{\mathcal{B}_\epsilon}$ through the expression

$$\log^3(2 + n \frac{\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{\mathcal{B}_\epsilon}}{\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{L^1}}).$$

Thus, if one changes $\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{\mathcal{B}_\epsilon}$ by a factor which is polynomial in n, this only changes the bound in Theorem 2.8 by a constant factor, and therefore does not change the result in Theorem 2.8. In this way, one can use any one of a variety of equivalent norms when defining $\|\cdot\|_{\mathcal{B}_\epsilon}$ (as long as one only changes the norm by a factor which is bounded by a polynomial in n) – we picked out the choice which is most natural for our purposes.

2.4.2. *The role of projective space.* Though it may not be apparent from the above definitions, the space $\mathbb{RP}^n$ plays a key role in the intuition behind our main results. In this section, we exhibit a special case where the role of $\mathbb{RP}^n$ is apparent, and we return to a more general version of these ideas in §4.5.

Recall that $\mathbb{RP}^n$ is defined as $\mathbb{R}^{n+1} \setminus \{0\}$ modulo the equivalence relation where we identify $\alpha, \beta \in \mathbb{R}^{n+1} \setminus \{0\}$ if there exists $c \in \mathbb{R} \setminus \{0\}$ with $\alpha = c\beta$. This sees $\mathbb{RP}^n$ has an n-dimensional manifold. Traditionally, there are $n+1$ standard coordinate charts on $\mathbb{RP}^n$. For these, we consider those points in $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ with $\alpha_j \neq 0$. Under the equivalence relation, α is equivalent to $\alpha_j^{-1}\alpha = (\alpha_j^{-1}\alpha_1, \ldots, \alpha_j^{-1}\alpha_{j-1}, 1, \alpha_j^{-1}\alpha_{j+1}, \ldots, \alpha_j^{-1}\alpha_{n+1})$. This identifies such points with a copy of $\mathbb{R}^n$ and yields a coordinate chart on $\mathbb{RP}^n$–every point in $\mathbb{RP}^n$ lies in the image of at least one of these charts. This sees a copy of $\mathbb{R}^n$ inside of $\mathbb{RP}^n$ given by $(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_{j-1}, 1, \alpha_{j+1}, \ldots, \alpha_n)$.

Functions on $\mathbb{RP}^n$ can be identified with functions $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ such that $f(c\alpha) = f(\alpha)$–i.e., functions which are homogeneous of degree 0 and are even. Suppose we are given $f : \mathbb{RP}^n \to \mathbb{C}$. We obtain a function $f_0 : \mathbb{R}^n \to \mathbb{C}$ by viewing $\mathbb{R}^n \hookrightarrow \mathbb{RP}^n$ via the map $(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_n, 1)$. Thus, given an even function $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ which is homogeneous of degree 0, we obtain a function $f : \mathbb{RP}^n \to \mathbb{C}$, and therefore a function $f_0 : \mathbb{R}^n \to \mathbb{C}$ (and f_0 uniquely determines f off of a set of lower dimension in $\mathbb{RP}^n$).

We consider here the special case when

$$K(\alpha, v) = \gamma(\alpha)\kappa(v)$$

and κ is a classical Calderón-Zygmund kernel which is *homogeneous of degree* $-d$ and smooth away from $v = 0$. For $\alpha \in \mathbb{R}^n$ and functions $b_1, \ldots, b_{n+2}$, consider

$$\begin{aligned} F_0(\alpha) &= \iint \kappa(x-y) b_{n+2}(x) b_{n+1}(y) \prod_{i=1}^n b_i(x - \alpha_i(x-y))\, dx\, dy \\ &= \iint \kappa(v) b_{n+2}(x) b_{n+1}(x-v) \prod_{i=1}^n b_i(x - \alpha_i v)\, dx\, dv. \end{aligned} \tag{2.15}$$

The multilinear form we wish to study (in this special case) is given by

$$\int \gamma(\alpha) F_0(\alpha)\, d\alpha.$$

One main aspect of our assumptions is that this operator should be of the same form when we permute the roles of $b_1, \ldots, b_{n+2}$. Many of these permutations are easy to understand: permuting the roles of $b_1, \ldots, b_n$ merely permutes the variables $\alpha_1, \ldots, \alpha_n$. Switching the roles of b_{n+1} and b_{n+2} changes α to $(1-\alpha_1, \ldots, 1-\alpha_n)$. Thus, the major difficulty in understanding adjoints can be reduced to understanding the question of switching the roles of b_j $(1 \leq j \leq n)$ and b_{n+1} (as every permutation of $\{1, \ldots, n+2\}$ can be generated by the these three types of permutations).

Define a new function $F : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ by

$$F(\alpha_1, \ldots, \alpha_{n+1}) = \iint \kappa(v) b_1(x - \alpha_1 v) \cdots b_n(x - \alpha_n v) b_{n+1}(x - \alpha_{n+1} v) b_{n+2}(x)\, dx\, dv\,.$$

Because of the homogeneity of κ, we see (for $c \in \mathbb{R} \setminus \{0\}$), $F(c\alpha) = F(\alpha)$. By the above discussion, F defines a function on $\mathbb{RP}^n$, and therefore induces a function $F_0 : \mathbb{R}^n \to \mathbb{C}$ as above. This induced function F_0 is exactly the function of the same name from (2.15). Thus, we have defined F_0 in a way which is symmetric in $b_1, \ldots, b_{n+1}$.

$F(\alpha)$ defines a function on $\mathbb{RP}^n$, and therefore if $\gamma(\alpha)d\alpha$ were a measure on $\mathbb{RP}^n$, it would make sense to write

$$\int \gamma(\alpha)F(\alpha)\, d\alpha.$$

Indeed, our main assumptions in this special case are equivalent to assuming that $\gamma(\alpha)d\alpha$ is a density which lies in the space $\bigcup_{0<\varepsilon<1} B^{\epsilon}_{1,\infty}(\mathbb{RP}^n)$ (where $B^{\epsilon}_{1,\infty}(\mathbb{RP}^n)$ denotes a *Besov space of densities* on $\mathbb{RP}^n$, see §4.5 for a proof of this remark). When we write the expression as

$$\int \gamma(\alpha)F_0(\alpha)\, d\alpha,$$

we are merely choosing the coordinate chart $\mathbb{R}^n \hookrightarrow \mathbb{RP}^n$ denoted above. With this formulation, the operator

$$\int \gamma(\alpha)F(\alpha)\, d\alpha$$

clearly remains of the same form when $b_1, \dots, b_{n+1}$ are permuted, and from here it is easy to see that the class of operators is "closed under adjoints."

Remark. In our more general setting, $K(\alpha, v)$ is not homogeneous in the v variable, and therefore we cannot define a function F on $\mathbb{RP}^n$ as was done above. Nevertheless, these ideas play an important role in our proofs, see §4.5 below.

3. Kernels

In this section, we prove various results announced in Section 2. We first show the independence of the space $\mathcal{K}_\varepsilon$ of the particular choice of η satisfying (2.2) and then give the proof of Propositions 3.2 and 3.3.

3.1. **Independence of η.** The following lemma shows that $\mathcal{K}_\varepsilon$ does not depend on the choice of $\eta \in \mathcal{S}(\mathbb{R}^d)$ satisfying (2.2).

Lemma 3.1. *Let $\eta, \eta' \in \mathcal{S}(\mathbb{R}^d)$ and η be as in* (2.2)*. Let $0 < \varepsilon \leq 1$. There exists $C = C(\eta, \eta')$ such that for all $K \in \mathcal{K}_\varepsilon$*

$$\|K\|_{\mathcal{K}^{\eta'}_\varepsilon} \leq C\|K\|_{\mathcal{K}^{\eta}_\varepsilon}$$

The constant C is independent of n.

Proof. Let $K \in \mathcal{K}_\varepsilon$. Only two of the terms of the definition of $\|K\|_{\mathcal{K}_\varepsilon}$ depend on the choice of η. Thus, the result will follow once we prove the following two estimates.

$$\sup_{\substack{1\leq i\leq n\\ t>0}} \int (1+|\alpha_i|)^\varepsilon \|\eta' * K^{(t)}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)}\, d\alpha \leq C \sup_{\substack{1\leq i\leq n\\ t>0}} \int (1+|\alpha_i|)^\varepsilon \|\eta * K^{(t)}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)}\, d\alpha, \tag{3.1}$$

and

$$\begin{aligned}\sup_{\substack{1\leq i\leq n\\ t>0\\ 0<h\leq 1}} h^{-\varepsilon} \int \|\eta' * [K^{(t)}(\alpha + he_i,\cdot) - K^{(t)}(\alpha,\cdot)]\|_{L^2(\mathbb{R}^d)}\, d\alpha \qquad\qquad \\ \leq C \sup_{\substack{1\leq i\leq n\\ t>0\\ 0<h\leq 1}} h^{-\varepsilon} \int \|\eta * [K^{(t)}(\alpha + he_i,\cdot) - K^{(t)}(\alpha,\cdot)]\|_{L^2(\mathbb{R}^d)}\, d\alpha.\end{aligned} \tag{3.2}$$

The proofs of these two equations are nearly identical, so we prove only (3.1).

Let $\chi \in C^\infty_0(\mathbb{R}^d)$ be supported in $\{\xi : \frac{1}{2} < |\xi| < 2\}$ with the property that $\sum_{k\in\mathbb{Z}}[\chi(2^{-k}\xi)]^2 = 1$, for $\xi \in \mathbb{R}^d \setminus \{0\}$. Since $\eta' \in \mathcal{S}(\mathbb{R}^d)$, we have $\|\chi(2^{-k}\cdot)\widehat{\eta'}(\cdot)\|_{L^\infty} \leq C_N \min\{2^{-kN}, 1\}$. By (2.2) and the compactness of $\{\xi : \frac{1}{2} \le |\xi| \le 2\}$ there is a finite index set Θ and real numbers $\tau_\nu > 0$ such that

$$\sum_{\nu\in\Theta} |\widehat{\eta}(\tau_\nu \xi)|^2 \ge c > 0 \ \text{ for } \ \frac{1}{2} \le |\xi| \le 2.$$

Let

$$m_\nu(\xi) = \frac{\overline{\widehat{\eta}(\tau_\nu\xi)}\chi(\xi)}{\sum_{\tilde\nu\in\Theta} |\widehat{\eta}(\tau_{\tilde\nu}\xi)|^2};$$

then $\|m_\nu\|_{L^\infty} \le C_\nu$ and we have

$$\widehat{\eta'}(\xi) = \sum_{k\in\mathbb{Z}} \chi(2^{-k}\xi)\widehat{\eta}(\xi) \sum_{\nu\in\Theta} m_\nu(2^{-k}\xi)\widehat{\eta}(2^{-k}\tau_\nu\xi).$$

Hence,

$$\|\eta' * K^{(t)}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)} \lesssim \sum_{k\in\mathbb{Z}} \min\{2^{-kN}, 1\} \sum_{\nu\in\Theta} \|m_\nu\|_\infty \|\widehat{\eta}(2^{-k}\tau_\nu\cdot)\widehat{K^{(t)}}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)},$$

where the implicit constant depends on N. Note

$$\|\widehat{\eta}(2^{-k}\tau_\nu\cdot)\widehat{K^{(t)}}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)} = (2^k/\tau_\nu)^{d/2}\|\eta * K^{(2^{-k}\tau_\nu t)}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)},$$

and so taking $N > d/2$ we obtain

$$\begin{aligned}
&\int (1+|\alpha_i|)^\varepsilon \|\eta' * K^{(t)}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)}\, d\alpha \\
&\lesssim \sum_{k\in\mathbb{Z}} \min\{2^{-k(N-d/2)},\, 2^{kd/2}\} \sum_{\nu\in\Theta} C_\nu \int (1+|\alpha_i|)^\varepsilon \|\eta * K^{(2^{-k}\tau_\nu t)}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)}\, d\alpha \\
&\lesssim \sup_{r>0} \int (1+|\alpha_i|)^\varepsilon \|\eta * K^{(r)}(\alpha,\cdot)\|_{L^2(\mathbb{R}^d)}\, d\alpha,
\end{aligned}$$

which completes the proof of (3.1). □

3.2. **Proof of Theorem 2.6.** The theorem follows from two propositions. In the first we prove an estimate for the ς_j as in (2.11), which arise in the decomposition of $K = \sum_j \varsigma_j^{(2^j)}$.

Proposition 3.2. *Suppose $\varepsilon \in (0,1]$, $0 < \delta < \varepsilon$. For every $K \in \mathcal{K}_\varepsilon$, let*

$$\varsigma_j = (Q_jK)^{(2^{-j})}.$$

Then $\{\varsigma_j : j \in \mathbb{Z}\}$ is a bounded subset of $\mathcal{B}_\delta(\mathbb{R}^n \times \mathbb{R}^d)$ satisfying

$$\int \varsigma_j(\alpha, v)\, dv = 0,$$

for all j and almost every $\alpha \in \mathbb{R}^n$ and

$$\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{\mathcal{B}_\delta} \le C_{\delta,\varepsilon,d}\|K\|_{\mathcal{K}_\varepsilon},$$

and such that

$$K = \sum_{j\in\mathbb{Z}} \varsigma_j^{(2^j)},$$

with the sum converging in the sense of the topology on $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$.

The second proposition provides $\mathcal{K}_\delta$-estimates for kernels that are given as sums $\sum_j \varsigma_j^{(2^j)}$, with uniform $\mathcal{B}_\varepsilon$-estimates for the ς_j.

Proposition 3.3. *Let $\varepsilon \in (0,1]$, and $0 < \delta < \varepsilon/2$. Suppose $\{\varsigma_j : j \in \mathbb{Z}\} \subset \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ is a bounded set satisfying $\int \varsigma_j(\alpha, v)\, dv = 0$, for all j. Then the sum*

$$K(\alpha, v) := \sum_{j\in\mathbb{Z}} \varsigma_j^{(2^j)}(\alpha, v)$$

converges in the sense of the topology on $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$, and $K \in \mathcal{K}_\delta$. Furthermore,

$$\|K\|_{\mathcal{K}_\delta} \leq C_{\delta,\varepsilon,d} \sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{\mathcal{B}_\varepsilon}.$$

The proofs of these propositions will be given in §3.2.1 and §3.2.2

3.2.1. *Proof of Proposition 3.2.* We need several lemmata.

Lemma 3.4. *Let $\varepsilon > 0$. Then, there exists $\delta = \delta(\varepsilon, d) > 0$ such that for $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$, we have*

$$\iint |v|^{-\delta} |\varsigma(\alpha, v)|\, d\alpha\, dv \leq C_{\varepsilon,d} \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Proof. Clearly $\iint_{|v|>1} |v|^{-\delta}|\varsigma(\alpha, v)|\, d\alpha\, dv \lesssim \|\varsigma\|_{L^1} \leq \|\varsigma\|_{\mathcal{B}_\varepsilon}$, so it suffices to prove

$$\iint_{|v|\leq 1} |v|^{-\delta}|\varsigma(\alpha, v)|\, d\alpha\, dv \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon}. \tag{3.3}$$

By a weak version of the Sobolev embedding theorem (see [35] or [39]), there exists $p = p(\varepsilon, d) > 1$ such that

$$\int \Big(\int |\varsigma(\alpha, v)|^p\, dv \Big)^{\frac{1}{p}} d\alpha \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Let p' be dual to p and let $\delta < 1/p'$. We have, by Hölder's inequality, and then Minkowski's inequality,

$$\begin{aligned} &\iint_{|v|\leq 1} |v|^{-\delta}|\varsigma(\alpha, v)|\, d\alpha\, dv \leq \Big(\int_{|v|\leq 1} |v|^{-\delta p'}\, dv \Big)^{\frac{1}{p'}} \Big(\int \Big(\int |\varsigma(\alpha, v)|\, d\alpha \Big)^p dv \Big)^{\frac{1}{p}} d\alpha \\ &\lesssim \Big(\int \Big(\int |\varsigma(\alpha, v)|\, d\alpha \Big)^p dv \Big)^{\frac{1}{p}} d\alpha \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon}. \end{aligned}$$

This shows (3.3) and completes the proof of the lemma. □

Lemma 3.5. *Let $\{\varsigma_j : j \in \mathbb{Z}\} \subset \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ be a bounded set with $\int \varsigma_j(\alpha, v)\, dv = 0$, for all $j \in \mathbb{Z}$. The sum*

$$\sum_{j\in\mathbb{Z}} \varsigma_j^{(2^j)}(\alpha, v)$$

converges in the sense of the topology on $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ (and a fortiori in the sense of tempered distributions).

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^d)$. We will show, for some $\delta > 0$,

$$\Big| \int \varsigma_j^{(2^j)}(\alpha, v) f(\alpha, v)\, d\alpha\, dv \Big| \lesssim 2^{-|j|\delta} \sup_{\alpha\in\mathbb{R}^n, x\in\mathbb{R}^d} \sum_{|\gamma|\leq 1} (1 + |x|)|\partial_x^\gamma f(\alpha, x)|,$$

and the result will follow by the completeness of $L\mathcal{S}'$.

First we consider the case $j \geq 0$. In this case, we have

$$\begin{aligned}
&\Big| \iint \varsigma_j^{(2^j)}(\alpha, v) f(\alpha, v)\, d\alpha\, dv \Big| = \Big| \iint \varsigma_j^{(2^j)}(\alpha, v)[f(\alpha, v) - f(\alpha, 0)]\, d\alpha\, dv \Big| \\
&\lesssim \Big(\sup_{\alpha \in \mathbb{R}^n, x \in \mathbb{R}^d} \sum_{|\gamma| \leq 1} |\partial_x^\gamma f(\alpha, x)| \Big) \iint |\varsigma_j^{(2^j)}(\alpha, v)| |v|^\varepsilon\, dv\, d\alpha \\
&\lesssim 2^{-j\varepsilon} \Big(\sup_{\alpha \in \mathbb{R}^n, x \in \mathbb{R}^d} \sum_{|\gamma| \leq 1} |\partial_x^\gamma f(\alpha, x)| \Big) \|\varsigma_j\|_{\mathcal{B}_\varepsilon},
\end{aligned}$$

as desired.

We now turn to $j < 0$. Take $\delta > 0$ as in Lemma 3.4. We have

$$\begin{aligned}
&\Big| \iint \varsigma_j^{(2^j)}(\alpha, v) f(\alpha, v)\, d\alpha\, dv \Big| \leq \Big(\sup_{\alpha \in \mathbb{R}^n, x \in \mathbb{R}^d} |x|^\delta |f(\alpha, x)| \Big) \iint |\varsigma_j^{(2^j)}(\alpha, v)|\, |v|^{-\delta}\, d\alpha\, dv \\
&= \big(\sup_{\alpha \in \mathbb{R}^n, x \in \mathbb{R}^d} |x|^\delta |f(\alpha, x)| \big)\, 2^{j\delta} \iint |\varsigma_j(\alpha, v)|\, |v|^{-\delta}\, d\alpha\, dv \\
&\lesssim \big(\sup_{\alpha \in \mathbb{R}^n, x \in \mathbb{R}^d} |x|^\delta |f(\alpha, x)| \big)\, 2^{j\delta} \|\varsigma_j\|_{\mathcal{B}_\varepsilon},
\end{aligned}$$

where in the last line we have used our choice of δ and Lemma 3.4. $\square$

Let $\phi \in C_0^\infty(B^d(1/2))$ be a radial, non-negative function with $\int \phi = 1$. For $j \in \mathbb{Z}$ let $\phi^{(2^j)}(v) = 2^{jd}\phi(2^j v)$. Let $\psi(x) = \phi(x) - \frac{1}{2}\phi(x/2) \in C_0^\infty(B^d(1))$. Let $Q_j f = f * \psi^{(2^j)}$. Note that $f = \sum_{j \in \mathbb{Z}} Q_j f$ for $f \in \mathcal{S}(\mathbb{R}^d)$ with convergence in the sense of tempered distributions.

The heart of the proof of Proposition 3.2 is the following lemma.

Lemma 3.6. *Suppose $0 < \varepsilon \leq 1$, $0 < \delta < \varepsilon$ and let $K \in \mathcal{K}_\varepsilon$. Let*

$$\varsigma(\alpha, v) = Q_0 K(\alpha, v).$$

Then, $\varsigma \in \mathcal{B}_\delta(\mathbb{R}^n \times \mathbb{R}^d)$ and

$$\|\varsigma\|_{\mathcal{B}_\delta} \leq C_{\delta, \varepsilon, d} \|K\|_{\mathcal{K}_\varepsilon}.$$

Proof of Proposition 3.2 given Lemma 3.6. Since $K^{(2^j)}$ is of the same form as K, the lemma also yields, with $\varsigma_j := (Q_j K)^{(2^{-j})}$,

$$\sup_{j \in \mathbb{Z}} \big\|\varsigma_j\big\|_{\mathcal{K}_\varepsilon} \leq C_{\delta, \varepsilon, d} \|K\|_{\mathcal{K}_\varepsilon}.$$

As $\int \varsigma_j(\alpha, x) dx = 0$ for all j it follows from standard estimates that $K = \sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}$, in the sense of tempered distributions. Since we know $\sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}$ converges in the sense of the topology on $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ it follows that the sum can be taken in that sense as well. The result now follows from Lemma 3.6. $\square$

Proof of Lemma 3.6. Note that, in light of Lemma 3.1, we may replace the test function η with ψ in the definition of $\|K\|_{\mathcal{K}_\varepsilon}$.

We begin by bounding $\|\varsigma\|_{\mathcal{B}_{\delta,1}}$ as in (2.9a) and split, for fixed $1 \leq i \leq n$,

$$\iint (1 + |\alpha_i|)^\delta |\varsigma(\alpha, x)|\, dx\, d\alpha = \iint_{|x| \leq 1} + \iint_{1 < |x| \leq 1 + |\alpha_i|} + \iint_{|x| > 1 + |\alpha_i|} =: (I) + (II) + (III).$$

For (I), we apply the Cauchy-Schwarz inequality to see

$$(I) = \iint_{|x|\le 1} (1+|\alpha_i|)^\delta |\varsigma(\alpha,x)|\,dx\,d\alpha \lesssim \int (1+|\alpha_i|)^\delta \Big(\int |\psi * K(\alpha,x)|^2\,dx\Big)^{\frac{1}{2}}\,d\alpha \le \|K\|_{\mathcal{K}^\psi_{\varepsilon,1}}.$$

For (II), we have

$$\begin{aligned}(II) &= \iint\limits_{1<|x|\le 1+|\alpha_i|} (1+|\alpha_i|)^\delta |\varsigma(\alpha,x)|\,dx\,d\alpha \lesssim \sum_{k\ge 0} \iint\limits_{\substack{1+|\alpha_i|>2^k\\ 2^k\le|x|\le 2^{k+1}}} (1+|\alpha_i|)^\delta |\psi * K(\alpha,x)|\,dx\,d\alpha \\ &\lesssim \sum_{k\ge 0} 2^{k(\delta-\varepsilon)} \iint\limits_{2^{k-1}\le|x|\le 2^{k+3}} (1+|\alpha_i|)^\varepsilon |K(\alpha,x)|\,dx\,d\alpha \lesssim \|K\|_{\mathcal{K}_{\varepsilon,3}}\end{aligned}$$

For (III), we use that $\int \psi = 0$ and $\mathrm{supp}(\psi) \subset B^d(0,1)$ to see

$$\begin{aligned}(III) &= \iint\limits_{|x|>1+|\alpha_i|} (1+|\alpha_i|)^\delta |\varsigma(\alpha,x)|\,dx\,d\alpha \\ &\lesssim \iint_{|x|>1+|\alpha_i|} (1+|\alpha_i|)^\delta \Big| \int \psi(y)[K(\alpha,x-y)-K(\alpha,x)] \Big|\,dx\,d\alpha \\ &\lesssim \sum_{k\ge 0} 2^{k\delta} \int |\psi(y)| \iint\limits_{\substack{2^k\le|\alpha_i|\le 2^{k+1}\\ |x|>2^k}} |K(\alpha,x-y)-K(\alpha,x)|\,dx\,d\alpha\,dy \\ &\lesssim \sum_{k\ge 0} 2^{k\delta} \int_{|y|\le 1} \iint_{|x|>2^k} |K(\alpha,x-y)-K(\alpha,x)|\,dx\,d\alpha\,dy \\ &\lesssim \sum_{k\ge 0} 2^{k(\delta-\varepsilon)} \|K\|_{\mathcal{K}_{\varepsilon,5}} \lesssim \|K\|_{\mathcal{K}_{\varepsilon,5}},\end{aligned}$$

as desired. Combining the estimates for (I), (II), (III) gives

$$\|\varsigma\|_{\mathcal{B}_{\delta,1}} \lesssim \|K\|_{\mathcal{K}^\psi_{\varepsilon,1}} + \|K\|_{\mathcal{K}_{\varepsilon,3}} + \|K\|_{\mathcal{K}_{\varepsilon,5}} \lesssim \|K\|_{\mathcal{K}_\varepsilon}.$$

We turn to bounding $\|\varsigma\|_{\mathcal{B}_{\delta,2}}$. Let $1 \le i \le n$ and $0 < h \le 1$ and split

$$\iint |\varsigma(\alpha+he_i,x)-\varsigma(\alpha,x)|\,dx\,d\alpha = \iint\limits_{|x|\le 2} + \iint\limits_{2\le|x|\le 10h^{-1}} + \iint\limits_{|x|<10h^{-1}} =: (IV)+(V)+(VI).$$

Our goal is to show $(IV),(V),(VI) \lesssim h^\delta \|K\|_{\mathcal{K}_\varepsilon}$. We have, by the Cauchy-Schwarz inequality,

$$\begin{aligned}(IV) &= \iint\limits_{|x|\le 2} |\varsigma(\alpha+he_i,x)-\varsigma(\alpha,x)|\,dx\,d\alpha \\ &\lesssim \int \Big(\int |\psi * [K(\alpha+he_i,\cdot)-K(\alpha,\cdot)](x)|^2\,dx \Big)^{\frac{1}{2}}\,d\alpha \le h^\varepsilon \|K\|_{\mathcal{K}^\psi_{\varepsilon,2}}.\end{aligned}$$

For (V), we have

$$\begin{aligned}
(V) &= \iint_{2<|x|\le 10h^{-1}} |\varsigma(\alpha+he_i,x)-\varsigma(\alpha,x)|\,dx\,d\alpha \\
&\lesssim \sum_{1\le 2^k\le 10h^{-1}} \iint_{2^{k-1}\le|x|\le 2^{k+2}} |K(\alpha+he_i,x)-K(\alpha,x)|\,dx\,d\alpha \\
&\lesssim \sum_{1\le 2^k\le 10h^{-1}} h^\varepsilon \|K\|_{\mathcal{K}_{\varepsilon,4}} \lesssim h^\varepsilon \log(2+h^{-1})\|K\|_{\mathcal{K}_{\varepsilon,4}}\,.
\end{aligned}$$

For (VI), we use that $\int\psi=0$ and $\operatorname{supp}(\psi)\subset B^d(0,1)$ and obtain

$$\begin{aligned}
(VI) &= \iint_{|x|\ge 10h^{-1}} |\varsigma(\alpha+he_i,x)-\varsigma(\alpha,x)|\,dx\,d\alpha \le 2\iint_{|x|>10h^{-1}} |\psi * K(\alpha,x)|\,dx\,d\alpha \\
&\lesssim \iint_{|x|>10h^{-1}} \Big|\int \psi(y)[K(\alpha,x-y)-K(\alpha,x)]\,dy\Big|\,dx\,d\alpha \\
&\lesssim \int |\psi(y)| \iint_{|x|\ge 10h^{-1}} |K(\alpha,x-y)-K(\alpha,x)|\,dx\,d\alpha\,dy \\
&\lesssim h^\varepsilon \|K\|_{\mathcal{K}_{\varepsilon,5}}.
\end{aligned}$$

Combining the estimates for (IV), (V), (VI) we get

$$\|\varsigma\|_{\mathcal{B}_{\delta,2}} \lesssim \|K\|_{\mathcal{K}^{\psi}_{\varepsilon,2}} + \|K\|_{\mathcal{K}_{\varepsilon,4}} + \|K\|_{\mathcal{K}_{\varepsilon,5}} \lesssim \|K\|_{\mathcal{K}_\varepsilon}.$$

We now turn to bounding $\|\varsigma\|_{\mathcal{B}_{\delta,3}}$. Fix $h\in\mathbb{R}^d$ with $0<|h|\le 1$. Using that $\int\psi=0$, we have

$$\begin{aligned}
&\iint |\varsigma(\alpha,x+h)-\varsigma(\alpha,x)|\,dx\,d\alpha \\
&\le \iint_{|x|\le 10} \Big|\int_0^1 \langle h, \nabla_x\psi * K(\alpha,x+sh)\rangle ds\Big|\,dx\,d\alpha \\
&\quad + \sum_{8\le 2^k\le 10|h|^{-1}} \iint_{2^k\le|x|\le 2^{k+1}} \Big|\int_0^1 \langle h, \nabla_x\psi * K(\alpha,x+sh)\rangle\,ds\Big|\,dx\,d\alpha \\
&\quad + 2\iint_{|x|\ge 9|h|^{-1}} \Big|\int \psi(y)[K(\alpha,x-y)-K(\alpha,x)]\,dy\Big|\,dx\,d\alpha \\
&=: (VII)+(VIII)+2(IX).
\end{aligned}$$

We need to show $(VII), (VIII), (IX) \lesssim |h|^\delta \|K\|_{\mathcal{K}_\varepsilon}$.

We begin with (VII) and use the Cauchy-Schwarz inequality to see

$$\begin{aligned}(VII) &= \iint_{|x|\le 10} \Big| \int_0^1 \langle h, \nabla_x \psi * K(\alpha, x+sh)\rangle \, ds \Big| \, dx \, d\alpha \\ &\le |h| \iint_{|x|\le 11} |\nabla \psi * K(\alpha,x)| \, dx \, d\alpha \\ &\lesssim |h| \int \Big(\int |\nabla\psi * K(\alpha,x)|^2 \, dx \Big)^{\frac12} d\alpha \\ &\lesssim |h| \|K\|_{\mathcal{K}^{\nabla\psi}_{\varepsilon,1}} .\end{aligned}$$

For $(VIII)$ we have

$$\begin{aligned}(VIII) &= \sum_{8\le 2^k \le 10|h|^{-1}} \iint_{2^k \le |x| \le 2^{k+1}} \Big| \int_0^1 \langle h, \nabla_x \psi * K(\alpha, x+sh)\rangle \, ds \Big| \, dx \, d\alpha \\ &\le |h| \sum_{8\le 2^k \le 10|h|^{-1}} \iint_{2^{k-1} \le |x| \le 2^{k+2}} |\nabla\psi * K(\alpha,x)| \, dx \, d\alpha \\ &\lesssim |h| \sum_{8\le 2^k \le 10|h|^{-1}} \iint_{2^{k-2} \le |x| \le 2^{k+3}} |K(\alpha,x)| \, dx \, d\alpha \\ &\lesssim |h| \sum_{8\le 2^k \le 10|h|^{-1}} \|K\|_{\mathcal{K}_{0,3}} \lesssim |h| \log(2+|h|^{-1}) \|K\|_{\mathcal{K}_{0,3}} .\end{aligned}$$

For (IX) we use $\mathrm{supp}(\psi) \subset B^d(0,1)$ and estimate

$$\begin{aligned}(IX) &= \iint_{|x|\ge 9|h|^{-1}} \Big| \int \psi(y)[K(\alpha,x-y) - K(\alpha,x)] \, dy \Big| \, dx \, d\alpha \\ &\le \int |\psi(y)| \iint_{|x|\ge 9|h|^{-1}} |K(\alpha,x-y) - K(\alpha,x)| \, dx \, d\alpha \, dy \\ &\lesssim h^{\varepsilon} \|K\|_{\mathcal{K}_{\varepsilon,5}},\end{aligned}$$

as desired. Summarizing,

$$\|\varsigma\|_{\mathcal{B}_{\delta,3}} \lesssim \|K\|_{\mathcal{K}^{\nabla\psi}_{\varepsilon,1}} + \|K\|_{\mathcal{K}_{0,3}} + \|K\|_{\mathcal{K}_{\varepsilon,5}} \lesssim \|K\|_{\mathcal{K}_{\varepsilon}}$$

where in the last inequality we have used Lemma 3.1.

Finally we estimate $\|\varsigma\|_{\mathcal{B}_{\varepsilon,4}}$ and split

$$\iint (1+|x|)^{\delta} |\varsigma(\alpha,x)| \, d\alpha \, dx = \iint_{|x|\le 10} + \iint_{|x|>10} =: (X) + (XI).$$

We have, by the Cauchy-Schwarz inequality,

$$(X) = \iint_{|x|\le 10} (1+|x|)^{\delta} |\varsigma(\alpha,x)| \, dx \, d\alpha \lesssim \int \Big(\int |\psi * K(\alpha,x)|^2 \, dx \Big)^{\frac12} d\alpha \lesssim \|K\|_{\mathcal{K}^{\psi}_{\varepsilon,1}} .$$

Using that $\int \psi = 0$ and $\mathrm{supp}(\psi) \subset B^d(0,1)$, we have

$$\begin{aligned}
(XI) &= \iint_{|x|>10} (1+|x|)^\delta |\varsigma(\alpha,x)|\,dx\,d\alpha \\
&\lesssim \sum_{k\geq 3} 2^{k\delta} \iint\limits_{2^k\leq|x|\leq 2^{k+1}} \Big|\int \psi(y)[K(\alpha,x-y)-K(\alpha,x)]\,dy\Big|\,dx\,d\alpha \\
&\lesssim \sum_{k\geq 3} 2^{k\delta} \int |\psi(y)| \iint_{|x|\geq 2^k} |K(\alpha,x-y)-K(\alpha,x)|\,dx\,d\alpha\,dy \\
&\lesssim \sum_{k\geq 3} 2^{k(\delta-\varepsilon)} \|K\|_{\mathcal{K}_{\varepsilon,5}} \lesssim \|K\|_{\mathcal{K}_{\varepsilon,5}}.
\end{aligned}$$

Hence

$$\|\varsigma\|_{\mathcal{B}_{\varepsilon,4}} \lesssim \|K\|_{\mathcal{K}^{\psi}_{\varepsilon,1}} + \|K\|_{\mathcal{K}_{\varepsilon,5}} \lesssim \|K\|_{\mathcal{K}_\varepsilon}.$$

This completes the proof. $\square$

3.2.2. *Proof of Proposition 3.3.* We begin with a preparatory lemma. Let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\int \Phi(x)dx = 1$, and let $\Psi(x) = \Phi(x) - \frac{1}{2}\Phi(\frac{x}{2})$. Define $Q_j f = f * \Psi^{(2^j)}$.

Lemma 3.7. *Let $\varepsilon > 0$ and $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$. Then, for $l > 0$,*

$$\iint |Q_l\varsigma(\alpha,x)|\,dx\,d\alpha + 2^{-l}\iint |\nabla_x Q_l\varsigma(\alpha,x)|\,dx\,d\alpha \lesssim 2^{-l\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}, \tag{3.4}$$

$$\iint\limits_{|x|\geq R} |Q_l\varsigma(\alpha,x)|\,dx\,d\alpha + 2^{-l}\iint\limits_{|x|\geq R} |\nabla_x Q_l\varsigma(\alpha,x)|\,dx\,d\alpha \lesssim R^{-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}, \tag{3.5}$$

and for $|h| \leq 1$,

$$\iint\limits_{|x|\geq R} \big|Q_l\varsigma(\alpha,x+h) - Q_l\varsigma(\alpha,x)\big|\,dx\,d\alpha \lesssim \min\{2^l|h|,1\}\min\{2^{-l\varepsilon},R^{-\varepsilon}\}\|\varsigma\|_{\mathcal{B}_\varepsilon}. \tag{3.6}$$

Let $0<\delta<\varepsilon$. Then for $R \geq 0$, $i = 1,\dots n$,

$$\iint_{|x|\geq R} (1+|\alpha_i|)^\delta |Q_l\varsigma(\alpha,x)|\,dx\,d\alpha \lesssim \min\{2^{-l(\varepsilon-\delta)}, R^{-(\varepsilon-\delta)}\}\|\varsigma\|_{\mathcal{B}_\varepsilon}, \tag{3.7}$$

and for all $0 < |\tau| \leq 1$, $j = 1,\dots,n$,

$$|\tau|^{-\delta}\iint_{|x|\geq R} \big|Q_l\varsigma(\alpha+\tau e_j,x) - Q_l\varsigma(\alpha,x)\big|\,dx\,d\alpha \lesssim \min\{2^{-l(\varepsilon-\delta)}, R^{-(\varepsilon-\delta)}\}\|\varsigma\|_{\mathcal{B}_\varepsilon}. \tag{3.8}$$

Proof. First observe that (3.4) is an immediate consequence of the definitions. Next, for the proof of (3.5) we may assume $R \geq 1$. Also, observe, for every $N \in \mathbb{N}$,

$$\begin{aligned}
&\iint\limits_{|x|\geq R} |Q_l\varsigma(\alpha,x)|\,dx\,d\alpha + 2^{-l}\iint\limits_{|x|\geq R} |\nabla_x Q_l\varsigma(\alpha,x)|\,dx\,d\alpha \\
&\leq C_N \iiint\limits_{|x|\geq R} \frac{2^{ld}}{(1+2^l|y|)^N}|\varsigma(\alpha,x-y)|\,dx\,d\alpha\,dy \\
&= C_N \iiint\limits_{\substack{|x|\geq R\\ |y|\leq R/2}} + C_N \iiint\limits_{\substack{|x|\geq R\\ |y|>R/2}} =: C_N\big((I)+(II)\big).
\end{aligned}$$

For (I) we have

$$(I) \lesssim R^{-\varepsilon} \iiint\limits_{\substack{|x|\geq R\\ |y|\leq R/2}} \frac{2^{ld}}{(1+2^l|y|)^N}(1+|x-y|)^\varepsilon |\varsigma(\alpha, x-y)|\, dx\, d\alpha\, dy \lesssim R^{-\varepsilon}\|\varsigma\|_{\mathcal{B}_{\varepsilon,4}}.$$

For (II), taking $N \geq d+1$, we have

$$(II) \lesssim \|\varsigma\|_{L^1} \int_{|y|>R/2} \frac{2^{ld}}{(1+2^l|y|)^N}\, dy \lesssim (2^l R)^{-1}\|\varsigma\|_{L^1} \leq R^{-\varepsilon}\|\varsigma\|_{\mathcal{B}_{0,4}},$$

and (3.5) follows. (3.6) follows by combining (3.4) and (3.5).

We now turn to (3.7) and we separate the proof into two cases, $R \leq 2^l$ and $R \geq 2^l$. For $R \leq 2^l$ we have, by (3.4),

$$\iint\limits_{|x|\geq R} (1+|\alpha_i|)^\delta |Q_l\varsigma(\alpha,x)|\, dx\, d\alpha \leq \iint\limits_{|\alpha_i|\leq 2^l} + \iint\limits_{|\alpha_i|>2^l} =: (III) + (IV).$$

For (III), we apply (3.4) to see

$$(III) \lesssim 2^{l\delta} \iint |Q_l\varsigma(\alpha,x)|\, dx\, d\alpha \lesssim 2^{-l(\varepsilon-\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Also, we have

$$\begin{aligned}(IV) &\lesssim 2^{-l(\varepsilon-\delta)} \int (1+|\alpha_i|)^\varepsilon \int |Q_l\varsigma(\alpha,x)|\, dx\, d\alpha\\ &\lesssim 2^{-l(\varepsilon-\delta)} \iint (1+|\alpha_i|)^\varepsilon |\varsigma(\alpha,x)|\, dx\, d\alpha \lesssim 2^{-l(\varepsilon-\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}.\end{aligned}$$

In the second case, $R \geq 2^l$, we have

$$\iint\limits_{|x|\geq R} (1+|\alpha_i|)^\delta |Q_l\varsigma(\alpha,x)|\, dx\, d\alpha \leq \iint\limits_{\substack{|\alpha_i|\leq R\\ |x|\geq R}} + \iint\limits_{|\alpha_i|>R} =: (V) + (VI).$$

Using (3.5),

$$(V) \lesssim R^\delta \iint_{|x|\geq R} |Q_l\varsigma(\alpha,x)|\, dx\, d\alpha \lesssim R^{\delta-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

And,

$$\begin{aligned}(VI) &\lesssim R^{\delta-\varepsilon} \int_{|\alpha_i|>R} (1+|\alpha_i|)^\varepsilon \int |Q_l\varsigma(\alpha,x)|\, dx\, d\alpha\\ &\lesssim R^{\delta-\varepsilon} \iint (1+|\alpha_i|)^\varepsilon |\varsigma(\alpha,x)|\, dx\, d\alpha \lesssim R^{\delta-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon},\end{aligned}$$

which completes the proof of (3.7).

Finally, we turn to (3.8). This we separate into four cases. In the first case, $R \leq 2^l$, $\tau \geq 2^{-l}$, we have

$$|\tau|^{-\delta} \iint_{|x|\geq R} |Q_l\varsigma(\alpha+\tau e_j, x) - Q_l\varsigma(\alpha,x)|\, dx\, d\alpha \lesssim 2^{l\delta} \iint |Q_l\varsigma(\alpha,x)|\, dx\, d\alpha \lesssim 2^{-l(\varepsilon-\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

In the second case, $R \le 2^l$, $|\tau| \le 2^{-l}$, we have

$$\begin{aligned}&|\tau|^{-\delta}\iint_{|x|\ge R}|Q_l\varsigma(\alpha+\tau e_j,x)-Q_l\varsigma(\alpha,x)|\,dx\,d\alpha\\&\lesssim 2^{-l(\varepsilon-\delta)}|\tau|^{-\varepsilon}\iint|\varsigma(\alpha+\tau e_j,x)-\varsigma(\alpha,x)|\,dx\,d\alpha\lesssim 2^{-l(\varepsilon-\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}.\end{aligned}$$

In the third case, $R \ge 2^l$, $|\tau| \ge R^{-1}$, we have

$$|\tau|^{-\delta}\iint_{|x|\ge R}|Q_l\varsigma(\alpha+\tau e_j,x)-Q_l\varsigma(\alpha,x)|\,dx\,d\alpha\lesssim R^{\delta}\iint_{|x|\ge R}|Q_l\varsigma(\alpha,x)|\,dx\,d\alpha\lesssim R^{\delta-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon},$$

where in the last inequality we have used (3.5). In the last case, $R \ge 2^l$, $|\tau| \le R^{-1}$,

$$\begin{aligned}&|\tau|^{-\delta}\iint_{|x|\ge R}|Q_l\varsigma(\alpha+\tau e_j,x)-Q_l\varsigma(\alpha,x)|\,dx\,d\alpha\\&\lesssim R^{\delta-\varepsilon}|\tau|^{-\varepsilon}\iint|\varsigma(\alpha+\tau e_j,x)-\varsigma(\alpha,x)|\,dx\,d\alpha\lesssim R^{\delta-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon},\end{aligned}$$

as desired. This completes the proof. □

Proof of Proposition 3.3, conclusion. Let ς_j be as in the statement of the proposition. By Lemma 3.5 we already know the sum $\sum_{j\in\mathbb{Z}}\varsigma_j^{(2^j)}$ converges in the topology on $L\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^d)$. Our goal is to show convergence of the sum $\|\sum_{j\in\mathbb{Z}}\varsigma_j^{(2^j)}\|_{\mathcal{K}_\delta}$ in $\mathcal{K}_\delta$ for $0<\delta<\varepsilon/2$. Fix $j_1, j_2\in\mathbb{Z}$, $j_1<j_2$. Define $K=\sum_{j_1\le j\le j_2}\varsigma_j^{(2^j)}$. We will show $\|K\|_{\mathcal{K}_\delta}\lesssim\sup_j\|\varsigma_j\|_{\mathcal{B}_\varepsilon}$, with the implicit constant independent of j_1, j_2. The result then follows y a limiting argument. In what follows, summations in j are taken over the range $j_1\le j\le j_2$. We assume, without loss of generality,

$$\sup_j\|\varsigma_j\|_{\mathcal{B}_\varepsilon}=1.$$

Let $\chi_0\in\mathcal{S}(\mathbb{R}^d)$ be so that $\widehat{\chi_0}(\xi)=1$ for $|\xi|\le 1$ and $\widehat{\chi_0}$ is supported in $\{\xi:|\xi|\le 2\}$. For $l\ge 1$ let $\chi_l=\chi_0^{(2^l)}-\chi_0^{(2^{l-1})}$, so that $\sup_{l\in\mathbb{Z}}\widehat{\chi}_l(\xi)=1$ for $\xi\ne 0$. We write

$$K=\sum_j\varsigma_j^{(2^j)}=\sum_{l\ge 0}\sum_j\varsigma_{j,l}^{(2^j)},$$

where

$$\varsigma_{j,l}(\alpha,\cdot)=\chi_l*\varsigma_j(\alpha,\cdot)$$

and the convolution is in $\mathbb{R}^d$. Let

$$K_l=\sum_j\varsigma_{j,l}^{(2^j)}.$$

The proof will be complete once we have shown $\|K_l\|_{\mathcal{K}_\delta}\lesssim 2^{-l(\varepsilon-2\delta)}$.

Our first goal is to show, for $1\le i\le n$, $t\in\mathbb{R}$,

$$\int(1+|\alpha_i|)^{\delta}\|\eta*K_l^{(t)}\|_{L^2(\mathbb{R}^d)}\,d\alpha\lesssim(1+l)2^{-l(\varepsilon-\delta)}\tag{3.9}$$

which gives $\|K_l\|_{\mathcal{K}^{\eta}_{\delta,1}}\lesssim(1+l)2^{-l(\varepsilon-\delta)}$. To prove (3.9), we will show

$$\int(1+|\alpha_i|)^{\delta}\big\|\eta*\varsigma_{j,l}^{(2^jt)}(\alpha,\cdot)\big\|_2\,d\alpha\lesssim\begin{cases}(2^{l(\varepsilon-\delta)}2^jt)^{-\frac{\varepsilon-\delta}{1+\varepsilon-\delta}} & \text{if } 2^jt\ge 2^l,\\ 2^{-l(\varepsilon-\delta)} & \text{if } 2^{-2l}\le 2^jt\le 2^l,\\ (2^{l+j}t)^{d/2} & \text{if } 2^jt\le 2^{-2l}.\end{cases}\tag{3.10}$$

Summing (3.10) in j yields (3.9), so we focus on (3.10).

First we consider the case when $2^j t \geq 2^l$. Letting $r \in [1, 2^j t]$ be chosen later, we use that $\int \varsigma_{j,l}(\alpha, x)\, dx = 0$ to see

$$\begin{aligned}
&\int (1+|\alpha_i|)^\delta \big\| \eta * \varsigma_{j,l}(\alpha,\cdot)^{(2^j t)} \big\|_2 \, d\alpha \\
&\lesssim \int (1+|\alpha_i|)^\delta \Big(\int \Big| \int [\eta(x-y) - \eta(x)] (2^j t)^d \varsigma_{j,l}(\alpha, 2^j t y)\, dy \Big|^2 dx \Big)^{\frac12} d\alpha \\
&\lesssim \int (1+|\alpha_i|)^\delta \int |\varsigma_{j,l}(\alpha, v)| \|\eta(\cdot - \tfrac{v}{2^j t}) - \eta(\cdot)\|_{L^2(\mathbb{R}^d)} \, dv \, d\alpha \\
&\lesssim \iint (1+|\alpha_i|)^\delta |\varsigma_{j,l}(\alpha, v)| \min\{\tfrac{|v|}{2^j t}, 1\} \, dv \, d\alpha \\
&= \iint_{|v|\leq r} + \iint_{|v|>r} =: (I) + (II).
\end{aligned}$$

We have, using (3.7) with $R = 0$,

$$(I) \lesssim \frac{r}{2^j t} \iint (1+|\alpha_i|)^\delta |\varsigma_{j,l}(\alpha, v)| \, dv \, d\alpha \lesssim \frac{r}{2^j t} 2^{-l(\varepsilon-\delta)}.$$

Using (3.7) with $R = r$,

$$(II) \lesssim \iint_{|x|\geq r} (1+|\alpha_i|)^\delta |\varsigma_{j,l}(\alpha, v)| \, dv \, d\alpha \lesssim r^{-(\varepsilon-\delta)}.$$

We choose r so that $r^{1+\varepsilon-\delta} = 2^{l(\varepsilon-\delta)} 2^j t$; this yields (3.10) in the case $2^j t \geq 2^l$ under consideration.

For $2^{-2l} \leq 2^j t \leq 2^l$ we use the trivial $L^1 \to L^2$ bound for convolution with η and a change of variables, combined with (3.7) (with $R = 0$) to see

$$\int (1+|\alpha_i|)^\delta \|\eta * \varsigma_{j,l}^{(2^j t)}(\alpha,\cdot)\|_2 \, d\alpha \lesssim \int (1+|\alpha_i|)^\delta \|\varsigma_{j,l}(\alpha,\cdot)\|_1 \, d\alpha \lesssim 2^{-l(\varepsilon-\delta)},$$

as desired.

Now assume $2^j t \leq 2^{-l}$. Let $u \in \mathcal{S}(\mathbb{R}^d)$ be such that $\widehat{u}(\xi) = 1$ for $|\xi| \leq 2$, so that $\widehat{u}(2^{-l}\cdot) = 1$ on the support of $\widehat{\varsigma}_{j,l}$. We then have, using $\|\widehat{u}(2^{-j-l}t^{-1}\cdot)\widehat{\eta}(\cdot)\|_2 \lesssim (2^{j+l}t)^{d/2}$,

$$\begin{aligned}
&\int (1+|\alpha_i|)^\delta \|\eta * \varsigma_{j,l}^{(2^j t)}(\alpha,\cdot)\|_2 \, d\alpha \lesssim \int (1+|\alpha_i|)^\delta \|\eta * u^{(2^{j+l}t)}\|_2 \|\varsigma_{j,l}(\alpha,\cdot)\|_1 \, d\alpha \\
&\lesssim \int (1+|\alpha_i|)^\delta \|\widehat{u}(2^{-j-l}t^{-1}\cdot)\widehat{\eta}(\cdot)\|_2 \|\varsigma_{j,l}(\alpha,\cdot)\|_1 \, d\alpha \lesssim (2^{j+l}t)^{d/2}.
\end{aligned}$$

This completes the proof of (3.10) and therefore of (3.9).

A simple modification of the above proof, using (3.8) in place of (3.7), gives for $|\tau| \leq 1$,

$$\int \big\| \eta * [\varsigma_{j,l}^{(2^j t)}(\alpha + \tau e_j, \cdot) - \varsigma_{j,l}^{(2^j t)}(\alpha,\cdot)] \big\|_2 \, d\alpha \lesssim |\tau|^\delta \cdot \begin{cases} (2^j t)^{-(\varepsilon-\delta)} & \text{if } 2^j R \geq 2^l, \\ 2^{-l(\varepsilon-\delta)} & \text{if } 2^{-2l} \leq 2^j R \leq 2^l, \\ (2^{l+j} R)^d & \text{if } 2^j R \leq 2^{-2l}. \end{cases}$$

Summing in j shows that for $0 < h \leq 1$,

$$h^{-\varepsilon} \int \big\| \eta * [K_l^{(t)}(\alpha + h e_i, \cdot) - K_l^{(t)}(\alpha,\cdot)] \big\|_2 \, d\alpha \lesssim (1+l) 2^{-l(\varepsilon-\delta)}$$

and hence $\|K_l\|_{\mathcal{K}^\eta_{\delta,2}} \lesssim (1+l) 2^{-l(\varepsilon-\delta)}$.

Next we wish to show $\|K_l\|_{\mathcal{K}_{\delta,3}} \lesssim (1+l)2^{-l(\varepsilon-\delta)}$, that is, for $1 \le i \le n$, $R > 0$,

$$\iint\limits_{R \le |x| \le 2R} (1+|\alpha_i|)^\delta |K_l(\alpha,x)|\,dx\,d\alpha \lesssim (1+l)2^{-(\varepsilon-\delta)l}. \tag{3.11}$$

To prove (3.11) we will show

$$\iint\limits_{R \le |x| \le 2R} (1+|\alpha_i|)^\delta |\varsigma_{j,l}^{(2^j)}(\alpha,x)|\,dx\,d\alpha \lesssim \begin{cases} (2^jR)^{-(\varepsilon-\delta)} & \text{if } 2^jR \ge 2^l, \\ 2^{-l(\varepsilon-\delta)} & \text{if } 2^{-2l} \le 2^jR \le 2^l, \\ (2^{l+j}R)^d & \text{if } 2^jR \le 2^{-2l}. \end{cases} \tag{3.12}$$

Summing (3.12) in j yields (3.11). Now, applying (3.7),

$$\iint\limits_{R \le |x| \le 2R} (1+|\alpha_i|)^\delta |\varsigma_{j,l}^{(2^j)}(\alpha,x)|\,dx\,d\alpha \le \iint\limits_{2^jR \le |x|} (1+|\alpha_i|)^\delta |\varsigma_{j,l}(\alpha,x)|\,dx\,d\alpha$$
$$\lesssim \begin{cases} (2^jR)^{-(\varepsilon-\delta)} & \text{if } 2^jR \ge 2^l, \\ 2^{-l(\varepsilon-\delta)} & \text{if } 2^jR \le 2^l. \end{cases}$$

Thus, to complete the proof of (3.12) we need only consider the case when $2^jR \le 2^{-2l}$. We have

$$\iint\limits_{R \le |x| \le 2R} (1+|\alpha_i|)^\delta |\varsigma_{j,l}^{(2^j)}(\alpha,x)|\,dx\,d\alpha = \iint\limits_{2^jR \le |x| \le 2^{j+1}R} (1+|\alpha_i|)^\delta |\varsigma_{j,l}(\alpha,x)|\,dx\,d\alpha$$
$$\lesssim (2^jR)^d \int (1+|\alpha_i|)^\delta \|\varsigma_{j,l}(\alpha,\cdot)\|_{L^\infty(\mathbb{R}^d)}\,d\alpha \lesssim (2^jR)^d 2^{ld} \int (1+|\alpha_i|)^\delta \|\varsigma_j(\alpha,\cdot)\|_{L^1(\mathbb{R}^d)}\,d\alpha$$
$$\lesssim (2^{j+l}R)^d,$$

competing the proof of (3.12) and therefore of (3.11).

A simple modification of the above yields, for $0 < |\tau| \le 1$,

$$\iint\limits_{R \le |x| \le 2R} |\varsigma_{j,l}^{(2^j)}(\alpha+\tau e_i,x) - \varsigma_{j,l}^{(2^j)}(\alpha,x)|\,dx\,d\alpha \lesssim |\tau|^\delta \cdot \begin{cases} (2^jR)^{-(\varepsilon-\delta)} & \text{if } 2^jR \ge 2^l, \\ 2^{-l(\varepsilon-\delta)} & \text{if } 2^{-2l} \le 2^jR \le 2^l, \\ (2^{l+j}R)^d & \text{if } 2^jR \le 2^{-2l}. \end{cases}$$

Summing in j yields, for $0 < h \le 1$, $R > 0$,

$$h^{-\delta} \iint\limits_{R \le |x| \le 2R} |K_l(\alpha+he_i,x) - K_l(\alpha,x)|\,dx\,d\alpha \lesssim (1+l)2^{-(\varepsilon-\delta)l}$$

and hence $\|K\|_{\mathcal{K}_{\varepsilon,4}} \lesssim (1+l)2^{-(\varepsilon-\delta)l}$.

Finally, we wish to show, for $R \ge 2$, $y \in \mathbb{R}^d$,

$$R^\delta \iint\limits_{|x| \ge R|y|} |K(\alpha,x-y) - K(\alpha,x)|\,dx\,d\alpha \lesssim 2^{-l(\varepsilon-2\delta)}. \tag{3.13}$$

First, estimate

$$R^\delta \iint\limits_{|x| \ge R|y|} |\varsigma_{j,l}^{(2^j)}(\alpha,x-y) - \varsigma_{j,l}^{(2^j)}(\alpha,x)|\,dx\,d\alpha = R^\delta \iint\limits_{|x| > 2^j|y|R} |\varsigma_{j,l}(\alpha,x-2^jy) - \varsigma_{j,l}(\alpha,x)|\,dx\,d\alpha$$
$$\lesssim R^\delta \min\{1, 2^{j+l}|y|\} \min\{2^{-l\varepsilon}, (2^j|y|R)^{-\varepsilon}\} =: \mathcal{E}(j,l,R).$$

Here we applied (3.6) with $2^j|y|$ in place of $|h|$ and $2^j|y|R$ in place of R. Note the left hand side of (3.13) is bounded by $\sum_j \mathcal{E}(j,l,R)$.

In the case $R \geq 2^{2l}$, we estimate

$$\sum_j \mathcal{E}(j,l,R) \lesssim$$
$$\sum_{2^j|y|\geq 2^{-l}} R^{\delta-\varepsilon}(2^j|y|)^{-\varepsilon} + \sum_{2^l/R\leq 2^j|y|\leq 2^{-l}} 2^l(2^j|y|)^{1-\varepsilon}R^{\delta-\varepsilon} + \sum_{2^j|y|\leq 2^l/R} R^\delta(2^j|y|)2^{l(1-\varepsilon)}.$$

The first two sums are $O(R^{\delta-\varepsilon}2^{l\varepsilon})$, and the third sum is $O(R^{\delta-1}2^{(2-\varepsilon)l})$; here we used $R \geq 2^{2l}$.

In the case $R \leq 2^{2l}$ we have

$$\sum_j \mathcal{E}(j,l,R) \lesssim \sum_{2^j|y|\geq 2^l/R} R^{\delta-\varepsilon}(2^j|y|)^{-\varepsilon} + \sum_{2^{-l}\leq 2^j|y|\leq 2^l/R} R^\delta 2^{-l\varepsilon} + \sum_{2^j|y|\leq 2^{-l}} R^\delta 2^j|y|2^{l(1-\varepsilon)}.$$

The first sum is $O(R^\delta 2^{-l\varepsilon})$, the second sum is $O(R^\delta 2^{-l\varepsilon}\log(1+2^{2l}/R))$, and since $R \leq 2^{2l}$ the third sum is $O(R^\delta 2^{-l\varepsilon})$. In both cases we obtain $\sum_j \mathcal{E}(j,l,R) \lesssim 2^{-l(\varepsilon-2\delta)}$. This completes the proof of (3.13). Combining all of the above inequalities completes the proof of the proposition. □

4. Adjoints

This section is devoted to studying the space $\mathcal{B}_\varepsilon$; in particular will give the proof of Theorem 2.9. It will be advantageous to work with a variant of this class, for functions on $\mathbb{R}^N$, with $N = n+d$.

Definition 4.1. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. We define a Banach space $\mathfrak{B}_\varepsilon(\mathbb{R}^N)$ to be the space of measurable functions $\gamma : \mathbb{R}^N \to \mathbb{C}$ such that the norm

$$\|\gamma\|_{\mathfrak{B}_\varepsilon} := \max_{1\leq i\leq N} \int (1+|s_i|)^\varepsilon|\gamma(s)|\,ds + \sup_{\substack{0<h\leq 1\\ 1\leq i\leq N}} h^{-\varepsilon}\int |\gamma(s+he_i)-\gamma(s)|\,ds,$$

is finite. Here $e_1, \dots, e_N$ denotes the standard basis of $\mathbb{R}^N$.

Remark 4.2. The spaces $\mathfrak{B}_\varepsilon(\mathbb{R}^{n+d})$ and $\mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ coincide; indeed, for $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n\times\mathbb{R}^d)$, we have the equivalence

$$\|\varsigma\|_{\mathfrak{B}_\varepsilon} \approx \|\varsigma\|_{\mathcal{B}_\varepsilon},$$

with implicit constants depending only on d. In this section we find it more useful to use the space $\mathfrak{B}_\varepsilon$ as it treats the α and x variables symmetrically.

The following two propositions involve operations on functions in $\mathfrak{B}_\varepsilon$ involving inversions and multiplicative shears. They are the main technical results needed for the proof of Theorem 2.9.

Proposition 4.3. *Let $\varepsilon > 0$ and $\delta < \varepsilon/3$. Let $\gamma \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$ and*

$$J_1\gamma(s_1,\dots,s_N) := s_1^{-2}\gamma(s_1^{-1}, s_2, \dots, s_N),$$

$\gamma \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$. Then $J_1\gamma \in \mathfrak{B}_\delta(\mathbb{R}^N)$ and

$$\|J_1\gamma\|_{\mathfrak{B}_\delta} \lesssim \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

Proposition 4.4. *Let $\varepsilon > 0$ and $\delta < \varepsilon/3$. Let $\gamma \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$, $n \in \{1,\dots,N\}$ and set*

$$M\gamma(s_1,\dots,s_N) := s_1^{n-1}\gamma(s_1, s_1s_2, \dots, s_1s_n, s_{n+1}, s_{n+2}, \dots, s_N).$$

Then $M\gamma \in \mathfrak{B}_{\varepsilon'}(\mathbb{R}^N)$ and

$$\|M\gamma\|_{\mathfrak{B}_\delta} \lesssim n\|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

For later use in §4.5 we state these results in a different form:

Corollary 4.5. *Let $1 \le n \le N$. For $\gamma \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$ define two functions*

$$\Gamma_1(s_1,\dots,s_N) := s_1^{-n-1}\gamma(s_1^{-1}, s_1^{-1}s_2,\dots,s_1^{-1}s_n, s_{n+1},\dots,s_N),$$

$$\Gamma_2(s_1,\dots,s_N) := s_1^{-(n-1)}\gamma(s_1, s_1^{-1}s_2,\dots,s_1^{-1}s_n, s_{n+1},\dots,s_N).$$

There exists $\varepsilon' = \epsilon'(\varepsilon) > 0$ (depending neither on N nor n) such that

$$\|\Gamma_1\|_{\mathfrak{B}_{\varepsilon'}} + \|\Gamma_2\|_{\mathfrak{B}_{\varepsilon'}} \le C_{\varepsilon,\varepsilon'} n \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

Proof. Notice that $\Gamma_1 = J_1 M\gamma$, $\Gamma_2 = J_1 M J_1 \gamma$ where J_1 and M are as in the propositions above. □

4.1. **Proof of Theorem 2.9.** We assume Proposition 4.3 and Proposition 4.4 and deduce Theorem 2.9. If $\varsigma \in L^1(\mathbb{R}^n \times \mathbb{R}^d)$ and ϖ is a permutation of $\{1,\dots,n+2\}$, we shall show

$$\Lambda[\varsigma](b_{\varpi(1)},\dots,b_{\varpi(n+2)}) = \Lambda[\ell_\varpi\varsigma](b_1,\dots,b_{n+2}),$$

such that $\|\ell_\varpi\varsigma\|_{L^1} = \|\varsigma\|_{L^1}$ and such that there exists $\varepsilon' > c\varepsilon$, with c independent of ϖ, and

$$\|\ell_\varpi\varsigma\|_{\mathcal{B}_{\varepsilon'}} \lesssim n^2 \|\varsigma\|_{\mathcal{B}_\varepsilon}$$

for $\varsigma \in \mathcal{B}_\varepsilon$.

Every permutation of $\{1,\dots,n+2\}$ is a composition of at most four permutations of the following three forms, with the permutation in (iii) occuring at most twice.

(i) A permutation of $\{1,\dots,n\}$, leaving $n+1$ and $n+2$ fixed.
(ii) The permutation which switches $n+1$ and $n+2$, leaving all other elements fixed.
(iii) The permutation which switches $n+1$ and 1, leaving all other elements fixed.

Case (i) If ϖ is a permutation of $\{1,\dots,n\}$, leaving $n+1$ and $n+2$ fixed, then it is immediate to verify

$$\ell_\varpi\varsigma(\alpha,v) = \varsigma(\alpha_{\varpi^{-1}(1)},\dots,\alpha_{\varpi^{-1}(n)}, v), \tag{4.1}$$

and thus $\|\ell_\varpi\varsigma\|_{\mathcal{B}_\varepsilon} = \|\varsigma\|_{\mathcal{B}_\varepsilon}$ and $\|\ell_\varpi\varsigma\|_{L^1} = \|\varsigma\|_{L^1}$.

Case (ii). If ϖ is the permutation which switches $n+1$ and $n+2$, leaving all other elements fixed, then it is immediate to verify that

$$\ell_\varpi\varsigma(\alpha,v) = \varsigma(1-\alpha_1,\dots,1-\alpha_n, v). \tag{4.2}$$

We have $\|\varsigma_\varpi\|_{\mathcal{B}_\varepsilon} \approx \|\varsigma\|_{\mathcal{B}_\varepsilon}$ and $\|\varsigma_\varpi\|_{L^1} = \|\varsigma\|_{L^1}$.

In both of the above cases, if $\int \varsigma(\alpha,v)\,dv = 0\ \forall\alpha$ then $\int \varsigma_\varpi(\alpha,v)\,dv = 0\ \forall\alpha$.

Case (iii). We compute

$$\begin{aligned}
&\Lambda[\varsigma](b_{n+1}, b_2,\dots,b_n, b_1, b_{n+2})\\
&= \iiint \varsigma(\alpha,v) b_{n+1}(x-\alpha_1 v)\Big(\prod_{i=2}^n b_i(x-\alpha_i v)\Big) b_1(x-v) b_{n+2}(x)\,dv\,dx\,d\alpha\\
&= \iiint |\alpha_1|^{-d}\varsigma(\alpha, \alpha_1^{-1}w) b_{n+1}(x-w)\Big(\prod_{i=2}^n b_i(x-\alpha_i\alpha_1^{-1}w)\Big) b_1(x-\alpha_1^{-1}w) b_{n+2}(x)\,dx\,dw\,d\alpha\\
&= \iiint \beta^{d-n-1}\varsigma(\beta_1^{-1}, \beta_1^{-1}\beta_2,\dots,\beta_1^{-1}\beta_n, \beta_1 w)\prod_{i=1}^n b_i(x-\beta_i v) b_{n+1}(x-w) b_{n+2}(x) dx\,dw\,d\beta
\end{aligned}$$

where we have first changed variables $v = \alpha_1^{-1}u$, then interchanged the order of integration, and changed variables $\alpha_1 = \beta_1^{-1}$, $\alpha_i = \beta_i\beta_1^{-1}$ for $i = 2, \dots, n$. Hence if ϖ is the transposition interchanging 1 and $n+1$ and leaving $2, \dots, n, n+2$ fixed then $\Lambda^{\varpi}[\varsigma] = \Lambda[\ell_{\varpi}\varsigma]$ with

$$\ell_{\varpi}\varsigma(\alpha_1, \dots, \alpha_n, v) = \varsigma(\alpha_1^{-1}, \alpha_1^{-1}\alpha_2, \dots, \alpha_1^{-1}\alpha_n, \alpha_1 v)\,. \tag{4.3}$$

Now if we define the inversion J, with respect to the α_1 variable, and multiplicative shears $M_{n-1}, \widetilde{M}_d$ by

$$\begin{aligned} Jg(\alpha_1, \dots, \alpha_n, v) &= \alpha_1^{-2} g(\alpha_1^{-1}, \alpha_2, \dots, \alpha_n, v) \\ M_{n-1}g(\alpha_1, \dots, \alpha_n, v) &= \alpha_1^{n-1} g(\alpha_1, \alpha_1\alpha_2, \dots, \alpha_1\alpha_n, v) \\ \widetilde{M}_d g(\alpha_1, \dots, \alpha_n, v) &= \alpha_1^{d} g(\alpha_1, \dots, \alpha_n, \alpha_1 v) \end{aligned}$$

then it is straightforward to check that the linear transformation ℓ_{ϖ} in (4.3) can be factorized as

$$\ell_{\varpi} = J \circ \widetilde{M}_d \circ J \circ M_{n-1} \circ J\,. \tag{4.4}$$

By Remark 4.2 the $\mathcal{B}_{\varepsilon}(\mathbb{R}^n \times \mathbb{R}^d)$ and the $\mathfrak{B}_{\varepsilon}(\mathbb{R}^{n+d})$ norms are equivalent with equivalence constants not depending on n. By Proposition 4.3 we have $\|Jg\|_{\mathcal{B}'_{\varepsilon}} \lesssim \|g\|_{\mathcal{B}_{\varepsilon}}$, and by Proposition 4.4 we have $\|M_{n-1}g\|_{\mathcal{B}'_{\varepsilon}} \lesssim n\|g\|_{\mathcal{B}_{\varepsilon}}$, and $\|M_d g\|_{\mathcal{B}'_{\varepsilon}} \lesssim \|g\|_{\mathcal{B}_{\varepsilon}}$, for $\varepsilon' < \varepsilon/3$. Hence $\|\ell_{\varpi}\varsigma\|_{\mathcal{B}_{\delta}} \lesssim n\|\varsigma\|_{\mathcal{B}_{\varepsilon}}$, at least when $\delta < 3^{-5}\varepsilon$.

Finally if ϖ is a general permutation than we can split $\varpi = \varpi_1 \circ \varpi_2 \circ \varpi_3 \circ \varpi_4$, each ϖ_i of the form in (i), (ii) or (iii), with at most two of the form in (iii). Hence we get $\Lambda^{\varpi}[\varsigma] = \Lambda[\ell_{\varpi}\varsigma]$ where $\|\ell_{\varpi}\varsigma\|_{\mathcal{B}_{\delta}} \lesssim n^2\|\varsigma\|_{\mathcal{B}_{\varepsilon}}$, at least for $\delta < 3^{-10}\varepsilon$. We remark that if we avoid the factorization (4.4) and use the formula for ℓ_{ϖ} directly we should get a better range for δ but this will be irrelevant for our final boundedness results on the forms Λ^{ϖ}. $\square$

4.2. **Proof of Propositions 4.3 and 4.4.** We first prove several preliminary lemmata, then give the proof of Proposition 4.3 in §4.2.2 and the proof of Proposition 4.4 in §4.2.3.

4.2.1. *Preparatory Results.* We first recall a standard fact about Besov spaces $B^{\varepsilon}_{1,q}(\mathbb{R})$; $1 \le q \le \infty$. If $0 < \varepsilon < 1$ then there the characterizations

$$\|f\|_{B^{\varepsilon}_{1,q}} \approx \|f\|_1 + \Big(\int_0^1 \|f(\cdot + h) - f\|_1^q \frac{dh}{h^{1+\varepsilon q}}\Big)^{1/q}, \quad 1 \le q < \infty, \tag{4.5a}$$

and

$$\|f\|_{B^{\varepsilon}_{1,\infty}} \approx \|f\|_1 + \sup_{0<h<1} h^{-\varepsilon}\|f(\cdot + h) - f\|_1\,. \tag{4.5b}$$

Moreover there are the continous embeddings

$$B^{\varepsilon}_{1,q_1} \subset B^{\varepsilon}_{1,q_2}, \quad q_1 < q_2. \tag{4.6}$$

For (4.5) and (4.6) we refer to [35, §V.5] or [39]. As a corollary we get

Lemma 4.6. *Let $0 < \delta < \varepsilon < 1$. Then for functions in $L^1(\mathbb{R})$ then there are constants $c, C > 0$ depending only on ε, δ such that*

$$\begin{aligned} &c\|f\|_1 + c\int_{0<h<1} h^{-\delta}\|f(\cdot + h) - f\|_1 \frac{dh}{h} \\ \le &\|f\|_1 + \sup_{0<h<1} h^{-\varepsilon}\|f(\cdot + h) - f\|_1 \\ \le &C\|f\|_1 + C\int_{0<h<1} h^{-\varepsilon}\|f(\cdot + h) - f\|_1 \frac{dh}{h}\,. \end{aligned}$$

We let e_i, $i = 1, \dots, N$, denote the standard basis vectors in $\mathbb{R}^N$ and let $e_i^\perp$ to be the orthogonal complement. For $g \in L^1(\mathbb{R}^N)$ and $w \in e_i^\perp$ define

$$\pi_i^w g(s) = g(se_i + w); \tag{4.7}$$

this is defined as an $L^1(\mathbb{R})$ function for almost every $w \in e_i^\perp$, and by Fubini $w \mapsto \int_\mathbb{R} |\pi_i^w g(s)| ds$ belongs to $L^1(e_i^\perp)$. Moreover if $g \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$ for some $\varepsilon > 0$ then for almost every $w \in e_i^\perp$ the function $h \mapsto \int_\mathbb{R} |\pi_i^w g(s+h) - \pi_i^w g(s)| ds$ is continuous.

Lemma 4.7. *Let $0 \le \delta < 1$. Then the following statements hold.*

(i)

$$\|g\|_{\mathfrak{B}_\delta(\mathbb{R}^N)} \le \max_{i=1,\dots,n} \int_{e_i^\perp} \left\|\pi_i^w g\right\|_{\mathfrak{B}_\delta(\mathbb{R})} dw\,.$$

(ii) If $0 < \delta < \varepsilon \le 1$ then there exists $C = C(\varepsilon, \delta) > 0$ (not depending on N) such that for all $f \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$

$$\max_{i=1,\dots,N} \int_{e_i^\perp} \left\|\pi_i^w g\right\|_{\mathfrak{B}_\delta(\mathbb{R})} dw \le C\|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}\,.$$

Proof. (i) follows immediately from the definitions of $\mathfrak{B}_\delta(\mathbb{R})$ and $\mathfrak{B}_\delta(\mathbb{R}^N)$. For (ii) fix $i \in \{1, \dots, N\}$ and split $\int_{e_i^\perp} \left\|\pi_i^w g\right\|_{\mathfrak{B}_\delta(\mathbb{R})} dw = I + II$ where

$$\begin{aligned}
I &= \int_{e_i^\perp} \int (1+|s|)^\delta |g(se_i + w)| ds\, dw \\
II &= \int_{e_i^\perp} \sup_{0\le h\le 1} |h|^{-\delta} \int |g((s+h)e_i + w) - g(se_i + w)| ds\, dw\,.
\end{aligned}$$

It is immediate that $I \le \|g\|_{\mathfrak{B}_\delta(\mathbb{R}^N)} \le \|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}$. For the second term we use Lemma 4.6 to estimate

$$\begin{aligned}
II &\le C_\delta \int_{e_i^\perp} \int_{0\le h\le 1} |h|^{-\delta} \int |g((s+h)e_i + w) - g(se_i + w)| ds \frac{dh}{h}\, dw \\
&= C_\delta \int_0^1 h^{\varepsilon-\delta} h^{-\varepsilon} \int_{\mathbb{R}^N} |g(x + he_i) - g(x)| dx \frac{dh}{h} \\
&\le C_\delta (\varepsilon - \delta)^{-1} \sup_{0<h<1} |h|^{-\varepsilon} \|g(\cdot + he_i) - g\|_{L^1(\mathbb{R}^N)}
\end{aligned}$$

and hence $II \le C(\varepsilon, \delta)\|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}$. □

Lemma 4.8. *Let $R \ge 1$ and let $\Omega_R^i = \{x \in \mathbb{R}^N : |x_i| \ge R\}$. Then*

$$\int_{\Omega_R^i} |g(x)| dx \le R^{-\varepsilon} \|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}.$$

Proof. This is immediate from

$$\int_{\Omega_R^i} |g(x)|\, dx \le R^{-\varepsilon} \int (1+|x_i|)^\varepsilon |g(x)|\, dx\,.$$ □

The following lemma is a counterpart to Lemma 4.8 which is used when integrating over sets whose projection to a coordinate axis has small measure. It can be seen as a standard application of a Sobolev embedding theorem for functions on the real line. For measurable $J \subset \mathbb{R}$ we denote by $|J|$ the Lebesgue measure.

Lemma 4.9. *Let $0 < \varepsilon \le 1$ and $f \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$, and let $0 < \varepsilon' < \varepsilon$. Let $E \subset \mathbb{R}^N$ and let*

$$\mathrm{proj}_i(E) = \{s \in \mathbb{R} : se_i + w \in E \text{ for some } w \in e_i^\perp\}.$$

Then

$$\int_E |f(x)|dx \le C_{\varepsilon,\varepsilon'} |\mathrm{proj}_i(E)|^{\varepsilon'} \|f\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)} .$$

Moreover for $i = 1, \dots, N$, $\delta < \varepsilon$,

$$\int_{e_i^\perp} \int_{|x_i|\le 1} |x_i|^{-\delta} |f(x)|dx \le C(\varepsilon,\delta) \|f\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}.$$

Proof. For $k \ge 0$ let $E_k = \{x \in \mathbb{R}^N : 2^{-k-1} \le |x_i| \le 2^{-k}\}$. The second inequality is a consequence of the first applied to the sets E_k.

To prove the first statement pick $p = (1-\varepsilon')^{-1} > 1$ so that $\varepsilon' = 1 - p^{-1}$. By Hölder's inequality,

$$\int_E |f(x)|dx \le |\mathrm{proj}_i(E)|^{\varepsilon'} \int_{e_i^\perp} \Big(\int |f(se_i + w)|^p ds \Big)^{1/p} dw .$$

Let $\pi_i^w f(s) = f(se_i + w)$. Let $\phi \in \mathcal{S}(\mathbb{R})$, $\int \phi(s)ds = 1$ such that the Fourier transform $\widehat{\phi}$ is supported in $\{|\xi| \le 1\}$. Let $\psi_k = 2^k \phi(2^k \cdot) - 2^{k-1}\phi(2^{k-1}\cdot)$. Choose $\tilde{\phi} \in \mathcal{S}(\mathbb{R})$ whose Fourier transform is equal to 1 on $\{|\xi| \le 2\}$ and let $\tilde{\phi}_k = 2^k \tilde{\phi}(2^k \cdot)$. Then

$$\pi_i^w f = \tilde{\phi} * \phi * \pi_i^w f + \sum_{k=1}^\infty \tilde{\phi}_k * \psi_k * \pi_i^w f$$

and thus, by Young's inequality,

$$\begin{aligned} \|\pi_i^w f\|_{L^p(\mathbb{R})} &\le \|\tilde{\phi}\|_{L^p(\mathbb{R})} \|\phi * \pi_i^w f\|_{L^p(\mathbb{R})} + \sum_{k=1}^\infty \|\tilde{\phi}_k\|_{L^p(\mathbb{R})} \|\psi_k * \pi_i^w f\|_{L^1(\mathbb{R})} \\ &\lesssim \|\phi * \pi_i^w f\|_{L^p(\mathbb{R})} + \sum_{k=1}^\infty 2^{k(1-1/p)} \|\psi_k * \pi_i^w f\|_{L^1(\mathbb{R})}. \end{aligned}$$

Since $\int \psi_k(s)ds = 0$ we have

$$\begin{aligned} \big|\psi_k * \pi_i^w f(s)\big| &= \Big| \int \psi_k(h) \big[\pi_i^w f(s-h) - \pi_i^w f(s)\big] dh \Big| \\ &\lesssim \int \frac{2^k}{(1+2^k|h|)^3} \big|\pi_i^w f(s-h) - \pi_i^w f(s)\big| dh . \end{aligned}$$

Using this in the above expression we get after integration in w

$$\begin{aligned} &\int_{e_i^\perp} \Big(\int |f(se_i + w)|^p ds \Big)^{1/p} dw \\ &\lesssim \|f\|_1 + \sum_{k=1}^\infty 2^{k(1-\frac{1}{p})} \int \frac{2^k}{(1+2^k|h|)^3} \big|\pi_i^w f(s-h) - \pi_i^w f(s)\big| ds\, dw\, dh \\ &\lesssim \|f\|_1 + \sum_{k=1}^\infty \int_{|h|\le 1} \frac{2^{k(2-\frac{1}{p})}|h|^\varepsilon}{(1+2^k|h|)^3} dh \sup_{|u|\le 1} \frac{\|f(\cdot + ue_i) - f(\cdot)\|_1}{|u|^\varepsilon} + \sum_{k=1}^\infty \int_{|h|\ge 1} \frac{2^{k(2-\frac{1}{p})}}{(1+2^k|h|)^3} dh\, \|f\|_1. \end{aligned}$$

The last term is $\lesssim \sum_{k=1}^\infty 2^{-k(1+1/p)} \|f\|_1 \lesssim \|f\|_1$. The middle term is $\lesssim \sum_{k=1}^\infty 2^{k(-\varepsilon+1-1/p)} \|f\|_{\mathfrak{B}_\varepsilon}$ and since $1 - 1/p = \varepsilon' < \varepsilon$ we obtain the required bound. □

4.2.2. *Proof of Proposition 4.3.* The main lemma needed in the proof is an estimate for functions on the real line.

Lemma 4.10. *For $g \in \mathfrak{B}_\varepsilon(\mathbb{R})$ let $Jg(s) = s^{-2}g(s^{-1})$. Then for $\delta < \varepsilon/3$*

$$\|Jg\|_{\mathfrak{B}_\delta(\mathbb{R})} \le C(\varepsilon,\delta)\|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R})}.$$

Proof. First observe that for $\varepsilon' < \varepsilon$

$$\int (1+|\sigma|)^{\varepsilon'}|Jg(\sigma)|d\sigma = \int (1+|s|^{-1})^{\varepsilon'}|g(s)|ds \lesssim \|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R})},$$

by Lemma 4.9. Thus, in light of Lemma 4.6 it remains to prove that for $\rho \le 1/2$,

$$\int_\rho^{2\rho}\int |Jg(\sigma+h) - Jg(\sigma)|d\sigma \frac{dh}{h} \lesssim \rho^{\delta'}\|g\|_{\mathfrak{B}_\varepsilon}, \tag{4.8}$$

for any $\varepsilon' < \delta' < \varepsilon/3$. Choose any $\beta \in (\delta'/\varepsilon, 1/3)$. We have by changes of variables

$$\int_\rho^{2\rho}\int_{|\sigma|\le\rho^\beta} |Jg(\sigma+h)| + |Jg(\sigma)|d\sigma\frac{dh}{h} \lesssim \int_{|\sigma|\le 3\rho^\beta}|Jg(\sigma)|d\sigma \le \int_{|s|\ge\rho^{-\beta}/3}|g(s)|ds \le \rho^{\beta\varepsilon}\|g\|_{\mathfrak{B}_\varepsilon}$$

by Lemma 4.8. Also

$$\int_\rho^{2\rho}\int_{|\sigma|\ge\rho^{-\beta}} |Jg(\sigma+h)| + |Jg(\sigma)|d\sigma\frac{dh}{h} \lesssim \int_{|\sigma|\le\rho^\beta/2}|Jg(\sigma)|d\sigma \le \int_{|s|\le 2\rho^\beta}|g(s)|ds \le \rho^{\beta\varepsilon}\|g\|_{\mathfrak{B}_\varepsilon},$$

by Lemma 4.9. It remains to consider

$$\begin{aligned}\int_\rho^{2\rho}\int_{\rho^\beta}^{\rho^{-\beta}} |Jg(\sigma+h) - Jg(\sigma)|d\sigma\frac{dh}{h} &= \int_\rho^{2\rho}\int_{\rho^\beta}^{\rho^{-\beta}} \big|\tfrac{s^{-2}}{(s^{-1}+h)^2}g\big(\tfrac{1}{s^{-1}+h}\big) - g(s)\big|ds\,\frac{dh}{h} \\ &= \int_\rho^{2\rho}\int_{\rho^\beta}^{\rho^{-\beta}} \big|\tfrac{1}{(1+hs)^2}g\big(\tfrac{s}{1+hs}\big) - g(s)\big|ds\,\frac{dh}{h};\end{aligned}$$

here we have performed the change of variable $s = \sigma^{-1}$. We now interchange the order of integration and then change variables $u = \frac{s}{1+hs} - s = -\frac{s^2h}{1+hs}$. Observe that $du/dh = s^2(1+hs)^{-2}$ and thus $\frac{|du|}{|u|} = |1+hs|^{-1}\frac{|dh|}{|h|}$. Therefore for $|h| \approx \rho$ and $\rho^\beta < |s| \le \rho^{-\beta}$ we can replace $|dh|/|h|$ by $|du|/|u|$. Also observe that $h = -u(su+s^2)^{-1}$ and $1 + hs = s(u+s)^{-1}$. Thus the last displayed expression can be written as

$$\int_{\rho^\beta\le|s|\le\rho^{-\beta}}\int_{-\frac{2\rho s^2}{1+2\rho s}}^{-\frac{\rho s^2}{1+\rho s}} \big|\big(\tfrac{u+s}{s}\big)^2 g(s+u) - g(s)\big|\frac{du}{|u|}ds \le (I) + (II)$$

where

$$(I) := \iint\limits_{\substack{\rho^\beta\le|s|\le\rho^{-\beta}\\ |u|\approx\rho s^2}} \big|\tfrac{(u+s)^2}{s^2} - 1\big||g(s+u)|\frac{du}{|u|}\,ds$$

$$(II) := \iint\limits_{\substack{\rho^\beta\le|s|\le\rho^{-\beta}\\ |u|\approx\rho s^2}} \big|g(s+u) - g(s)\big|\frac{du}{|u|}\,ds\,.$$

First estimate

$$
\begin{aligned}
(I) &\lesssim \int_{\rho^\beta \le |s| \le \rho^{-\beta}} \int_{|u|\approx \rho s^2} |g(u+s)| \frac{u^2+2|us|}{s^2} \frac{du}{|u|}\, ds \\
&\lesssim \int_0^{C\rho^{1-2\beta}} \int_{c\rho^\beta}^{C\rho^{-\beta}} (\rho + |s|^{-1}) |g(s)| ds\, du \\
&\lesssim \rho^{1-2\beta} \|g\|_1 + \sum_{\substack{k\ge 0\\ 2^{-k}\ge c\rho^\beta}} 2^k \int_{2^{-k}\le |s| \le 2^{1-k}} |g(s)| ds
\end{aligned}
$$

and, since by Lemma 4.9 $\int_{|s|\le 2^{-k}} |g(s)|ds \lesssim 2^{-k\varepsilon''}\|g\|_{\mathfrak{B}_\varepsilon}$ for $\varepsilon'' < \varepsilon$, we get

$$
(I) \lesssim \rho^{1-2\beta}\Big(\|g\|_1 + \sum_{\substack{k\ge 0\\ 2^{-k}\ge c\rho^\beta}} 2^{k(1-\varepsilon'')} \|g\|_{\mathfrak{B}_\varepsilon} \Big) \lesssim \rho^{1-3\beta+\beta\varepsilon''} \|g\|_{\mathfrak{B}_\varepsilon}.
$$

Finally,

$$
\begin{aligned}
(II) &\le \sum_{k: 2^{-k}\le C\rho^{1-2\beta}} \int_{2^{-k}\le |u| \le 2^{1-k}} \|g(\cdot + u) - g\|_1 \frac{du}{|u|} \\
&\le \sum_{k: 2^{-k}\le C\rho^{1-2\beta}} 2^{-k\varepsilon} \|g\|_{B^\varepsilon_{1,\infty}} \lesssim \rho^{(1-2\beta)\varepsilon} \|g\|_{\mathfrak{B}_\varepsilon}.
\end{aligned}
$$

Now collect the estimates and keep in mind that $\beta < 1/3$ is chosen close to $1/3$. We may choose ε'' above so that $3\delta' < \varepsilon'' < \varepsilon$. Then the asserted estimate (4.8) follows, and the lemma is proved. □

Proof of Proposition 4.3, concluded. Let $\pi_i^w g(s) = g(se_i + w)$ be as in (4.7). We have

$$
\|J_1\gamma\|_{\mathfrak{B}_\delta} \le \max_{1\le i\le N} \int_{e_i^\perp} \|\pi_i^w (J_1 g)\|_{\mathfrak{B}_\delta(\mathbb{R})} dw.
$$

By Lemma 4.7 and a change of variable $w_1 \mapsto w_1^{-1}$ we obtain for $2\le i\le n$, $\delta_1 > \delta$,

$$
\int_{e_i^\perp} \|\pi_i^w (J_1 g)\|_{\mathfrak{B}_\delta(\mathbb{R})} dw = \int_{e_i^\perp} \|\pi_i^w g\|_{\mathfrak{B}_\delta(\mathbb{R})} dw \lesssim \|g\|_{\mathfrak{B}_{\delta_1}(\mathbb{R}^N)}.
$$

Let $3\delta < \tilde{\varepsilon} < \varepsilon$. For the main term with $i=1$ we use Lemma 4.10 and then Lemma 4.7 to get

$$
\int_{e_1^\perp} \|\pi_i^w (J_1 g)\|_{\mathfrak{B}_\delta(\mathbb{R})} dw = \int_{e_1^\perp} \|J_1(\pi_i^w g)\|_{\mathfrak{B}_\delta(\mathbb{R})} dw \lesssim \int_{e_1^\perp} \|\pi_i^w g\|_{\mathfrak{B}_{\tilde{\varepsilon}}(\mathbb{R})} dw \lesssim \|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}.
$$

This concludes the proof of the proposition. □

4.2.3. *Proof of Proposition 4.4.* We now turn to Proposition 4.4. Fix $\varepsilon > 0$, $n \in \{1,\dots,N\}$, $\gamma \in \mathfrak{B}_\varepsilon(\mathbb{R}^N)$ and recall the definition

$$
M\gamma(s) = s_1^{n-1}\gamma(s_1, s_1 s_2, \dots, s_1 s_n, s_{n+1}, \dots, s_N).
$$

We separate the proof into three lemmata. The most straightforward one is

Lemma 4.11. *Let* $0<\varepsilon<1$. *For* $\delta < \varepsilon/2$, $i = 1,\dots,N$,

$$
\int (1+|s_i|)^\delta |M\gamma(s)|\, ds \lesssim \|\gamma\|_{\mathfrak{B}_\varepsilon}.
$$

Proof. Let $\varepsilon' > 0$ be a number, to be chosen later. If $i = 1$ or $n+1 \le i \le N$, we have, by a change of variable,

$$\int (1+|\sigma_i|)^{\varepsilon'} |M\gamma(\sigma)|\, d\sigma = \int (1+|s_i|)^{\varepsilon'} |\gamma(s)|\, ds \lesssim \|\gamma\|_{\mathfrak{B}_\varepsilon}, \quad \varepsilon' \le \varepsilon.$$

Let $2 \le i \le n$. We have by a change of variable

$$\int (1+|\sigma_i|)^{\varepsilon'} |M\gamma(\sigma)|\, d\sigma = \int (1+|\frac{s_i}{s_1}|)^{\varepsilon'} |\gamma(s)|\, ds.$$

Let $\Omega_1 = \{s : |s_1| \ge 3\}$, $\Omega_2 = \{s : |s_1| \le 3, |s_i| \ge |s_1|^{-1}\}$, $\Omega_3 = \{s : |s_1| \le 3, |s_i| \le |s_1|^{-1}\}$, and bound the integrals over the three regions separately. First, for $\varepsilon' \le \varepsilon$,

$$\int_{\Omega_1} (1+|\frac{s_i}{s_1}|)^{\varepsilon'} |\gamma(s)|\, ds \lesssim \int (1+|s_i|)^{\varepsilon'} |\gamma(s)|\, ds \le \|\gamma\|_{\mathfrak{B}_\varepsilon},$$

Next, for $\varepsilon' \le \varepsilon/2$,

$$\int_{\Omega_2} (1+|\frac{s_i}{s_1}|)^{\varepsilon'} |\gamma(s)|\, ds \lesssim \int (1+|s_i|)^{2\varepsilon'} |\gamma(s)|\, ds \le \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

Finally, for the third term we use Lemma 4.9 to estimate, for $\varepsilon' < \varepsilon/2$,

$$\int_{\Omega_3} (1+|\frac{s_i}{s_1}|)^{\varepsilon'} |\gamma(s)|\, ds \lesssim_{\varepsilon'} \int_{|s_1|\le 3} (1+|s_1|^{-2\varepsilon'}) |\gamma(s)|\, ds \le \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

The asserted estimate follows. □

Lemma 4.12. *(i) For $n+1 \le i \le N$, $\varepsilon > 0$*

$$\sup_{0<h\le 1} h^{-\varepsilon} \|M\gamma(\cdot + he_i) - M\gamma\|_1 \le \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

(ii) For $2 \le i \le n$, $\delta < \varepsilon/2$

$$\sup_{0<h\le 1} h^{-\delta} \|M\gamma(\cdot + he_i) - M\gamma\|_1 \lesssim \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

Proof. In the case $n+1 \le i \le N$ a change of variables shows,

$$\int_{\mathbb{R}^N} |M\gamma(\sigma + he_i) - M\gamma(\sigma)|\, d\sigma = \int_{\mathbb{R}^N} |\gamma(s+he_i) - \gamma(s)|\, ds,$$

and the result follows.

Now consider the case $2 \le i \le n$. By Lemma 4.6 it suffices to show that for $\rho \le 1$

$$\int_\rho^{2\rho} \int_{\mathbb{R}^N} |M\gamma(\sigma + he_i) - M\gamma(\sigma)|\, d\sigma \frac{dh}{h} \lesssim \rho^{\varepsilon'} \|\gamma\|_{\mathfrak{B}_\varepsilon}, \quad \varepsilon' \le \varepsilon/2. \tag{4.9}$$

Our assumptions are symmetric in $s_2, \dots, s_n$, and thus it suffices to prove (4.9) for $i = 2$. The result is trivial for $10^{-2} \le \rho \le 1$, so we may assume $\rho \le 10^{-2}$. In the inner integral we change variables, setting $(s_1, \dots, s_N) = (\sigma_1, \sigma_1\sigma_2, \dots, \sigma_1, \sigma_n, \sigma_{n+1}, \dots, \sigma_N)$ and the left hand side of (4.9) becomes

$$\begin{aligned} &\int_\rho^{2\rho} \int_{\mathbb{R}^N} |\gamma(s_1, s_2 + s_1 h, s_1 s_3, \dots, s_n, s_{n+1}, \dots, s_N) - \gamma(s)|\, ds \frac{dh}{h} \\ &= \iint\limits_{\substack{\rho \le h \le 2\rho \\ |s_1| \ge \rho^{-\beta}}} + \iint\limits_{\substack{\rho \le h \le 2\rho \\ |s_1| \le \rho^{-\beta}}} =: (I) + (II) \end{aligned}$$

where $\beta \in (0,1)$ is to be determined. We have the following estimate for the first term:

$$(I) \le 2 \iint\limits_{\substack{\rho \le h \le 2\rho \\ |s_1| \ge \rho^{-\beta}}} |\gamma(s)|\, ds\, \frac{dh}{h} \lesssim \int\limits_{|s_1| \ge \rho^{-\beta}} |\gamma(s)|\, ds \lesssim \rho^{\beta\varepsilon} \int (1+|s_1|)^{\varepsilon} |\gamma(s)|\, ds \lesssim \rho^{\beta\varepsilon} \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

For the term (II) we interchange the order of integration and put for fixed s_1, $\tilde{h} = s_1 h$ so that $d\tilde{h}/\tilde{h} = dh/h$. Also, on the domain of integration of (II), we have $|\tilde{h}| \le 2\rho^{1-\beta}$. Thus we may estimate

$$(II) \le \int_{|\tilde{h}| \le 2\rho^{1-\beta}} \|\gamma(\cdot + \tilde{h}e_2) - \gamma(\cdot)\|_1 |\tilde{h}|^{-1} d\tilde{h} \le \|\gamma\|_{\mathfrak{B}_\varepsilon} \int_0^{2\rho^{1-\beta}} \tilde{h}^{\varepsilon - 1}\, d\tilde{h} \lesssim \rho^{\varepsilon(1-\beta)} \|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

If we choose $\beta = 1/2$ then (4.9) follows from the estimates for (I) and (II). $\square$

Remark. One can replace the application of Lemma 4.6 by a more careful argument to show that (4.9) implies that the statement (ii) in the lemma holds even for the endpoint $\delta = \varepsilon/2$. However this is not important for the purposes of this paper.

The main technical estimate in the proof of Proposition 4.4 is an analogue of Lemma 4.12 for regularity in the first variable, given as Lemma 4.14 below. We first give an auxiliary estimate for functions of two variables.

Lemma 4.13. *Let $\beta < 1/2$, $\varepsilon' < \varepsilon$. For $g \in \mathfrak{B}_\varepsilon(\mathbb{R}^2)$, and $0 < \rho \le 1$,*

$$\iiint\limits_{\substack{\rho^\beta \le |s_1| \le \rho^{-\beta} \\ \rho \le h \le 2\rho}} \Big| (1 + \tfrac{h}{s_1}) g(s_1 + h, (1 + \tfrac{h}{s_1}) s_2) - g(s_1 + h, s_2) \Big|\, ds_1 ds_2 \frac{dh}{h}$$
$$\le C(\beta, \varepsilon') \big(\rho^{\varepsilon' \beta} + \rho^{1-2\beta} \big) \|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^2)}\,.$$

Proof. We may assume that $\rho \le 10^{-2/\beta}$, since otherwise the bound is trivial. We wish to discard the contributions of the integral where $|s_2| \le \rho^\beta$ or $|s_2| \ge \rho^{-\beta}$. We estimate the left hand side by $A + I_1 + I_2 + II_1 + II_2$ where

$$A = \iiint\limits_{\substack{\rho^\beta \le |s_1|, s_2 \le \rho^{-\beta} \\ \rho \le h \le 2\rho}} \Big| (1 + \tfrac{h}{s_1}) g(s_1 + h, (1 + \tfrac{h}{s_1}) s_2) - g(s_1 + h, s_2) \Big|\, ds_1 ds_2 \frac{dh}{h}\,,$$

$$I_1 + II_1 = \iiint\limits_{\substack{\rho^\beta \le |s_1| \le \rho^{-\beta} \\ |s_2| \le \rho^\beta \\ \rho \le h \le 2\rho}} + \iiint\limits_{\substack{\rho^\beta \le |s_1| \le \rho^{-\beta} \\ |s_2| \ge \rho^{-\beta} \\ \rho \le h \le 2\rho}} \Big| (1 + \tfrac{h}{s_1}) g(s_1 + h, (1 + \tfrac{h}{s_1}) s_2) \Big|\, ds_1 ds_2 \frac{dh}{h}\,,$$

$$I_2 + II_2 = \iiint\limits_{\substack{\rho^\beta \le |s_1| \le \rho^{-\beta} \\ |s_2| \le \rho^\beta \\ \rho \le h \le 2\rho}} + \iiint\limits_{\substack{\rho^\beta \le |s_1| \le \rho^{-\beta} \\ |s_2| \ge \rho^{-\beta} \\ \rho \le h \le 2\rho}} |g(s_1 + h, s_2)|\, ds_1 ds_2 \frac{dh}{h}\,.$$

To bound I_1 we change (for fixed h, s_1) variables as $\sigma_2 = (1 + h/s_1) s_2$ and observe that $(1 + h/s_1) \approx 1$. Thus the σ_2 integration is extended over $\sigma_2 \lesssim \rho^\beta$, and we may apply Lemma 4.9. A similar argument applies to I_2, and we get

$$I_1 + I_2 \lesssim \rho^{\beta\varepsilon'} \|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^2)}.$$

The same argument applies to the terms II_1, II_2, with the σ_2 integration now extended over $|\sigma_2| \geq \rho^{-\beta} - 2\rho \geq c\rho^{-\beta}$ for $c > 0$. Now we apply Lemma 4.8 instead and the result is

$$II_1 + II_2 \lesssim \rho^{\beta\varepsilon}\|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^2)}.$$

We now consider the term A and estimate $A \leq III + IV$ where

$$III = \iiint\limits_{\substack{\rho^\beta \leq |s_1|, |s_2| \leq \rho^{-\beta} \\ \rho \leq h \leq 2\rho}} \left|1 + \tfrac{h}{s_1}\right| \left|g(s_1 + h, (1 + \tfrac{h}{s_1})s_2) - g(s_1 + h, s_2)\right| ds_1 ds_2 \frac{dh}{h}\,,$$

$$IV = \iiint\limits_{\substack{\rho^\beta \leq |s_1|, |s_2| \leq \rho^{-\beta} \\ \rho \leq h \leq 2\rho}} \tfrac{|h|}{|s_1|} |g(s_1 + h, s_2)|\, ds_1 ds_2 \frac{dh}{h}\,.$$

Since $h \approx \rho$ and $|s_1| \gtrsim \rho^\beta$ in the domain of integration we immediately get

$$IV \lesssim \rho^{1-\beta}\|g\|_{L^1(\mathbb{R}^2)}\,.$$

In the estimation of III we may ignore the factor $1 + h/s_1$ which is $O(1)$. We make the change of variable $\sigma_1 = s_1 + h$ which does not substantially change the domain of integration since $\frac{1}{2}\rho^\beta \leq |\sigma_1| \leq 2\rho^{-\beta}$ for the ranges of ρ we consider here. We see that

$$III \lesssim \iiint\limits_{\substack{\frac{1}{2}\rho^\beta \leq |\sigma_1|, |s_2| \leq 2\rho^{-\beta} \\ \rho \leq h \leq 2\rho}} \left|g(\sigma_1, (1 + \tfrac{h}{\sigma_1 - h})s_2) - g(\sigma_1, s_2)\right| d\sigma_1 ds_2 \frac{dh}{h}$$

We now interchange the order of integration, and then, for fixed σ_1, s_2 change variables $u = u(h) = \frac{hs_2}{\sigma_1 - h}$. Then observe that

$$\frac{\partial u}{\partial h} = \frac{\sigma_1 s_2}{(\sigma_1 - h)^2}, \qquad \frac{du}{u} = \frac{\sigma_1}{\sigma_1 - h}\frac{dh}{h};$$

moreover the range of $|u|$ is contained in $[\frac{1}{4}\rho^{1+2\beta}, 4\rho^{1-2\beta}]$. Since $|du|/|u| \approx |dh|/|h|$ we get the estimate

$$\begin{aligned} III &\lesssim \sum_{2^{-k-1} \leq 4\rho^{1-2\beta}} \int_{2^{-k-1}}^{2^{-k}} \iint \left|g(\sigma_1, s_2 + u) - g(\sigma_1, s_2)\right| d\sigma_1 ds_2 \frac{du}{|u|} \\ &\lesssim \sum_{2^{-k-1} \leq 4\rho^{1-2\beta}} 2^{-k\varepsilon}\|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^2)} \lesssim \rho^{1-2\beta}\|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^2)}\,. \end{aligned}$$

We collect the estimates and obtain the desired bound. □

Lemma 4.14. *For $0 < \varepsilon \leq 1$, $\delta < \varepsilon/3$,*

$$\sup_{0 < h \leq 1} h^{-\delta}\|M\gamma(\cdot + he_1) - M\gamma\|_1 \lesssim n\|\gamma\|_{\mathfrak{B}_\varepsilon}.$$

Proof. Let $\tilde{\varepsilon} < \varepsilon$, $\delta_1 > \delta$ be such that $\delta < \delta_1 < \tilde{\varepsilon}/3$. By Lemma 4.6 it suffices to show for $\rho \leq 1$ the inequality

$$\int_\rho^{2\rho} \|M\gamma(\cdot + he_1) - M\gamma\|_1 \frac{dh}{h} \lesssim \rho^{\delta_1} n\|\gamma\|_{\mathfrak{B}_\varepsilon}. \tag{4.10}$$

We let $\beta < 1/2$ to be chosen later; a suitable choice will be $\beta \in (\delta_1/\tilde{\varepsilon}, 1/3)$. We may assume $\rho \leq 10^{-2/\beta}$ since otherwise the result is obvious. We first discard the contributions of the integral for $|s_1| \le \rho^{\beta}$ or $|s_1| \ge \rho^{-\beta}$. We estimate

$$\int_{\rho}^{2\rho} \|M\gamma(\cdot + he_1) - M\gamma\|_1 \frac{dh}{h} \lesssim \rho^{\delta_1} \le (A) + (I_1) + (I_2) + (II_1) + (II_2)$$

where

$$\begin{aligned}
(A) &= \int_{\rho}^{2\rho} \int_{s:\rho^{\beta} \le |s_1| \le \rho^{-\beta}} |M\gamma(s+he_1) - M\gamma(s)|\, ds \frac{dh}{h}\,, \\
(I_1) + (I_2) &= \int_{\rho}^{2\rho} \int_{s:|s_1| \le \rho^{\beta}} |M\gamma(s+he_1)|\, ds \frac{dh}{h} + \int_{\rho}^{2\rho} \int_{s:|s_1| \le \rho^{\beta}} |M\gamma(s)|\, ds \frac{dh}{h}\,, \\
(II_1) + (II_2) &= \int_{\rho}^{2\rho} \int_{s:|s_1| \ge \rho^{-\beta}} |M\gamma(s+he_1)|\, ds \frac{dh}{h} + \int_{\rho}^{2\rho} \int_{s:|s_1| \ge \rho^{-\beta}} |M\gamma(s)|\, ds \frac{dh}{h}\,.
\end{aligned}$$

We make a change of variable $\sigma = (s_1+h, (s_1+h)s_2, \dots, (s_1+h)s_n, s_{n+1}, \dots, s_N)$ and estimate

$$(I_1) \le \int_{\rho}^{2\rho} \int_{\sigma:|\sigma_1| \le \rho^{\beta}+2\rho} |\gamma(\sigma)|\, d\sigma \frac{dh}{h} \lesssim \rho^{\beta\varepsilon} \|\gamma\|_{\mathfrak{B}_{\varepsilon}(\mathbb{R}^N)}.$$

where we have used Lemma 4.9. Similarly

$$(II_1) \le \int_{\rho}^{2\rho} \int_{\sigma:|\sigma_1| \ge \rho^{-\beta}-2\rho} |\gamma(\sigma)|\, d\sigma \frac{dh}{h} \lesssim \rho^{\beta\varepsilon} \|\gamma\|_{\mathfrak{B}_{\varepsilon}(\mathbb{R}^N)}.$$

by Lemma 4.8 and the estimate $2\rho \le \frac{1}{2}\rho^{-\beta}$ which holds in the range of ρ under consideration. The bound $(I_2) + (II_2) \lesssim \rho^{\beta\varepsilon}\|\gamma\|_{\mathfrak{B}_{\varepsilon}(\mathbb{R}^N)}$ follows in the same way.

It thus remains to estimate (A). We change variables and write

$$\begin{aligned}
(A) &= \int_{\rho}^{2\rho} \int_{s:\rho^{\beta} \le |s_1| \le \rho^{-\beta}} \big| (s_1+h)^{n-1}\gamma(s_1+h, (s_1+h)s_2, \dots, (s_1+h)s_n, s_{n+1}, \dots, s_N) \\
&\qquad\qquad - s_1^{n-1}\gamma(s_1, s_1s_2, \dots, s_1s_n, s_{n+1}, \dots, s_N)\big|\, ds\, \frac{dh}{h} \\
&= \int_{\rho}^{2\rho} \int_{s:\rho^{\beta} \le |s_1| \le \rho^{-\beta}} \big| \big(1+\tfrac{h}{s_1}\big)^{n-1}\gamma(s_1+h, (1+\tfrac{h}{s_1})s_2, \dots, (1+\tfrac{h}{s_1})s_n, s_{n+1}, \dots, s_N) \\
&\qquad\qquad - \gamma(s_1, s_2, \dots, s_n, s_{n+1}, \dots, s_N)\big|\, ds\, \frac{dh}{h}.
\end{aligned}$$

We split the integrand as a sum of n differences $\Delta_k(s,h)$, $k = 0, \dots, n-1$, where

$$\Delta_0(s,h) = \gamma(s+he_1) - \gamma(s)$$

and, for $k = 1, \dots, n-1$,

$$\begin{aligned}
\Delta_k(s,h) =& \big(1+\tfrac{h}{s_1}\big)^{k}\gamma(s_1+h, (1+\tfrac{h}{s_1})s_2, \dots, (1+\tfrac{h}{s_1})s_k, (1+\tfrac{h}{s_1})s_{k+1}, s_{k+2} \dots, s_N) \\
&- \big(1+\tfrac{h}{s_1}\big)^{k-1}\gamma(s_1+h, (1+\tfrac{h}{s_1})s_2, \dots, (1+\tfrac{h}{s_1})s_k, s_{k+1}, \dots, s_N).
\end{aligned}$$

Then $(A) \le \sum_{k=0}^{n-1}(A_k)$ where

$$(A_k) = \int_{\rho}^{2\rho} \int_{s:\rho^{\beta} \le |s_1| \le \rho^{-\beta}} |\Delta_k(s,h)|\, ds \frac{dh}{h}.$$

It is immediate that

$$(A_0) \lesssim \rho^{\varepsilon}\|\gamma\|_{\mathfrak{B}_{\varepsilon}}\,.$$

For the estimation of (A_k) we make a change of variable in the s_i variables where $2 \le i \le k$; this replaces $(1 + h/s_1)s_i$ by s_i (i.e. there is no change of variable if $k = 1$). This gives, for $1 \le k \le n-1$,

$$(A_k) = \int_\rho^{2\rho} \int_{s:\rho^\beta \le |s_1| \le \rho^{-\beta}} \Big|(1+\tfrac{h}{s_1})\gamma(s_1+h, s_2, \dots, s_k, (1+\tfrac{h}{s_1})s_{k+1}, s_{k+2} \dots, s_N) \\ - \gamma(s_1+h, s_2, \dots, s_k, s_{k+1}, \dots, s_N)\Big|\, ds \frac{dh}{h}\,.$$

By symmetry considerations we may assume $k = 1$. We may now freeze the $s_3, \dots, s_N$-variables, apply the auxiliary Lemma 4.13 for functions of (s_1, s_2) and obtain for $\varepsilon' < \tilde\varepsilon$

$$(A_k) \lesssim \big(\rho^{\varepsilon'\beta} + \rho^{1-2\beta}\big) \int \cdots \int \big\|g(\cdot, \cdot, s_3, \dots, s_N)\big\|_{\mathfrak{B}_{\tilde\varepsilon}(\mathbb{R}^2)} ds_3 \cdots ds_N.$$

Since $\tilde\varepsilon < \varepsilon$ this also implies, by Lemma 4.7,

$$(A_k) \lesssim \big(\rho^{\varepsilon'\beta} + \rho^{1-2\beta}\big)\|g\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}.$$

We collect estimates we see that the quantity on the left hand side of (4.10) is estimated by

$$C(\beta, \varepsilon', \varepsilon) n \big(\rho^{\beta\varepsilon'} + \rho^{1-2\beta}\big)\|f\|_{\mathfrak{B}_\varepsilon(\mathbb{R}^N)}$$

and with the correct choice of $\varepsilon' \in (3\delta, \varepsilon)$ and then $\beta \in (\delta/\varepsilon', 1/3)$ we see that (4.10) is established. □

4.3. **A decomposition lemma.** Later in the paper, we will need a decomposition result for $\mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$, which we present here.

Lemma 4.15. *Fix $0 < \varepsilon < 1$ and $0 < \delta < \varepsilon/2$. If $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$. Then there are $\varsigma_m \in \mathcal{B}_\delta(\mathbb{R}^n \times \mathbb{R}^d)$, $m \in \mathbb{N}$, with $\operatorname{supp}(\varsigma_m) \subseteq \{(\alpha, v) : |v| \le 1/4\}$ and*

$$\varsigma = \sum_{m \ge 0} \varsigma_m^{(2^{-m})},$$

such that

$$\|\varsigma_m\|_{\mathcal{B}_\delta} \lesssim 2^{-m(\varepsilon - 2\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Proof. Let $\eta_0 \in C_0^\infty$ be supported in $\{|x| \le 1/4\}$ such that with $0 \le \eta_0 \le 1$ and $\eta_0(x) = 1$ for $|x| \le 1/8$. Set $\eta_1(v) = \eta_0(v) - \eta_0(2v)$, so that $0 \le |\eta_1| \le 1$, $\operatorname{supp}(\eta_1) \subseteq \{\frac{1}{16} \le |v| \le \frac14\}$ and $1 = \eta_0(v) + \sum_{m \ge 1}\eta_1(2^{-m}v)$. For $m \in \mathbb{N}$, define

$$\varsigma_m(v) = \begin{cases} \eta_0(v)\varsigma(\alpha, v) & \text{if } m = 0, \\ \eta_1(v) 2^{md}\varsigma(\alpha, 2^m v) & \text{if } m \ge 1. \end{cases}$$

Then $\varsigma_m(x) = 0$ for $|x| \ge 1/4$ and $\varsigma = \sum_{m \ge 0}\varsigma_m^{(2^{-m})}$. Clearly $\|\varsigma_0\|_{\mathcal{B}_\varepsilon} \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon}$. It remains to bound $\|\varsigma_m\|_{\mathcal{B}_\delta}$ for $m \ge 1$.

We show

$$\iint (1 + |\alpha_i|)^\delta |\varsigma_m(\alpha, v)|\, d\alpha\, dv + \iint (1 + |v|)^\delta |\varsigma_m(\alpha, v)|\, d\alpha\, dv \lesssim 2^{-m(\varepsilon - \delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}, \tag{4.11}$$

$$\sup_{|h| \le 1} |h|^{-\delta} \iint |\varsigma_m(\alpha + he_i, v) - \varsigma_m(\alpha, v)|\, d\alpha\, dv \lesssim 2^{-m(\varepsilon-\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}\,. \tag{4.12}$$

We change variables and see that the left hand side of (4.11) is bounded by

$$\iint (1 + |\alpha_i|)^\delta |\varsigma(\alpha, v)||\eta_1(2^{-m}v)|\, d\alpha\, dv + \iint (1 + |2^{-m}v|)^\delta |\varsigma(\alpha, v)||\eta_1(2^{-m}v)|\, d\alpha\, dv\,.$$

We estimate

$$\iint_{\substack{|\alpha_i|\le 2^m\\|v|\approx 2^m}} (1+|\alpha|)^\delta |\varsigma(\alpha,v)| d\alpha dv \lesssim 2^{-m(\varepsilon-\delta)} \iint (1+|v|)^\varepsilon |\varsigma(\alpha,v)| d\alpha dv \lesssim 2^{-m(\varepsilon-\delta)} \|\varsigma\|_{\mathcal{B}_\varepsilon},$$

$$\iint_{\substack{|\alpha_i|\ge 2^m\\|v|\approx 2^m}} (1+|\alpha|)^\delta |\varsigma(\alpha,v)| d\alpha dv \lesssim 2^{-m(\varepsilon-\delta)} \iint (1+|\alpha_i|)^\varepsilon |\varsigma(\alpha,v)| d\alpha dv \lesssim 2^{-m(\varepsilon-\delta)} \|\varsigma\|_{\mathcal{B}_\varepsilon},$$

$$\iint_{|v|\approx 2^m} (1+2^{-m}|v|)^\delta |\varsigma(\alpha,v)| d\alpha dv \lesssim 2^{-m(\varepsilon-\delta)} \iint (1+|v|)^\varepsilon |\varsigma(\alpha,v)| d\alpha dv \lesssim 2^{-m(\varepsilon-\delta)} \|\varsigma\|_{\mathcal{B}_\varepsilon},$$

and (4.11) follows.

Next, we consider, for $|h| \le 1$, the expression

$$\iint |\varsigma_m(\alpha+he_i,v)-\varsigma_m(\alpha,v)|\, d\alpha\, dv \lesssim \iint |\eta_1(2^{-m}v)||\varsigma(\alpha+he_i,v)-\varsigma(\alpha,v)|\, d\alpha\, dv$$

and distinguish the cases $2^m|h| \le 1$ and $2^m|h| \ge 1$. If $2^m \ge |h|^{-1}$ then we estimate

$$\iint_{|v|\approx 2^m} |\varsigma(\alpha+he_i,v)-\varsigma(\alpha,v)|\, d\alpha\, dv \lesssim 2^{-m\varepsilon} \int (1+|v|)^\varepsilon |\varsigma(\alpha,v)|\, d\alpha dv \lesssim |h|^\delta 2^{-m(\varepsilon-\delta)} \|f\|_{\mathcal{B}_\varepsilon}$$

and if $2^m \le |h|^{-1}$,

$$\iint_{|v|\approx 2^m} |\varsigma(\alpha+he_i,v)-\varsigma(\alpha,v)|\, d\alpha\, dv \lesssim |h|^\varepsilon \|\varsigma\|_{\mathcal{B}_\varepsilon} \lesssim |h|^\delta 2^{-m(\varepsilon-\delta)} \|f\|_{\mathcal{B}_\varepsilon}\,.$$

Now (4.12) follows. Note that so far we have only used $\delta < \varepsilon$.

For our last estimate we need $\delta < \varepsilon/2$, and we need to show

$$\iint |\varsigma_m(\alpha,v+h)-\varsigma_m(\alpha,v)| d\alpha dv \lesssim |h|^\delta 2^{-m(\varepsilon-2\delta)} \|\varsigma\|_{\mathcal{B}_\varepsilon}. \tag{4.13}$$

The left hand side is estimated by $(I)+(II)$ where

$$(I) = \iint |\eta(v+h)-\eta(v)| 2^{md} |\varsigma(\alpha, 2^m(v+h))|\, d\alpha\, dv\,,$$
$$(II) = \iint |\eta(v)| 2^{md} |\varsigma(\alpha, 2^m(v+h)) - \varsigma(\alpha, 2^m v)|\, d\alpha\, dv\,.$$

Note that $|\eta(v+h)-\eta(v)| \lesssim \chi_{\{\frac{1}{32}\le|v|\le\frac12\}} |h|$ and so the first term is estimated as

$$(I) \lesssim |h| \iint_{\frac{1}{64}\le|v|\le 1} 2^{md} |\varsigma(\alpha, 2^m v)|\, d\alpha\, dv = |h| \iint_{2^{m-8}\le|v|\le 2^m} |\varsigma(\alpha,v)|\, d\alpha\, dv$$
$$\lesssim 2^{-m\varepsilon}|h| \iint (1+|v|)^\varepsilon |\varsigma(\alpha,v)|\, d\alpha\, dv \lesssim |h| 2^{-m\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}$$

which is a better bound than the one in (4.13). More substantial is the estimate for (II). Here we first consider the case $|h| \geq 2^{-2m}$ and bound

$$\begin{aligned}(II) &\lesssim \iint_{2^{-4}\leq|v|\leq 2^{-2}} 2^{md}|\varsigma(\alpha,2^m(v+h))-\varsigma(\alpha,2^m v)|\,d\alpha\,dv \leq 2 \iint_{2^{-8}\leq|v|\leq 2^{-1}} 2^{md}|\varsigma(\alpha,2^m v)|\,d\alpha\,dv \\ &\lesssim \iint_{2^{m-8}\leq|v|\leq 2^{m-1}} |\varsigma(\alpha,v)|\,d\alpha\,dv \lesssim 2^{-m\varepsilon}\iint(1+|v|)^{\varepsilon}|\varsigma(\alpha,v)|\,d\alpha\,dv \\ &\lesssim 2^{-m\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon} \lesssim |h|^{\delta}2^{-m(\varepsilon-2\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}.\end{aligned}$$

Finally for the case $|h| \leq 2^{-2m}$ we get

$$(II) \lesssim \iint |\varsigma(\alpha,v+2^m h)-\varsigma(\alpha,v)|\,d\alpha\,dv \lesssim (2^m|h|)^{\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon} \lesssim |h|^{\delta}2^{-m(\varepsilon-2\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

This yields (4.13) and the proof is complete. □

4.4. **Invariance properties.** We state certain identities concerning the behavior of our multilinear forms with respect to scalings and translations. These will be used repeatedly. The straightforward proofs are omitted.

Lemma 4.16. *Let $\varsigma \in L^1(\mathbb{R}^n \times \mathbb{R}^d)$, and $\varsigma^{(2^j)}(\alpha,\cdot) = 2^{jd}\varsigma(\alpha,2^j\cdot)$. Let $b_i \in L^{p_i}(\mathbb{R}^d)$, for $i = 1,\dots,n+2$. Then*

(i) Let $\tau_h f = f(\cdot - h)$. Then

$$\Lambda[\varsigma](\tau_h b_1,\dots,\tau_h b_{n+2}) = \Lambda[\varsigma](b_1,\dots,b_{n+2})\,.$$

(ii)

$$\Lambda[\varsigma^{(2^j)}](b_1,\dots,b_{n+2}) = 2^{-jd}\Lambda[\varsigma](b_1(2^{-j}\cdot),\dots,b_{n+2}(2^{-j}\cdot))\,.$$

(iii)

$$\Lambda[\varsigma^{(2^j)}](b_1,\dots,b_{n+2}) = \int b_{n+2}(x)\int 2^{jd}k_j(2^j x,2^j y)b_{n+1}(y)\,dy\,dx$$

where

$$k_j(x,y) = \int \varsigma(\alpha,x-y)\prod_{i=1}^{n} b_i(2^{-j}(x-\alpha_i(x-y)))\,d\alpha.$$

(iv) If $g_i = 2^{-jd/p_i}b_i(2^{-j}\cdot)$ then $\|g_i\|_{p_i} = \|b_i\|_{p_i}$, and

$$\Lambda[\varsigma^{(2^j)}](b_1,\dots,b_{n+2}) = \Lambda[\varsigma](g_1,\dots,g_{n+2}) \;\text{ if } \sum_{i=1}^{n+2} p_i^{-1} = 1\,.$$

(v) Let $\kappa_1,\dots,\kappa_{n+2}$ be bounded Borel measures and $\kappa_i^{(t)} = t^d\kappa(t\cdot)$. Set $\tilde{b}_i(x) = b_i(2^{-j}x)$. Then

$$\Lambda[\varsigma^{(2^j)}](\kappa_1 * b_1,\dots,\kappa_{n+2} * b_{n+2}) = 2^{-jd}\Lambda[\varsigma](\kappa_1^{(2^{-j})} * \tilde{b}_1,\dots,\kappa_{n+2}^{(2^{-j})} * \tilde{b}_{n+2}).$$

(vi)

$$\Lambda[\varsigma^{(2^j)}](\kappa_1 * b_1,\dots,\kappa_{n+2} * b_{n+2}) = \int 2^{jd}\widetilde{k}_j(2^j x,2^j y)b_{n+1}(y)\,b_{n+2}(x)\,dx$$

where

$$\widetilde{k}_j(x,y) = \iint \varsigma(\alpha,w-z)\prod_{i=1}^{n}\kappa_i^{(2^{-j})} * [b_i(2^{-j}\cdot)](w-\alpha_i(w-z))d\kappa_{n+2}((x-w)d\kappa_{n+1}(z-y)\,.$$

4.5. **The role of projective space, revisited.** A particular special case of Theorems 2.9 and 2.8 involve the case when

$$K(\alpha, v) = \gamma_0(\alpha) K_0(v),$$

K_0 is a classical Calderón-Zygmund convolution kernel which is *homogeneous* of degree $-d$, smooth away from 0, and $\gamma_0 \in \mathfrak{B}_\epsilon(\mathbb{R}^n)$ for some $\epsilon > 0$. We saw in Section 2.4.2 that such operators would be closed under adjoints provided we could see the space of γ_0 as a space of densities on $\mathbb{RP}^n$ in an appropriate way. Indeed, this is the case, and this section is devoted to discussing that fact. These results are not used in the sequel, and are intended as motivation for our main results.

For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, and $0 < \epsilon < 1$, we set

$$\|f\|_{B^\epsilon_{1,\infty}(\mathbb{R}^n)} := \|f\|_{L^1} + \max_{i=1,\dots,n} \sup_{0<h_i\leq 1} |h_i|^{-\epsilon} \int |f(s + h_i e_i) - f(s)|\, ds,$$

where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$.

Let M be a compact manifold of dimension n, without boundary. Let μ be a measure on M. Take a finite open cover $V_1, \dots, V_L$ of M such that each V_j is diffeomorphic to $B^n(1)$–the open ball of radius 1 in $\mathbb{R}^n$. Let $\Phi_j : B^n(1) \to V_j$ be this diffeomorphism and let $\phi_1, \dots, \phi_L$ be a C^∞ partition of unity subordinate to this cover. On each neighborhood V_j, let $\Phi_j^\# \mu$ denote the pull back of μ via Φ_j. We suppose $\Phi_j^\# \mu$ is absolutely continuous with respect to Lebesgue measure on $B^n(1)$ and we write $d\Phi_j^\# \mu =: \gamma_j(x)\, dx$ where dx denotes Lebesgue measure.

Remark 4.17. γ_j is called a density, because of the way it transforms under diffeomorphisms.

Definition 4.18. For $0 < \epsilon < 1$ we define $B^\epsilon_{1,\infty}(M)$ to be the space of those measures μ such that the following norm is finite:

$$\|\mu\|_{B^\epsilon_{1,\infty}(M)} := \sum_{j=1}^L \|\phi_j \circ \Phi_j(\cdot)\gamma_j(\cdot)\|_{B^\epsilon_{1,\infty}(\mathbb{R}^n)}.$$

Remark 4.19. The norm $\|\cdot\|_{B^\epsilon_{1,\infty}(M)}$ depends on various choices we made: the finite open cover, the diffeomorphisms Φ_j, and the partition of unity ϕ_j. However, the *equivalence class* of the norm $\|\cdot\|_{B^\epsilon_{1,\infty}(M)}$ does not depend on any of these choices, and therefore the Banach space $B^\epsilon_{1,\infty}(M)$ does not depend on any of these choices.

We now turn to the case $M = \mathbb{RP}^n$. Given a measure $\mu \in B^\epsilon_{1,\infty}(\mathbb{RP}^n)$, we consider the map taking $\mathbb{R}^n \hookrightarrow \mathbb{RP}^n$ induced by the map $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 1)$. Pulling μ back via this map, we obtain a measure on $\mathbb{R}^n$–since $\mu \in B^\epsilon_{1,\infty}(\mathbb{RP}^n)$ this pulled back measure is absolutely continuous with respect to Lebesgue measure and we write this pulled back measure as $\gamma_0(x)\, dx$. This induces a map taking measures in $B^\epsilon_{1,\infty}(\mathbb{RP}^n)$ to functions $\mathbb{R}^n$ given by $\mu \mapsto \gamma_0$.

Theorem 4.20. *The map $\mu \mapsto \gamma_0$ is a bijection $\bigcup_{0<\epsilon<1} B^\epsilon_{1,\infty}(\mathbb{RP}^n) \to \bigcup_{0<\epsilon<1} \mathfrak{B}_\epsilon(\mathbb{R}^n)$ in the following sense:*

(i) $\forall \epsilon \in (0,1)$, $\exists \epsilon' \in (0, \epsilon]$, and $C = C(\epsilon, n) < \infty$ such that $\forall \mu \in B^\epsilon_{1,\infty}(\mathbb{RP}^n)$, $\gamma_0 \in \mathfrak{B}_{\epsilon'}(\mathbb{R}^n)$ and $\|\gamma_0\|_{\mathfrak{B}_{\epsilon'}} \leq C \|\mu\|_{B^\epsilon_{1,\infty}(\mathbb{RP}^n)}$.

(ii) $\forall \epsilon \in (0,1)$, $\exists \epsilon' \in (0, \varepsilon]$, $\forall \gamma_0 \in \mathfrak{B}_\epsilon(\mathbb{R}^n)$, there exists a unique $\mu \in B^{\epsilon'}_{1,\infty}(\mathbb{RP}^n)$ with $\mu \mapsto \gamma_0$ under this map. Furthermore, $\exists C = C(\epsilon, n)$ such that $\|\mu\|_{B^{\epsilon'}_{1,\infty}(\mathbb{RP}^n)} \leq C \|\gamma_0\|_{\mathfrak{B}_\epsilon}$.

Proof. Fix $\epsilon \in (0,1)$ and let $\mu \in B^\epsilon_{1,\infty}(\mathbb{RP}^n)$. We define an open cover of $\mathbb{RP}^n$. For $j = 1, \dots, n+1$, let V_j denote those points $\{(x_1, \dots, x_{j-1}, 1, x_j, \dots, x_n) : x \in \mathbb{R}^n, |x| < 2\}$, written in homogenous coordinates on $\mathbb{RP}^n$. V_j is an open subset of $\mathbb{RP}^n$ which is diffeomorphic to $B^n(2)$, and $\cup_{j=1}^{n+1} V_j = \mathbb{RP}^n$.

Let ϕ_j, $1 \le j \le n+1$ be a smooth partition of unity subordinate to the cover $V_1, \dots, V_{n+1}$. $\mu = \sum_{j=1}^{n} \phi_j \mu$. By the assumption that $\mu \in B^\epsilon_{1,\infty}(\mathbb{RP}^n)$, it follows that $\phi_j \mu = \gamma_j(x)\,dx$, when written in the standard coordinates on V_j, and $\|\gamma_j\|_{B^\epsilon_{1,\infty}(\mathbb{R}^n)} \lesssim \|\mu\|_{B^\epsilon_{1,\infty}(\mathbb{RP}^n)}$. Since γ_j has compact support, we have $\|\gamma_j\|_{\mathfrak{B}_\epsilon} \lesssim \|\gamma_j\|_{B^\epsilon_{1,\infty}(\mathbb{R}^n)} \lesssim \|\mu\|_{B^\epsilon_{1,\infty}(\mathbb{RP}^n)}$. Finally,

$$\gamma_0(x)\,dx = \gamma_{n+1}(x)\,dx + \sum_{j=1}^{n} x_j^{-n-1} \gamma_j(x_j^{-1}x_1, x_j^{-1}x_2, \dots, x_j^{-1}x_{j-1}, x_j^{-1}x_{j+1}, \dots, x_j^{-1}x_n, x_j^{-1})\,dx\,.$$

It follows from Corollary 4.5, applied to each term of the sum, that $\|\gamma_0\|_{\mathfrak{B}_{\epsilon'}} \le C_n \|\mu\|_{B^\epsilon_{1,\infty}(\mathbb{RP}^n)}$, and part (i) is proved.

Because γ_0 uniquely determines μ except at those point which cannot be written in homogeneous coordinates as $(x_1, \dots, x_n, 1)$, it follows that there is at most one $\mu \in \cup_{\epsilon>0} B^\epsilon_{1,\infty}(\mathbb{RP}^n)$ which maps to a given γ_0 (because such a μ is absolutely continuous with respect to Lebesgue measure in every coordinate chart, and gives such points measure 0). Hence, given $\gamma_0 \in \mathfrak{B}_\epsilon(\mathbb{R}^n)$ there is at most one μ such that $\mu \mapsto \gamma$. We wish to construct such a μ.

Let ϕ_j be the coordinate charts from above. Given $\gamma_0 \in \mathfrak{B}_\epsilon(\mathbb{R}^n)$ define $\gamma_{n+1}(x)\,dx := \phi_{n+1}(x)\gamma_0(x)\,dx$ and for $1 \le j \le n$,

$$\gamma_j(x)\,dx := \phi_j(x) x_n^{-n-1} \gamma_0(x_n^{-1}x_1, \dots, x_n^{-1}x_{j-1}, x_n^{-1}, x_n^{-1}x_j, \dots, x_n^{-1}x_{n-1})\,dx.$$

Define $d\mu_j := \gamma_j(x)\,dx$ on V_j. By Corollary 4.5, there exists $\epsilon' > 0$ with $\|\gamma_j\|_{\mathfrak{B}_{\epsilon'}} \le C\|\gamma\|_{\mathfrak{B}_\epsilon}$. We set $\mu = \sum_{j=1}^{n+1} \mu_j$. We have $\|\mu\|_{B^{\epsilon'}_{1,\infty}(\mathbb{RP}^n)} \le C'\|\gamma_0\|_{\mathfrak{B}_\epsilon}$ and $\mu \mapsto \gamma_0$, as desired. □

Remark 4.21. In this section we were not explicit about how each constant depends on n. The above can be set up in such a way that all constants are polynomial in n, which is natural for our purposes–see §2.4.1. In fact, it would be hard to avoid this polynomial dependance on n, since there are naturally $n+1$ coordinate charts in the definition of $\mathbb{RP}^n$.

Remark 4.22. Corollary 4.5 implies that the space $\bigcup_{\epsilon>0} \mathfrak{B}_\epsilon(\mathbb{R}^n)$ (when thought of as densities on $\mathbb{RP}^n$) is closed under the action of a particular diffeomorphism of $\mathbb{RP}^n$. Namely, if $\gamma \in \bigcup_{\epsilon>0} \mathfrak{B}_\epsilon(\mathbb{R}^n)$, then

$$s_1^{-n-1}\gamma(s_1^{-1}, s_1^{-1}s_2, \dots, s_1^{-1}s_n) \in \bigcup_{\epsilon>0} \mathfrak{B}_\epsilon(\mathbb{R}^n).$$

Theorem 4.20 tells us that more is true: $\bigcup_{\epsilon>0} \mathfrak{B}_\epsilon(\mathbb{R}^n)$ is closed under the action of *any* smooth diffeomorphism of $\mathbb{RP}^n$ (as $\bigcup_{\epsilon>0} B^\epsilon_{1,\infty}(\mathbb{RP}^n)$ clearly is). It is not hard to see that, when taking adjoints of our multilinear operator in the special case when $K(\alpha, x) = \gamma_0(\alpha)K_0(x)$ where K_0 is a homogenous Calderón-Zygmund kernel, each permutation of $b_1, \dots, b_{n+2}$ corresponds to the action of a diffeomorphism of $\mathbb{RP}^n$ on γ_0. In fact, each permutation corresponds to an action of an element of $\mathrm{GL}(n+1, \mathbb{R})$ on $\mathbb{RP}^n$ (where the action of $\mathrm{GL}(n+1, \mathbb{R})$ on $\mathbb{RP}^n$ is defined in the usual way).

5. Outline of the proof of boundedness

In this section, we begin the proof of Theorem 2.10 on the boundedness of our multilinear forms. Let ϕ be an even C_0^∞ function supported in $\{|x| < 1\}$ such that $\phi \ge 0$ and $\int \phi = 1$. For $j \in \mathbb{Z}$ define $\phi^{(2^j)}(x) := 2^{jd}\phi(2^j x)$ and define the operator $P_j f = f * \phi^{(2^j)}$. Furthermore, we choose ϕ to be even so that $P_j^* = P_j = {}^tP_j$ (here P_j^* is the adjoint of P_j and ${}^tP_j$ is the transpose). There are two key facts to note about P_j. First, for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{j\to+\infty} P_j f = f, \quad \lim_{j\to-\infty} P_j f = 0, \tag{5.1}$$

with convergence in $\mathcal{S}'$. Secondly, by the nonnegativity of ϕ the operator norm on L^∞ is bounded by 1:

$$\|P_j\|_{L^\infty\to L^\infty} = 1\,. \tag{5.2}$$

In Theorem 2.10 we are given a *bounded* family in $\mathcal{B}_\varepsilon$,

$$\vec{\varsigma} = \{\varsigma_j : j \in \mathbb{Z}\}. \tag{5.3}$$

For (parts of the) proof of Theorem 2.10 we shall also need to assume the cancellation condition

$$\int \varsigma_j(\alpha, v)\, dv = 0 \tag{5.4}$$

for all $j \in \mathbb{Z}$. Of particular interest are the choices in Proposition 3.2, namely $\varsigma_j = (Q_j K)^{(2^{-j})}$, given $K \in \mathcal{K}_\alpha$ for some $\alpha > \varepsilon$. Theorem 2.10 concerns the sum

$$\Lambda(b_1,\dots,b_{n+2}) = \lim_{N\to\infty} \sum_{j=-N}^{N} \Lambda[\varsigma_j^{(2^j)}](b_1,\dots,b_{n+2}), \tag{5.5}$$

where $b_1,\dots,b_n \in L^\infty(\mathbb{R}^d)$, $b_{n+1} \in L^p(\mathbb{R}^d)$, and $b_{n+2} \in L^{p'}(\mathbb{R}^d)$, with $p \in (1,2]$ and $p' \in [2,\infty)$ is the dual exponent to p. We have not yet shown that this sum converges in any reasonable sense though it is easy to see that it converges if all b_j belong to $C_0^\infty(\mathbb{R}^d)$. One first establishes estimates for the partial sums $\sum_{j=-N}^{N} \Lambda[\varsigma_j^{(2^j)}](b_1,\dots,b_{n+2})$ which are independent of N. Thus, in order to state a priori results one should first assume that all but finitely many of the ς_j are zero. In the general case we shall establish convergence in the operator topology of multilinear functionals (or in slightly stronger convergence modes). Throughout we take $n \geq 1$, as the result for $n = 0$ is classical. Our estimates will involve quantities depending on the family $\vec{\varsigma}$. It will be convenient to use the following notation. Let

$$\Gamma_\varepsilon \equiv \Gamma_\varepsilon[\vec{\varsigma}] := \frac{\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}}{\sup_j \|\varsigma_j\|_{L^1}}, \tag{5.6}$$

and for $n \geq 1$, $\nu \geq 0$ set

$$\mathfrak{M}_\nu^{n,\varepsilon} \equiv \mathfrak{M}_\nu^{n,\varepsilon}[\vec{\varsigma}] := \sup_j \|\varsigma_j\|_{L^1} \log^\nu(1 + n\,\Gamma_\varepsilon(\vec{\varsigma}))\,. \tag{5.7}$$

We split the sum (5.5) into various terms which we study separately. For $1 \leq l_1 < l_2 \leq n+2$, we define

$$\begin{aligned}&\Lambda^1_{l_1,l_2}(b_1,\dots,b_{n+2})\\ &:= \sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](b_1,\dots,b_{l_1-1},(I-P_j)b_{l_1},P_jb_{l_1+1},\dots,P_jb_{l_2-1},(I-P_j)b_{l_2},P_jb_{l_2+1},\dots,P_jb_{n+2}).\end{aligned} \tag{5.8}$$

For $1 \leq l \leq n+2$, we define

$$\Lambda^2_l(b_1,\dots,b_{n+2}) := \sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](P_jb_1,\dots,P_jb_{l-1},(I-P_j)b_l,P_jb_{l+1},\dots,P_jb_{n+2}). \tag{5.9}$$

Finally, we define

$$\Lambda^3(b_1,\dots,b_{n+2}) := \sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](P_jb_1,\dots,P_jb_{n+2}). \tag{5.10}$$

One verifies (by induction on n) that

$$\begin{aligned}(5.11)\quad &\Lambda(b_1,\dots,b_{n+2})\\ &= \sum_{1\le l_1<l_2\le n+2} \Lambda^1_{l_1,l_2}(b_1,\dots,b_{n+2}) + \sum_{1\le l\le n+2} \Lambda^2_l(b_1,\dots,b_{n+2}) + \Lambda^3(b_1,\dots,b_{n+2}).\end{aligned}$$

For $b_1,\dots,b_n \in L^\infty(\mathbb{R}^d)$ fixed, we can identify the multilinear form Λ with an operator $T \equiv T[b_1,\dots,b_n]$ defined by

$$\int g(x)\, T[b_1,\dots,b_n]f(x)\,dx := \Lambda(b_1,\dots,b_n,f,g). \tag{5.12}$$

In this way we associate operators $T^1_{l_1,l_2}$, T^2_l and T^3 to the forms $\Lambda^1_{l_1,l_2}$, Λ^2_l and Λ^3. We shall see that the sums defining these operators converge in the strong operator topology as operators $L^p \to L^p$ (for fixed $b_1,\dots,b_n \in L^\infty(\mathbb{R}^d)$), see §1.3 for the definitions.

The main estimates. We separate the proof of Theorem 2.10 into the following five parts.

Theorem 5.1. *Let $p \in (1,2]$ and $p' \in [2,\infty)$ with $\frac{1}{p}+\frac{1}{p'}=1$.*

(a) Suppose that $\varsigma_j = 0$ for all but finitely many j. Then

(I)

$$\big|\Lambda^1_{n+1,n+2}(b_1,\dots,b_{n+2})\big| \lesssim \mathfrak{M}_2^{n,\varepsilon}[\vec\varsigma]\Big(\prod_{i=1}^n \|b_i\|_{L^\infty}\Big)\|b_{n+1}\|_{L^p}\|b_{n+2}\|_{L^{p'}}.$$

(II) For $1\le l_1\le n$, $l_2\in\{n+1,n+2\}$,

$$|\Lambda^1_{l_1,l_2}(b_1,\dots,b_{n+2})| \lesssim \mathfrak{M}_{5/2}^{n,\varepsilon}[\vec\varsigma]\,\Big(\prod_{i=1}^n \|b_i\|_\infty\Big)\|b_{n+1}\|_p\|b_{n+2}\|_{p'}.$$

(III) For $1\le l_1<l_2\le n$,

$$|\Lambda^1_{l_1,l_2}(b_1,\dots,b_{n+2})| \lesssim \mathfrak{M}_3^{n,\varepsilon}[\vec\varsigma]\,\Big(\prod_{i=1}^n \|b_i\|_\infty\Big)\|b_{n+1}\|_p\|b_{n+2}\|_{p'}.$$

(IV) Under the additional cancellation condition (5.4) we have, for $1\le l\le n+2$,

$$|\Lambda^2_l(b_1,\dots,b_{n+2})| \lesssim n\,\mathfrak{M}_3^{n,\varepsilon}[\vec\varsigma]\,\Big(\prod_{i=1}^n \|b_i\|_\infty\Big)\|b_{n+1}\|_p\|b_{n+2}\|_{p'}.$$

(V) Suppose that (5.4) holds. Then

$$|\Lambda^3(b_1,\dots,b_{n+2})| \lesssim n^2\,\mathfrak{M}_3^{n,\varepsilon}[\vec\varsigma]\,\Big(\prod_{i=1}^n \|b_i\|_\infty\Big)\|b_{n+1}\|_p\|b_{n+2}\|_{p'}.$$

In the above inequalities the implicit constants depend only on $p\in(1,2]$, $d\in\mathbb{N}$, and $\epsilon>0$.

(b) For general families $\vec\varsigma = \{\varsigma_j : j\in\mathbb{Z}\}$, bounded in $\mathcal{B}_\varepsilon$, the sums defining the above five functionals converge in the operator topology of multilinear functionals and the limits satisfy the above estimates.

(c) The sums defining the operators $T^1_{l_1,l_2}[b_1,\dots,b_n]$, $T^2_l[b_1,\dots,b_n]$ and $T^3[b_1,\dots,b_n]$ associated to the forms $\Lambda^1_{l_1,l_2}$, Λ^2_l and Λ^3 via (5.12) converge in the strong operator topology as operators from $L^p\to L^p$.

Summing up the estimates for the five parts yields Theorem 2.10.

6. Some auxiliary operators

In this section, we introduce some auxiliary operators which play a role in the proof of Theorems 2.10, 5.1. Recall that in §5 we introduced the operator P_j, which was defined as $P_j f = f * \phi^{(2^j)}$, where $\phi \in C_0^\infty(B^d(1))$ was a fixed even function with $\int \phi = 1$, $\phi \geq 0$, and $\phi^{(2^j)}(x) = 2^{jd}\phi(2^j x)$.

Define $\psi(x) := \phi(x) - 2^{-d}\phi(x/2) \in C_0^\infty(B^d(0,2))$, and let $Q_k f = f * \psi^{(2^k)}$ so that

$$Q_k = P_k - P_{k-1}\,. \tag{6.1}$$

Note that, in the sense of distributions, we have the following identities

$$I = \sum_{j\in\mathbb{Z}} Q_j, \quad P_j = \sum_{k\leq j} Q_k, \quad I - P_j = \sum_{k>j} Q_k \tag{6.2}$$

with convergence in the strong operator topology (as operators $L^p \to L^p$, $1 < p < \infty$).

Remark 6.1. There is one subtlety that we must consider. While $\lim_{j\to-\infty} P_j f = 0$ for $f \in C_0^\infty(\mathbb{R}^d)$ (or even $f \in L^p$, $p \neq \infty$) is it not the case that $\lim_{j\to-\infty} P_j f = 0$ for $f \in L^\infty$. Indeed, this is not true for a constant function. Thus, the first two identities in (6.2) do not hold when thought of as operators on L^∞. However, the third identity does hold (with the limit taken almost everywhere), which we shall use.

Let $\chi_0 \in \mathcal{S}(\mathbb{R}^d)$ so that $\chi_0(\xi) = 1$ for $|\xi| < 1/2$ and χ_0 is supported in $\{|\xi| < 1\}$. For $j \geq 1$ let η_j be defined via

$$\widehat{\eta_j}(\xi) = \chi_0(2^{-j}\xi) - \chi_0(2^{1-j}\xi) \tag{6.3}$$

so that $\widehat{\eta_j}$ is supported in the annulus $\{\xi : 2^{j-2} \leq |\xi| \leq 2^j\}$ and $\sum_{j\in\mathbb{Z}} \widehat{\eta_j}(\xi) = 1$ for $\xi \neq 0$. Let $\widetilde{\eta}_0$ be a Schwartz function so that its Fourier transform vanishes in a neighborhood of the origin and is compactly be supported, and equal to 1 on the support of $\widehat{\eta}_0$. Let $\widetilde{\eta}_j = \widetilde{\eta}_0^{(2^j)}$. Note that η_j, $\widetilde{\eta}_j$ belong to $\mathcal{S}_0(\mathbb{R}^d)$ – the space of Schwartz functions, all of whose moments vanish. Define

$$\mathcal{Q}_j f = f * \eta_j, \quad \widetilde{\mathcal{Q}}_j f = f * \widetilde{\eta}_j. \tag{6.4}$$

and note that

$$\mathcal{Q}_j = \mathcal{Q}_j \widetilde{\mathcal{Q}}_j = \widetilde{\mathcal{Q}}_j \mathcal{Q}_j \tag{6.5}$$

and $I = \sum_{j\in\mathbb{Z}} \mathcal{Q}_j = \sum_{j\in\mathbb{Z}} \mathcal{Q}_j \widetilde{\mathcal{Q}}_j = \sum_{j\in\mathbb{Z}} \widetilde{\mathcal{Q}}_j \mathcal{Q}_j$, where this identity holds in the weak (distributional sense) and also in the strong operator topology, as operators on L^p, if $1 < p < \infty$. We also have the following well known estimates for the associated Littlewood-Paley square functions: for $1 < p < \infty$, $f \in L^p(\mathbb{R}^d)$,

$$\|f\|_p \approx \Big\| \Big(\sum_{j\in\mathbb{Z}} |\mathcal{Q}_j f|^2 \Big)^{\frac12} \Big\|_p \approx \Big\| \Big(\sum_{j\in\mathbb{Z}} |\widetilde{\mathcal{Q}}_j f|^2 \Big)^{\frac12} \Big\|_p \tag{6.6}$$

with implicit constants depending only on p and d. The same estimates hold with $\mathcal{Q}_k$ and $\widetilde{\mathcal{Q}}_k$ replaced by their adjoints.

We introduce a class of operators generalizing Q_j, $\mathcal{Q}_j$, and $\widetilde{\mathcal{Q}}_j$.

Definition 6.2. $\mathcal{U}$ is defined to be the space of those functions $u \in C^1(\mathbb{R}^d)$ such that the norm

$$\|u\|_{\mathcal{U}} := \sup_{x\in\mathbb{R}^d} (1 + |x|^{d+\frac12})(|u(x)| + |\nabla u(x)|)$$

is finite and such that

$$\int u(x)\, dx = 0.$$

Definition 6.3. For $u \in \mathfrak{U}$ and $j \in \mathbb{Z}$, define $\overline{Q}_j[u]f := f * u^{(2^j)}$.

Remark 6.4. Note that $\psi, \eta_0, \widetilde{\eta}_0 \in \mathfrak{U}$ and $Q_j = \overline{Q}_j[\psi]$, $\mathcal{Q}_j = \overline{Q}_j[\eta_0]$, and $\widetilde{\mathcal{Q}}_j = \overline{Q}_j[\widetilde{\eta}_0]$.

The class $\mathfrak{U}$ comes up through the following proposition (which is very close to a similar one in [7]).

Proposition 6.5. *If $\{f_j\}_{j\in\mathbb{Z}} \subset L^2(\mathbb{R}^d)$, then*

$$\Big\| \sum_{j\in\mathbb{Z}} Q_j f_j \Big\|_2 \lesssim \sup_{\substack{u\in\mathfrak{U}\\ \|u\|_{\mathfrak{U}}=1}} \Big(\sum_{j\in\mathbb{Z}} \|\overline{Q}_j[u] f_j\|_2^2 \Big)^{\frac12},$$

in the sense that $\sum_j Q_j f_j$ converges unconditionally in the L^2 norm if the right hand side is finite.

6.1. **Proof of Proposition 6.5.** We need several lemmata.

Lemma 6.6. *For $\ell \le 0$, $\phi \in C_0^\infty(B^d(2))$, $u \in \mathcal{S}(\mathbb{R}^d)$ if we define $\gamma_{-\ell} := \phi * u^{(2^{-\ell})}$, we have $\gamma_{-\ell} \in \mathfrak{U}$ and $\|\gamma_{-\ell}\|_{\mathfrak{U}} \lesssim 2^{\ell/2}$.*

Proof. It is clear that $\gamma_{-\ell} \in C^\infty(\mathbb{R}^d)$, so it suffices to prove the bound on $\|\gamma_{-\ell}\|_{\mathfrak{U}}$. Because, for $\nu = 1, \dots, d$, $\partial_{x_\nu}\gamma_{-\ell}$ is of the same form as $\gamma_{-\ell}$, it suffices to show $|\gamma_{-\ell}(x)| \lesssim 2^{\ell/2}(1+|x|^{d+1/2})^{-1}$. This is evident for $|x| \le 4$, since $|\gamma_{-\ell}| \le \|\phi\|_\infty \|u\|_1 \lesssim 1$.

Since $\phi(x-y)$ is supported on $|x-y| \le 2$, we have for $|x| \ge 4$ and any m,

$$|\gamma_{-\ell}(x)| \lesssim \int_{|x-y|\le 2} 2^{-\ell d}(1+2^{-\ell}|y|)^{-m}\,dy \approx \int_{|x-y|\le 2} 2^{-\ell d}(1+2^{-\ell}|x|)^{-m}\,dy \lesssim 2^{-\ell d}(1+2^{-\ell}|x|)^{-m}.$$

Taking $m = d + 1/2$, we have

$$|\gamma_{-\ell}(x)| \lesssim 2^{-\ell d}(1+2^{-\ell}|x|)^{-d-1/2} \lesssim 2^{\ell/2}(1+|x|^{d+1/2})^{-1}, \quad |x| \ge 4,$$

as desired. □

Lemma 6.7. *Suppose $u_1 \in \mathcal{S}(\mathbb{R}^d)$, $u_2 \in \mathcal{S}_0(\mathbb{R}^d)$. For $j \ge 0$, let $u_j := u_1 * u_2^{(2^j)}$. Then, for $m = 0, 1, 2, \dots,$*

$$\sum_{|\alpha|\le m} |\partial_x^\alpha u_j(x)| \lesssim 2^{-jm}(1+|x|)^{-m}.$$

Proof. The goal is to show, for every m, $\{2^{jm}u_j : j \ge 0\} \subset \mathcal{S}(\mathbb{R}^d)$ is a bounded set. To do this, we show $\{2^{jm}\widehat{u}_j : j \ge 0\} \subset \mathcal{S}(\mathbb{R}^d)$ is a bounded set. We have, for every α,

$$\begin{aligned}\Big|\partial_\xi^\alpha \widehat{u}_j(\xi)\Big| &= \big|\sum_{\beta+\gamma=\alpha} C_{\beta,\gamma}\partial_\xi^\beta \widehat{u}_1(\xi)\partial_\xi^\beta \widehat{u}_2(2^{-j}\xi)\big| \lesssim \sum_{\beta+\gamma=\alpha} 2^{-j|\gamma|}|\partial_\xi^\beta \widehat{u}_1(\xi)(\partial_\xi^\gamma \widehat{u}_2)(2^{-j}\xi)| \\ &\lesssim \sum_{\beta+\gamma=\alpha} 2^{-j|\gamma|}(1+|\xi|)^{-2m}|2^{-j}\xi|^m(1+|2^j\xi|)^{-2m} \lesssim 2^{-mj}(1+|\xi|)^{-m}.\end{aligned}$$

The result follows. □

Lemma 6.8. *There exists functions $\varphi_1, \dots, \varphi_d \in C_0^\infty(B^d(2))$ such that $\psi = \sum_{\nu=1}^d \partial_{x_\nu}\varphi_\nu$.*

Proof. Indeed, write

$$\psi(x) = \phi(x) - 2^{-d}\phi(2^{-1}x) = \sum_{\nu=1}^d \psi_\nu(x),$$

where $\psi_\nu(x)$ is given by

$$2^{-(\nu-1)}\phi(x_1/2, x_2/2, \dots, x_{\nu-1}/2, x_\nu, x_{\nu+1}, \dots, x_d) - 2^{-\nu}\phi(x_1/2, x_2/2, \dots, x_\nu/2, x_{\nu+1}, \dots, x_d).$$

Letting $\varphi_\nu(x) = \int_{-\infty}^{x_\nu} \psi_\nu(x_1, \dots, x_{\nu-1}, y_\nu, x_{\nu+1}, \dots, x_d)\, dy_\nu$, the result follows. $\square$

Lemma 6.9. *For $j, k \in \mathbb{Z}$, $\widetilde{\mathcal{Q}}_{j+k} Q_j = 2^{-|k|/2} \overline{Q}_j[u_k]$, where $u_k \in \mathfrak{U}$ and $\|u_k\|_{\mathfrak{U}} \lesssim 1$.*

Proof. By scale invariance, it suffices to consider the case $j = 0$; then $u_k = \psi * \widetilde{\eta}_0^{(2^k)}$. When $k \le 0$, we use Lemma 6.8 to see

$$u_k(x) = \sum_{\nu=1}^{d} \int (\partial_{x_\nu} \varphi_\nu)(y) \widetilde{\eta}_0^{(2^k)}(x-y)\, dy = -2^k \sum_{\nu=1}^{d} \int \varphi_\nu(y) (\partial_{x_\nu} \widetilde{\eta}_0)^{(2^k)}(x-y)\, dy.$$

From here, the desired estimate follows from Lemma 6.6. For $k \ge 0$, the result follows immediately from Lemma 6.7. $\square$

Proof of Proposition 6.5, conclusion. Let $\{f_j : j \in \mathbb{Z}\} \subset L^2(\mathbb{R}^d)$ and let $g \in L^2(\mathbb{R}^d)$ with $\|g\|_{L^2} = 1$. Let $\langle \cdot, \cdot \rangle$ denote the inner product in L^2. We have, letting u_k be as in Lemma 6.9,

$$\begin{aligned}
\Big| \Big\langle g, \sum_{j=J_1}^{J_2} Q_j f_j \Big\rangle \Big| &= \Big| \Big\langle g, \sum_{j=J_1}^{J_2} \sum_{k \in \mathbb{Z}} \mathcal{Q}_{j+k} \widetilde{\mathcal{Q}}_{j+k} Q_j f_j \Big\rangle \Big| \le \sum_{k \in \mathbb{Z}} \sum_{j=J_1}^{J_2} \Big| \Big\langle \mathcal{Q}_{j+k}^* g, \widetilde{\mathcal{Q}}_{j+k} Q_j f_j \Big\rangle \Big| \\
&\lesssim \sum_{k \in \mathbb{Z}} \Big(\sum_{j=J_1}^{J_2} \big\|\mathcal{Q}_{j+k}^* g\big\|_2^2 \Big)^{\frac12} \Big(\sum_{j=J_1}^{J_2} \big\|\widetilde{\mathcal{Q}}_{j+k} Q_j f_j\big\|_2^2 \Big)^{\frac12} \lesssim \sum_{k \in \mathbb{Z}} 2^{-|k|/2} \Big(\sum_{j=J_1}^{J_2} \big\|\overline{Q}_j[u_k] f_j\big\|_2^2 \Big)^{\frac12}.
\end{aligned}$$

The result follows easily. $\square$

6.2. **A decomposition result for functions in $\mathfrak{U}$.** The proof of the following result follows closely a similar result in [33].

Proposition 6.10. *Let $u \in \mathfrak{U}$. Then there exists $u_j \in C_0^1(B^d(\frac14))$ with $\|u_j\|_{C^0} \lesssim \|u\|_{\mathfrak{U}}$, $\int u_j = 0$, and*

$$u = \sum_{j \le 0} 2^{j/2} u_j^{(2^j)}.$$

Proof. Let $\chi_0 \in C_0^\infty$, supported in $\{|x| \le 1/4\}$. with $0 \le \chi_0 \le 1$ and $\chi_0(x) = 1$ for $|x| \le 1/8$. For $j \ge 1$ define $\chi_j(x) = \chi_0(2^{-j}x) - \chi_0(2^{1-j}x)$ so that that for $j \ge 1$, $\mathrm{supp}(\chi_j) \subseteq \{2^{j-4} \le |x| \le 2^{j-2}\}$, and

$$1 = \sum_{j=0}^{\infty} \chi_j(x).$$

Observe that

$$\int \chi_j(x)\, dx = (2^{jd} - 1) \int \chi_0(x) dx \gtrsim 2^{jd}. \tag{6.7}$$

Also let

$$\tilde{\chi}_j(x) = \frac{\chi_j(x)}{\int \chi_j(y)\, dy}.$$

Set $a_j = \int u(x) \chi_j(x)\, dx$ and $A_j = \sum_{k \ge j} a_k = -\sum_{0 \le k < j} a_k$ (where the second equality follows from the fact that $\sum a_j = \int u = 0$).

Note $|a_0| \lesssim 1$, and for $j \ge 1$,

$$|a_j| \le \int |u(x)||\chi_j(x)|\, dx \le \int_{2^{j-4} \le |x| \le 2^{j-2}} (1 + |x|^{d+1/2})^{-1}\, dx \|u\|_{\mathfrak{U}} \lesssim 2^{-j/2} \|u\|_{\mathfrak{U}}. \tag{6.8}$$

Thus,

$$|A_j| \le \sum_{k\ge j} |a_k| \lesssim 2^{-j/2}\|u\|_{\mathfrak{U}}. \tag{6.9}$$

Notice, $A_0 = 0$. We have,

$$\begin{aligned} u(x) &= \sum_{j\ge 0} u(x)\chi_j(x) = \sum_{j\ge 0}(u(x)\chi_j(x) - a_j\tilde{\chi}_j(x)) + \sum_{j\ge 0}(A_j - A_{j+1})\tilde{\chi}_j(x) \\ &= \sum_{j\ge 0}(u(x)\chi_j(x) - a_j\tilde{\chi}_j(x)) + \sum_{j\ge 1} A_j(\tilde{\chi}_j(x) - \tilde{\chi}_{j-1}(x)) =: \sum_{j\ge 0} B_j(x), \end{aligned}$$

where $B_j(x) = u(x)\chi_j(x) - a_j\tilde{\chi}_j(x) + (A_j(\tilde{\chi}_j(x) - \tilde{\chi}_{j-1}(x)))\epsilon_j$ and $\epsilon_j = 1$ if $j \ge 1$, $\epsilon_0 = 0$. Here we have used $A_0 = 0$ and $\lim_{j\to\infty} A_j = 0$. Clearly $\int B_j = 0$, and $\operatorname{supp}(B_j) \subseteq \{|x| \le 2^{j-2}\}$. We have

$$|B_j(x)| \le |u(x)\chi_j(x)| + |a_j||\tilde{\chi}_j(x)| + |A_j|(|\tilde{\chi}_j(x)| + |\tilde{\chi}_{j-1}(x)|)\epsilon_j.$$

(6.7) shows $|\tilde{\chi}_j(x)| \lesssim 2^{-jd}$. The support of χ_j shows $|u(x)\chi_j(x)| \lesssim 2^{-j(d+\frac12)}\|u\|_{\mathfrak{U}}$. Combining this with (6.8) and (6.9) shows $|B_j(x)| \lesssim 2^{-j(d+\frac12)}\|\varsigma\|_{\mathfrak{U}}$. Setting, for $j \ge 0$, $u_{-j}(x) = 2^{jd}2^{j/2}B_j(2^jx)$, the result follows easily. □

7. Basic L^2 estimates

7.1. **An L^2 estimate for rough kernels.** An essential part to many of our estimates is the following L^2 estimate.

Theorem 7.1. *Let u be a continuous function supported in $\{y \in \mathbb{R}^d : |y| \le 1/4\}$ such that $\|u\|_\infty \le 1$ and*

$$\int u(y)dy = 0.$$

Let $\mathfrak{Q}_k$ be the operator of convolution with $u^{(2^k)}$. Let $0 < \varepsilon < 1$, $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ and assume that $\operatorname{supp}(\varsigma) \subset \{(\alpha, v) : |v| \le 1/4\}$. *Then for all $k \in \mathbb{N}$, for $b_{n+1}, b_{n+2} \in L^2(\mathbb{R}^d)$, $b_i \in L^\infty(\mathbb{R}^d)$, $i = 1, \dots, n$,*

$$|\Lambda[\varsigma](b_1, \dots, \mathfrak{Q}_k b_{n+1}, b_{n+2})| \lesssim 2^{-k\varepsilon/(3d+3)} n \|\varsigma\|_{B_\varepsilon} \|b_{n+1}\|_2 \|b_{n+2}\|_2 \prod_{i=1}^n \|b_j\|_\infty.$$

In §7.2 below we shall prove a similar theorem without the support assumptions on ς and u. In what follows we give the proof of Theorem 7.1.

7.1.1. *Applying the Leibniz rule.* We have

$$\Lambda[\varsigma](b_1, \dots, \mathfrak{Q}_k b_{n+1}, b_{n+2}) = \iint F_k[\varsigma](x,y)\, b_{n+1}(y) b_{n+2}(x)\, dx\, dy, \tag{7.1}$$

where, using the cancellation of u we have

$$\begin{aligned} F_k[\varsigma](x,y) &= \iint \varsigma(\alpha, x-z) \prod_{i=1}^n b_i(x - \alpha_i(x-z))\, u^{(2^k)}(z-y)\, dz\, d\alpha \\ &= \iint \Big[\varsigma(\alpha, x-z)\prod_{i=1}^n b_i(x-\alpha_i(x-z)) - \varsigma(\alpha, x-y)\prod_{i=1}^n b_i(x-\alpha_i(x-y))\Big]\, u^{(2^k)}(z-y)\, dz\, d\alpha\,. \end{aligned}$$

We let $T_k[\varsigma]$ denote the operator with Schwartz kernel $F_k[\varsigma]$.

For further decomposition we use a Leibniz rule for differences

$$\prod_{j=0}^{n} A_j - \prod_{j=0}^{n} B_j =$$
$$(A_0 - B_0)\Big(\prod_{j=1}^{n} A_j\Big) + \sum_{i=1}^{n-1}\Big(\Big(\prod_{j=0}^{i-1} B_j\Big)(A_i - B_i)\Big(\prod_{j=i+1}^{n} A_j\Big)\Big) + \Big(\prod_{j=0}^{n-1} B_j\Big)(A_n - B_n).$$

Thus

$$F_k[\varsigma] = \sum_{i=0}^{n} F_{k,i}[\varsigma]$$

where

$$F_{k,0}[\varsigma](x,y) = \iint \big[\varsigma(\alpha, x-z) - \varsigma(\alpha, x-y)\big] \prod_{j=1}^{n} b_j(x - \alpha_j(x-z))\, u^{(2^k)}(z-y)\, dz\, d\alpha\,,$$

$$F_{k,i}[\varsigma](x,y) = \iint \varsigma(\alpha, x-y)\big] \prod_{j=1}^{i-1} b_i(x - \alpha_i(x-y)) \times$$
$$\Big(b_i(x - \alpha_i(x-z) - b_i(x - \alpha_i(x-y))\Big) \prod_{j=i+1}^{n} b_j(x - \alpha_j(x-z))\, u^{(2^k)}(z-y)\, dz\, d\alpha\,,$$

with the convention that the products $\prod_{j=1}^{0}$ and $\prod_{j=n+1}^{n}$ stand for the number 1. We thus have to estimate the $L^2 \to L^2$ operator norms for the operators $T_{k,i}[\varsigma]$ with Schwartz kernels $F_{k,i}[\varsigma]$. For $i = 0$ we may use the standard Schur test and the condition $\varsigma \in \mathcal{B}_\varepsilon$

$$\begin{aligned} &\sup_x \int |F_{k,0}[\varsigma](x,y)|\, dy \\ &\le \sup_x \prod_{j=1}^{n} \|b_j\|_\infty \int_{|h|\le 2^{-k}} |u^{(2^k)}(h)| \int |\varsigma(\alpha, x-y-h) - \varsigma(\alpha, x-y)|\, dy\, d\alpha\, dh \\ &\lesssim \prod_{j=1}^{n} \|b_j\|_\infty \sup_{|h|\le 2^{-k}} \int \|\varsigma(\alpha, \cdot - h) - \varsigma(\alpha, \cdot)\| d\alpha \lesssim 2^{-k\varepsilon} \prod_{j=1}^{n} \|b_j\|_\infty \|\varsigma\|_{\mathcal{B}_\varepsilon} \end{aligned} \tag{7.2a}$$

and similarly

$$\sup_y \int |F_{k,0}[\varsigma](x,y)| dx \lesssim 2^{-k\varepsilon} \prod_{j=1}^{n} \|b_j\|_\infty \|\varsigma\|_{\mathcal{B}_\varepsilon}. \tag{7.2b}$$

Hence

$$\|T_{k,0}[\varsigma]\|_{L^2\to L^2} \lesssim 2^{-k\varepsilon} \prod_{j=1}^{n} \|b_j\|_\infty \|\varsigma\|_{\mathcal{B}_\varepsilon}. \tag{7.3}$$

We shall now turn to the operators $T_{k,i}[\varsigma]$, $i = 1, \dots, n$. We start with a trivial bound.

Lemma 7.2. *For* $1 \le p \le \infty$

$$\|T_{k,i}[\varsigma]\|_{L^p\to L^p} \lesssim \|\varsigma\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)} \prod_{j=1}^{n} \|b_i\|_\infty.$$

Proof. This follows immediately from Schur's test since

$$\sup_x \int |F_{k,i}[\varsigma](x,y)|\,dy + \sup_y \int |F_{k,i}[\varsigma](x,y)|\,dx \lesssim \|\varsigma\|_{L^1(\mathbb{R}^n\times\mathbb{R}^d)} \prod_{j=1}^n \|b_i\|_\infty. \qquad \square$$

We begin with a regularization of ς, in the x and the α_i variables, depending on a parameter R to be chosen later. Here $1 \ll R \ll 2^k$ (we shall see that $R = 2^{k/(3d+3)}$ will be a good choice).

Let $\phi \in C^\infty(\mathbb{R}^d)$ supported in $\{x : |x| \le 1/2\}$ so that $\int \phi(x)dx = 1$. Let $\varphi \in C^\infty(\mathbb{R})$ be supported in $\{u : |u| \le 1/2\}$ so that $\int \varphi(u)du = 1$. Define

$$\varsigma_R^i(\alpha, v) = \iint \chi_{[-R,R]}(\alpha - se_i)\varsigma(\alpha - se_i, v - z)R\varphi(Rs)R^d\phi(Rz)\,dz\,ds.$$

Lemma 7.3. *For $i = 1, \dots, n$,*

(i)

$$\|\varsigma - \varsigma_R^i\|_{L^1(\mathbb{R}^n\times\mathbb{R}^d)} \lesssim R^{-\varepsilon}$$

(ii)

$$\|T_{k,i}[\varsigma - \varsigma_R^i]\|_{L^2\to L^2} \lesssim R^{-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}\,.$$

Proof. We expand $\varsigma - \varsigma_R^i = I + II + III$ where

$$\begin{aligned} I(\alpha, v) &= \int \big[\varsigma(\alpha, v) \,-\, \varsigma(\alpha, v - z)\big]R^d\phi(Rz)\,dz, \\ II(\alpha, v) &= \iint \big[\varsigma(\alpha, v - z) \,-\, \varsigma(\alpha - se_i, v - z)\big]R\varphi(Rs)R^d\phi(Rz)\,dz\,ds, \\ III(\alpha, v) &= \iint \chi_{[-R,R]^\complement}(\alpha - se_i)\varsigma(\alpha - se_i, v - z)R\varphi(Rs)R^d\phi(Rz)\,dz\,ds\,. \end{aligned}$$

Then

$$\|I\|_{L^1(\mathbb{R}^n\times\mathbb{R}^d)} \lesssim \int R^d|\phi(Rz)| \iint \big|\varsigma(\alpha, v) \,-\, \varsigma(\alpha, v - z)\big|\,d\alpha\,dv\,|R^d\phi(Rz)|\,dz \lesssim R^{-\varepsilon}\|\varsigma\|_{\mathcal{B}_{\varepsilon,3}}.$$

For the second term,

$$\|II\|_{L^1(\mathbb{R}^n\times\mathbb{R}^d)} \lesssim \int R|\varphi(Rs)| \iint \big|\varsigma(\alpha, v) \,-\, \varsigma(\alpha - se_i, v)\big|\,d\alpha\,dv\,ds \lesssim R^{-\varepsilon}\|\varsigma\|_{\mathcal{B}_{\varepsilon,2}}.$$

Finally

$$\|III\|_{L^1(\mathbb{R}^n\times\mathbb{R}^d)} \lesssim \int \int_{[-R,R]^\complement} |\varsigma(\alpha, v)|\,d\alpha\,dv \lesssim R^{-\varepsilon}\|\varsigma\|_{\mathcal{B}_{\varepsilon,1}}$$

and part (i) follows. The second part follows from Lemma 7.2 applied to $\varsigma - \varsigma_R^i$, and the first part. $\square$

For the more regular term ς_R^i we shall need the inequalities

Lemma 7.4. *Let $0 < \varepsilon < 1$, $d \ge 2$. Then*

(i)

$$\int \Big(\int \big|\varsigma_R^i(\alpha, v)\big|^2 dv\Big)^{\frac{1}{2}} d\alpha \lesssim R^{\frac{d}{2}-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

(ii) Let $\theta \in S^{d-1}$ and let $\theta^\perp$ the orthogonal complement of $\mathbb{R}\theta$. Then

$$\int \sup_\theta \Big(\int_{\theta^\perp} \sup_{s\in\mathbb{R}} |\varsigma_R^i(\alpha, v^\perp + s\theta)|^2 dv_{\theta^\perp}\Big)^{\frac12} d\alpha \lesssim R^{\frac{d+1}{2}-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}$$

and

$$\int \sup_\theta \Big(\int_{\theta^\perp} \sup_{s\in\mathbb{R}} |\partial_{\alpha_i}\varsigma_R^i(\alpha, v^\perp + s\theta)|^2 dv_{\theta^\perp}\Big)^{\frac12} d\alpha \lesssim R^{\frac{d+3}{2}-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon} .$$

Proof. Let $\beta_0 \in \mathcal{S}(\mathbb{R}^d)$ so that $\widehat{\beta}_0(\xi) = 1$ for $|\xi| < 1/2$ and $\widehat{\beta}_0$ is supported in $\{|\xi| < 1\}$. Let $\beta_1 = \beta_0^{(2)} - \beta_0$ and $\beta_k = \beta_1^{(2^{k-1})}$ so that $\widehat{\beta}_k$ has support in an annulus $\{|\xi| \approx 2^k\}$, and $f = \sum_{k=0}^\infty \beta_k * f$ in the sense of distributions. Let $\widetilde{\beta}_0 \in \mathcal{S}(\mathbb{R}^d)$ be such that its Fourier transform equals 1 on the support of $\widehat{\beta}_0$. Let $\widetilde{\beta}_1$ be a Schwartz function so that its Fourier transform vanishes in a neighborhood of the origin and is compactly be supported, and equal to 1 on the support of $\widehat{\beta}_1$. Let $\widetilde{\beta}_k = \widetilde{\beta}_1^{(2^{k-1})}$.

Let

$$\widetilde{\varsigma}_R^i(\alpha, v) = \iint \chi_{[-R,R]}(\alpha - se_i)\varsigma(\alpha - se_i, v) R\varphi(Rs)\, ds$$

so that $\widetilde{\varsigma}_R^i(\alpha,\cdot) * \phi_R = \varsigma_R^i$ (the definition of φ was given right before the statement of Lemma 7.3). Then

$$\varsigma_R^i(\alpha,\cdot) = \sum_{l=0}^\infty \beta_l * \widetilde{\varsigma}_R^i(\alpha,\cdot) * \phi_R * \widetilde{\beta}_l.$$

By Young's inequality

$$\|\varsigma_R^i(\alpha, v) * \widetilde{\beta}_l * \beta_l\|_2 \le \|\widetilde{\varsigma}_R^i(\alpha,\cdot) * \beta_l\|_1 \|\widetilde{\beta}_l * \Phi_R\|_2$$

and it is easy to see that

$$\|\widetilde{\beta}_l * \Phi_R\|_2 \le C_M 2^{ld/2} \min\{1, (R2^{-l})^M\} .$$

Thus

$$\begin{aligned}
&\int \Big(\int \big|\varsigma_R^i(\alpha, v)\big|^2 dv\Big)^{\frac12} d\alpha \\
&\lesssim \sum_{l=0}^\infty 2^{ld/2} \min\{1, (R2^{-l})^M\} \iint \Big| \int \beta_l(v-w)\varsigma_R^i(\alpha, w) dw\Big|\, dv\, d\alpha \\
&\lesssim \sum_{l=0}^\infty 2^{ld/2} \min\{1, (R2^{-l})^M\} 2^{-l\varepsilon} \|\widetilde{\varsigma}_R^i\|_{\mathcal{B}_\varepsilon} \lesssim R^{\frac{d}{2}-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}.
\end{aligned}$$

The first inequality in (ii) is proved similarly, except that we first use the one-dimensional version of Young's inequality in the θ-direction. Since the Fourier transform of β_l is supported on a set of diameter $O(2^l)$ we have, for fixed θ and almost every α,

$$\Big(\int_{\theta^\perp} \sup_{s\in\mathbb{R}} |\beta_l * \varsigma_R^i(\alpha, v^\perp + s\theta)|^2 dv^\perp\Big)^{\frac12} \lesssim 2^{l/2} \Big(\int_{\theta^\perp} \int_{-\infty}^\infty |\beta_l * \varsigma_R^i(\alpha, v^\perp + s\theta)|^2 ds\, dv^\perp\Big)^{\frac12}.$$

Notice that the double integral on the right hand side is just the $L^2(\mathbb{R}^d)$ norm of $\varsigma_R^i(\alpha,\cdot)$ and thus does not depend on θ. Take the sup over θ, then integrate in α, and sum in l. Arguing as above we obtain:

$$\begin{aligned}
&\int \sup_{\theta} \Big(\int_{\theta^\perp} \sup_{s\in\mathbb{R}} |\beta_l * \varsigma_R^i(\alpha, v^\perp + s\theta)|^2 dv^\perp \Big)^{\frac12} d\alpha \\
&\lesssim \sum_{l\ge 0} 2^{l/2} \int \Big(|\beta_l * \varsigma_R^i(\alpha, v)|^2 dv \Big)^{1/2} d\alpha \\
&\lesssim \sum_{l\ge 0} 2^{l(d+1)/2} \min\{1, (R2^{-l})^M\} \iint |\beta_l * \tilde\varsigma_R^i(\alpha, v)|\, dv\, d\alpha \\
&\lesssim \sum_{l\ge 0} 2^{l(d+1)/2} \min\{1, (R2^{-l})^M\} 2^{-l\varepsilon} \|\tilde\varsigma_R^i\|_{\mathcal{B}_\varepsilon} \lesssim R^{\frac{d+1}{2}-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}.
\end{aligned}$$

The second inequality in (ii) is proved in the same way. The differentiation in α_i hitting the mollifier $R\varphi(R\cdot)$ produces an additional factor of R. $\square$

By the support assumptions on ς and u, we have

$$\mathrm{supp}(F_{k,i}[\varsigma_R^i]) \subseteq \{(x,y) : |x-y| < 1\}.$$

We shall use the following lemma to obtain the bound $C(R)2^{-k\varepsilon'}$ of the L^2 operator norms.

Lemma 7.5. *Suppose $V(x,y) \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ is supported in the strip $\{(x,y) : |x-y| \le 1\}$ and let $\mathcal{V}$ be the operator with Schwartz kernel V. Then,*

$$\|\mathcal{V}\|^2_{L^2\to L^2} \lesssim \sup_z \iint_{\substack{|x-z|<1\\ |y-z|<1}} |V(x,y)|^2\, dx\, dy.$$

Proof. Let A denote the quantity on the right hand side. For $\mathfrak{z} \in \mathbb{Z}^d$ let $q_\mathfrak{z}$ be the cube $\mathfrak{z} + [0,1]^d$ and $f_\mathfrak{z} = \chi_{q_\mathfrak{z}}$. Then $f = \sum_\mathfrak{z} f_\mathfrak{z}$ and for each $\mathfrak{z}$, $Vf_\mathfrak{z}$ is supported in the union $q_\mathfrak{z}^*$ of cubes which have a common side with $q_\mathfrak{z}$. By Hölder's inequality it is immediate that

$$\|\mathcal{V} f_\mathfrak{z}\|_2 \lesssim \Big(\iint_{q_\mathfrak{z}^* \times q_\mathfrak{z}} |V(x,y)|^2 dx\, dy \Big)^{1/2} \|f_\mathfrak{z}\|_2 \le C(d) A \|f_\mathfrak{z}\|_2,$$

and then

$$\|\mathcal{V} f\|_2 = \Big\| \sum_\mathfrak{z} \mathcal{V} f_\mathfrak{z} \Big\|_2 \le 3^{d/2} \Big(\sum_\mathfrak{z} \big\|\mathcal{V} f_\mathfrak{z}\big\|_2^2 \Big)^{1/2} \le C'(d) A \Big(\sum_\mathfrak{z} \big\| f_\mathfrak{z}\big\|_2^2 \Big)^{1/2} \le C'(d) A \|f\|_2 . \quad \square$$

In light of Lemma 7.5 the following proposition gives a basic L^2 bound for the operators $T_{k,i}[\varsigma_R]$.

Proposition 7.6. *For $k \ge 0$*

$$\Big(\sup_{y_0} \iint_{\substack{|x-y_0|<1\\ |y-y_0|<1}} |F_{k,i}[\varsigma_R^i](x,y)|^2\, dx\, dy \Big)^{\frac12} \lesssim 2^{-k/3} R^{d+1} \prod_{i=1}^n \|b_i\|_\infty. \tag{7.4}$$

7.1.2. *Proof of Proposition 7.6.* Note that the class of operators is invariant under translations. That is, if $\tau_a f := f(x-a)$, then the kernel of $\tau_a T_{k,i}[\varsigma_R^i]\tau_{-a}$, i.e. $F_{k,i}[\varsigma_R^i](x-a, y-a)$, is of the same form of $F_{k,i}$, with the functions b_j replaced by $\tau_a b_j$. Therefore we may take $y_0 = 0$ in Proposition 7.6. We may also assume

$$\|b_j\|_\infty \le 1, \quad 1 \le j \le n\,. \tag{7.5}$$

As in §4 we decompose α as $\alpha = \alpha_i e_i + \alpha_i^\perp$ where $\alpha_i^\perp = (..., \alpha_{i-1}, \alpha_{i+1}, ...) \in \mathbb{R}^{n-1}$. We bound, using the Cauchy-Schwarz inequality in the z-variable, and then Minkowski's inequality in the $\alpha_i^\perp$ variables, as well as (7.5) for $j \neq i$,

$$\iint_{\substack{|x|<1\\|y|<1}} |F_{k,i}[\varsigma_R^i](x,y)|^2\, dx\, dy\Big)^{\frac12}$$

$$\lesssim \int \Big(\iiint_{\substack{|x|,|y|\le 1\\|y-z|\le 2^{-k}}} 2^{kd} \Big| \int \varsigma_R^i(\alpha, x-y)\big[b_i(x-\alpha_i(x-z)) - b_i(x-\alpha_i(x-y))\big] d\alpha_i \Big]^2 dz\, dx\, dy\Big)^{1/2} d\alpha_i^\perp$$

$$\lesssim \iiint_{\substack{|x|,|v|,|w|\le 2\\|v-w|\le 2^{-k}}} \Big| \int \varsigma_R^i(\alpha, v)\big[b_i(x-\alpha_i v) - b_i(x-\alpha_i w)\big] \Big|^2 dv\, dw\, dx\Big)^{1/2} d\alpha_i^\perp$$

where for the last integral we have changed variables to $v = x - z$, $w = x - y$. The proof of Proposition 7.6 will be complete after the following lemma is proved.

Lemma 7.7. *Let ς_R^i be as in Proposition 7.6. Then for $g \in L^\infty(\mathbb{R}^d)$ and $k > 0$,*

$$\Big(2^{kd} \iiint_{\substack{|x|<2\\|v|,|w|<2\\|v-w|<2^{-k}}} \Big| \int \varsigma_R^i(\alpha, v)\big(g(x-\alpha_i v) - g(x-\alpha_i w)\big) d\alpha \Big|^2 dx\, dv\, dw\Big)^{\frac12} \lesssim R^{d+1-\varepsilon} 2^{-k\varepsilon/3} \|\varsigma\|_{\mathcal{B}_\varepsilon} \|g\|_\infty.$$

Proof. We may and shall assume $\|g\|_{L^\infty} = 1$. Let $g_R(x) = g(x)$ if $|x| \le 2R+2$ and $g_R(x) = 0$ if $|x| > 2R+2$. We first observe that since $\varsigma_R^i(\alpha, v) = 0$ for $|\alpha_i| \ge R+1$ we may replace g by g_R in the above expression. Note that

$$\|g_R\|_2 \lesssim R^{d/2}. \tag{7.6}$$

We interchange the (v,w)- and x-integrations, then apply Plancherel's theorem, and interchange integrals again to get

$$\int \Big(2^{kd} \iiint_{\substack{|x|<2\\|v|,|w|<2\\|v-w|<2^{-k}}} \Big| \int \varsigma_R^i(\alpha, v)\big(g_R(x-\alpha_i v) - g_R(x-\alpha_i w)\big) d\alpha \Big|^2 dx\, dv\, dw\Big)^{\frac12} d\alpha_i^\perp$$

$$= \int \Big(\int |\widehat{g}_R(\xi)|^2 2^{kd} \iint_{\substack{|v|,|w|<2\\|v-w|<2^{-k}}} \Big| \int \varsigma_R^i(\alpha, v)\big(e^{2\pi \imath \alpha_i \langle v, \xi\rangle} - e^{2\pi \imath \alpha_i \langle w, \xi\rangle}\big) d\alpha_i \Big|^2 dv\, dw\, d\xi\Big)^{\frac12} d\alpha_i^\perp.$$

For a constant $U \ge 1$ (to be determined) we split the ξ-integration into the parts for $|\xi| \le U$ and $|\xi| \ge 1$.

For $|\xi|\le U$ we bound $|e^{2\pi\imath\alpha_i\langle v,\xi\rangle}-e^{2\pi\imath\alpha_i\langle w,\xi\rangle}|\lesssim RU2^{-k}$ since $|\alpha_i|\le(R+1)$ and $|v-w|\le 2^{-k}$. Hence we obtain

$$\begin{aligned}(7.7)\quad &\int\Big(\int_{|\xi|\le U}|\widehat{g}_R(\xi)|^2 2^{kd}\iint_{\substack{|v|,|w|<2\\|v-w|<2^{-k}}}\Big|\int\varsigma_R^i(\alpha,v)\big(e^{2\pi\imath\alpha_i\langle v,\xi\rangle}-e^{2\pi\imath\alpha_i\langle w,\xi\rangle}\big)d\alpha_i\Big|^2\,dv\,dw\,d\xi\Big)^{\frac12}d\alpha_i^{\perp}\\
&\lesssim RU2^{-k}\|\widehat{g}_R\|_2\int\Big(\int|\varsigma_R^i(\alpha,v)|^2dv\Big)^{1/2}d\alpha\\
&\lesssim R^{\frac{d+2}{2}-\varepsilon}U2^{-k}\|g_R\|_2\|\varsigma\|_{\mathcal{B}_\varepsilon}\end{aligned}$$

where in the last inequality we have used part (i) of Lemma 7.4.

Next we consider the part when $|\xi|>U$. Using the symmetry in v,w we may estimate

$$\begin{aligned}&\int\Big(\int_{|\xi|\ge U}|\widehat{g}_R(\xi)|^2 2^{kd}\iint_{\substack{|v|,|w|<2\\|v-w|<2^{-k}}}\Big|\int\varsigma_R^i(\alpha,v)\big(e^{2\pi\imath\langle v,\xi\rangle\alpha_i}-e^{2\pi\imath\langle w,\xi\rangle\alpha_i}\big)d\alpha_i\Big|^2\,dv\,dw\,d\xi\Big)^{\frac12}d\alpha_i^{\perp}\\
&\le 2\int\Big(\int_{|\xi|\ge U}|\widehat{g}_R(\xi)|^2 2^{kd}\iint_{\substack{|v|,|w|<2\\|v-w|<2^{-k}}}\Big|\int\varsigma_R^i(\alpha,v)e^{2\pi\imath\langle v,\xi\rangle\alpha_i}d\alpha_i\Big|^2\,dv\,dw\,d\xi\Big)^{\frac12}d\alpha_i^{\perp}\\
&\lesssim\int\Big(\int_{|\xi|\ge U}|\widehat{g}_R(\xi)|^2\int\Big|\int\varsigma_R^i(\alpha,v)e^{2\pi\imath\langle v,\xi\rangle\alpha_i}d\alpha_i\Big|^2\,dv\,d\xi\Big)^{\frac12}d\alpha_i^{\perp}\,.\end{aligned}$$

For fixed $\xi=|\xi|\theta$ ($\theta\in S^{d-1}$) we separate the v-integral into two parts. Let $0<b<1$ (which will be optimally chosen later). For fixed $\theta=\xi/|\xi|$, $\alpha_i^{\perp}$ we have $v=\pi_{\theta^\perp}v+s\theta$ where $\pi_{\theta^\perp}v$ is the projection of v to the orthogonal complement of $\mathbb{R}\theta$ and $s=\langle\theta,v\rangle$. We split

$$\begin{aligned}\int\Big|\int\varsigma_R^i(\alpha,v)e^{2\pi\imath\langle v,\xi\rangle\alpha_i}d\alpha_i\Big|^2\,dv\,&=\int\int\Big|\int\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+s\theta))e^{2\pi\imath s|\xi|\alpha_i}d\alpha_i\Big|^2dsdv_{\theta^\perp}\\
&=:I_b(\alpha_i^{\perp},|\xi|\theta)+II_b(\alpha_i^{\perp},|\xi|\theta)\end{aligned}$$

where

$$\begin{aligned}I_b(\alpha_i^{\perp},|\xi|\theta)&:=\int\int_{[-b,b]}\Big|\int\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+s\theta))e^{2\pi\imath s|\xi|\alpha_i}d\alpha_i\Big|^2dsdv_{\theta^\perp}\\
II_b(\alpha_i^{\perp},|\xi|\theta)&:=\int\int_{[-b,b]^{\complement}}\Big|\int\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+s\theta))e^{2\pi\imath s|\xi|\alpha_i}d\alpha_i\Big|^2dsdv_{\theta^\perp}\end{aligned}$$

so that

$$\begin{aligned}&\int\Big(\int_{|\xi|\ge U}|\widehat{g}_R(\xi)|^2\int\Big|\int\varsigma_R^i(\alpha,v)e^{2\pi\imath\langle v,\xi\rangle\alpha_i}d\alpha_i\Big|^2\,dv\,d\xi\Big)^{\frac12}d\alpha_i^{\perp}\\
&\lesssim\int\Big(\int_{|\xi|\ge U}|\widehat{g}_R(\xi))|^2[I_b(\alpha_i^{\perp},\xi)+II_b(\alpha_i^{\perp},\xi)]d\xi\Big)^{\frac12}d\alpha_i^{\perp}\,.\end{aligned}$$

The expression I_b is estimated as

$$I_b(\alpha_i^{\perp},|\xi|\theta)|\le 2b\int\sup_{|s|\le b}\Big[\int|\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+s\theta))|d\alpha_i\Big]^2dv_{\theta^\perp}$$

and we get using part (ii) of Lemma 7.4

$$\begin{aligned}
&\int \Big(\int_{|\xi|\geq U} |\widehat{g}_R(\xi))|^2 I_b(\alpha_i^\perp,\xi)d\xi\Big)^{\frac12} d\alpha_i^\perp\\
&\lesssim b^{1/2}\|g_R\|_2 \int \Big(\sup_\theta \int \sup_s \Big[\int |\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+s\theta))|d\alpha_i\Big]^2 dv_{\theta^\perp}\Big)^{1/2} d\alpha_i^\perp\\
&\lesssim b^{1/2}R^{\frac{d+1}{2}-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}\|g_R\|_2\,. \qquad (7.8)
\end{aligned}$$

To estimate $II_b(\alpha_\perp,\xi)$ we observe that the function $\alpha_i\mapsto\varsigma_R^i(\alpha,v)$ is smooth and compactly supported. We use integration by parts to write

$$\int \varsigma_R^i(\alpha,\pi_{\theta^\perp}v+s\theta))e^{2\pi\imath s|\xi|\alpha_i}d\alpha_i = -\int \partial_{\alpha_i}\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+s\theta))(2\pi\imath|\xi|)^{-1}s^{-1}e^{2\pi\imath s\alpha_i}d\alpha_i$$

and thus for $|\xi|\geq U$

$$\begin{aligned}
II_b(\alpha_i^\perp,|\xi|\theta)| &\leq \int_b^\infty |\xi|^{-2}|s|^{-2}ds\int \sup_t\Big[\int |\partial_{\alpha_i}\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+t\theta))|d\alpha_i\Big]^2 dv_{\theta^\perp}\\
&\lesssim U^{-2}b^{-1}\int \sup_t\Big[\int |\partial_{\alpha_i}\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+t\theta))|d\alpha_i\Big]^2 dv_{\theta^\perp}\,.
\end{aligned}$$

Hence, by the second inequality in part (ii) of Lemma 7.4,

$$\begin{aligned}
&\int \Big(\int_{|\xi|\geq U} |\widehat{g}_R(\xi))|^2 II_b(\alpha_i^\perp,\xi)d\xi\Big)^{\frac12} d\alpha_i^\perp\\
&\lesssim U^{-1}b^{-1/2}\|g_R\|_2 \int \Big(\sup_\theta\int \sup_t\Big[\int |\partial_{\alpha_i}\varsigma_R^i(\alpha,\pi_{\theta^\perp}v+t\theta)|d\alpha_i\Big]^2 dv_{\theta^\perp}\Big)^{1/2} d\alpha_i^\perp\\
&\lesssim U^{-1}b^{-1/2}R^{\frac{d+3}{2}-\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}\|g_R\|_2. \qquad (7.9)
\end{aligned}$$

We combine (7.7), (7.8), (7.9) to deduce

$$\begin{aligned}
&\int \Big(2^{kd}\iiint_{\substack{|x|<2\\|v|,|w|<2\\|v-w|<2^{-k}}} \Big|\int \varsigma_R^i(\alpha,v)\big(g_R(x-\alpha_i v)-g_R(x-\alpha_i w)\big)d\alpha\Big|^2 dx\,dv\,dw\Big)^{\frac12} d\alpha_i^\perp \qquad (7.10)\\
&\lesssim \big(R^{\frac{d+2}{2}-\varepsilon}U2^{-k} + R^{\frac{d+1}{2}-\varepsilon}b^{1/2} + R^{\frac{d+3}{2}-\varepsilon}U^{-1}b^{-1/2}\big)\|\varsigma\|_{\mathcal{B}_\varepsilon}\|g_R\|_2\,.
\end{aligned}$$

We choose b,U so that the three terms are comparable, i.e. $b=RU^{-1}$, $U=2^{2k/3}$. The result is that the left hand side of (7.10) is bounded by a constant times

$$R^{\frac{d+2}{2}-\varepsilon}2^{-k/3}\|\varsigma\|_{\mathcal{B}_\varepsilon}\|g_R\|_2 \lesssim R^{d+1-\varepsilon}2^{-k/3}\|\varsigma\|_{\mathcal{B}_\varepsilon},$$

by (7.6), and the proof is complete. □

7.1.3. *Proof of Theorem 7.1.* By (7.3),

$$\|T_{k,0}[\varsigma]\|_{L^2\to L^2}\lesssim 2^{-k\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}\prod_{l=1}^n\|b_l\|_\infty.$$

By Lemma 7.3 and Proposition 7.6 we have for $i=1,\dots,n$,

$$\begin{aligned}
\|T_{k,i}[\varsigma]\|_{L^2\to L^2} &\leq \|T_{k,i}[\varsigma-\varsigma_R^i]\|_{L^2\to L^2}+\|T_{k,i}[\varsigma_R^i]\|_{L^2\to L^2}\\
&\lesssim R^{-\varepsilon}(1+2^{-k/3}R^{d+1})\|\varsigma\|_{\mathcal{B}_\varepsilon}\prod_{l=1}^n\|b_l\|_\infty\,.
\end{aligned}$$

Choosing $R = 2^{k/(3d+3)}$ yields the bound

$$\sum_{i=0}^{n} \|T_{k,i}[\varsigma]\|_{L^2\to L^2} \lesssim (n+1)2^{-k\varepsilon/(3d+3)}\|\varsigma\|_{\mathcal{B}_\varepsilon}\prod_{l=1}^{n}\|b_l\|_\infty$$

and thus the estimates for the multilinear forms claimed in Theorem 7.1. $\square$

7.2. **Generalizations of Theorem 7.1.** We shall now drop the support assumptions on $x \mapsto \varsigma(\alpha, x)$ and on u in Theorem 7.1. Moreover, we extend to L^p estimates and replace ς by the scaled versions $\varsigma^{(2^j)}$ (with the scaling in the x variables).

Theorem 7.8. *There exists $c > 0$, independent of n and ε, so that the following statement holds for all $1 \le p \le \infty$. For all $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$, for all $j, k \in \mathbb{Z}$, $1 \le l_1 \ne l_2 \le n+2$, $b_{l_1} \in L^2(\mathbb{R}^d)$, $b_{l_2} \in L^2(\mathbb{R}^d)$, $b_l \in L^\infty(\mathbb{R}^d)$ for $l \ne l_1, l_2$, and $u \in \mathfrak{U}$,*

$$\begin{aligned}&|\Lambda[\varsigma^{(2^j)}](b_1, \dots, b_{l_2-1}, \overline{Q}_k[u]b_{l_2}, b_{l_2+1}, \dots, b_{n+2})| \\ &\qquad\lesssim \min\{n2^{c\varepsilon(j-k)}\|\varsigma\|_{\mathcal{B}_\varepsilon},\, \|\varsigma\|_{L^1}\}\|u\|_{\mathfrak{U}}\Big(\prod_{l\ne l_1,l_2}\|b_l\|_{L^\infty}\Big)\|b_{l_1}\|_2\|b_{l_2}\|_2\,.\end{aligned}$$

Proof. In light of Theorem 2.9, Theorem 7.8 follows immediately from Lemma 2.7 and the estimate (for some $c' > 0$, independent of n)

$$\Lambda[\varsigma^{(2^j)}](b_1, \dots, b_n, \overline{Q}_k[u]b_{n+1}, b_{n+2}) \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon} n2^{-c'\varepsilon(k-j)}\|u\|_{\mathfrak{U}}\Big(\prod_{l=1}^{n}\|b_l\|_\infty\Big)\|b_{n+1}\|_2\|b_{n+2}\|_2\,. \tag{7.11}$$

By scaling (Lemma 4.16) it suffices to prove (7.11) for $j = 0$. Theorem 7.1 covers the case of ς supported in $\mathbb{R}^n \times \{|x| \le 1/4\}$. To cover the general case we apply Proposition 6.10 to write $u = \sum_{l\ge0} 2^{-l/2}u_l^{(2^{-l})}$ where u_l is continuous and supported in $\{|x| \le 1/4\}$, $\int u_l = 0$, and $\|u_l\|_{C^0} \lesssim \|u\|_{\mathfrak{U}}$. We apply Theorem 4.15 to write $\varsigma = \sum_{m\ge0} 2^{-mc_1\varepsilon}\varsigma_m^{(2^m)}$ for some $c_1 > 0$, where $\varsigma_m \in \mathcal{B}_{c_1\varepsilon}$, $\|\varsigma_m\|_{\mathcal{B}_{c_1\varepsilon}} \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon}$, and $\mathrm{supp}(\varsigma_m) \subset \{(\alpha, v) : |v| \le \frac14\}$. We then have

$$\begin{aligned}&\big|\Lambda[\varsigma]\big(b_1, \dots, b_n, \overline{Q}_k[u]b_{n+1}, b_{n+2}\big)\big| \\ &\le \sum_{l\ge0}\sum_{m\ge0} 2^{-l/2}2^{-mc_1\varepsilon}\big|\Lambda[\varsigma_m^{(2^{-m})}]\big(b_1, \dots, b_n, \overline{Q}_k[u_l^{(2^{-l})}]b_{n+1}, b_{n+2}\big)\big| \\ &= \sum_{l\ge0}\sum_{m\ge0} 2^{-l/2}2^{-mc_1\varepsilon}\big|\Lambda[\varsigma_m]\big(g_1, \dots, g_n, \overline{Q}_{k-l+m}[u_l]g_{n+1}, g_{n+2}\big)\big|\end{aligned}$$

where $g_l = b_l(2^m\cdot)$, $l = 1, \dots, n$, $g_{n+1} = 2^{md/2}b_{n+1}(2^m\cdot)$, $g_{n+2} = 2^{md/2}b_{n+2}(2^m\cdot)$ (see Lemma 4.16). By Theorem 7.1 we have, for some $c_2 > 0$

$$\begin{aligned}&\big|\Lambda[\varsigma_m]\big(g_1, \dots, g_n, \overline{Q}_{k-l+m}[u_l](g_{n+1}), g_{n+2}\big)\big| \\ &\qquad\lesssim \min\{1, n2^{-(k-l+m)c_2\varepsilon}\}\|u\|_{\mathfrak{U}}\Big(\prod_{i=1}^{n}\|g_i\|_\infty\Big)\|g_{n+1}\|_2\|g_{n+2}\|_2\,.\end{aligned}$$

Now $\sum_{l\ge0}\sum_{m\ge0} 2^{-l/2}2^{-mc_1\varepsilon}\min\{1, n2^{-(k-l+m)c_2\varepsilon}\} \lesssim n2^{-kc_3\varepsilon}$ for some c_3 with $0 < c_3 < \min\{1/2, c_2\}$ and (7.11) for $j = 0$ follows easily. $\square$

8. Some results from Calderón-Zygmund theory

In this section, we present some essentially well known results from the Calderón-Zygmund theory which do not seem to be stated in the literature in the precise form we need them. We begin by recalling some classical results (see [36]).

Consider kernels $K \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ such that K is locally integrable on $(\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta$; here $\Delta = \operatorname{diag}(\mathbb{R}^d \times \mathbb{R}^d) = \{(x,x) : x \in \mathbb{R}^d\}$. Let $T_K : C_0^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ be the operator with Schwartz kernel K. Then the expression

$$\langle T_K f, g\rangle = \iint K(x,y) f(y) g(x)\, dy\, dx$$

makes sense for bounded functions f, g with compact and disjoint supports. For such kernels K we define the singular integral semi-norms

$$\mathrm{SI}^1[K] := \sup_{y,y'} \int_{|x-y|\ge 2|y-y'|} |K(x,y) - K(x,y')|\, dx, \tag{8.1}$$

$$\mathrm{SI}^\infty[K] := \sup_{x,x'} \int_{|y-x|\ge 2|x-x'|} |K(x,y) - K(x',y)|\, dy. \tag{8.2}$$

Let $1 < q < \infty$. It is a standard and classical theorem (see [36]) that if T_K extends as a bounded operator on $L^q(\mathbb{R}^d)$ and $\mathrm{SI}^1[K] < \infty$ then T_K extends as an operator of weak type $(1,1)$, as an operator mapping the Hardy space $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ and as a bounded operator on L^p, $1 \le p < 2$, and one has the following estimates for the operator norms (or quasi-norms).

$$\|T_K\|_{H^1\to L^1} + \|T_K\|_{L^1\to L^{1,\infty}} \lesssim \|T_K\|_{L^q\to L^q} + \mathrm{SI}^1[K]. \tag{8.3}$$

We note that in order to prove the $H^1 \to L^1$ result, it suffices to check $\|T_K a\|_1 \le \|T_K\|_{L^q\to L^q} + \mathrm{SI}^1[K]$ for q-atoms, see [29]. Let L^∞_0 be the subspace of L^∞ consisting of functions with compact support (in the sense of distributions). Then we also have for $q \ge 1$

$$\|T_K\|_{L_0^\infty\to BMO} \lesssim \|T_K\|_{L^q\to L^q} + \mathrm{SI}^\infty[K]. \tag{8.4}$$

Furthermore (taking $q = 2$), by interpolation

$$\|T_K\|_{L^p\to L^p} \le C_{p,d}(\|T_K\|_{L^2\to L^2} + \|T_K\|_{L^2\to L^2}^{2-\frac{2}{p}} (\mathrm{SI}^1[K])^{\frac{2}{p}-1}), \quad 1 < p < 2, \tag{8.5}$$

and

$$\|T_K\|_{L^p\to L^p} \le C_{p,d}(\|T_K\|_{L^2\to L^2} + \|T_K\|_{L^2\to L^2}^{\frac{2}{p}} (\mathrm{SI}^\infty[K])^{1-\frac{2}{p}}), \quad 2 < p < \infty. \tag{8.6}$$

We will apply these results to singular integral kernels given by

$$K = \sum_j \mathrm{Dil}_{2^j} \tau_j \equiv \sum_j 2^{jd} \tau_j(2^j\cdot, 2^j\cdot) \tag{8.7}$$

where τ_j satisfy suitable uniform Schur and regularity conditions.

8.1. **Classes of kernels.**

8.1.1. *Schur Norms and Regularity Conditions.* In what follows we consider complex-valued locally integrable functions $(x,y) \mapsto k(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$.

We formulate conditions related to the usual Schur test, involving integrability conditions in the x and y variables. We let Int^1 be the class of kernels $k \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ for which

$$\mathrm{Int}^1[k] = \sup_{y\in\mathbb{R}^d} \int |k(x,y)|\, dx \tag{8.8}$$

is finite. Here and in what follows $\sup_y$ is used synonymously with essential supremum (or L^∞-norm). We let Int^∞ be the class of kernels $k \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ for which

$$\mathrm{Int}^\infty[k] = \sup_{x\in\mathbb{R}^d} \int |k(x,y)|\, dy \tag{8.9}$$

is finite. Here the supremum is interpreted as essential supremum (i.e. the L^∞ norm with respect to y). The notation is motivated by the fact that for $k \in \mathrm{Int}^1$ the integral operator with kernel k is bounded on $L^\infty(\mathbb{R}^d)$, with operator norm $\mathrm{Int}^1[k]$, and for $k \in \mathrm{Int}^\infty$ this operator is bounded on $L^\infty(\mathbb{R}^d)$, with operator norm $\mathrm{Int}^\infty[k]$.

Next we need stronger conditions, which add some weights in terms of the distance of (x,y) to the diagonal Δ. Define

$$\mathrm{Int}^1_\varepsilon[k] := \sup_{y\in\mathbb{R}^d} \int (1+|x-y|)^\varepsilon |k(x,y)|\, dx, \tag{8.10}$$

$$\mathrm{Int}^\infty_\varepsilon[k] := \sup_{x\in\mathbb{R}^d} \int (1+|x-y|)^\varepsilon |k(x,y)|\, dy. \tag{8.11}$$

Let

$$k^{\mathrm{dual}}(x,y) = k(y,x)$$

and note that $\mathrm{Int}^\infty_\varepsilon[k] = \mathrm{Int}^1_\varepsilon[k^{\mathrm{dual}}]$.

In Calderón-Zygmund theory we also need some variants involving regularity, in either the left (x-) or right (y-)variable. We define

$$\mathrm{Reg}^1_{\varepsilon,\mathrm{lt}}[k] := \sup_{0<|h|\le 1} \sup_y |h|^{-\epsilon} \int |k(x+h,y) - k(x,y)|\, dx, \tag{8.12}$$

$$\mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}[k] := \sup_{0<|h|\le 1} \sup_y |h|^{-\epsilon} \int |k(x,y+h) - k(x,y)|\, dx, \tag{8.13}$$

and

$$\mathrm{Reg}^\infty_{\varepsilon,\mathrm{lt}}[k] := \sup_{0<|h|\le 1} \sup_x |h|^{-\epsilon} \int |k(x+h,y) - k(x,y)|\, dy, \tag{8.14}$$

$$\mathrm{Reg}^\infty_{\varepsilon,\mathrm{rt}}[k] := \sup_{0<|h|\le 1} \sup_x |h|^{-\epsilon} \int |k(x,y+h) - k(x,y)|\, dy, \tag{8.15}$$

so that $\mathrm{Reg}^\infty_{\varepsilon,\mathrm{lt}}[k] = \mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}[k^{\mathrm{dual}}]$ and $\mathrm{Reg}^\infty_{\varepsilon,\mathrm{lt}}[k] = \mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}[k^{\mathrm{dual}}]$.

8.1.2. *Singular Integral Kernels.* We now consider distributions $K \in \mathcal{D}'((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$ which are locally integrable in $(\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta$. We define variants of (8.1), (8.2) with more decay away from the diagonal (here $\varepsilon \ge 0$)

$$\mathrm{SI}^1_\varepsilon[K] := \sup_{y,y'} \sup_{R\ge 2} R^\varepsilon \int_{|x-y|\ge R|y-y'|} |K(x,y) - K(x,y')|\, dx\,, \tag{8.16}$$

$$\mathrm{SI}^\infty_\varepsilon[K] := \sup_{x,x'} \sup_{R\ge 2} R^\varepsilon \int_{|y-x|\ge R|x-x'|} |K(x,y) - K(x',y)|\, dy\,. \tag{8.17}$$

Note that for $\varepsilon = 0$ we recover the norms defined in (8.1), (8.2).

Remark. We shall also use the alternative notation $\|K\|_{\mathrm{SI}^1_\varepsilon} = \mathrm{SI}^1_\varepsilon[K]$ etc. We will say $K \in \mathrm{SI}^1_\varepsilon$ if $\mathrm{SI}^1_\varepsilon[K] < \infty$ etc.

We say that $K \in L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$ satisfies one of the *uniform annular integrability conditions* Ann^1, Ann^∞ if the respective expressions

$$\mathrm{Ann}^1[K] := \sup_{R>0} \sup_y \int_{x: R \le |x-y| \le 2R} |K(x,y)|\, dx, \tag{8.18}$$

$$\mathrm{Ann}^\infty[K] := \sup_{R>0} \sup_x \int_{y: R \le |x-y| \le 2R} |K(x,y)|\, dy \tag{8.19}$$

are finite.

We say that K satisfies the *averaged annular integrability condition* $\mathrm{Ann}_{\mathrm{av}}$ if

$$\mathrm{Ann}_{\mathrm{av}}[K] = \sup_{a \in \mathbb{R}^d} \sup_{R>0} R^{-d} \iint\limits_{\substack{|x-a| \le R \\ R \le |x-y| \le 2R}} |K(x,y)|\, dy\, dx \tag{8.20}$$

is finite.

The last notion will be used in §8.2 below.

Lemma 8.1. *Let $K \in L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta\})$. Then*

$$\mathrm{Ann}_{\mathrm{av}}[K] \approx \mathrm{Ann}_{\mathrm{av}}[K^{\mathrm{dual}}].$$

Moreover,

$$\mathrm{Ann}_{\mathrm{av}}[K] \lesssim \min\{\mathrm{Ann}^1[K], \mathrm{Ann}^\infty[K]\}\,.$$

Proof. Immediate from the definitions. □

Lemma 8.2. *Let $K \in L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$. Suppose that for some $\varepsilon > 0$,*

$$\mathrm{SI}^1_\varepsilon[K] \le B, \quad \mathrm{Ann}[K] \le A.$$

Then

$$\mathrm{SI}^1_0[K] \lesssim A \log(2 + \varepsilon^{-1} B/A).$$

Proof. Fix $y \ne y'$ and split

$$\int_{|x-y| \ge 2|y-y'|} |K(x,y) - K(x,y')|\, dx = I + II$$

where

$$I = \int_{2|y-y'| \le |x-y| \le R|y-y'|} |K(x,y) - K(x,y')|\, dx\,,$$

$$II = \int_{|x-y| \ge R|y-y'|} |K(x,y) - K(x,y')|\, dx\,.$$

Then if we apply condition Ann_1 with $O(\log R)$ annuli to estimate

$$I \lesssim A \log R;$$

moreover we have

$$II \lesssim BR^{-\varepsilon}.$$

If we choose $R = 2 + (B/A)^{1/\varepsilon}$ the assertion follows. □

8.1.3. *Integral conditions for singular integrals.* We formulate a proposition which is used to verify the condition $\mathrm{SI}^1_\varepsilon$, $\mathrm{SI}^\infty_\varepsilon$ for kernels of the form (8.7).

Proposition 8.3. *Suppose that* $\tau_j \in \mathrm{Int}^1_\varepsilon \cap \mathrm{Reg}^1_{\varepsilon,R}$ *and*

$$\sup_j \mathrm{Int}^1_0[\tau_j] \le A,$$

$$\sup_j \mathrm{Int}^1_\varepsilon[\tau_j] + \sup_j \mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}[\tau_j] \le B.$$

Then the sum (8.7) *converges in the sense of* $L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$ *and the limit* K *satisfies*

$$\mathrm{SI}^1_{\varepsilon/2}[K] \lesssim B. \tag{8.21}$$

Moreover,

$$\mathrm{SI}^1_0[K] \lesssim A \log(2 + B/A)\,. \tag{8.22}$$

Proof. We fix y, y' and $R \ge 0$ and consider

$$\begin{aligned} I^R_j(y,y') &= \int_{x:|x-y|\ge R|y-y'|} |\mathrm{Dil}_{2^j}\tau_j(x,y) - \mathrm{Dil}_{2^j}\tau_j(x,y')|\,dx \\ &= \int_{x:|x-2^jy|\ge R|2^jy-2^jy'|} |\tau_j(x,2^jy) - \tau_j(x,2^jy')|\,dx. \end{aligned}$$

Clearly $I^R_j(y,y') \le 2A$. We now give two estimates, the first valid when $2^j|y-y'| \ge 1/R$, the second valid when $2^j|y-y'| \le 1$; thus both estimates will be valid when $1/R \le 2^j|y-y'| \le 1$.

For $2^j|y-y'| \ge 1/R$ we have

$$\begin{aligned} &\int_{x:|x-2^jy|\ge R|2^jy-2^jy'|} |\tau_j(x,2^jy)|dx \le \int |\tau_j(x,2^jy)| \frac{(1+|x-2^jy|)^\varepsilon}{(R2^j|y-y'|)^\varepsilon} dx \\ &\le (2^j|y-y'|R)^{-\varepsilon} \mathrm{Int}^1_\varepsilon[\tau_j] \le B(2^j|y-y'|R)^{-\varepsilon}. \end{aligned}$$

Also note that if $|x-2^jy| \ge R|2^jy-2^jy'|$ then also $|x-2^jy'| \ge (R-1)|2^jy-2^jy'|$. Thus the last argument also gives (for $R \ge 2$)

$$\int_{x:|x-2^jy|\ge R|2^jy-2^jy'|} |\tau_j(x,2^jy')|dx \le B(2^j|y-y'|(R-1))^{-\varepsilon}$$

and hence

$$I^R_j(y,y') \lesssim B(2^j|y-y'|)^{-\varepsilon}R^{-\varepsilon} \text{ if } 2^j|y-y'| \ge 1/R\,.$$

For $2^j|y-y'| \le 1$ we obtain

$$I^R_j(y,y') \le \int |\tau_j(x,2^jy) - \tau_j(x,2^jy')|\,dx \le \mathrm{Reg}^1_\varepsilon[\tau_j](2^j|y-y'|)^\varepsilon \le B(2^j|y-y'|)^\varepsilon.$$

Hence

$$\sum_{j\in\mathbb{Z}} I^R_j(y,y') \lesssim \sum_{j:2^j|y-y'|\le R^{-1/2}} B(2^j|y-y'|)^\varepsilon + \sum_{j:2^j|y-y'|>R^{-1/2}} B(R2^j|y-y'|)^{-\varepsilon} \lesssim BR^{-\varepsilon/2}$$

and (8.21) follows. The same argument gives

$$\sum_{j\in\mathbb{Z}} I^R_j(y,y') \lesssim \sum_{j\in\mathbb{Z}} \min\{A, B(2^j|y-y'|)^\varepsilon, B(2^j|y-y'|)^\varepsilon \lesssim A(\log(2+B/A))$$

which yields (8.22).

$\square$

The following proposition is useful for verifying membership in the classes Ann^1, Ann^∞ for kernels of the form (8.7).

Proposition 8.4. *Suppose that $\tau_j \in \mathrm{Int}^1_\varepsilon \cap \mathrm{Reg}^1_{\varepsilon,\mathrm{lt}}$ such that*

$$\sup_j \mathrm{Int}^1_0[\tau_j] \le A\,,$$
$$\sup_j \mathrm{Int}^1_\varepsilon[\tau_j] + \sup_j \mathrm{Reg}^1_{\varepsilon,\mathrm{lt}}[\tau_j] \le B\,.$$

Then the sum $K = \sum_j \mathrm{Dil}_{2^j}\tau_j$ converges in the sense of $L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta)$ and

$$\mathrm{Ann}^1[K] \lesssim A\log(2 + B/A)\,.$$

This follows from the following lemma regarding functions in $L^1(\mathbb{R}^d)$.

Lemma 8.5. *Let $0 < \varepsilon < 1$, $g_j \in L^1(\mathbb{R}^d)$ such that*

$$\int |g_j(x)|\,dx \le A,$$

$$\int |g_j(x)|(1+|x|)^\varepsilon\,dx \le B_1,$$

and

$$\sup_{|h|<1} |h|^{-\varepsilon} \int |g_j(x+h) - g_j(x)|\,dx \le B_2\,.$$

Then for every compact set $K \subset \mathbb{R}^d \setminus \{0\}$, the series $G(x) = \sum_{j\in\mathbb{Z}} 2^{jd} g_j(2^j x)$ converges in $L^1(K)$, so that $G \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$. Moreover, if $K_R = \{x : R \le |x| \le 2R\}$,

$$\sup_{R>0} \int_{K_R} |G(x)|dx \lesssim A\log(1 + \frac{B_1 + B_2}{A})\,.$$

Proof. It suffices to consider the case $K = K_R$. Let $G_j = 2^{jd} g_j(2^j\cdot)$ then

$$\|G_j\|_{L^1(K_R)} = \|g_j\|_{L_1(K_{2^jR})} \le A.$$

First assume that $2^jR \ge 1$. In this case

$$\|g_j\|_{L_1(K_{2^jR})} \lesssim (2^jR)^{-\varepsilon} B_1.$$

For $2^jR \le 1$ we have by Hölder's inequality

$$\|g_j\|_{L_1(K_{2^jR})} \lesssim (2^jR)^{d/p'}\|g_j\|_p,$$

and by Sobolev imbedding it follows $\|g_j\|_p \lesssim B_1$ provided that $d/p' < \varepsilon$. Hence we obtain for $0 < \varepsilon' < \varepsilon$ we get

$$\|G\|_{L^1(K_R)} \lesssim \sum_{j\in\mathbb{Z}} \min\{A, B_1(2^jR)^{-\varepsilon}, B_2(2^jR)^{\varepsilon'}\} \lesssim A\log\big(1 + \frac{B_1+B_2}{A}\big)\,. \qquad \square$$

Proof of Proposition 8.4. Apply Lemma 8.5 to the functions $v \mapsto K(y+v, y)$. $\square$

8.1.4. *Kernels with cancellation.* We state a standard estimates involving the Schur test for compositions with operators exhibiting some cancellation; this will be used when proving L^2 estimates in §11.

Lemma 8.6. *Fix $0<\varepsilon\le 1$. Let $\ell\in\mathbb{Z}$ with $\ell\le 0$. Suppose ρ, $\sigma_\ell:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ are measurable functions satisfying*

$$\operatorname{Int}^1[\rho]\le A_1,\quad \operatorname{Int}^\infty_\varepsilon[\rho]\le A_{\varepsilon,\infty}, \tag{8.23a}$$

$$\operatorname{Int}^1[\sigma_\ell]\le B_1,\quad \operatorname{Int}^\infty[\sigma_\ell]\le B_\infty, \tag{8.23b}$$

and

$$\operatorname{Int}^\infty[\nabla_x\sigma_\ell]\le 2^{-\ell}\widetilde{B}_\infty. \tag{8.23c}$$

Assume

$$\int\rho(x,y)\,dy=0 \text{ for almost every } x\in\mathbb{R}^d. \tag{8.24}$$

Let R, S_ℓ be the integral operators with Schwartz kernels $\rho(x,y)$, $\sigma_\ell(x,y)$. Then

$$\|RS_\ell\|_{L^2\to L^2}\lesssim 2^{-\ell\varepsilon/2}\sqrt{A_1A_{\varepsilon,\infty}B_1(B_\infty+\widetilde{B}_\infty)}.$$

Proof. Let k_ℓ be the Schwartz kernel of RS_ℓ. Then, by the cancellation assumption,

$$k_\ell(x,y)=\int\rho(x,z)\big(\sigma_\ell(z,y)-\sigma_\ell(x,y)\big)\,dz$$

Clearly for a.e. $y\in\mathbb{R}^d$

$$\int|k_\ell(x,y)|dx\le\int|\sigma_\ell(z,y)|\int|\rho(x,z)|dx\,dz\lesssim B_1A_1.$$

Moreover,

$$\int|k_\ell(x,y)|dy\le(I_x)+(II_x)$$

where

$$(I_x):=\int_{|x-z|\le 2^\ell}|\rho(x,z)|\int\big|\sigma_\ell(z,y)-\sigma_\ell(x,y)\big|\,dy\,dz,$$

$$(II_x):=\int_{|x-z|\ge 2^\ell}|\rho(x,z)|\int\big(|\sigma_\ell(z,y)|+|\sigma_\ell(x,y)|\big)\,dy\,dz.$$

Now by assumption, for fixed x,z

$$\int|\sigma_\ell(z,y)|\,dy\;+\int|\sigma_\ell(x,y)|dy\lesssim B_\infty$$

and

$$\begin{aligned}\int\big|\sigma_\ell(z,y)-\sigma_\ell(x,y)\big|\,dy&=\int\Big|\int_0^1\langle z-x,\nabla_x\sigma_\ell((1-s)x+sz),y\rangle\,ds\Big|\,dy\\&\le|x-z|\int_0^1\int|\nabla_x\sigma_\ell((1-s)x+sz),y)|\,dy\,d\tau\lesssim\widetilde{B}_\infty 2^{-\ell}|x-z|.\end{aligned}$$

For (I_x) we then get

$$(I_x)\le\widetilde{B}_\infty\int_{|z-x|\le 2^\ell}|\rho(x,z)|2^{-\ell}|x-z|\,dz$$

and estimate (using $\varepsilon \le 1$)

$$\int_{|z-x|\le 2^\ell} |\rho(x,z)| 2^{-\ell}|x-z|\,dz \le \int_{|z-x|\le 2^\ell} |\rho(x,z)|[2^{-\ell}|x-z|]^\varepsilon\,dz$$
$$\lesssim 2^{-\ell\varepsilon}\int |\rho(x,z)(1+|x-z|)^\varepsilon dz \lesssim 2^{-\ell\varepsilon} A_{\varepsilon,\infty}.$$

Hence $(I_x) \lesssim 2^{-\ell\varepsilon}\widetilde{B}_\infty A_{\varepsilon,\infty}$. For (II_x) we have

$$(II_x) \le B_\infty \int_{|z-x|\ge 2^\ell} |\rho(x,z)|\,dz \lesssim B_\infty 2^{-\ell\varepsilon}\int_{|z-x|\ge 2^\ell} |\rho(x,z)|(1+|x-z|)^\varepsilon\,dz$$

and thus $(II_x) \lesssim 2^{-\ell\varepsilon}B_\infty A_{\varepsilon,\infty}$. Finally, we obtain by Schur's test

$$\|RS_\ell\|_{L^2\to L^2} \le \sqrt{\mathrm{Int}_1[k_\ell]}\sqrt{\mathrm{Int}_\infty[k_\ell]} \lesssim \sqrt{A_1B_1}\sqrt{(B_\infty+\widetilde{B}_\infty)A_{\varepsilon,\infty}2^{-\ell\varepsilon}}.$$

The assertion is proved. □

8.1.5. *On operator topologies.* We finish this section by stating a version of the uniform boundedness principle which is used for the partial sums of operators defined by kernels of the form (8.7).

Lemma 8.7. *Let X, Y be Banach spaces and let $\Sigma_N : X \to Y$ be bounded operators. Assume that Σ_N converges in the weak operator topology, i.e. there is a linear operator $\Sigma : X \to Y$ so that for every $f \in X$ and every linear functional $g \in Y'$,*

$$\lim_{N\to\infty}\langle \Sigma_N f, g\rangle = \langle \Sigma f, g\rangle.$$

Then $\Sigma : X \to Y$ is bounded, and there exists $B < \infty$ so that

$$\|\Sigma\|_{X\to Y} \le \sup_N \|\Sigma_N\|_{X\to Y} \le B.$$

Proof. We have $\sup_N \|\langle \Sigma_N f, g\rangle| \le C_{f,g} < \infty$ for every $f, \in X$, $g \in Y'$. By the uniform boundedness principle this implies $\sup_N \|\Sigma_N f\|_Y \le C_f < \infty$ for all $f \in X$. By the uniform boundedness principle again there is $A < \infty$ so that $A := \sup_N \|\Sigma_N\|_{X\to Y} < \infty$. Thus $C_{f,g} \le A\|f\|_X\|g\|_{Y'}$. Passing to the limit we see $|\langle \Sigma f, g\rangle| \le A\|f\|_X\|g\|_{Y'}$ which implies $\|\Sigma\|_{X\to Y} \le A$. □

Given a formal series $\sum_{j\in\mathbb{Z}} T_j$ of bounded operators we say that $\sum_{j\in\mathbb{Z}} T_j$ converges in the weak operator topology as operators $X \to Y$ if the partial sums $\Sigma_N = \sum_{j=-N}^N T_j$ satisfy the assumptions in Lemma 8.7.

Lemma 8.8. *Let X, Y be Banach spaces, let W be a linear subspace of X which is dense in X. Let $\Sigma_N : X \to Y$ be bounded operators. Assume that*

$$\sup_N \|\Sigma_N\|_{X\to Y} \le A$$

and that for every $f \in W$, and every $g \in Y'$

$$\lim_{N\to\infty}\langle \Sigma_N f, g\rangle = \langle \Sigma f, g\rangle$$

where $\Sigma : W \to Y$ is a linear operator. Then Σ_N converges to Σ in the weak operator topology (as operators $X \to Y$) and we have $\|\Sigma\|_{X\to Y} \le A$.

Proof. The assumptions imply that $\|\Sigma f\|_Y \le \|f\|_X$ for all $f \in W$, and Σ extends uniquely to a bounded operator $X \to Y$ with operator norm at most A. Moreover, using $\|\Sigma_N\|_{X\to Y} \le A$ it follows easily that $\Sigma_N \to \Sigma$ in the weak operator topology. □

8.1.6. *Consequences for sums of dilated kernels.* We now formulate some consequences of the propositions above and the boundedness result (8.5).

Proposition 8.9. *Let $\tau_j \in \mathrm{Int}^1_\varepsilon \cap \mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}$, so that*

$$\mathrm{Int}^1_0[\tau_j] \lesssim A, \quad \mathrm{Int}^1_\varepsilon[\tau_j] + \mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}[\tau_j] \leq B.$$

Let T_j denote the integral operator with kernel $\mathrm{Dil}_{2^j}\tau_j$.

(i) Suppose that $T = \sum_{j\in\mathbb{Z}} T_j$ converges in the weak operator topology as operators $L^2 \to L^2$. Then, for $1 < p \leq 2$, T extends to an operator bounded on L^p such that

$$\|T\|_{L^p\to L^p} \leq C_{d,p,\epsilon}\big(\|T\|_{L^2\to L^2} + \|T\|^{2-\frac{2}{p}}_{L^2\to L^2}\big(A\log(2+B/A)\big)^{\frac{2}{p}-1}\big).$$

Moreover T extends to an operator bounded from H^1 to L^1 and

$$\|T\|_{H^1\to L^1} \leq C_{d,\epsilon}\big(\|T\|_{L^2\to L^2} + A\log(2+B/A)\big).$$

(ii) Suppose that $T = \sum_{j\in\mathbb{Z}} T_j$ converges in the strong operator topology, as operators $L^2 \to L^2$. Then the sum also converges in the strong operator topology as operators $L^p \to L^p$, $1 < p < 2$ and in the strong operator topology as operators $H^1 \to L^1$.

Proof. By Proposition 8.3 we have for K as in (8.7) $\mathrm{SI}^1_0[K] \leq \log(2+B/A)$ and the assertion (i) follows from (8.5) and (8.3).

For (ii) we examine the proof of $H^1 \to L^1$ boundedness. Let a be a 2-atom supported in a cube Q with center y_Q, i.e. we have $\|a\|_2 \leq |Q|^{-1/2}$, $\int a(x)dx = 0$. Let Q^* be the double cube with the same center. By assumption $\sum_{j=-N}^N T_j a$ converges in $L^2(Q^*)$ and by Hölder's inequality in $L^1(Q^*)$. Also, by the argument in the proof of Proposition 8.3,

$$\begin{aligned}\|T_j a\|_{L^1(\mathbb{R}^d\setminus Q^*)} &\lesssim \int |a(y)| \int_{\mathbb{R}^d\setminus Q^*} |\mathrm{Dil}_{2^j}\tau_j(x,y) - \mathrm{Dil}_{2^j}\tau_j(x,y_Q)|\,dx\,dy \\ &\lesssim B\min\{(2^j\mathrm{diam}(Q))^\varepsilon, (2^j\mathrm{diam}(Q))^{-\varepsilon}\}\end{aligned}$$

and clearly $\sum_{j=-N}^N T_j a$ converges in $L^1(\mathbb{R}^d\setminus Q^*)$ as well.

Let $f \in H^1$; we need to establish convergence of $\sum_j T_j f$ in L^1. By the atomic decomposition $f = \sum_{\nu=1}^\infty c_\nu a_\nu$ where a_ν are 2-atoms and $\sum_\nu |c_\nu| \lesssim \|f\|_{H^1}$. Given $\varepsilon > 0$ take M so that $\sum_{\nu=M}^\infty |c_\nu| \leq \varepsilon$. Then there is C independent of M, ε so that for all N we have

$$\Big\|\sum_{j=-N}^N T\Big(\sum_{\nu=M}^\infty c_\nu a_\nu\Big)\Big\|_1 < C\varepsilon.$$

It is now straightforward to combine the arguments and deduce the convergence of $\sum_j T_j f$ in L^1.

In order to prove convergence in the strong operator topology as operators $L^p \to L^p$, $1 < p < 2$, we apply the interpolation inequality $\|h\|_p \leq \|h\|_1^{\frac{2}{p}-1}\|h\|_2^{2-\frac{2}{p}}$ to $h = \sum_{j\in\mathcal{J}} T_j g$ where $g \in H^1 \cap L^2$. This yields that $\sum_j T_j g$ converges in L^p. Since $H^1 \cap L^2$ is dense and since the operator norms $\sum_{j\in\mathcal{J}} T_j$ are bounded uniformly in $\mathcal{J}$, it is now straightforward to show convergence of $\sum_j T_j f$ for every $f \in L^p$. □

In our applications we work with the following setting. Let $\phi \in C_0^\infty(B^d(1))$ have $\int\phi = 1$ and define $P_j f = f * \phi^{(2^j)}$. Set $\psi(x) = \phi(x) - 2^{-d}\phi(2^{-1}x)$, and set $Q_j f = f * \psi^{(2^j)}$. We have $I = \sum_{j\in\mathbb{Z}} Q_j$, $P_j = \sum_{k\leq j} Q_k$ and $I - P_j = \sum_{k>j} Q_k$ in the sense of distributions.

Corollary 8.10. *Let $s_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a sequence of locally integrable kernels and assume that*

$$\sup_j \mathrm{Int}^1_0[s_j] \le A, \quad \sup_j \mathrm{Int}^1_\varepsilon[s_j] \le B\,.$$

Let S_j be the integral operator with integral kernel $\mathrm{Dil}_{2^j} s_j$. Suppose the sum $S = \sum_{j\in\mathbb{Z}} S_j P_j$ converges in the weak operator topology as operators $L^2 \to L^2$. Then, for $1 < p \le 2$, $S : L^p \to L^p$ is bounded and

$$\|S\|_{L^p\to L^p} \le C_{d,p,\epsilon}\big(\|S\|_{L^2\to L^2} + \|S\|^{2-\frac{2}{p}}_{L^2\to L^2}(A\log(2+B/A))^{\frac{2}{p}-1}\big).$$

Proof. The kernel of S_jP_j is equal to $\mathrm{Dil}_{2^j}\tau_j$ where

$$\tau_j(x,y) = \int s_j(x,z)\phi(z-y)\,dz\,.$$

Clearly $\mathrm{Int}^1_\varepsilon[\tau_j] \lesssim \mathrm{Int}^1_\varepsilon[s_j]$ for $\varepsilon \ge 0$ and in view of the regularity and support of ϕ we also have

$$\mathrm{Reg}^1_{\delta,\mathrm{rt}}[\tau_j] \lesssim \mathrm{Int}^1_0[s_j]$$

for $\delta \le 1$. The assertion now follows from Corollary 8.9. □

Corollary 8.11. *Let s_j, S_j be as in Corollary 8.10 For $k \in \mathbb{N}$ define $S^k := \sum_{j\in\mathbb{Z}} S_jQ_{j+k}$. Suppose that this sum converges in the weak operator topology as operators $L^2 \to L^2$, and suppose that for some $\varepsilon' > 0$*

$$D_{\varepsilon'} := \sup_{k>0} 2^{k\epsilon'}\|S^k\|_{L^2\to L^2} < \infty.$$

Also define $D_0 := \sup_{k>0}\|S^k\|_{L^2\to L^2}$. Then, for $1 < p \le 2$,

$$\|S^k\|_{L^p\to L^p} \le C_{p,d,\epsilon}\Big(\min\{2^{-k\epsilon'}D_{\epsilon'}, D_0\} + \big(\min\{2^{-k\epsilon'}D_{\epsilon'}, D_0\}\big)^{2-\frac{2}{p}}\big(A\log(2^k+B/A)\big)^{\frac{2}{p}-1}\Big).$$

Proof. By definition $\|S^k\|_{L^2\to L^2} \le \min\{2^{-k\epsilon'}D_{\epsilon'}, D_0\}$. The integral kernel of S_jQ_{j+k} is given by $\mathrm{Dil}_{2^j}\tau_{j,k}$ where

$$\tau_{j,k}(x,y) = \int s_j(x,z)2^{kd}\psi(2^k(z-y))\,dz\,.$$

We have $\mathrm{Int}^1_\varepsilon[\tau_{j,k}] \lesssim \mathrm{Int}^1_\varepsilon[s_j]$ for $\varepsilon \ge 0$ and now

$$\mathrm{Reg}^1_{\delta,\mathrm{rt}}[\tau_{j,k}] \lesssim 2^k\mathrm{Int}^1_0[s_j] \lesssim 2^kA$$

for $\delta \le 1$. The assertion follows from Corollary 8.9. □

Corollary 8.12. *Let s_j, S_j, S^k be as in Corollary 8.11. Define $\widetilde{S} := \sum_{j\in\mathbb{Z}} S_j(I-P_j) = \sum_{k>0} S^k$. For $1 < p \le 2$,*

$$\|\widetilde{S}\|_{L^p\to L^p} \le C_{p,d,\epsilon,\epsilon'}\Big(D_0\log\big(2+\frac{D_{\epsilon'}}{D_0}\big) + D_0^{2-\frac{2}{p}}A^{\frac{2}{p}-1}\log\big(2+\frac{D_{\epsilon'}}{D_0}\big)\log^{\frac{2}{p}-1}\big(2+\frac{D_{\epsilon'}}{D_0}+\frac{B}{A}\big)\Big).$$

Proof. By Corollary 8.11, we have

$$\|\widetilde{S}\|_{L^p\to L^p} \lesssim \sum_{k>0}\min\{2^{-k\epsilon'}D_{\epsilon'}, D_0\} + \sum_{k>0}\big(\min\{2^{-k\epsilon'}D_{\epsilon'}, D_0\}\big)^{2-\frac{2}{p}}\big(A\log(2^k+B/A)\big)^{\frac{2}{p}-1}.$$

Clearly, $\sum_{k>0}\min\{2^{-k\epsilon'}D_{\epsilon'}, D_0\} \lesssim D_0\log(2+D_{\varepsilon'}/D_0)$. Also, the second sum equals

$$D_0^{2-\frac{2}{p}}A^{\frac{2}{p}-1}\sum_{k>0}\min\big\{2^{-k\epsilon'}\frac{D_{\epsilon'}}{D_0}, 1\big\}^{2-\frac{2}{p}}\big(\log(2^k+\frac{B}{A})\big)^{\frac{2}{p}-1}.$$

To conclude apply the following Lemma 8.13 with $\beta = -1+2/p$. □

Lemma 8.13. *Fix $\epsilon > 0$, $\alpha > 0$, $\beta \geq 0$. Let $U, V \geq 1$, then*

$$\sum_{k\geq 0} (\min\{2^{-k\epsilon}U, 1\})^{\alpha} \log^{\beta}(2^k + V) \leq C_{\epsilon,\alpha,\beta} \log(1+U) \log^{\beta}(1+U+V).$$

Proof. Let $J_k(U,V) = (\min\{2^{-k\epsilon}U, 1\})^{\alpha} \log^{\beta}(2^k + V)$.

We first consider the terms with $2^{-k\varepsilon/2}U \leq 1$. Observe

$$\sum_{\substack{2^{-k\varepsilon/2}U\leq 1\\ 2^k \leq V}} J_k(U,V) \lesssim \log^{\beta}(1+V) \sum_{2^{k\varepsilon/2}\leq U} (U2^{-k\varepsilon})^{\alpha} \lesssim \log^{\beta}(1+V)$$

and

$$\sum_{\substack{2^{-k\varepsilon/2}U\leq 1\\ 2^k > V}} J_k(U,V) \lesssim \sum_{2^{-k\varepsilon/2}U\geq 1} (U2^{-k\varepsilon})^{\alpha} k^{\beta} \lesssim \sum_{k:2^{-k\varepsilon/2}U\leq 1} (U2^{-k\varepsilon/2})^{\alpha} \lesssim 1\,.$$

The main contribution comes from the terms with $2^{-k\varepsilon/2}U \geq 1$; here we use

$$\sum_{\substack{2^{-k\varepsilon/2}U\geq 1\\ 2^k \leq V}} J_k(U,V) \lesssim \log^{\beta}(1+V) \sum_{2^{k\varepsilon/2}\leq U} 1 \lesssim \log(1+U) \log^{\beta}(1+V)$$

and

$$\sum_{\substack{k:2^{-k\varepsilon/2}U\geq 1\\ 2^k \geq V}} J_k(U,V) \lesssim \sum_{k:2^{k\varepsilon/2}\leq U} k^{\beta} \lesssim \log^{\beta+1}(1+U)\,.$$

Clearly, all four terms are $\lesssim \log(1+U)\log^{\beta}(1+U+V)$ and the asserted bound follows. □

8.2. **On a result of Journé.** For a cube Q let Q^* be the double cube with same center.

Definition 8.14. Let $T : C_0^{\infty}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ be an operator with Schwartz kernel K. We say that T satisfies a Carleson condition if there is a constant C so that for all cubes Q and for all bounded functions f supported in Q, $Tf \in L^1(Q^*)$ and the inequality

$$\int_{Q^*} |Tf(x)|dx \leq C|Q|\|f\|_{\infty}$$

is satisfied. We denote by $\|T\|_{\mathrm{Carl}}$ the best constant in the displayed inequality.

Journé [28] considered a class of operators associated with regular singular integral kernels satisfying, say, $|K(x,y)| \lesssim |x-y|^{-d}$, $|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \lesssim |x-y|^{-d-1}$ and showed that the following conditions are equivalent.

- T satisfies a Carleson condition.
- T maps H^1 to L^1.
- T maps L_0^{∞} to BMO.

He then used an interpolation theorem to show that each condition is equivalent with

- T maps L^2 to L^2.

We now give versions of Journé's theorem for larger classes of kernels which arise in our main result.

Definition 8.15. (i) A integrable function is called an ∞-atom associated to a cube Q if a is supported on Q, and satisfies $\|a\|_\infty \le |Q|^{-1}$ and $\int a(x)dx = 0$.

(ii) A linear operator defined on compactly supported functions with integral zero satisfies the *atomic boundedness condition* if

$$\|T\|_{\mathrm{At}} := \sup \|Ta\|_1 < \infty$$

where the sup is taken over all ∞-atoms.

Remark 8.16. One can also make a definition of a class $\mathrm{At}(q)$ where one works with q-atoms satisfying $\mathrm{supp}(a) \subset Q$, $\|a\|_q \le |Q|^{-1+1/q}$ and $\int a(x)dx = 0$. Define $\|T\|_{\mathrm{At}(q)} = \sup \|Ta\|_1$ where the supremum is taken over all q-atoms. For the case $1 < q < \infty$ one has $T \in \mathrm{At}(q)$ if and only if T extends to a bounded operator $H^1 \to L^1$, and $\|T\|_{\mathrm{At}(q)} \approx \|T\|_{H^1 \to L^1}$. This is a special case of a result by Meda, Sjögren and Vallarino [29]. The equivalence may fail for the case $q = \infty$, as was shown by Bownik [3]. We remark that for special classes of Calderón-Zygmund operators the equivalence holds true even for $q = \infty$ (see [30, §7.2], and the proof of Theorem 8.20 below). For most situations in harmonic analysis the use of ∞-atoms (instead of q-atoms) does not yield a significant advantage, but in our application it will be crucial to work with ∞-atoms.

In the following three propositions $T : C_0^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ will denote a linear operator with Schwartz kernel $K \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d) \cap L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) setminus \Delta)$. The proofs use the arguments of Journé [28, §4.2].

Proposition 8.17. *Suppose that T satisfies the atomic boundedness condition and the averaged annular integrability condition. Then*

$$\|T\|_{\mathrm{Carl}} \lesssim \|T\|_{\mathrm{At}} + \mathrm{Ann}_{\mathrm{av}}[K]\,.$$

Proposition 8.18. *Suppose that* $\mathrm{SI}^\infty[K] < \infty$, $\mathrm{Ann}_{\mathrm{av}}[K] < \infty$ *and that T satisfies a Carleson condition. Then T extends to a bounded operator from L_0^∞ to BMO satisfying*

$$\|T\|_{L_0^\infty \to BMO} \lesssim \|T\|_{\mathrm{Carl}} + \mathrm{SI}^\infty[K].$$

Proposition 8.19. *Suppose that* $\mathrm{SI}^1[K] < \infty$ *and that T extends to a bounded operator T : $L_0^\infty \to BMO$. Then T satisfies the atomic boundedness condition and*

$$\|T\|_{\mathrm{At}} \lesssim \|T\|_{L_0^\infty \to BMO} + \mathrm{SI}^1[K].$$

For the convenience of the reader we give the proof of the three propositions. In what follows Q will denote a cube, x_Q its center, and as above Q^* will be the double cube with same center.

Proof of Proposition 8.17. Let f be a bounded function supported in a cube Q. We need to establish the estimate

$$\int_{Q^*} |Tf|dx \lesssim C|Q|\|f\|_\infty \big(\|T\|_{\mathrm{At}} + \mathrm{Ann}_{\mathrm{av}}[K]\big). \tag{8.25}$$

Let Q_1 be a cube with the same sidelength of Q^* and of distance $\mathrm{diam}(Q^*)$ to Q^*. Let f_1 be a function supported in $Q \cup Q_1$ so that $f_1(y) = f(y)$ for $y \in Q$, $\|f_1\|_\infty \le \|f\|_\infty$ and $\int f_1(y)dy = 0$. Then, if

$$a(x) = |Q|^{-1}\|f\|_\infty^{-1} f_1(x)$$

then there is $C_d > 0$ so that $C_d^{-1} a$ is an ∞-atom. Set $f_2 = f - f_1$ so that f_2 is supported in Q_1 and split

$$\int_{Q^*} |Tf|dx \lesssim \int_{Q^*} |Tf_1|dx + \int_{Q^*} |Tf_2|dx.$$

We estimate

$$\int_{Q^*} |Tf_1|dx \lesssim |Q| \, \|T\|_{\mathrm{At}} \|f\|_\infty . \tag{8.26}$$

Since $\mathrm{dist}(Q^*, Q_1) \approx \mathrm{diam}(Q_1) \approx \mathrm{diam}(Q^*) \approx \mathrm{diam}(Q)$ we may use the averaged annular integrability condition and estimate

$$\frac{1}{|Q|}\int_{Q_1}\int_{Q^*} |K(x,y)|dy\, dx \lesssim \mathrm{Ann}_{\mathrm{av}}[K].$$

This yields

$$\int_{Q^*} |Tf_2|dx \lesssim \int_{Q^*}\int_{Q_1} |K(x,y)||f_2(y)|dx\, dy \lesssim \|f_2\|_\infty |Q| \mathrm{Ann}_{\mathrm{av}}[K]. \tag{8.27}$$

Since $\|f_2\|_\infty \le 2\|f\|_\infty$, (8.25) follows from (8.26) and (8.27). □

Proof of Proposition 8.18. Let $g \in L^\infty_0$ and let Q be any cube with center x_Q. We have to verify

$$\inf_C \fint_Q |Tg(x) - C|dx \le \|T\|_{\mathrm{Carl}} + \mathrm{SI}^\infty[K] \tag{8.28}$$

where the slashed integral denotes the average over Q.

Let $g_1 = g\mathbb{1}_{Q^*}$, $g_2 = g\mathbb{1}_{\mathbb{R}^d\setminus Q^*}$, so that $g = g_1 + g_2$. Since g has compact support it is immediate by the assumed finiteness of $\mathrm{Ann}_{\mathrm{av}}[K]$ that $Tg_2(w)$ is finite for almost every w in

$$B_Q := \{w : |w - x_Q| \le (2d)^{-1}\mathrm{diam}(Q)\}.$$

Now

$$\inf_C \fint_Q |Tg(x) - C|dx \lesssim \fint_{B_Q}\Big[\fint_Q |Tg_1(x)|dx + \fint_Q |Tg_2(x) - Tg_2(w)|dx\Big]dw.$$

From the Carleson condition we get

$$\fint_Q |Tg_1(x)|dx \le 4^d \|T\|_{\mathrm{Carl}} \|g_1\|_\infty \lesssim \|T\|_{\mathrm{Carl}} \|g\|_\infty .$$

Moreover,

$$\begin{aligned}\fint_{B_Q}\fint_Q |Tg_2(x) - Tg_2(w)|dx\, dw &\le \|g_2\|_\infty \sup_{w\in B_Q} \fint_Q \int_{\mathbb{R}^d\setminus Q^*} |K(x,y) - K(w,y)|dy\, dx \\ &\lesssim \mathrm{SI}^\infty[K]\|g\|_\infty .\end{aligned}$$

and (8.28) follows. □

Proof of Proposition 8.19. Let a be an ∞-atom, associated with the cube Q. We need to verify

$$\|Ta\|_1 \lesssim \|T\|_{L^\infty_0 \to BMO} + \mathrm{SI}^\infty[K] . \tag{8.29}$$

First estimate Ta in the complement of Q^*, using the cancellation of a:

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus Q^*} |Ta(x)|\, dx \lesssim \int_{\mathbb{R}^d\setminus Q^*} \Big| \int_Q [K(x,y) - K(x,x_Q)]a(y)dy\Big|\, dx \\ &\le \int_Q |a(y)| \int\limits_{|x-x_Q|\ge 2|y-x_Q|} |K(x,y) - K(x,x_Q)|\, dx\, dy \\ &\le \mathrm{SI}^1[K]\|a\|_1 \lesssim \mathrm{SI}^1[K] .\end{aligned}$$

Let $\widetilde{Q}$ be a cube which is contained in $CQ^* \setminus Q^*$ and has distance $O(\mathrm{diam}(Q))$ to Q^*, say, a cube adjacent to Q^* and of same sidelength. The above calculation also yields

$$\int_{\widetilde{Q}} |Ta(x)|dx \lesssim \mathrm{SI}^1[K]\,. \tag{8.30}$$

We choose such a cube $\widetilde{Q}$ and estimate

$$\int_{Q^*} |Ta(x)|dx \lesssim I_Q + II_Q + III_Q$$

where

$$\begin{aligned} I_Q &= \int_{Q^*} \Big|Ta(x) - \fint_{Q^*} Ta(y)dy\Big|\,dx\,,\\ II_Q &= |Q^*|\Big|\fint_{Q^*} Ta(y)dy - \fint_{\widetilde{Q}} Ta(y)dy\Big|\,,\\ III_Q &= |Q^*|\Big|\fint_{\widetilde{Q}} Ta(y)dy\Big|\,. \end{aligned}$$

Clearly

$$|I_Q| \le |Q^*|\|Ta\|_{BMO} \le \|T\|_{L^\infty\to BMO}|Q^*|\|a\|_{L^\infty} \lesssim \|T\|_{L^\infty\to BMO}.$$

To estimate II_Q we let Q^{**} be a cube containing both Q^* and $\widetilde{Q}$, and of comparable sidelength. Then

$$\begin{aligned} &\Big|\fint_{Q^*} Ta(y)dy - \fint_{\widetilde{Q}} Ta(y)dy\Big|\\ &\le \fint_{Q^*} \Big|Ta(y) - \fint_{Q^{**}} Ta(z)dz\Big|\,dy + \fint_{\widetilde{Q}} \Big|Ta(y) - \fint_{Q^{**}} Ta(z)dz\Big|\,dy\\ &\lesssim \fint_{Q^{**}} \Big|Ta(y) - \fint_{Q^{**}} Ta(z)dz\Big|\,dy \lesssim \|Ta\|_{BMO} \end{aligned}$$

and thus

$$|II_Q| \lesssim \|T\|_{L^\infty_0\to BMO}|Q|\|a\|_\infty \lesssim \|T\|_{L^\infty_0\to BMO}|Q|\|a\|_\infty \lesssim \|T\|_{L^\infty_0\to BMO}\,.$$

Finally,

$$|III_Q| \le |Q^*|\Big|\fint_{\widetilde{Q}} Ta(y)dy\Big| \lesssim \|Ta\|_{L^1(\widetilde{Q})} \lesssim A\,,$$

by (8.30), and the proof of (8.29) is finished. □

Theorem 8.20. *Let $T : C_0^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ and assume that the Schwartz kernel K is locally integrable in $(\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta$. Assume that*

$$\mathrm{SI}[K] := \mathrm{Ann}_{\mathrm{av}}[K] + \mathrm{SI}^1[K] + \mathrm{SI}^\infty[K] < \infty.$$

(i) Let $1 < q < \infty$. The following statements are equivalent.

- *T satisfies a Carleson condition.*
- *T maps $L^\infty_0 \to BMO$.*
- *T satisfies the atomic boundedness condition.*
- *T extends to a bounded operator $H^1 \to L^1$.*
- *T extends to an operator bounded on L^q.*

(ii) We have the following equivalences of norms.

$$\|T\|_{\mathrm{Carl}}+\mathrm{SI}[K]\approx\|T\|_{L^\infty_0\to BMO}+\mathrm{SI}[K]\approx\|T\|_{\mathrm{At}}+\mathrm{SI}[K]\approx_q\|T\|_{L^q\to L^q}+\mathrm{SI}[K]. \tag{8.31}$$

Moreover,

$$\|T\|_{\mathrm{At}}\approx\|T\|_{H^1\to L^1}\,. \tag{8.32}$$

Proof. The first three equivalences are immediate from a combination of Propositions 8.17, 8.18 and 8.19. Since ∞-atoms satisfy $\|a\|_{H^1}\le C$ it is clear that

$$\|T\|_{\mathrm{At}}\lesssim\|T\|_{H^1\mapsto L^1}\,.$$

The converse

$$\|T\|_{H^1\to L^1}\lesssim\|T\|_{\mathrm{At}} \tag{8.33}$$

is not obvious (and the inequality without the term $\mathrm{SI}[K]$ might not hold if we drop our assumption $\mathrm{SI}[K]<\infty$, see [3]). By the Coifman-Latter theorem about the atomic decomposition (see [36, §III.2]) we may write $f=\sum_Q\lambda_Q a_Q$, with $\sum_Q|\lambda_Q|\lesssim\|f\|_{H^1}$ and a_Q being ∞-atoms; here the convergence of the series is understood in the L^1 sense. We immediately get

$$\Big\|\sum_Q\lambda_Q Ta_Q\Big\|_1\le\sum_Q|\lambda_Q|\|T\|_{\mathrm{At}}\|a_Q\|_1\lesssim\|T\|_{\mathrm{At}}\|f\|_{H_1}.$$

However the decomposition $f=\sum_Q\lambda_Q a_Q$ is not unique and in order to prove that the expression $\sum_Q\lambda_Q Ta_Q$ can be used as a definition for Tf we need to show the following consistency condition for a sequence of atoms $\{a_\nu\}_{\nu=1}^\infty$,

$$\sum_Q|c_\nu|<\infty\,,\ \sum_\nu c_\nu a_\nu=0\implies\sum_\nu c_\nu Ta_\nu=0. \tag{8.34}$$

Fortunately, a version of an approximation (or weak compactness) argument in [30, §7.2] applies to our situation. As stated above the atomic boundedness condition implies the Carleson condition. Let $\phi\in C^\infty_0$ be supported in a ball of radius 1/2 such that $\int\phi(x)dx=1$. Set $P_mf=\phi^{(2^m)}*f$. Let K_m be the distribution kernel for P_mTP_m. Note that we have

$$|K_m(x,y)|\lesssim 2^{md}\mathrm{Ann}_{\mathrm{av}}[K]\text{ if }|x-y|\ge 2^{2-m}$$

and

$$|K_m(x,y)|\lesssim 2^{md}\|T\|_{\mathrm{Carl}}\text{ if }|x-y|\le 2^{2-m}.$$

Hence $K_m\in L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ and thus P_mTP_m maps L^1 to L^∞. This implies $\sum_\nu c_\nu P_mTP_ma_\nu=P_mTP_m(\sum c_\nu a_\nu)=0$. Now, since the P_m form an approximation of the identity, it is clear that, for each atom a_ν, we have $\|P_mTP_ma_\nu-Ta_\nu\|_1\to0$ as $\nu\to\infty$. Taking in account that $\sum_\nu|a_\nu|<\infty$, a straightforward limiting argument yields $\sum_\nu c_\nu Ta_\nu=0$. Note that the condition $\mathrm{SI}[K]<\infty$ is used to establish (8.32) only in order to verify the implication (8.34) (via the boundedness of K_m); it does not enter in (8.32) itself.

We still have to show the equivalence of the first three conditions in (8.31) with the fourth condition. Assume first that T is L^q-bounded. Then we have the standard estimates (8.3), (8.4) and thus the $H^1\to L^1$ operator norms and $L^\infty_0\to L^\infty$ operator norms of T are bounded by $\|T\|_{L^q\to L^q}+\mathrm{SI}[K]$. The other direction uses the interpolation result (cf. the remarks below)

$$\|T\|_{L^q\to L^q}\le C_q\|T\|_{H^1\to L^1}^{1/q}\|T\|_{L^\infty_0\to BMO}^{1-1/q}$$

together with the equivalence of the first three conditions in (8.31) and the equivalence (8.32). □

Remarks on interpolation of H^1 and BMO. In the above interpolation one uses the interpolation formulas $[H^1, BMO]_{\theta,q} = L^{p,q}$, $[H^1, BMO]_\theta = L^p$ for $1-\theta = 1/p$, $1 < p < \infty$, or a direct interpolation result for operators in §3.III of Journé's monograph [28]. One also has $[L^1, BMO]_{\theta,q} = L^{p,q}$, $[L^1, BMO]_\theta = L^p$ for $1-\theta = 1/p$, $1 < p < \infty$.

The result for complex interpolation can be obtained from the results $[H^1, L^{p_1}]_\vartheta = L^p$, $1/p = 1-\vartheta+\vartheta/p_1$, $1 < p_1 < \infty$, (or its respective standard counterpart $[L^1, L^{p_1}]_\vartheta = L^p$), together with $[L^{p_0}, BMO]_\vartheta = L^p$, $1/p = (1-\theta)/p_0$, $1 < p_0 < \infty$ which can be found in Fefferman and Stein [16], see also the discussion in Janson and Jones [27]. The stated interpolation formula for H^1 and BMO follows then from Wolff's four space reiteration theorem for the complex method [40]. One can also use the results by Fefferman, Rivière, Sagher [15] for the real method, and then combine it with Wolff's result [40] for the real method. From the above remarks we also get an interpolation inequality for functions $g \in L^1 \cap BMO$,

$$\|g\|_p \le C_p \|g\|_{L^1}^{1/p} \|f\|_{BMO}^{1-1/p}, \quad 1 < p < \infty \tag{8.35}$$

which will be useful in the proof of Theorem 8.22 below.

8.3. **Sums of dilated kernels.** We shall now formulate some corollaries for operators of the form (8.7) or its relatives. We use norms combining the various Schur and regularity norms.

For each $j \in \mathbb{Z}$, let $\tau_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a measurable function. Let $0 < \varepsilon \le 1$. Set, for $0 < \varepsilon \le 1$,

$$\|\tau\|_{\mathrm{Op}_\varepsilon} = \mathrm{Int}_\varepsilon^1[\tau] + \mathrm{Int}_\varepsilon^\infty[\tau] + \mathrm{Reg}^1_{\varepsilon,\mathrm{lt}}[\tau] + \mathrm{Reg}^\infty_{\varepsilon,\mathrm{lt}}[\tau] + \mathrm{Reg}^1_{\varepsilon,\mathrm{rt}}[\tau] + \mathrm{Reg}^\infty_{\varepsilon,\mathrm{rt}}[\tau],$$

and set

$$\|\tau\|_{\mathrm{Op}_0} := \mathrm{Int}_0^1[\tau] + \mathrm{Int}_0^\infty[\tau].$$

This means for $\varepsilon > 0$

(8.36)

$$\begin{aligned}
\|\tau\|_{\mathrm{Op}_\varepsilon} =& \sup_x \int (1+|x-y|)^\epsilon |\tau(x,y)|\, dy + \sup_y \int (1+|x-y|)^\epsilon |\tau(x,y)|\, dx \\
&+ \sup_{\substack{y \\ 0<|h|\le 1}} |h|^{-\epsilon} \int |\tau(x+h,y)-\tau(x,y)|\, dx + \sup_{\substack{x \\ 0<|h|\le 1}} |h|^{-\epsilon} \int |\tau(x+h,y)-\tau(x,y)|\, dy \\
&+ \sup_{\substack{y \\ 0<|h|\le 1}} |h|^{-\epsilon} \int |\tau(x,y+h)-\tau(x,y)|\, dx + \sup_{\substack{x \\ 0<|h|\le 1}} |h|^{-\epsilon} \int |\tau(x,y+h)-\tau(x,y)|\, dy
\end{aligned}$$

and, for $\varepsilon = 0$,

$$\|\tau\|_{\mathrm{Op}_0} = \sup_x \int |\tau(x,y)|\, dy + \sup_y |\tau(x,y)|\, dx\,. \tag{8.37}$$

We shall consider families $\{\tau_j\}$ for which the Op_ε norm is uniformly bounded in j. We let T_j be the operator with kernel $\mathrm{Dil}_{2^j}\tau_j$, i.e.

$$T_j f(x) = \int 2^{jd} \tau_j(2^j x, 2^j y) f(y) dy\,. \tag{8.38}$$

Theorem 8.21. *Suppose that* $\sup_j \|\tau_j\|_{\mathrm{Op}_\varepsilon} \le \mathcal{C}_\varepsilon$ *for some* $\varepsilon \in (0,1)$ *and that* $\sup_j \|\tau_j\|_{\mathrm{Op}_0} \le \mathcal{C}_0$. *Let* T_j *be the operator with kernel* $\mathrm{Dil}_{2^j}\tau_j$ *and suppose that* $\sum_j T_j$ *converges to an operator* $T : L^\infty_{\mathrm{comp}} \to L^1_{\mathrm{loc}}$ *in the sense that for compactly supported* L^∞ *functions* f *and* g

$$\Big\langle \sum_{j=-N}^{N} T_j f, g \Big\rangle \to \langle Tf, g\rangle$$

as $N \to \infty$ and assume that there exists $A > 0$ such that for all $x \in \mathbb{R}^d$, $t > 0$, $N \in \mathbb{N}$,

$$\Big|\Big\langle \sum_{j=-N}^{N} T_j f, g\Big\rangle\Big| \le At^d \|f\|_{L^\infty} \|g\|_{L^\infty} \quad \text{if } \operatorname{supp}(f) \cup \operatorname{supp}(g) \subset B^d(x,t). \tag{8.39}$$

Then T extends to an operator bounded on $L^2(\mathbb{R}^d)$ and

$$\|T\|_{L^2\to L^2} \le C_{d,\epsilon}\Big(A + \mathcal{C}_0 \log\big(1 + \frac{\mathcal{C}_\epsilon}{\mathcal{C}_0}\big)\Big).$$

Proof. The inequality (8.39) implies $\| \sum_{j=-N}^{N} T_j \|_{\mathrm{Carl}} \lesssim A$. This inequality extends to the limit T. Let K_N, K be the Schwartz kernels of the operators $\sum_{j=-N}^{N} T_j$ and T respectively. Then by Propositions 8.3 and 8.4, applied to both τ_j and its adjoint version we have $\mathrm{SI}[K_N]$, $\mathrm{SI}[K] \lesssim \mathcal{C}_0 \log(2 + \mathcal{C}_\varepsilon/\mathcal{C}_0)$. The assertion follows now from Theorem 8.20. □

Theorem 8.22. *Suppose that $\sup_j \|\tau_j\|_{\mathrm{Op}_\varepsilon} \le \mathcal{C}_\varepsilon$ for some $\varepsilon \in (0,1)$ and that $\sup_j \|\tau_j\|_{\mathrm{Op}_0} \le \mathcal{C}_0$. Let T_j be the operator with kernel $\mathrm{Dil}_{2^j}\tau_j$ and suppose that the sum $T = \sum T_j$ converges in the sense of distributions on $C^\infty_{0,0}$ (test functions with vanishing integrals), i.e. for every $f \in C^\infty_{0,0}$ and every $g \in C^\infty_0$ we have*

$$\lim_{N\to\infty} \sum_{j=-N}^{N} \langle T_j f, g\rangle = \langle Tf, g\rangle. \tag{8.40}$$

Then the following statements hold.

(i) If $\sup_N \| \sum_{j=-N}^{N} T_j \|_{H^1\to L^1} \le A$, for some $A < \infty$, then we also have

$$\sup_N \Big\| \sum_{j=-N}^{N} T_j \Big\|_{L^2\to L^2} \lesssim A + \mathcal{C}_0 \log\big(1 + \frac{\mathcal{C}_\varepsilon}{\mathcal{C}_0}\big).$$

Moreover, T extends to a bounded operator on L^2, $\sum_{j=-N}^{N} T_j$ converges to T in the weak operator topology and $\|T\|_{L^2\to L^2} \lesssim A + \mathcal{C}_0 \log\big(1 + \mathcal{C}_\varepsilon/\mathcal{C}_0\big)$.

(ii) If $\sup_N \| \sum_{j=-N}^{N} T_j \|_{L^2\to L^2} \le B$, for some $B < \infty$, then we also have

$$\sup_N \Big\| \sum_{j=-N}^{N} T_j \Big\|_{H^1\to L^1} \lesssim B + \mathcal{C}_0 \log\big(1 + \frac{\mathcal{C}_\varepsilon}{\mathcal{C}_0}\big).$$

Moreover T extends to an operator bounded from H^1 to L^1, $\sum_{j=-N}^{N} T_j \to T$ converges in the weak operator topology (as operators $H^1 \to L^1$) and $\|T\|_{H^1\to L^1} \lesssim B + \mathcal{C}_0 \log\big(1 + \mathcal{C}_\varepsilon/\mathcal{C}_0\big)$.

(iii) The sum $T = \sum_{j\in\mathbb{Z}} T_j$ converges in the strong operator topology as operators $H^1 \to L^1$ if and only if it converges in the strong operator topology as operators $L^2 \to L^2$.

Proof. The assertions on the operators $\sum_{j=-N}^{N} T_j$ follow immediately from Theorem 8.20. Note that $C^\infty_{0,0}$ is dense in both H^1 and L^p, $1 < p < \infty$. The uniform bounds for the operator norms of $\sum_{j=-N}^{N} T_j$ and the convergence hypothesis (8.40) imply convergence in the respective weak operator topologies.

Now we prove (iii). If $T = \sum_{j\in\mathbb{Z}} T_j$ converges in the strong operator topology as operators $L^2 \to L^2$ then it is immediate from Proposition 8.9 that $T = \sum_{j\in\mathbb{Z}} T_j$ converges in the strong operator topology as operators $H^1 \to L^1$.

Vice versa assume that $T=\sum_{j\in\mathbb{Z}} T_j$ converges in the strong operator topology as operators $H^1\to L^1$. By the interpolation inequality (8.35) we have for any finite set $\mathcal{J}\in\mathbb{Z}$ and any $f\in C^\infty_{0,0}$.

$$\Big\|\sum_{j\in\mathcal{J}} T_j f\Big\|_2 \le C\Big\|\sum_{j\in\mathcal{J}} T_j f\Big\|_1^{1/2}\Big\|\sum_{j\in\mathcal{J}} T_j f\Big\|_{BMO}^{1/2} \le C\Big\|\sum_{j\in\mathcal{J}} T_j f\Big\|_1^{1/2}\Big\|\sum_{j\in\mathcal{J}} T_j\Big\|_{L^\infty\to BMO}^{1/2}\|f\|_\infty$$

and since $\|\sum_{j\in\mathcal{J}} T_j\|_{L^\infty\to BMO}$ is bounded independently of $\mathcal{J}$ we see that $\sum_j T_j f$ converges in L^2 for any $f\in C^\infty_{0,0}$. Since $C^\infty_{0,0}$ is dense in L^2 we conclude that $\sum_j T_j$ converges in the strong operator topology as operators $L^2\to L^2$. □

We now formulate a version of Theorem 8.21 which has a convergence statement with respect to the strong operator topology.

Theorem 8.23. *Suppose that* $\sup_j \|\tau_j\|_{\mathrm{Op}_\varepsilon}\le \mathcal{C}_\varepsilon$ *for some* $\varepsilon\in(0,1)$ *and that* $\sup_j\|\tau_j\|_{\mathrm{Op}_0}\le \mathcal{C}_0$. *Let* T_j *be the operator with kernel* $\mathrm{Dil}_{2^j}\tau_j$. *Suppose that* $\sum_j T_j$ *converges to an operator* $T: L^\infty_{\mathrm{comp}}\to L^1_{\mathrm{loc}}$ *in the strong sense that for any compactly supported* L^∞ *function* f *and for any compact set* K

$$\lim_{N\to\infty}\int_K\Big|\sum_{j=-N}^{N} T_j f(x)-Tf(x)\Big|dx=0.$$

Suppose that there exists $A>0$ *such that for all* $x\in\mathbb{R}^d$, $t>0$, $N\in\mathbb{N}$,

$$\int_{B_d(x,t)}\Big|\sum_{j=-N}^{N} T_j f(w)\Big|dw\le At^d\|f\|_\infty \quad \text{if } \mathrm{supp}(f)\subset B^d(x,t). \tag{8.41}$$

Then the sum $T=\sum_{j\in\mathbb{Z}}T_j$ *converges in the strong operator topology as operators* $L^2\to L^2$ *and and*

$$\|T\|_{L^2\to L^2}\le C_{d,\epsilon}\Big(A+\mathcal{C}_0\log\big(1+\frac{\mathcal{C}_\epsilon}{\mathcal{C}_0}\big)\Big).$$

Proof. If a is an L^∞ atom supported on a cube Q and Q^* is the double cube, we see that $\sum_{j=-N}^N Ta\to Ta$ in $L^1(Q^*)$. Standard arguments using the cancellation of a yield

$$\int_{\mathbb{R}^d\setminus Q^*}|T_j a(x)|\lesssim\begin{cases}\mathrm{Int}^1_\varepsilon[\tau_j]\,(2^j\mathrm{diam}(Q))^{-\varepsilon} & \text{if } 2^j\mathrm{diam}(Q)\ge 1,\\ \mathrm{Reg}^1_{\varepsilon,\mathrm{lt}}[\tau_j]\,(2^j\mathrm{diam}(Q))^{\varepsilon} & \text{if } 2^j\mathrm{diam}(Q)\le 1.\end{cases}$$

Altogether we see that $\sum_{j=-N}^N T_j a\to Ta$ in L^1. By Theorem 8.21 we also have the uniform bounds $\|Ta\|_1\le C_{d,\epsilon}\Big(A+\mathcal{C}_0\log\big(1+\frac{\mathcal{C}_\epsilon}{\mathcal{C}_0}\big)\Big)$ for L^∞ atoms. Now, writing $f\in H^1$ as $f=\sum_\nu c_\nu a_\nu$ where the a_ν are ∞-atoms and $\sum_\nu|c_\nu|<\infty$, we easily derive that $\sum_{j=-N}^N T_j f\to Tf$ in L^1. Thus we see that $\sum_j T_j$ converges in the strong operator topology as operators $H^1\to L^1$ and we have the uniform bound

$$\Big\|\sum_{j=-N}^{N} T_j\Big\|_{H^1\to H^1}\lesssim\Big(A+\mathcal{C}_0\log\big(1+\frac{\mathcal{C}_\epsilon}{\mathcal{C}_0}\big)\Big).$$

We apply parts (i) and (iii) of Theorem 8.22 to see that that $\sum_j T_j$ converges in the strong operator topology as operators $L^2\to L^2$, and obtain the asserted bounds on the $L^2\to L^2$ operator norms. □

The following lemma allows us to apply Theorems 8.21, 8.22 and 8.23 to sums of the form $\sum_j P_j S_j P_j$ where $P_j f = f * \phi^{(2^j)}$, and S_j is an integral operator with kernel $\mathrm{Dil}_{2^j} s_j$, with $\sup_j(\mathrm{Int}^1_\varepsilon[s_j] + \mathrm{Int}^\infty_\varepsilon[s_j]) < \infty$.

Lemma 8.24. *Suppose that* $\mathrm{Int}^1_\varepsilon[s] + \mathrm{Int}^\infty_\varepsilon[s] \le C_\varepsilon$ *and* $\mathrm{Int}^1[s] + \mathrm{Int}^\infty[s] \le C_0$. *Let* $\phi \in C_0^\infty$ *supported in* $\{v : |v| \le 10\}$. *Let*

$$\widetilde{s}(x,y) = \iint \phi(x-w)s(w,z)\phi(z-y)\,dw\,dz.$$

Then $\|\widetilde{s}\|_{\mathrm{Op}_\varepsilon} \lesssim C_\varepsilon$ *and* $\|\widetilde{s}\|_{\mathrm{Op}_0} \lesssim C_0$.

Proof. Left to the reader. □

We also have

Lemma 8.25. *Let* $s \in \mathrm{Op}_\varepsilon$, $0 \le \varepsilon \le 1$. *Let* $\phi \in C^1$ *supported in* $\{v : |v| \le 10\}$ *and let*

$$s_1(x,y) = \int \phi(x-w)s(w,y)\,dw,$$
$$s_2(x,y) = \int s(x,z)\phi(z-y)\,dz.$$

Then $\|s_1\|_{\mathrm{Op}_\varepsilon} \lesssim \|s\|_{\mathrm{Op}_\varepsilon}\|\phi\|_{C^1}$, $\|s_2\|_{\mathrm{Op}_\varepsilon} \lesssim \|s\|_{\mathrm{Op}_\varepsilon}\|\phi\|_{C^1}$.

Proof. Immediate from the definition. □

9. Almost orthogonality

We shall repeatedly use a rather standard almost orthogonality lemma which involves the Littlewood-Paley operators $\mathcal{Q}_k$ introduced in (6.4).

Lemma 9.1. *Let* $\mathcal{I}$ *be an index set. Suppose that for each* $j \in \mathbb{Z}$, $\nu \in \mathcal{I}$, $V_j^\nu : L^2 \to L^2$ *is a bounded operator such that for* $k_1, k_2 \in \mathbb{Z}$,

$$\big\|\mathcal{Q}_{k_1} V^\nu_{j+k_1} \mathcal{Q}_{j+k_1+k_2}\big\|_{L^2\to L^2} \lesssim A_{j,k_2}, \tag{9.1}$$

where

$$\sum_{j,k_2} A_{j,k_2} < \infty.$$

Then the sum $V^\nu := \sum_{j\in\mathbb{Z}} V^\nu_j$, *converges in the strong operator topology (as operators on* L^2*), with equiconvergence with respect to* $\mathcal{I}$, *and we have*

$$\sup_\nu \|V^\nu\|_{L^2\to L^2} \lesssim \sum_{j,k_2} A_{j,k_2}. \tag{9.2}$$

Proof. Recall, from §6, $\sum_k \widetilde{\mathcal{Q}}_k \mathcal{Q}_k = \sum_k \mathcal{Q}_k \widetilde{\mathcal{Q}}_k = I$. Let $f, g \in L^2(\mathbb{R}^d)$ with $\|f\|_2 = \|g\|_2 = 1$. By (6.6), we have

$$\Big(\sum_k \|\widetilde{\mathcal{Q}}_k f\|_2^2\Big)^{\frac12} = \Big\|\Big(\sum_k |\widetilde{\mathcal{Q}}_k f|^2\Big)^{\frac12}\Big\|_2 \approx 1, \quad \Big(\sum_k \|\widetilde{\mathcal{Q}}_k^* g\|_2^2\Big)^{\frac12} = \Big\|\Big(\sum_k |\widetilde{\mathcal{Q}}_k^* g|^2\Big)^{\frac12}\Big\|_2 \approx 1.$$

First observe, for integers $J_1 \le J_2$,

$$\begin{aligned}
&\Big| < g, \sum_{j=J_1}^{J_2} V_j^\nu f >_{L^2} \Big| = \Big| < g, \sum_{k_1,k_2\in\mathbb{Z}} \sum_{j=J_1}^{J_2} \widetilde{\mathcal{Q}}_{k_1} \mathcal{Q}_{k_1} V_j^\nu \mathcal{Q}_{k_2} \widetilde{\mathcal{Q}}_{k_2} f >_{L^2} \Big| \\
&= \Big| < g, \sum_{j=J_1}^{J_2} \sum_{k_1,k_2\in\mathbb{Z}} \widetilde{\mathcal{Q}}_{k_1} \mathcal{Q}_{k_1} V_j^\nu \mathcal{Q}_{k_2} \widetilde{\mathcal{Q}}_{k_2} f >_{L^2} \Big| \\
&\le \sum_{k_1\in\mathbb{Z}} \Big| < \widetilde{\mathcal{Q}}_{k_1}^* g, \sum_{j=J_1-k_1}^{J_2-k_1} \sum_{k_2\in\mathbb{Z}} \mathcal{Q}_{k_1} V_{j+k_1}^\nu \mathcal{Q}_{j+k_1+k_2} \widetilde{\mathcal{Q}}_{j+k_1+k_2} f >_{L^2} \Big| \\
&\le \Big(\sum_{k_1\in\mathbb{Z}} \|\widetilde{\mathcal{Q}}_{k_1}^* g\|_2^2 \Big)^{\frac12} \Big(\sum_{k_1\in\mathbb{Z}} \Big\| \sum_{j=J_1-k_1}^{J_2-k_1} \sum_{k_2\in\mathbb{Z}} \mathcal{Q}_{k_1} V_{j+k_1}^\nu \mathcal{Q}_{j+k_1+k_2} \widetilde{\mathcal{Q}}_{j+k_1+k_2} f \Big\|_2^2 \Big)^{\frac12}.
\end{aligned}$$

Now

$$\begin{aligned}
&\Big(\sum_{k_1\in\mathbb{Z}} \Big\| \sum_{j=J_1-k_1}^{J_2-k_1} \sum_{k_2\in\mathbb{Z}} \mathcal{Q}_{k_1} V_{j+k_1}^\nu \mathcal{Q}_{j+k_1+k_2} \widetilde{\mathcal{Q}}_{j+k_1+k_2} f \Big\|_2^2 \Big)^{\frac12} \\
&\lesssim \sum_{j\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} \Big(\sum_{k_1=J_1-j}^{J_2-j} \big\| \mathcal{Q}_{k_1} V_{j+k_1}^\nu \mathcal{Q}_{j+k_1+k_2} \widetilde{\mathcal{Q}}_{j+k_1+k_2} f \big\|_2^2 \Big)^{\frac12} \\
&\le \sum_{j\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} A_{j,k_2} \Big(\sum_{k_1=J_1-j}^{J_2-j} \|\widetilde{\mathcal{Q}}_{j+k_1+k_2} f\|_2^2 \Big)^{\frac12}.
\end{aligned}$$

We take the sup over g with $\|g\|_2 = 1$ and obtain from the two previous displays

$$\Big\| \sum_{j=J_1}^{J_2} V_j^\nu f \Big\|_2 \lesssim \sum_{j\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} A_{j,k_2} \Big(\sum_{k_1=J_1-j}^{J_2-j} \|\widetilde{\mathcal{Q}}_{j+k_1+k_2} f\|_2^2 \Big)^{\frac12} \lesssim \sum_{j\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} A_{j,k_2} \|f\|_2. \tag{9.3}$$

The first inequality in (9.3) implies that for fixed $f \in L^2$ the partial sums of $\Sigma_N^\nu f = \sum_{j=-N}^N V_j^\nu f$ form a Cauchy sequence, more precisely, for each $\varepsilon > 0$ there is $N(\varepsilon, f) \in \mathbb{N}$ (independent of $\mathcal{I}$) such that $\|\Sigma_{N_1} f - \Sigma_{N_2} f\|_2 < \varepsilon$ for $N_1, N_2 > N(\varepsilon, f)$. By completeness of L^2, $\Sigma_N^\nu f$ converge to a limit $\Sigma^\nu f$ and Σ^ν defines a linear bounded operator on L^2. Thus $\Sigma_N^\nu \to \Sigma^\nu$ in the strong operator topology, and, by the above, we get equiconvergence with respect to $\mathcal{I}$. □

10. Proof of Theorem 5.1: Part I

We are given a family $\vec{\varsigma} = \{\varsigma_j\}$ with $\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} < \infty$. In this and the following sections we use the notation

$$\Gamma_\varepsilon = \frac{\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}}{\sup_j \|\varsigma_j\|_{L^1}}$$

introduced in (5.6). Notice that always $\Gamma_\varepsilon \ge 1$.

Recall,

$$\Lambda_{n+1,n+2}^1(b_1, \dots, b_{n+2}) = \sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](b_1, \dots, b_n, (I - P_j)b_{n+1}, (I - P_j)b_{n+2}).$$

Given $\varepsilon > 0$ and $\vec{\varsigma}$, it is our goal to prove Part I of Theorem 5.1, i.e. for $1 < p \le 2$, the estimate

$$|\Lambda_{n+1,n+2}^1(b_1, \dots, b_{n+2})| \le C_{d,p,\varepsilon} (\sup_j \|\varsigma_j\|_{L^1}) \log^2(1 + n\Gamma_\varepsilon) \Big(\prod_{l=1}^n \|b_l\|_\infty \Big) \|b_{n+1}\|_p \|b_{n+2}\|_{p'}. \tag{10.1}$$

We formulate a stronger result which will also be useful in other parts of the paper. For this, we need some new notation. Let $1 \le l_1 \neq l_2 \le n+2$ and let $\{b_l^j : j \in \mathbb{Z},\, l \neq l_1, l_2\} \subset L^\infty(\mathbb{R}^d)$ be a bounded subset of $L^\infty(\mathbb{R}^d)$. Let $k_1, k_2 \in \mathbb{N}$, and fix $u_1, u_2 \in \mathfrak{U}$.

Define an operator $S^{l_1,l_2}_{k_1,k_2,j}$ (which implicitly depends on $\{b_l^j : j \in \mathbb{Z}, l \neq l_1, l_2\}$, u_1, and u_2) by the formula

$$\int g(x)(S^{l_1,l_2}_{k_1,k_2,j} f)(x)\,dx$$
$$:= \Lambda[\varsigma_j^{(2^j)}](b_1^j, \dots, b_{l_1-1}^j, \overline{Q}_{j+k_1}[u_1]f, b_{l_1+1}^j, \dots, b_{l_2-1}^j, \overline{Q}_{j+k_2}[u_2]g, b_{l_2+1}^j, \dots, b_{n_2}^j).$$

Theorem 10.1. *Let $0<\varepsilon<1$ and suppose that $\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} < \infty$. Then*

$$S^{l_1,l_2}_{k_1,k_2} = \sum_{j\in\mathbb{Z}} S^{l_1,l_2}_{k_1,k_2,j}$$

converges in the strong operator topology, as bounded operators on L^2. Moreover there is $c>0$ such that

$$\|S^{l_1,l_2}_{k_1,k_2}\|_{L^2\to L^2} \lesssim \|u_1\|_{\mathfrak{U}} \|u_2\|_{\mathfrak{U}} \sup_j \|\varsigma_j\|_{L^1} \min\{1,\, n2^{-(k_1+k_2)\varepsilon'}\Gamma_\varepsilon\}\Big(\prod_{l\neq l_1,l_2} \sup_j \|b_j^l\|_\infty\Big).$$

Proof that Theorem 10.1 implies inequality (10.1). For this, fix $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$ with

$$\|b_j\|_\infty = 1, \quad j = 1, \dots, n. \tag{10.2}$$

For $k_1, k_2 \in \mathbb{N}$, define operators $\mathcal{V}$, $\mathcal{V}_{k_1}$, and $\mathcal{V}_{k_1,k_2}$ by the following formulas.

$$\int g(x)(\mathcal{V} f)(x)\,dx := \sum_j \Lambda[\varsigma_j^{(2^j)}](b_1, \dots, b_n, (I-P_j)f, (I-P_j)g),$$

$$\int g(x)(\mathcal{V}_{k_1} f)(x)\,dx := \sum_j \Lambda[\varsigma_j^{(2^j)}](b_1, \dots, b_n, Q_{j+k_1}f, (I-P_j)g),$$

$$\int g(x)(\mathcal{V}_{k_1,k_2} f)(x)\,dx := \sum_j \Lambda[\varsigma_j^{(2^j)}](b_1, \dots, b_n, Q_{j+k_1}f, Q_{j+k_2}g).$$

The estimate (10.1) is equivalent to

$$\|\mathcal{V}\|_{L^p\to L^p} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^2(1+n\Gamma_\varepsilon). \tag{10.3}$$

In light of (6.2), we have the following identities,

$$\mathcal{V} = \sum_{k_1>0} \mathcal{V}_{k_1}, \quad \mathcal{V}_{k_1} = \sum_{k_2>0} \mathcal{V}_{k_1,k_2}.$$

To see (10.3) we first use Theorem 10.1 to deduce

$$\|\mathcal{V}_{k_1,k_2}\|_{L^2\to L^2} \lesssim \min\{\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} n2^{-(k_1+k_2)c\varepsilon},\, \sup_j \|\varsigma_j\|_{L^1}\}.$$

Thus, by Lemma 8.13,

$$\|\mathcal{V}_{k_1}\|_{L^2\to L^2} \lesssim \sum_{k_1>0} \min\big\{\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} n2^{-(k_1+k_2)c\varepsilon},\ \sup_j \|\varsigma_j\|_{L^1}\big\}$$

which implies

$$\|\mathcal{V}_{k_1}\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \min\{n\Gamma_\varepsilon 2^{-k_2 c\varepsilon},\, \log(1+n\Gamma_\varepsilon)\}\,. \tag{10.4}$$

We turn to the proof of (10.3). Define an operator W_j by

$$\Lambda[\varsigma_j^{(2^j)}](b_1,\dots,b_n,b_{n+1},b_{n+2}) = \int W_j b_{n+1}(x) b_{n+2}(x)\,dx.$$

The Schwartz kernel of W_j is $\mathrm{Dil}_{2^j} w_j(x,y)$ where

$$w_j(x,y) = \int \varsigma_j(\alpha, x-y) \prod_{i=1}^{n} b_i(2^{-j}(x-\alpha_i(x-y)))\,d\alpha\,. \tag{10.5}$$

We observe that $\mathcal{V}_{k_1} = \sum_j (I-P_j) W_j Q_{j+k_1}$. If we set $S_j = (I-P_j)W_j$ then the Schwartz kernel of S_j is $\mathrm{Dil}_{2^j} s_j$ where $s_j(x,y) = w_j(x,y) - \int \phi(x-x') w_l(x',y)$. It is easy to see that $\mathrm{Int}^1(s_j) \lesssim \|\varsigma\|_{L^1} =: A$ and $\mathrm{Int}^1_\varepsilon(s_j) \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon} =: B$.

We wish to apply Corollary 8.12, with $S^{k_1} \equiv \sum S_j Q_{j+k_1} = \mathcal{V}_{k_1}$. By Lemma 10.4, we have $D_{\varepsilon'} \lesssim \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}$ and $D_0 \lesssim (\sup_j \|\varsigma_j\|_{L^1}) \log(1+n\frac{\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}}{\sup_j \|\varsigma_j\|_{L^1}})$. Plugging this into the conclusion of Corollary 8.12, (10.3) follows, and the proof is complete. □

Proof of Theorem 10.1. In light of Theorem 2.9, it suffices to prove Theorem 10.1 in the case $l_1 = n+1$, $l_2 = n+2$. We may also assume the normalizations

$$\begin{aligned} &\sup_j \|b_j^l\|_\infty = 1, \quad 1 \le l \le n,\\ &\|u_1\|_{\mathfrak{U}} = 1 = \|u_2\|_{\mathfrak{U}}. \end{aligned} \tag{10.6}$$

With these reductions, our goal is to show

$$\|S^{n+1,n+2}_{k_1,k_2}\|_{L^2\to L^2} \lesssim \max\big\{ \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} n 2^{-(k_1+k_2)c\varepsilon},\, \sup \|\varsigma_j\|_{L^1} \big\}\,. \tag{10.7}$$

To finish the proof we define, for $j \in \mathbb{Z}$, $k_1, k_2 \in \mathbb{N}$, an operator $S_{j,k_1,k_2} \equiv S^{n+1,n+2}_{j,k_1,k_2}$ by

$$\int g(x)\, S_{j,k_1,k_2} f(x)\,dx = \Lambda[\varsigma_j^{(2^j)}](b_1^j,\dots,b_n^j, \overline{Q}_{j+k_1}[u_1]f, \overline{Q}_{j+k_2}[u_2]g),$$

so that $S^{n+1,n+2}_{k_1,k_2} = \sum_{j\in\mathbb{Z}} S_{j,k_1,k_2}$.

We claim that there is $c>0$ such that for $j, k_1', k_2' \in \mathbb{Z}$, $k_1, k_2 \in \mathbb{N}$,

$$\begin{aligned} &\big\|\mathcal{Q}_{k_1'} S_{j+k_1',k_1,k_2} \mathcal{Q}_{j+k_1'+k_2'}\big\|_{L^2\to L^2}\\ &\qquad \lesssim \min\big\{ \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} n 2^{-(k_1+k_2)c\varepsilon},\, 2^{-|k_2-k_2'|-|k_1+j|} \sup_j \|\varsigma_j\|_{L^1} \big\}. \end{aligned} \tag{10.8}$$

To see this observe first that using

$$\big\|\mathcal{Q}_{k_1'}\, {}^t\overline{Q}_{j+k_1'+k_1}[u_1]\big\|_{L^2\to L^2} \lesssim 2^{-|k_1+j|}, \qquad \|\overline{Q}_{j+k_1'+k_2}[u_2] \mathcal{Q}_{j+k_1'+k_2'}\|_{L^2\to L^2} \lesssim 2^{-|k_2-k_2'|},$$

it follows from the simple Lemma 2.7 that

$$\big\|\mathcal{Q}_{k_1'} S_{j+k_1',k_1,k_2} \mathcal{Q}_{j+k_1'+k_2'}\big\|_{L^2\to L^2} \lesssim 2^{-|k_2-k_2'|-|k_1+j|} \|\varsigma_{j+k_1'}\|_{L^1}.$$

Using $\|\mathcal{Q}_{k_1'}\|_{L^2\to L^2}, \|\mathcal{Q}_{j+k_1'+k_2'}\|_{L^2\to L^2} \lesssim 1$, it follows from the main L^2-estimate, Theorem 7.8, that

$$\big\|\mathcal{Q}_{k_1'} S_{j+k_1',k_1,k_2} \mathcal{Q}_{j+k_1'+k_2'}\big\|_{L^2\to L^2} \lesssim \|\varsigma_{j+k_1'}\|_{\mathcal{B}_\varepsilon} n 2^{-(k_1+k_2)c\varepsilon}$$

for some $c>0$ (independent of n). Inequality (10.8) follows from a combination of the two bounds.

To prove (10.7) we use Lemma 9.1 and inequality (10.8) to conclude

$$\begin{aligned}\|S^{n+1,n+2}_{k_1,k_2}\|_{L^2\to L^2} &\lesssim \sum_{j,k_2'\in\mathbb{Z}} \min\big\{\sup_{j'}\|\varsigma_{j'}\|_{\mathcal{B}_\varepsilon} n2^{-(k_1+k_2)c\varepsilon},\, 2^{-|k_2-k_2'|-|k_1+j|}\sup_{j'}\|\varsigma_{j'}\|_{L^1}\big\}\\ &\lesssim \min\big\{\sup_j\|\varsigma_j\|_{\mathcal{B}_\varepsilon} n^{1/2}2^{-(k_1+k_2)c\varepsilon/2},\, \sup_j\|\varsigma_j\|_{L^1}\big\},\end{aligned}$$

where we have used $\|\varsigma_j\|_{L^1}\le\|\varsigma_j\|_{\mathcal{B}_\varepsilon}$. This completes the proof (with c replaced by $c/2$). □

11. Proof of Theorem 5.1: Part II

This section is devoted to the boundedness of the multilinear forms $\Lambda^1_{l,n+2}$ and $\Lambda^1_{l,n+1}$. In §11.1 we shall formulate and prove a crucial L^2 bound for a useful generalization of the form of $\Lambda^1_{l,n+2}$ and then deduce the asserted estimates for $\Lambda^1_{l,n+2}$, and $\Lambda^1_{l,n+2}$. The proof of the main L^2 bound will be given in §11.2.

11.1. **The main L^2 estimate.** For $2\le l\le n$, fix bounded sets $\{b_l^j : j\in\mathbb{Z}\}\subset L^\infty(\mathbb{R}^d)$ with

$$\sup_j \|b_l^j\|_\infty \le 1, \quad l=2,\dots,n.$$

For $b_1\in L^\infty(\mathbb{R}^d)$, $j\in\mathbb{Z}$ define an operator

$$W_j[\varsigma_j,b_1]\equiv W_j[\varsigma_j,b_1,b_2^j,\dots,b_n^j]$$

by

$$\int g(x)\,W_j[\varsigma_j,b_1]f(x)\,dx = \Lambda[\varsigma_j^{(2^j)}](b_1,b_2^j,\dots,b_n^j,f,g),$$

and we denotes its transpose by ${}^tW_j[b_1]$:

$$\int f(x)\,{}^tW_j[\varsigma_j,b_1]g(x)\,dx = \Lambda[\varsigma_j^{(2^j)}](b_1,b_2^j,\dots,b_n^j,f,g),$$

Define an operator $\mathcal{T}_N=\mathcal{T}_N[\vec\varsigma,b_1]$ by

$$\mathcal{T}_N=\sum_{j=-N}^{N}(I-P_j)W_j[\varsigma_j,(I-P_j)b_1]P_j.$$

Using $I-P_j=\sum_{k>0}Q_{j+k}$ for the operator on the left we decompose $\mathcal{T}_N=\sum_{k>0}\mathcal{T}_N^k$ where

$$\mathcal{T}_N^k=\sum_{j=-N_1}^{N_2}Q_{j+k}W_j[\varsigma_j,(I-P_j)b_1]P_j.$$

We now state our main estimate and give the proof that it implies Part III of Theorem 5.1 in §11.3 below.

Theorem 11.1. *Let $0<\varepsilon\le 1$, and $\sup_j\|\varsigma_j\|_{\mathcal{B}_\varepsilon}<\infty$. Let Γ_ε be as in (5.6). Then $\mathcal{T}_N^k$ converges to an operator $\mathcal{T}^k$, and $\mathcal{T}_N$ converges to an operator $\mathcal{T}$, in the strong operator topology as operators $L^2\to L^2$. Moreover,*

$$\|\mathcal{T}^k\|_{L^2\to L^2}\le C_{d,\varepsilon}\|b_1\|_\infty\sup_j\|\varsigma_j\|_{L^1}\min\{2^{-\varepsilon_1 k}n\Gamma_\varepsilon^2,\log^{3/2}(1+n\Gamma_\varepsilon)\}.$$

and

$$\|\mathcal{T}\|_{L^2\to L^2}\le C_{d,\varepsilon}\|b_1\|_\infty\sup_j\|\varsigma_j\|_{L^1}\log^{5/2}(1+n\Gamma_\varepsilon).$$

11.2. **Proof of Theorem 11.1.** For fixed $k > 0$, in order to bound $\mathcal{T}^k$ we need to prove that for $f \in L^2$ the limit

$$\sum_{j=-N}^{N} Q_{j+k} W_j[\varsigma_j, (I - P_j)b_1] P_j f$$

exists in L^2 as $N \to \infty$ and that the estimate

(11.1)
$$\Big\| \sum_{j=-N}^{N} Q_{j+k} W_j[\varsigma_j, (I - P_j)b_1] P_j \Big\|_{L^2 \to L^2} \lesssim \|b_1\|_\infty \sup_j \|\varsigma_j\|_{L^1} \min\{2^{-\varepsilon_1 k} n \Gamma_\varepsilon^2, \log^{3/2}(1 + n\Gamma_\varepsilon)\}$$

holds uniformly in N. By Proposition 6.5, both statements are a consequence of a square-function estimate, namely, for $k > 0$

$$(11.2) \quad \Big(\sum_{j \in \mathbb{Z}} \|\overline{Q}_{j+k}[u] W_j[\varsigma_j, (I - P_j)b_1] P_j f\|_2^2 \Big)^{1/2} \lesssim \|b_1\|_\infty \|f\|_2 \|u\|_{\mathfrak{U}} \sup_j \|\varsigma_j\|_{L^1} \min\{2^{-\varepsilon_1 k} n \Gamma_\varepsilon^2, \log^{3/2}(1 + n\Gamma_\varepsilon)\}.$$

To show (11.2) one establishes the following two inequalities:

(11.3)
$$\Big(\sum_{j \in \mathbb{Z}} \int \big|\overline{Q}_{j+k}[u] W_j[\varsigma_j, (I - P_j)b_1] P_j f(x) - \overline{Q}_{j+k}[u] W_j[\varsigma_j, (I - P_j)b_1] 1(x) \cdot P_j f(x)\big|^2 \Big)^{1/2} \lesssim \|f\|_2 \|b_1\|_\infty \|u\|_{\mathfrak{U}} \sup_j \|\varsigma_j\|_{L^1} \min\{2^{-\varepsilon_1 k} n \Gamma_\varepsilon^2, \log(1 + n\Gamma_\varepsilon)\}.$$

and

$$(11.4) \quad \Big(\sum_{j \in \mathbb{Z}} \int \big|\overline{Q}_{j+k}[u] W_j[\varsigma_j, (I - P_j)b_1] 1(x) \cdot P_j f(x)\big|^2 \Big)^{1/2} \lesssim \|f\|_2 \|b_1\|_\infty \|u\|_{\mathfrak{U}} \sup_j \|\varsigma_j\|_{L^1} \min\{2^{-\varepsilon_1 k} n \Gamma_\varepsilon^2, \log^{3/2}(1 + n\Gamma_\varepsilon)\}.$$

For the proof of (11.4) we need the notion of a *Carleson function.*

Definition 11.2. We say a measurable function $w : \mathbb{R}^d \times \mathbb{Z} \to \mathbb{C}$ is a Carleson function if there is a constant c such that for all $k \in \mathbb{Z}$ and balls B of radius 2^{-k} ($k \in \mathbb{Z}$),

$$\Big(\frac{1}{|B|} \int_B \sum_{j=k}^{\infty} |w(x,j)|^2 \, dx \Big)^{\frac{1}{2}} \le c < \infty.$$

The smallest such c is denoted by $\|w\|_{carl}$.

Remark. w is a Carleson function if the measure $d\mu(x,t) = \sum_{j \in \mathbb{Z}} |w(x,j)|^2 dx \, d\delta_{2^{-j}}(t)$ is a Carleson measure on the upper half plane (in the usual sense) and the norm $\|w\|_{carl}$ is equivalent with the square root of the Carleson norm of μ.

Carleson measures or Carleson functions can be used to prove L^2-boundedness of nonconvolution operators. This idea goes back to Coifman and Meyer [11, ch. VI] and was crucial in the proof of the David-Journé theorem [13]. One uses Carleson functions via the following special case of the *Carleson embedding theorem.* A proof can be found e.g. in [28, §6.III] or [36, §II.2].

Theorem. *Let w be a Carleson function. Then,*

$$\Big(\sum_{j\in\mathbb{Z}}\int |P_jf(x)|^2|w(x,j)|^2\,dx\Big)^{\frac12}\le C_d\|w\|_{carl}\|f\|_2.$$

Note that (11.4) is an immediate consequence of this theorem and the following proposition.

Proposition 11.3. *The function*

$$w_k(x,j)=\overline{Q}_{j+k}[u]W_j[\varsigma_j,(I-P_j)b_1]1(x)$$

defines a Carleson function and there is $C\lesssim 1$ so that for $0<\varepsilon'\le C^{-1}\varepsilon^2$ we have the estimate

$$\|w_k\|_{carl}\lesssim \overline{Q}_{j+k}[u]\|b_1\|_\infty\|u\|_{\mathfrak{U}}\sup_j\|\varsigma_j\|_{L^1}\min\{2^{-k\varepsilon'}n\Gamma_\varepsilon^2,\log^{3/2}(1+n\Gamma_\varepsilon)\}. \tag{11.5}$$

Our next proposition is a restatement of the other square-function estimate (11.3), in a slightly more general form.

Proposition 11.4. *Let $0<\varepsilon\le 1$. There exists $C\lesssim 1$ so that for $0<\varepsilon'\le C^{-1}\varepsilon$*

$$\Big(\sum_{j\in\mathbb{Z}}\int \big|\overline{Q}_{j+k}[u]W_j[\varsigma_j,b_1^j]P_jf(x)-\overline{Q}_{j+k}[u]W_j[\varsigma_j,b_1^j]1(x)\cdot P_jf(x)\big|^2\Big)^{1/2}$$
$$\lesssim \|f\|_2\sup_j\|b_1^j\|_\infty\|u\|_{\mathfrak{U}}\sup_j\|\varsigma_j\|_{L^1}\min\{2^{-k\varepsilon'}n\Gamma_\varepsilon^2,\log(1+n\Gamma_\varepsilon)\}.$$

We emphasize that the implicit constants in the above propositions are independent of n and independent of the choices of b_i^j with $\|b_i^j\|_\infty=1$.

11.2.1. *Proof of Proposition 11.3.* We need to prove for $x_0\in\mathbb{R}^d$, $\ell\in\mathbb{Z}$,

$$\Big(\sum_{j\ge-\ell}\frac{1}{|B^d(x_0,2^\ell)|}\int_{B^d(x_0,2^\ell)}\big|\overline{Q}_{j+k}[u]W_j[\varsigma_j,(I-P_j)b_1,b_2^j,\dots,b_n^j]1(x)\big|\,dx\Big)^{1/2} \tag{11.6}$$
$$\lesssim\|b_1\|_\infty\|u\|_{\mathfrak{U}}\sup_j\|\varsigma_j\|_{L^1}\min\{2^{-k\varepsilon'}n\Gamma_\varepsilon^2,\log^{3/2}(1+n\Gamma_\varepsilon)\}.$$

Now

$$\frac{1}{|B^d(x_0,2^\ell)|}\int_{B^d(x_0,2^\ell)}\big|\overline{Q}_{j+k}[u]W_j[\varsigma_j,(I-P_j)b_1,b_2^j,\dots,b_n^j]f(x)\big|\,dx$$
$$=\frac{1}{|B^d(0,1)|}\int_{B^d(0,1)}\big|\overline{Q}_{j+k}[u]W_j[\varsigma_j,(I-P_j)b_1,b_2^j,\dots,b_n^j]f(x_0+2^\ell x)\big|\,dx$$

and we have by changes of variables

$$\overline{Q}_{j+k}[u]W_j[\varsigma_j,(I-P_j)b_1,b_2^j,\dots,b_n^j]f(x_0+2^\ell x) \tag{11.7}$$
$$=\overline{Q}_{j+\ell+k}[u]W_{j+\ell}[\varsigma_j,(I-P_{j+\ell})\tilde b_1,\tilde b_2^j,\dots,\tilde b_n^j]\tilde f(x)$$

where $\tilde b_1(x)=b_1(x_0+2^\ell x)$, $\tilde b_i^j(x)=b_i^j(x_0+2^\ell x)$, $\tilde f(x)=f(x_0+2^\ell x)$. Applying this with $f=1$ we see that it suffices to prove (11.6) with $x_0=0$, $\ell=0$.

The somewhat lengthy proof will be given in a series of lemmata, partially relying on the L^2 boundedness results in §7. Our first lemma is a restatement of such a result.

Lemma 11.5. *Let $0<\varepsilon<1$. There is $C\lesssim 1$ so that for all $\varepsilon'\le C^{-1}\varepsilon$ we have for all $k\ge 0$, and for all $u\in\mathfrak{U}$,*

$$\|\overline{Q}_{j+k}[u]W_j[\varsigma_j,b_1]\|_{L^2\to L^2}\lesssim\min\big\{n2^{-k\varepsilon'}\|\varsigma_j\|_{\mathcal{B}_\varepsilon},\,\|\varsigma_j\|_{L^1}\big\}\,\|b_1\|_\infty\|u\|_{\mathfrak{U}}.$$

Proof. For $f, g \in L^2$, we have

$$\int g(x)(\overline{Q}_{j+k}[u]W_j[\varsigma_j, b_1]f(x))\,dx = \Lambda[\varsigma_j^{(2^j)}](b_1, b_2^j, \ldots, b_n^j, f, {}^t\overline{Q}_{j+k}[u]g).$$

From here, the result follows immediately from Theorem 7.8. □

We now give an estimate on $\Lambda[\varsigma^{(2^j)}](b_1, \ldots, b_{n+2})$ under the assumptions that the supports of b_1 and b_{n+2} are separated.

Lemma 11.6. *Let $0 < \varepsilon \le 1$. For all $j, k \ge 0$, $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$, $u \in \mathcal{U}$, $R \ge 5$, $b_1, \ldots, b_{n+1} \in L^\infty(\mathbb{R}^d)$, $b_{n+2} \in L^1(\mathbb{R}^d)$, with*

$$\mathrm{supp}(b_1) \subseteq \{|v| \ge R\}, \quad \mathrm{supp}(b_{n+2}) \subseteq \{|v| \le 1\},$$

we have

$$\left|\Lambda[\varsigma_j^{(2^j)}](b_1, \ldots, b_{n+1}, \overline{Q}_{j+k}[u]b_{n+2})\right| \lesssim \|u\|_{\mathcal{U}}\Big(\prod_{l=1}^{n+1} \|b_l\|_\infty\Big)\|b_{n+2}\|_{L^1} \min\left\{(2^jR)^{-\varepsilon/4}\|\varsigma\|_{\mathcal{B}_\varepsilon}, \|\varsigma\|_{L^1}\right\}.$$

Proof. Without loss of generality, we take $\|b_l\|_{L^\infty} = 1$, $1 \le l \le n+1$, $\|b_{n+2}\|_{L^1} = 1$, and $\|u\|_{\mathcal{U}} = 1$. The bound

$$|\Lambda[\varsigma_j^{(2^j)}](b_1, \ldots, b_{n+1}, \overline{Q}_{j+k}[u]b_{n+2})| \lesssim \|\varsigma\|_{L^1}$$

follows immediately from Lemma 2.7, so we prove only the estimate

$$|\Lambda[\varsigma_j^{(2^j)}](b_1, \ldots, b_{n+1}, \overline{Q}_{j+k}[u]b_{n+2})| \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon}(2^jR)^{-\varepsilon/4}. \tag{11.8}$$

We estimate

$$\begin{aligned}
&|\Lambda[\varsigma_j^{(2^j)}](b_1, \ldots, b_{n+1}, \overline{Q}_{j+k}[u]b_{n+2})| \\
&= \Big|\iiiint \varsigma^{(2^j)}(\alpha, v)\Big(\prod_{i=1}^n b_i(x - \alpha_i v)\Big)b_{n+1}(x - v)u^{(2^{j+k})}(x - x')b_{n+2}(x')\,dx\,dx'\,d\alpha\,dv\Big| \\
&\le \sup_{|x'| \le 1} \iiint |\varsigma^{(2^j)}(\alpha, v)||b_1(x - \alpha_1 v)||u^{(2^{j+k})}(x - x')|\,dx\,d\alpha\,dv.
\end{aligned}$$

Fix $x' \in \mathbb{R}^d$ with $|x'| \le 1$. Then

$$\begin{aligned}
&\iiint |\varsigma^{(2^j)}(\alpha, v)||b_1(x - \alpha_1 v)||u^{(2^{j+k})}(x - x')|\,dx\,d\alpha\,dv \\
&\le \iiint |\varsigma(\alpha, v)||b_1(x - \alpha_1 2^{-j}v)|2^{d(j+k)}(1 + 2^{j+k}|x - x'|)^{-d-\frac{1}{2}}\,dx\,d\alpha\,dv \\
&= \sum_{l_1=0}^\infty \sum_{l_2=0}^\infty \iiint\limits_{\substack{2^{l_1} \le 1+|v| \le 2^{l_1+1} \\ 2^{l_2} \le 1+2^{j+k}|x-x'| \le 2^{l_2+1}}} |\varsigma(\alpha, v)||b_1(x - \alpha_1 2^{-j}v)|2^{d(j+k)}(1 + 2^{j+k}|x - x'|)^{-d-\frac{1}{2}}\,dx\,d\alpha\,dv \\
&= \sum_{l_1=0}^\infty \sum_{2^{l_2} \ge R2^{j+k-2}} + \sum_{l_1=0}^\infty \sum_{2^{l_2} < R2^{j+k-2}} =: (I) + (II).
\end{aligned}$$

We begin with (I). We have, provided $\varepsilon' \le \varepsilon$,

$$(I) \le \sum_{l_1=0}^{\infty} \sum_{2^{l_2} \ge R2^{j+k-2}} 2^{-l_1\varepsilon' - l_2/4} \times$$
$$\iiint\limits_{\substack{2^{l_1} \le 1+|v| \le 2^{l_1+1} \\ 2^{l_2} \le 1+2^{j+k}|x-x'| \le 2^{l_2+1}}} (1+|v|)^{\varepsilon'} |\varsigma(\alpha,v)||b_1(x-\alpha_1 2^{-j}v)| \frac{2^{d(j+k)}}{(1+2^{j+k}|x-x'|)^{d+\frac14}}\, dx\, d\alpha\, dv$$
$$\lesssim \sum_{l_1=0}^{\infty} \sum_{2^{l_2} \ge R2^{j+k-2}} 2^{-l_1\varepsilon' - l_2/4} \|\varsigma\|_{\mathcal{B}_\varepsilon} \lesssim (2^{j+k}R)^{-1/4} \|\varsigma\|_{\mathcal{B}_\varepsilon} \lesssim (2^j R)^{-1/4} \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

We now turn to (II). We have

$$(II) = \sum_{l_1=0}^{\infty} \sum_{2^{l_2} < R2^{j+k-2}} 2^{-l_1\varepsilon' - l_2/4} \times$$
$$\iiint\limits_{\substack{2^{l_1} \le 1+|v| \le 2^{l_1+1} \\ 2^{l_2} \le 1+2^{j+k}|x-x'| \le 2^{l_2+1}}} (1+|v|)^{\varepsilon'} |\varsigma(\alpha,v)||b_1(x-\alpha_1 2^{-j}v)| \frac{2^{d(j+k)}}{(1+2^{j+k}|x-x'|)^{d+\frac14}}\, dx\, d\alpha\, dv\,.$$

On the support of the integral, $|x-\alpha_1 2^{-j}v| \ge R$ (by the support of b_1). Since $1+2^{j+k}|x-x'| \le 2^{l_2+1}$, we have $|x-x'| \le 2^{l_2+1-j-k}$. Thus, $|x| \le 2^{l_2+1-j-k}+1 \le \frac{R}{2}+1 \le \frac{R}{2}+\frac{R}{5} \le \frac34 R$. Thus, $|\alpha_1 2^{-j}v| \gtrsim R$ and therefore $|\alpha_1| \gtrsim 2^j \frac{R}{|v|} \gtrsim 2^{j-l_1}R$. Plugging this in, we have for $\varepsilon' = \varepsilon/2$,

$$(II) \lesssim \sum_{l_1=0}^{\infty} \sum_{2^{l_2} < R2^{j+k-2}} 2^{-l_1\varepsilon' - l_2/4} (1+2^{j-l_1}R)^{-\frac{\varepsilon'}{2}} \iiint\limits_{\substack{2^{l_1} \le 1+|v| \le 2^{l_1+1} \\ 2^{l_2} \le 1+2^{j+k}|x-x'| \le 2^{l_2+1}}} (1+|v|)^{\varepsilon'} \times$$
$$(1+|\alpha_1|)^{\frac{\varepsilon'}{2}} |\varsigma(\alpha,v)||b_1(x-\alpha_1 2^{-j}v)| \frac{2^{d(j+k)}}{(1+2^{j+k}|x-x'|)^{d+\frac14}}\, dx\, d\alpha\, dv$$
$$\lesssim \sum_{l_1=0}^{\infty} \sum_{2^{l_2} < R2^{j+k-2}} 2^{-l_1\varepsilon' - l_2/4} (1+2^{j-l_1}R)^{-\frac{\varepsilon'}{2}} \|\varsigma\|_{\mathcal{B}_\varepsilon} \lesssim (2^j R)^{-\varepsilon'/2}\,.$$

Combine the estimates for (I) and (II) to obtain (11.8) and the proof of the lemma is complete. ∎

Lemma 11.7. *Let $0 < \varepsilon \le 1$. Then for all $j, k \ge 0$, $u \in \mathfrak{U}$, $R \ge 5$, and $b_1 \in L^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(b_1) \subseteq \{|v| \ge R\}$ we have*

$$\Big(\int_{|x|\le 1} |(\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1]1)(x)|^2\, dx \Big)^{1/2} \lesssim \|u\|_{\mathfrak{U}} \|b_1\|_\infty \min\big\{ (2^j R)^{-\varepsilon/4} \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \|\varsigma_j\|_{L^1} \big\}.$$

Proof. Let $B = \{x : |x| \le 1\}$. We have, by the previous lemma,

$$\Big(\int_B |(\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1]1)(x)|^2\, dx \Big)^{1/2} \le \sup_{\substack{\|b_{n+2}\|_1 = 1 \\ \operatorname{supp}(b_{n+2}) \subseteq B}} \big| \Lambda[\varsigma_j^{(2^j)}](b_1, b_2^j, \dots, b_n^j, 1, {}^t\overline{Q}_{j+k}[u] b_{n+2}) \big|$$
$$\lesssim \sup_{\substack{\|b_{n+2}\|_1 = 1 \\ \operatorname{supp}(b_{n+2}) \subseteq B}} \|u\|_{\mathfrak{U}} \|b_1\|_\infty \|b_{n+2}\|_1 \min\big\{ (2^j R)^{-\varepsilon/4} \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \|\varsigma_j\|_{L^1} \big\}$$

and the assertion follows. ∎

For $j, k_1, k_2 \ge 0$ and $u \in \mathcal{U}$, define an operator $V_{j,k_1,k_2} \equiv V^{\varsigma_j,u}_{j,k_1,k_2}$ by

$$\int f(x)\, V_{j,k_1,k_2} g(x)\, dx = \int g(x)\, (\overline{Q}_{j+k_1}[u] W_j[\varsigma_j, Q_{j+k_2} f]1)(x)\, dx$$
$$= \Lambda[\varsigma_j^{(2^j)}](Q_{j+k_2} f, b_2^j, \dots, b_n^j, 1, {}^t\overline{Q}_{j+k_1}[u] g).$$

Lemma 11.8. *Let $0 < \varepsilon \le 1$. There exists $c > 0$ (independent of n and ε) such that for $\varepsilon' \le c\varepsilon$, $k_1, k_2 \ge 0$, and for all $f \in L^2(\mathbb{R}^d)$,*

$$\Big(\int \sum_{j\ge 0} |{}^t V_{j,k_1,k_2} f(x)|^2\, dx \Big)^{1/2} \lesssim \|f\|_{L^2} \|u\|_{\mathcal{U}} \sup_j \|\varsigma_j\|_{L^1} \min\big\{1, n 2^{-\varepsilon'(k_1+k_2)} \Gamma_\varepsilon\big\}.$$

Proof. From Theorem 10.1 we get the bound

$$\Big\| \sum_{j\ge 0} V_{j,k_1,k_2} \Big\|_{L^2\to L^2} \lesssim \|u\|_{\mathcal{U}} \min\big\{1, n 2^{-\varepsilon'(k_1+k_2)} \Gamma_\varepsilon\big\}. \tag{11.9}$$

Let δ_j be any sequence of ± 1. Note that $\delta_j V_{j,k_1,k_2}$ is of the same form as V_{j,k_1,k_2} with ς_j replaced by $\delta_j \varsigma_j$. Thus, by (11.9),

$$\Big\| \sum_{j\ge 0} \delta_j {}^t V_{j,k_1,k_2} f \Big\|_2 \lesssim \|f\|_{L^2} \|u\|_{\mathcal{U}} \min\big\{1, n 2^{-\varepsilon'(k_1+k_2)} \Gamma_\varepsilon\big\},$$

where the implicit constant does not depend on the particular sequence δ_j. By Khinchine's inequality

$$\Big(\int \sum_{j\ge 0} |{}^t V_{j,k_1,k_2} f(x)|^2\, dx \Big)^{1/2} \lesssim \sup \Big\| \sum_{j\ge 0} \delta_j\, {}^t V_{j,k_1,k_2} f \Big\|_2,$$

where the sup is taken over all ± 1-sequences $\{\delta_j\}$. The result follows. $\square$

Lemma 11.9. *Let $0 < \varepsilon < 1$. There exists $c > 0$ (independent of n and ε) so that for $\varepsilon' \le c\varepsilon^2$, for all $b_1 \in L^\infty(\mathbb{R}^d)$, for all $u \in \mathcal{U}$,*

$$\Big(\sum_{j\ge 0} \int_{|x|\le 1} \big| (\overline{Q}_{j+k_1}[u] W_j[\varsigma_j, (I - P_j) b_1]1)(x) \big|^2\, dx \Big)^{\frac{1}{2}}$$
$$\le C(\varepsilon, d) \|u\|_{\mathcal{U}} \|b_1\|_\infty \sup_j \|\varsigma_j\|_{L^1} \min\{2^{-k_1\varepsilon'} n \Gamma_\varepsilon^2, \log^{3/2}(1 + n\Gamma_\varepsilon)\}.$$

Proof. Fix $b_1 \in L^\infty(\mathbb{R}^d)$ and $u \in \mathcal{U}$. We may assume $\|b_1\|_{L^\infty} = 1$ and $\|u\|_{\mathcal{U}} = 1$. Fix $0 < \beta \le 1$ and $\delta > 0$ to be chosen later, see (11.11) below. Given $k_1, k_2 \ge 0$ we decompose $b_1 = b_{1,\infty}^{k_1,k_2} + b_{1,0}^{k_1,k_2}$ where

$$b_{1,\infty}^{k_1,k_2}(y) := \begin{cases} b_1(y) & \text{if } |y| \ge \max\{10, \beta\, 2^{1+\delta(k_1+k_2)}\} \\ 0 & \text{if } |y| < \max\{10, \beta\, 2^{1+\delta(k_1+k_2)}\} \end{cases},$$
$$b_{1,0}^{k_1,k_2}(y) := b_1(y) - b_{1,\infty}^{k_1,k_2}(y).$$

We expand $I - P_j = \sum_{k_2} Q_{j+k_2}$ and then have

$$\Big(\sum_{j\ge 0} \int_B |(\overline{Q}_{j+k_1}[u] W_j[\varsigma_j, (I - P_j) b_1]1)(x)|^2\, dx \Big)^{1/2} \le (I) + (II)$$

where

$$(I) := \sum_{k_2>0} \Big(\sum_{j\geq 0} \int_B |(\overline{Q}_{j+k_1}[u] W_j[\varsigma_j, Q_{j+k_2} b_{1,\infty}^{k_1,k_2}]1)(x)|^2 \, dx \Big)^{1/2},$$

$$(II) := \sum_{k_2>0} \Big(\sum_{j\geq 0} \int_B |(\overline{Q}_{j+k_1}[u] W_j[\varsigma_j, Q_{j+k_2} b_{1,0}^{k_1,k_2}]1)(x)|^2 \, dx \Big)^{1/2}.$$

We begin by estimating (I). Because $j, k_2 \geq 0$,

$$\mathrm{supp}(Q_{j+k_2} b_{1,\infty}^{k_1,k_2}) \subseteq \{y : |y| \geq R_{k_1,k_2}\} \text{ where } R_{k_1,k_2} := \max\{5,\ \beta 2^{(k_1+k_2)\delta}\},$$

we may apply Lemma 11.7 to see

$$\begin{aligned}
(I) &= \sum_{k_2>0} \Big(\sum_{j\geq 0} \int_B |(\overline{Q}_{j+k_1}[u] W_j[\varsigma_j, Q_{j+k_2} b_{1,\infty}^{k_1,k_2}]1)(x)|^2 \, dx \Big)^{1/2} \\
&\lesssim \sum_{k_2>0} \Big(\sum_{j\geq 0} \min\big\{ (2^j R_{k_1,k_2})^{-2\varepsilon/4} \|\varsigma_j\|_{\mathcal{B}_\varepsilon}^2, \sup_{j'} \|\varsigma_{j'}\|_{L^1}^2 \big\} \Big)^{1/2} \\
&\leq \sup_{j'} \|\varsigma_{j'}\|_{L^1} \sum_{k_2>0} \Big(\sum_{j\geq 0} \min\{1, 2^{-j\varepsilon/2-(k_1+k_2)\varepsilon\delta/2} \beta^{-\varepsilon/2} \Gamma_\varepsilon^2 \Big)^{1/2} \\
&\lesssim \sup_j \|\varsigma_j\|_{L^1} \sum_{k_2>0} \min\{1,\ 2^{-(k_1+k_2)\varepsilon\delta/4} \beta^{-\varepsilon/4} \Gamma_\varepsilon\} \log^{1/2}(1+\beta^{-2\varepsilon/4}\Gamma_\varepsilon^2) \\
&\lesssim \sup_j \|\varsigma_j\|_{L^1} \min\{1,\ 2^{-k_1\varepsilon\delta/4} \beta^{-\varepsilon/4} \Gamma_\varepsilon\} \log^{3/2}(1+\beta^{-\varepsilon/4}\Gamma_\varepsilon).
\end{aligned}$$

We now turn to (II). We have $\|b_{1,0}^{k_1,k_2}\|_2 \lesssim R_{k_1,k_2}^{d/2} \|b_{1,0}^{k_1,k_2}\|_\infty \lesssim R_{k_1,k_2}^{d/2}$ and use Lemma 11.8 to estimate, for some $c_1 \in (0,1)$,

$$\begin{aligned}
(II) &= \sum_{k_2>0} \Big(\sum_{j\geq 0} \int_B |(\overline{Q}_{j+k_1}[u] W_j[\varsigma_j, Q_{j+k_2} b_{1,0}^{k_1,k_2}]1)(x)|^2 \, dx \Big)^{\frac12} \\
&= \sum_{k_2>0} \Big(\sum_{j\geq 0} \int_B |{}^t V_{j,k_1,k_2} b_{1,0}^{k_1,k_2}(x)|^2 \, dx \Big)^{\frac12} \\
&\lesssim \sup_j \|\varsigma_j\|_{L^1} \sum_{k_2>0} \|b_{1,0}^{k_1,k_2}\|_{L^2} \min\{1,\ n 2^{-c_1\varepsilon(k_1+k_2)} \Gamma_\varepsilon\} \\
&\lesssim \sup_j \|\varsigma_j\|_{L^1} \sum_{k_2>0} \big(1+\beta 2^{k_1\delta+k_2\delta}\big)^{d/2} \min\{1,\ n 2^{-c_1\varepsilon(k_1+k_2)} \Gamma_\varepsilon\} \\
&= \sum_{\substack{k_2>0 \\ 1<\beta\, 2^{k_1\delta+k_2\delta}}} + \sum_{\substack{k_2>0 \\ 1\geq\beta\, 2^{k_1\delta+k_2\delta}}} =: (II_1) + (II_2).
\end{aligned} \tag{11.10}$$

We take

$$\beta = (n\Gamma_\varepsilon)^{-1/d}, \qquad \delta = \frac{c_1\varepsilon}{2d}. \tag{11.11}$$

Notice that since $\beta\, 2^{k_1\delta+k_2\delta} \geq 1$ in the sum (II_1) we may replace the power $d/2$ by d and get, with the choice (11.11),

$$\begin{aligned}
(\beta 2^{k_1\delta+k_2\delta})^{d/2} (n 2^{-\varepsilon' k_1 - \varepsilon' k_2} \Gamma_\varepsilon) &\leq \beta^d n \Gamma_\varepsilon 2^{(k_1+k_2)(\delta d - c_1\varepsilon)} \\
&\leq 2^{-(k_1+k_2)c_1\varepsilon/2}
\end{aligned}$$

and thus
$$(II_1) \lesssim \sum_{k_2>0} 2^{-(k_1+k_2)c_1\varepsilon/2} \sup_j \|\varsigma_j\|_{L^1} \lesssim 2^{-k_1c_1\varepsilon/2} \sup_j \|\varsigma_j\|_{L^1}.$$
Next,
$$\begin{aligned}(II_2) &\lesssim \sup_j \|\varsigma_j\|_{L^1} \sum_{k_2>0} \min\{1,\, n2^{-(k_1+k_2)c_1\varepsilon}\Gamma_\varepsilon\} \\ &\lesssim \sup_j \|\varsigma_j\|_{L^1} \times \begin{cases} \log(2+2^{-c_1\varepsilon k_1}\Gamma_\varepsilon n) & \text{if } 2^{-c_1\varepsilon k_1}\Gamma_\varepsilon n \geq 1 \\ 2^{-c_1\varepsilon k_1}\Gamma_\varepsilon n & \text{if } 2^{-c_1\varepsilon k_1}\Gamma_\varepsilon n \leq 1 \end{cases} \\ &\lesssim \sup_j \|\varsigma_j\|_{L^1} \min\{2^{-c_1\varepsilon k_1} n\Gamma_\varepsilon,\, \log(1+n\Gamma_\varepsilon)\}.\end{aligned}$$

Finally we use the choice (11.11) in the above estimate for (I) and get
$$\begin{aligned}(I) &\lesssim \sup_j \|\varsigma_j\|_{L^1} \min\{1,\, 2^{-k_1\frac{c_1\varepsilon^2}{8d}} n^{\frac{\varepsilon}{4d}} \Gamma_\varepsilon^{1+\frac{\varepsilon}{4d}}\} \log^{3/2}(1+n^{\frac{\varepsilon}{4d}}\Gamma_\varepsilon^{1+\frac{\varepsilon}{4d}}) \\ &\lesssim \sup_j \|\varsigma_j\|_{L^1} \min\{1,\, 2^{-k_1c\varepsilon^2} n\Gamma_\varepsilon^2\} \log^{3/2}(1+n\Gamma_\varepsilon)\end{aligned}$$
with $c = c_1/8d$. Combining this estimate with the above estimates for (II_1) and (II_2) yields the assertion. □

Proof of Proposition 11.3, conclusion. The lemma is just a restatement of (11.6) for $x_0 = 0$ and $\ell = 0$ and by (11.7) we reduced the proof of (11.6) to this special case. □

11.2.2. *Proof of Proposition 11.4.* We start with an elementary observation for $f \in L^\infty$.

Lemma 11.10. *For all $k \geq 0$, $j \in \mathbb{Z}$, $b_1 \in L^\infty(\mathbb{R}^d)$, and $u \in \mathfrak{U}$,*
$$\|\overline{Q}_{j+k}[u]W_j[\varsigma_j, b_1]f\|_{L^\infty} \lesssim \|u\|_{\mathfrak{U}} \|\varsigma_j\|_{L^1} \|b_1\|_\infty \|f\|_\infty.$$

Proof. For $g \in L^1$ with $\|g\|_1 = 1$ we have, using Lemma 2.7,
$$\begin{aligned}\Big|\int g(x)(\overline{Q}_{j+k}[u]W_j[b_1]1)(x)\,dx\Big| &= \big|\Lambda[\varsigma_j^{(2^j)}](b_1, b_2^j, \dots, b_n^j, f, {}^t\overline{Q}_{j+k}[u]g)\big| \\ &\lesssim \|b_1\|_\infty \|f\|_\infty \|{}^t\overline{Q}_{j+k}[u]g\|_1 \|\varsigma_j\|_{L^1} \lesssim \|b_1\|_\infty \|u\|_{\mathfrak{U}} \|\varsigma_j\|_{L^1},\end{aligned}$$
completing the proof. □

Lemma 11.11. *There is $c \in (0,1)$ (independent of n and ε) so that for $\varepsilon' \leq c\varepsilon^2$, and all $k \geq 0$, $j \in \mathbb{Z}$, $u \in \mathfrak{U}$, $b_1 \in L^\infty(\mathbb{R}^d)$, $f \in L^2(\mathbb{R}^d)$ we have*
$$\Big(\int \big|\overline{Q}_{j+k}[u]W_j[\varsigma_j, b_1]1(x)P_jf(x)\big|^2 dx\Big)^{1/2} \lesssim \|f\|_2 \|u\|_{\mathfrak{U}} \|b_1\|_{L^\infty} \min\{\|\varsigma_j\|_{L^1},\, n2^{-k\varepsilon'}\|\varsigma_j\|_{\mathcal{B}_\varepsilon}\}.$$

Proof. We may normalize and assume $\|b_1\|_\infty = 1$. We may assume, by scale invariance of the result, that $j = 0$ (see (11.7)). The assertion follows then from the inequality
$$\Big(\int \big|\overline{Q}_k[u]W_0[\varsigma, b_1]1(x)P_0f(x)\big|^2 dx\Big)^{1/2} \lesssim \|u\|_{\mathfrak{U}} \|b_1\|_\infty \|f\|_2 \min\{\|\varsigma\|_{L^1}, n2^{-k\varepsilon'}\|\varsigma\|_{\mathcal{B}_\varepsilon}\}. \tag{11.12}$$
Because the convolution kernel of P_0 is supported in $B^d(0,1)$, it suffices to show (11.12) for functions supported in a ball B of radius 1. We may assume (by translating the functions b_i) that B is centered at the origin. Let B^* be the ball of double radius.

Now $\|P_0f\|_\infty \lesssim \|f\|_2$ for f supported in B, and therefore it suffices to show
$$\|\overline{Q}_k[u]W_0[\varsigma, b_1]1\|_{L^2(B^*)} \lesssim \|u\|_{\mathfrak{U}} \min\{n2^{-k\varepsilon'}\|\varsigma\|_{\mathcal{B}_\varepsilon},\, \|\varsigma_j\|_{L^1}\}. \tag{11.13}$$

To show (11.13) we split $1 = \mathbb{1}_{\Omega_{k\delta}} + \mathbb{1}_{\Omega_{k\delta}^{\complement}}$ where $\Omega_{k\delta} = \{x : |x| \le 5 \cdot 2^{k\delta}\}$, with a choice of $\delta \ll \varepsilon$ to be determined.

It follows from Lemma 11.5 (or directly from Theorem 7.8) that for some $c > 0$ (independent of n)

$$\begin{aligned} \|\overline{Q}_k[u]W_0[\varsigma, b_1]\mathbb{1}_{\Omega_{k\delta}}\|_{L^2(B^*)} &\lesssim \|\mathbb{1}_{\Omega_{k\delta}}\|_2 \|u\|_{\mathfrak{U}} \min\{n2^{-kc\varepsilon}\|\varsigma\|_{\mathcal{B}_\varepsilon}, \|\varsigma\|_{L^1}\} \\ &\lesssim \|u\|_{\mathfrak{U}} \min\{n2^{-k(c\varepsilon - d\delta)}\|\varsigma\|_{\mathcal{B}_\varepsilon}, \|\varsigma\|_{L^1}\} \end{aligned} \tag{11.14}$$

and thus we want to choose $\delta \le c\varepsilon(2d)^{-1}$.

Next we estimate the $L^2(B^*)$ norm of $\overline{Q}_k[u]W_0[\varsigma, b_1]\mathbb{1}_{\Omega_{k\delta}^{\complement}}$. Let $\tilde{\varsigma}(\alpha, v) = \varsigma(1-\alpha_1, \cdots, 1-\alpha_n, v)$ so that $\|\tilde{\varsigma}\|_{\mathcal{B}_\varepsilon} \lesssim \|\varsigma_j\|_{\mathcal{B}_\varepsilon}$ and $\|\tilde{\varsigma}\|_{L^1} = \|\varsigma_j\|_{L^1}$. We have, for $\|g\|_{L^2(B^*)} = 1$,

$$\begin{aligned} &\Big| \int g(x)(\overline{Q}_k[u]W_0[\varsigma, b_1]\mathbb{1}_{\Omega_{\delta k}^{\complement}})(x)\, dx \Big| \\ &= \big|\Lambda[\varsigma](b_1, b_2^0, \dots, b_n^0, \mathbb{1}_{\Omega_{k\delta}^{\complement}}, {}^t\overline{Q}_k[u]g)\big| = \big|\Lambda[\tilde{\varsigma}](b_1, b_2^0, \dots, b_n^0, {}^t\overline{Q}_k[u]g, \mathbb{1}_{\Omega_{k\delta}^{\complement}})\big| \\ &= \Big| \iiiint \tilde{\varsigma}(\alpha, v) b_1(x - \alpha_1 v)\big(\prod_{i=2}^n b_i^0(x - \alpha_i v)\big)\mathbb{1}_{\Omega_{\delta k}^{\complement}}(x) u^{(2^k)}(y - x + v) g(y)\, dx\, dy\, dv\, d\alpha \Big| \end{aligned}$$

and this is estimated by

$$\begin{aligned} &\iiiint_{|x| > 5 \cdot 2^{\delta k}} |\tilde{\varsigma}(\alpha, v)||u^{(2^k)}(y - x + v) g(y)|\, dx\, dv\, d\alpha\, dy \\ &\le \sum_{2^{l_1} \ge 2 \cdot 2^{k\delta}} \sum_{l_2=0}^{\infty} \iiiint_{\substack{2^{l_1} \le |x| \le 2^{l_1+1} \\ 2^{l_2} \le 1 + |v| \le 2^{l_2+1}}} |\tilde{\varsigma}(\alpha, v)||u^{(2^k)}(y - x + v) g(y)|\, dx\, dv\, d\alpha\, dy \\ &= \sum_{2^{l_1} \ge 2 \cdot 2^{k\delta}} \sum_{l_2=(l_1-3)\vee 0}^{\infty} + \sum_{2^{l_1} \ge 2 \cdot 2^{k\delta}} \sum_{l_2=0}^{l_1-3} =: (I) + (II). \end{aligned}$$

We estimate $(I) \lesssim$

$$\begin{aligned} &\sum_{2^{l_1} \ge 2 \cdot 2^{k\delta}} \sum_{l_2=(l_1-3)\vee 0}^{\infty} 2^{-\varepsilon l_2} \iiiint_{\substack{2^{l_1} \le |x| \le 2^{l_1+1} \\ 2^{l_2} \le 1 + |v| \le 2^{l_2+1}}} (1 + |v|)^\varepsilon |\tilde{\varsigma}(\alpha, v)||u^{(2^k)}(y - x + v) g(y)|\, dx\, dv\, d\alpha\, dy \\ &\lesssim \sum_{2^{l_1} \ge 2 \cdot 2^{k\delta}} \sum_{l_2=(l_1-3)\vee 0}^{\infty} 2^{-\varepsilon l_2} \|\tilde{\varsigma}\|_{\mathcal{B}_\varepsilon} \|u\|_{\mathfrak{U}} \|g\|_1 \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon} \|u\|_{\mathfrak{U}} \|g\|_1 2^{-k\delta\varepsilon} \lesssim \|\varsigma\|_{\mathcal{B}_\varepsilon} \|u\|_{\mathfrak{U}} 2^{-k\delta\varepsilon}, \end{aligned}$$

where the last inequality uses the support of g to see $\|g\|_1 \lesssim \|g\|_2 = 1$.

For (II), we use the fact that $l_2 \le l_1 - 3$ to see that on the support of the integral, since $|y| \le 1$ (due to the support of g), we have $|y - x + v| \approx 2^{l_1}$. Thus, we have

$$
\begin{aligned}
(II) &\lesssim \sum_{2^{l_1} \ge 2\cdot 2^{k\delta}} \sum_{l_2=0}^{l_1-3} \|u\|_{\mathfrak{U}} \iiiint\limits_{\substack{2^{l_1}\le|x|\le 2^{l_1+1}\\ 2^{l_2}\le 1+|v|\le 2^{l_2+1}}} |\tilde{\varsigma}(\alpha,v)| \frac{2^{kd}}{(1+2^k|x-v-y|)^{d+\frac12}} |g(y)|\, dx\, dv\, dy\, d\alpha \\
&\lesssim \sum_{2^{l_1} \ge 2\cdot 2^{k\delta}} \sum_{l_2=0}^{l_1-3} 2^{(-k-l_1)/4} \|u\|_{\mathfrak{U}} \iiiint\limits_{\substack{2^{l_1}\le|x|\le 2^{l_1+1}\\ 2^{l_2}\le 1+|v|\le 2^{l_2+1}}} |\tilde{\varsigma}(\alpha,v)| \frac{2^{kd}}{(1+2^k|x-v-y|)^{d+\frac14}} |g(y)|\, dx\, dv\, dy\, d\alpha \\
&\lesssim \sum_{2^{l_1} \ge 2\cdot 2^{k\delta}} \sum_{l_2=0}^{l_1-3} 2^{(-k-l_1)/4} \|u\|_{\mathfrak{U}} \|\tilde{\varsigma}\|_{L^1} \|g\|_1 \lesssim \|u\|_{\mathfrak{U}} \|\varsigma\|_{L^1} \|g\|_1 2^{-k/4} \lesssim 2^{-k/4} \|u\|_{\mathfrak{U}} \|\varsigma\|_{L^1}.
\end{aligned}
$$

Finally, we have, by Lemma 11.10 applied to $f = \mathbb{1}_{\Omega_{k\delta}^{\complement}}$,

$$\Big| \int g(x) (\overline{Q}_k[u] W_0[b_1] \mathbb{1}_{\Omega_{k\delta}^{\complement}}(x)\, dx \Big| \lesssim \|\varsigma\|_{L^1} \|u\|_{\mathfrak{U}},$$

where the last inequality uses the support of g again to see $\|g\|_1 \lesssim \|g\|_2 = 1$. If we take $\delta = c\varepsilon/(4d)$ then a combination of the estimates for (I) and (II), and (11.14) , yields (11.13) for $\varepsilon' \le c\varepsilon^2/(4d)$. This completes the proof. □

In what follows we find it convenient to occasionally use the notation

$$\mathrm{Mult}\{g\} f = fg \tag{11.15}$$

for the operator of pointwise multiplication with g.

Lemma 11.12. *Let $0 < \varepsilon \le 1/2$. Then there is $c > 0$ (independent of n, ε) such that for $\varepsilon' \le c\varepsilon^2$, for all $k \ge 0$, $j, l \in \mathbb{Z}$, $\varsigma_j \in \mathcal{B}_\varepsilon$, $u \in \mathfrak{U}$, $b_1 \in L^\infty(\mathbb{R}^d)$,*

$$
\begin{aligned}
&\big\|\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1] P_j \mathcal{Q}_{j+l} - \mathrm{Mult}\{\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1] 1\} P_j \mathcal{Q}_{j+l}\big\|_{L^2\to L^2} \\
&\qquad \lesssim \begin{cases} \|u\|_{\mathfrak{U}} \|b_1\|_\infty \min\{n\|\varsigma_j\|_{\mathcal{B}_\varepsilon} 2^{-k\varepsilon'},\ 2^{-l}\|\varsigma\|_{L^1}\} & \text{if } l \ge 0, \\ \|u\|_{\mathfrak{U}} \|b_1\|_\infty \min\{n\|\varsigma_j\|_{\mathcal{B}_\varepsilon} 2^{l\varepsilon/4} 2^{-k\varepsilon'},\ \|\varsigma\|_{L^1}\} & \text{if } l \le 0. \end{cases}
\end{aligned}
$$

Proof. We may assume $\|\varsigma\|_{\mathfrak{U}} = 1$ and $\|b_1\|_{L^\infty} = 1$. We have

$$\|P_j \mathcal{Q}_{j+l}\|_{L^2\to L^2} \lesssim \min\{2^{-l}, 1\}\,. \tag{11.16}$$

Now, by Lemma 11.5,

$$\|\overline{Q}_{j+k}[u] W_j[b_1]\|_{L^2\to L^2} \lesssim \|\varsigma_j\|_{L^1} \tag{11.17}$$

and, by Lemma 11.10 and (11.16),

$$\|\mathrm{Mult}\{\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1] 1\} P_j \mathcal{Q}_{j+l}\|_{L^2\to L^2} \lesssim \min\{1, 2^{-l}\} \|\varsigma\|_{L^1}; \tag{11.18}$$

moreover, by Lemma 11.11,

$$\|\mathrm{Mult}\{\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1] 1\} P_j \mathcal{Q}_{j+l}\|_{L^2\to L^2} \lesssim n 2^{-k\varepsilon'} \|\varsigma\|_{\mathcal{B}_\varepsilon}. \tag{11.19}$$

A combination of (11.17), (11.18), and (11.18) immediately gives the assertion for $l \ge 0$, and also the second estimate for $l < 0$. It remains to show that

$$
\begin{aligned}
&\|(\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1] P_j \mathcal{Q}_{j+l} - \mathrm{Mult}\{\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1] 1\} P_j \mathcal{Q}_{j+l}\|_{L^2\to L^2} \\
&\qquad \lesssim n \|\varsigma_j\|_{\mathcal{B}_\varepsilon} \max\{2^{l\varepsilon/2}, 2^{l/4}\} \ \text{ if } l \le 0;
\end{aligned} \tag{11.20}
$$

indeed the assertion follows by taking a geometric mean of the bounds in (11.19) and (11.20).

By scale invariance (see (11.7)) it suffices to show (11.20) for $j = 0$, i.e.

$$\big\|(R_1 - R_2)\mathcal{Q}_l\big\|_{L^2\to L^2} \lesssim n\|\varsigma_j\|_{\mathcal{B}_\varepsilon} \max\{2^{l\varepsilon/2}, 2^{l/4}\} \text{ if } l \le 0; \tag{11.21}$$

for $R_1 = \overline{Q}_k[u]W_0[\varsigma, b_1]P_0$ and $R_2 = \mathrm{Mult}\{\overline{Q}_k[u]W_0[\varsigma, b_1]1\}P_0$. Let ρ_1, ρ_2, ρ be the Schwartz kernels of R_1, R_2, $R_1 - R_2$, and let σ_{-l} be the Schwartz kernel of $\mathcal{Q}_l$. We wish to apply Lemma 8.6 (note the notation $l = -\ell$ in that lemma). It is immediate that σ_ℓ satisfies assumptions (8.23b) and (8.23c) with $B_1, B_\infty, \widetilde{B}_\infty \lesssim 1$. The function ρ satisfies the crucial cancellation condition (8.24) since

$$(\overline{Q}_k[u]W_0[\varsigma, b_1]P_0 - \mathrm{Mult}\{\overline{Q}_k[u]W_0[\varsigma, b_1]1\}P_0)1 = 0\,.$$

It remains to check the size conditions (8.23a). We have

$$|\rho_1(x,y)| \le \iiint |u^{(2^k)}(x-x')||\varsigma(\alpha, x'-y')||\phi(y'-y)|\,dx'\,d\alpha\,dy'$$

and thus clearly

$$\sup_y \int |\rho_1(x,y)|dx \le \|u\|_1\|\varsigma\|_{L^1}\|\phi\|_1 \lesssim 1$$

since $\|u\|_1 \le \|u\|_{\mathfrak{U}}$. Also for some $M > d+1$,

$$\begin{aligned}
&\int |\rho_1(x,y)|(1+|x-y|)^\varepsilon dy\\
&\lesssim \int (1+|x-y|)^\varepsilon \iiint \Big|\frac{2^{kd}}{(1+2^k|x-x'|)^{d+\frac12}} \frac{|\varsigma(\alpha, x'-y')|}{(1+|y'-y|)^M}\,dx'\,d\alpha\,dy'\,dy\\
&\lesssim \iiint |\varsigma(\alpha, x'-y')|(1+|x'-y'|)^\varepsilon \omega(x,x',y')\,d\alpha\,dy'\,dx'\,,
\end{aligned}$$

where

$$\omega(x,x',y') = \frac{2^{kd}}{(1+2^k|x-x'|)^{d+\frac12}} \int \frac{1}{(1+|y'-y|)^M} \frac{(1+|x-y|)^\varepsilon}{(1+|x'-y'|)^\varepsilon}\,dy\,.$$

We have

$$\sup_{x'} \int |\varsigma(\alpha, x'-y')|(1+|x'-y'|)^\varepsilon\,d\alpha\,dy' \le \|\varsigma\|_{\mathcal{B}_\varepsilon}$$

and thus it suffices to show that

$$\sup_{x,y} \int \omega(x,x',y)dx' \lesssim 1. \tag{11.22}$$

Now by the triangle inequality $(1+|x-y|)^\varepsilon \le (1+|x-x'|)^\varepsilon(1+|x'-y'|)^\varepsilon(1+|y'-y|)^\varepsilon$ and hence

$$\begin{aligned}
\int \omega(x,x',y)dx' &\le \int \frac{2^{kd}(1+|x-x'|)^\varepsilon}{(1+2^k|x-x'|)^{d+\frac12}} \int \frac{1}{(1+|y'-y|)^{M-\varepsilon}} dy\,dx'\\
&\lesssim \int \frac{2^{kd}(1+|x-x'|)^\varepsilon}{(1+2^k|x-x'|)^{d+\frac12}}\,dx'
\end{aligned}$$

and (11.22) follows easily, provided that $\varepsilon < 1/2$. Thus condition (8.23a) is satisfied for ρ_1. By Lemma 11.10 it is immediate that condition (8.23a) is satisfied for ρ_2 as well. Thus we have verified the assumptions of Lemma 8.6 and (11.21) follows. This completes the proof of the lemma. $\square$

Proof of Proposition 11.4, conclusion. We may assume $\|u\|_{\mathfrak{U}} = 1$, $\|f\|_2 = 1$, and $\sup_j \|b_1^j\|_\infty = 1$. For $k \geq 0$, define

$$R_{k,j} := \overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1^j] P_j - \mathrm{Mult}\{\overline{Q}_{j+k}[u] W_j[\varsigma_j, b_1^j] 1\} P_j.$$

The proof is complete if we can show, for $k \geq 0$,

$$\Big(\sum_j \|R_{j,k} f\|_2^2\Big)^{1/2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \min\big\{2^{-\varepsilon_1 k} n\Gamma_\varepsilon,\, \log(1+n\Gamma_\varepsilon)\big\}. \tag{11.23}$$

Lemma 11.12 implies

$$\|R_{k,j}\mathcal{Q}_{j+l}\|_{L^2\to L^2} \lesssim \begin{cases} \sup_j \|\varsigma\|_{L^1} \min\{n\Gamma_\varepsilon 2^{-k\varepsilon'}, 2^{-l}\}, & \text{if } l \geq 0,\\ \sup_j \|\varsigma\|_{L^1} \min\{n\Gamma_\varepsilon 2^{l\varepsilon/4} 2^{-k\varepsilon'},\, 1\} & \text{if } l < 0. \end{cases}$$

Now

$$\begin{aligned}
\Big(\sum_j \|R_{k,j} f\|_2^2\Big)^{1/2} &= \Big(\sum_j \Big\|R_{k,j} \sum_{l\in\mathbb{Z}} \mathcal{Q}_{j+l} \widetilde{\mathcal{Q}}_{j+l} f\Big\|_2^2\Big)^{\frac12}\\
&\lesssim \sum_{l\in\mathbb{Z}} \Big(\sum_j \Big\|R_{k,j}\mathcal{Q}_{j+l}\widetilde{\mathcal{Q}}_{j+l} f\Big\|_2^2\Big)^{\frac12} \lesssim \sum_{l\in\mathbb{Z}} \sup_{j'} \|R_{k,j'}\mathcal{Q}_{j'+l}\|_{L^2\to L^2} \Big(\sum_j \|\widetilde{\mathcal{Q}}_{j+l} f\|_2^2\Big)^{1/2}\\
&\lesssim \sup_j \|\varsigma_j\|_{L^1} \Big[\sum_{l\geq 0} \min\{n\Gamma_\varepsilon 2^{-k\varepsilon'}, 2^{-l}\} + \sum_{l<0} \min\{n\Gamma_\varepsilon 2^{l\varepsilon/4} 2^{-k\varepsilon'}, 1\}\Big]\\
&\lesssim \sup_j \|\varsigma_j\|_{L^1} \min\big\{2^{-\varepsilon_1 k} n\Gamma_\varepsilon,\, \log(1+n\Gamma_\varepsilon)\big\}
\end{aligned}$$

for some sufficiently small $\varepsilon_1 > 0$, and the proof is complete. □

11.3. **Proof that Theorem 11.1 implies Part II of Theorem 5.1.** Let $1 < p \leq 2$. The asserted result follows from

$$\begin{aligned}
&\Big|\sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](b_1^j, \dots, b_{l-1}^j, (I-P_j)b_l, b_{l+1}^j, \dots, b_n^j, (I-P_j)b_{n+1}, P_j b_{n+2})\Big|\\
&\qquad \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^{5/2}(1+n\Gamma_\varepsilon)\Big(\prod_{\substack{i=1,\dots,n\\ i\neq l}} \sup_j \|b_i^j\|_\infty\Big)\|b_l\|_\infty \|b_{n+1}\|_p \|b_{n+2}\|_{p'}
\end{aligned} \tag{11.24}$$

and

$$\begin{aligned}
&\Big|\sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](b_1^j, \dots, b_{l-1}^j, (I-P_j)b_l, b_{l+1}^j, \dots, b_n^j, P_j b_{n+1}, (I-P_j)b_{n+2})\Big|\\
&\qquad \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^{5/2}(1+n\Gamma_\varepsilon)\Big(\prod_{\substack{i=1,\dots,n\\ i\neq l}} \sup_j \|b_i^j\|_\infty\Big)\|b_l\|_\infty \|b_{n+1}\|_p \|b_{n+2}\|_{p'}.
\end{aligned} \tag{11.25}$$

Once (11.24) and (11.25) are established we use them for the choices $b_i^j = b_i$, if $i < l$, $b_i^j = P_j b_i$, if $l < i \leq n$. Now it is crucial that $\|P_j\|_{L^\infty\to L^\infty} \leq 1$ (here $\phi_j \geq 0$, and $\int \phi_j = 1$ are used). Hence the two inequalities for $\Lambda^1_{l,n+1}$ and $\Lambda^1_{l,n+2}$ claimed in Theorem 5.1 are an immediate consequence of (11.24) and (11.25).

In order to establish (11.24) and (11.25) we may assume without loss of generality that $l = 1$. This is because we can permute the first n entries of the multilinear form and replace ς_j by $\ell_\varpi \varsigma_j$ as in (4.1). We may also assume that

$$\|b_1\|_\infty \leq 1, \qquad \|b_i^j\|_\infty = 1,\ 2 \leq i \leq n.$$

Now, in what follows let

$$\tilde{\varsigma}_j(\alpha, v) = \varsigma(1-\alpha_1, \dots, 1-\alpha_n, v)$$

(as in (4.2)). To prove (11.24) for $l = 1$ we observe

$$\sum_j \Lambda[\varsigma_j^{(2^j)}]((I-P_j)b_1, b_2^j, \dots, b_n^j, (I-P_j)b_{n+1}, P_j b_{n+2}) = \int b_{n+2}(x)\, {}^t\mathcal{T} b_{n+1}(x) dx$$

where

$${}^t\mathcal{T} = \sum_j P_j {}^tW_j[\varsigma_j, (I-P_j)b_1](I-P_j) = \sum_j P_j W_j[\tilde{\varsigma}_j, (I-P_j)b_1](I-P_j).$$

Now we expand $I - P_j = \sum_{k>0} Q_{j+k}$ and we get ${}^t\mathcal{T} = \sum_{k>0} {}^t\mathcal{T}^k$ where

$${}^t\mathcal{T}^k = \sum_j S_j Q_{j+k}, \text{ with } S_j = P_j W_j[\tilde{\varsigma}_j, (I-P_j)b_1].$$

The Schwartz kernel of S_j is equal to $\mathrm{Dil}_{2^j} s_j$ where

$$s_j(x,y) = \int \phi(x-x')\sigma_j(x',y) dy$$

with

$$\sigma_j(x,y) = \int \varsigma_j(\alpha, x-y)(I-P_0)b_1(2^{-j}(x-\alpha_i(x-y)) \prod_{i=2}^n b_i(2^{-j}(x-\alpha_i(x-y))\, d\alpha. \tag{11.26}$$

We wish to apply Corollary 8.12. It is easy to check that

$$\mathrm{Int}^1[s_j] \lesssim \sup_j \|\varsigma_j\|_{L^1} =: A, \qquad \mathrm{Int}^1_\varepsilon[s_j] \lesssim \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} =: B.$$

Now $\|\sum_j S_j Q_{j+k}\|_{L^2\to L^2} = \|\mathcal{T}^k\|_{L^2\to L^2}$ and by Theorem 11.1

$$\Big\|\sum_j S_j Q_{j+k}\Big\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^{3/2}(1+n\Gamma_\varepsilon) := D_0\,,$$

$$2^{\varepsilon_1 k}\|\sum_j S_j Q_{j+k}\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} n\Gamma_\varepsilon^2 := D_{\varepsilon_1}\,.$$

Now we easily obtain from Corollary 8.12 that

$$\Big\|\sum_{k>0}\sum_j S_j Q_{j+k}\Big\|_{L^p\to L^p} \lesssim C_p \sup_j \|\varsigma_j\|_{L^1} \log^{5/2}(1+n\Gamma_\varepsilon)$$

and (11.24) is proved.

Finally we turn to (11.25), for $l = 1$. The case $p = 2$ follows immediately from (11.24), by duality replacing ς_j with $\tilde{\varsigma}_j$. For $p < 2$ we observe that

$$\sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}]((I-P_j)b_1, b_2^j, \dots, b_n^j, P_j b_{n+1}, (I-P_j)b_{n+2}) = \int b_{n+2}(x)\, S_j P_j b_{n+1}(x) dx$$

with $S_j = (I-P_j)W_j[\varsigma_j, b_1]$. The Schwartz kernel of S_j is equal to $\mathrm{Dil}_{2^j} s_j$ where

$$s_j(x,y) = \sigma_j(x,y) - \int \phi(x-x')\sigma_j(x',y) dy$$

with σ_j as in (11.26). Then s_j satisfies $\mathrm{Int}^1[s_j] \lesssim \|\varsigma_j\|_{L^1}$ and $\mathrm{Int}^1_\varepsilon[s_j] \lesssim \|\varsigma_j\|_{\mathcal{B}_\varepsilon}$ and (11.25) for $p < 2$ follows immediately from the case $p = 2$ and Corollary 8.10.

12. Proof of Theorem 5.1: Part III

Let $n \geq 2$ and $1 \leq l_1 < l_2 \leq n$. In this section, we consider the multilinear functional

$$\begin{aligned}(12.1)\quad &\Lambda^1_{l_1,l_2}(b_1,\dots,b_{n+2}) := \\ &\sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](b_1,\dots,b_{l_1-1},(I-P_j)b_{l_1},P_jb_{l_1+1},\dots,P_jb_{l_2-1},(I-P_j)b_{l_2},P_jb_{l_2+1},\dots,P_jb_{n+2}),\end{aligned}$$

where, for some fixed $\varepsilon > 0$, $\vec{\varsigma} = \{\varsigma_j : j \in \mathbb{Z}\} \subset \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ is a bounded set. The goal of this section is to prove, for $p \in (1,2]$, $b_1,\dots,b_n \in L^\infty(\mathbb{R}^d)$, $b_{n+1} \in L^p(\mathbb{R}^d)$, $b_{n+2} \in L^{p'}(\mathbb{R}^d)$, the inequality

$$(12.2)\qquad \big|\Lambda^1_{l_1,l_2}(b_1,\dots,b_{n+2})\big| \leq C_{d,p,\varepsilon} \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon)\big(\prod_{l=1}^n \|b_l\|_\infty\big)\|b_{n+1}\|_p\|b_{n+2}\|_{p'},$$

together with convergence of the sum (12.1) in the operator topology of multilinear functionals. Moreover the operator sum $T^1_{l_1,l_2}$ associated to $\Lambda^1_{l_1,l_2}$ converges in the strong operator topology.

It will be convenient to prove a slightly more general theorem. Let $\{b_l^j : 3 \leq l \leq n, j \in \mathbb{Z}\} \subset L^\infty(\mathbb{R}^d)$ be a bounded set, with $\sup_{j\in\mathbb{Z}} \|b_l^j\|_{L^\infty} = 1$, for $3 \leq l \leq n$. For $b_1, b_2 \in L^\infty(\mathbb{R}^d)$ define an operator $S_j[b_1,b_2]$ by

$$\int g(x)(S_j[b_1,b_2]f)(x)\,dx := \Lambda[\varsigma_j^{(2^j)}]((I-P_j)b_1,(I-P_j)b_2,b_3^j,\dots,b_n^j,f,g).$$

Theorem 12.1. *With the above assumptions, for $1 < p \leq 2$, the sums $\sum_{j=-\infty}^{\infty} P_jS_j[b_1,b_2]P_j$ converge to $S[b_1,b_2]$, in the strong operator topology as operators $L^p \to L^p$, and $S[b_1,b_2]$ satisfies the estimate*

$$(12.3)\qquad \|S[b_1,b_2]\|_{L^p\to L^p} \leq C_{d,p,\varepsilon} \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon)\|b_1\|_\infty\|b_2\|_\infty.$$

Proof of (12.2) *given Theorem 12.1.* Using Theorem 2.9 we see that Theorem 12.1 also implies the inequality

$$\begin{aligned}\Big|\sum_j \Lambda[\varsigma_j^{(2^j)}](b_1^j,\dots,b_{l_1-1}^j,(I-P_j)b_{l_1},b_{l_1+1}^j,\dots,b_{l_2-1}^j,(I-P_j)b_{l_2}^j,b_{l_2+1}^j,\dots,b_n^j,b_{n+1},b_{n+2})\Big| \\ \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon)\|b_{l_1}\|_\infty\|b_{l_2}\|_\infty\big(\prod_{\substack{1\leq i\leq n\\ i\neq l_1,l_2}} \|b_i^j\|_\infty\big)\|b_{n+1}\|_p\|b_{n+2}\|_{p'}.\end{aligned}$$

Since $\|P_jb_l\|_q \leq \|b_l\|_q$ we may replace b_i by P_jb_i for $l_1+1 \leq i \leq l_2-1$, $i \geq l_2+1$, and if we use also $P_j = {}^tP_j$ then (12.2) follows. □

The rest of this section is devoted to the proof of Theorem 12.1. Thus, we consider sequences $b_l^j \in L^\infty(\mathbb{R}^d)$ fixed $(3 \leq l \leq n)$ with $\sup_j \|b_l^j\|_{L^\infty} = 1$. The L^2 estimates in §10 will be crucial. We restate them as

Proposition 12.2. *There is $C \lesssim 1$ such that for $\varepsilon' \leq \varepsilon/C$, and for all collections*

$$\{b_{n+1}^j : j \in \mathbb{Z}\}, \{b_{n+2}^j : j \in \mathbb{Z}\} \subset L^\infty(\mathbb{R}^d),\ \textit{with}\ \sup_j \|b_{n+1}^j\|_\infty = 1,\quad \sup_j \|b_{n+2}^j\|_\infty = 1,$$

we have for $f, g \in L^2(\mathbb{R}^d)$ and $k_1, k_2 \in \mathbb{N}$,

$$\Big|\sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](Q_{j+k_1}f, Q_{j+k_2}g, b_3^j, \dots, b_{n+2}^j)\Big| \lesssim \|f\|_2\|g\|_2 \min\big\{2^{-\varepsilon' k_1-\varepsilon' k_2} n \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \sup_j \|\varsigma_j\|_{L^1}\big\}.$$

Let $\mathcal{T}_{k_1,k_2}$ be defined by

$$\Lambda[\varsigma_j^{(2^j)}](Q_{j+k_1}f, Q_{j+k_2}g, b_3^j, \dots, b_{n+2}^j) = \int g(x)\, \mathcal{T}_{k_1,k_2,j} f(x)\, dx. \tag{12.4}$$

Then $\sum_j \mathcal{T}_{k_1,k_2,j}$ and $\sum_j {}^t\mathcal{T}_{k_1,k_2,j}$ converge in the strong operator topology as operators $L^2 \to L^2$, with equiconvergence with respect to $b_3^j, \dots, b_{n+2}^j$.

Proof. This follows from Theorem 10.1. □

Proposition 12.3. *Let $\{b_1^j, b_2^j : j \le -1\} \subset L^\infty(\mathbb{R}^d)$ be a bounded set with $\sup_{j\le -1} \|b_l^j\|_{L^\infty} = 1$, $l = 1, 2$, and let b_{n+1}, b_{n+2} be L^∞ functions supported in $\{y : |y| \le 1\}$.*

$$\sum_{j=-\infty}^{-1} \big|\Lambda[\varsigma_j^{(2^j)}](b_1^j, \dots, b_n^j, P_j b_{n+1}, P_j b_{n+2})\big| \lesssim \|b_{n+1}\|_{L^\infty}\|b_{n+2}\|_{L^\infty} \sup_j \|\varsigma_j\|_{L^1}.$$

Proof. We may assume $\|b_{n+1}\|_{L^\infty} = \|b_{n+2}\|_{L^\infty} = 1$. Then by Lemma 2.7

$$\begin{aligned}
&\big|\Lambda[\varsigma_j^{(2^j)}](b_1^j, \dots, b_n^j, P_j b_{n+1}, P_j b_{n+2})\big| \lesssim \sup_j \|\varsigma_j\|_{L^1} \|P_j b_{n+1}\|_2 \|P_j b_{n+2}\|_2 \\
&\lesssim \sup_j \|\varsigma_j\|_{L^1} 2^{jd} \|b_{n+1}\|_1 \|P_j b_{n+2}\|_1 \lesssim \sup_j \|\varsigma_j\|_{L^1} 2^{jd}
\end{aligned}$$

where we have used $\|P_j\|_{L^1\to L^2} \lesssim 2^{jd/2}$ and then the support assumption on b_{n+1}, b_{n+2}. Now sum over $j \le -1$ and the proof is complete. □

Lemma 12.4. *Let $0 < \varepsilon \le 1$. For all $R \ge 5$, all $j \ge 0$, $b_{n+1}, b_{n+2} \in L^\infty$ supported in $\{x : |x| \le 4\}$, $b_1, b_2 \in L^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(b_1) \subseteq \{v : |v| \ge R\}$, we have*

$$|\Lambda[\varsigma_j^{(2^j)}](b_1, b_2, b_3^j, \dots, b_n^j, b_{n+1}, b_{n+2})| \lesssim \min\big\{(2^j R)^{-\varepsilon/2} \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \|\varsigma_j\|_{L^1}\big\} \prod_{l\in\{1,2,n+1,n+2\}} \|b_l\|_\infty.$$

Proof. We may assume $\|b_l\|_{L^\infty} = 1$, $l = 1, 2, n+1, n+2$. The bound

$$|\Lambda[\varsigma_j^{(2^j)}](b_1, b_2, b_3^j, \dots, b_n^j, b_{n+1}, b_{n+2})| \lesssim \|\varsigma_j\|_{L^1} \tag{12.5}$$

follows immediately from Lemma 2.7 and the assumptions on the supports of b_{n+1} and b_{n+2}.

In order to establish the bound $(2^jR)^{-\varepsilon/2}\|\varsigma_j\|_{\mathcal{B}_\varepsilon}$ we estimate, using the assumption on $\operatorname{supp}(b_1)$,

$$\begin{aligned}
&\big|\Lambda[\varsigma_j^{(2^j)}](b_1,b_2,b_3^j,\dots,b_n^j,b_{n+1},b_{n+2})\big|\\
&= \Big|\iiint \varsigma_j^{(2^j)}(\alpha,v)b_1(x-\alpha_1 v)b_2(x-\alpha_2 v)\big(\prod_{i=3}^n b_i^j(x-\alpha_3 v)\big)b_{n+1}(x-v)b_{n+2}(x)\,dx\,d\alpha\,dv\Big|\\
&\le \int_{|x|\le 4}\int_{|v|\le 8}\int_{|\alpha_1|\ge\frac{R-|x|}{|v|}} |\varsigma_j^{(2^j)}(\alpha,v)||b_1(x-\alpha_1 v)|\,d\alpha\,dv\,dx\\
&\lesssim \int_{|w|\le 2^{j+3}}\int_{|\alpha_1|\ge\frac{R-|4|}{2^{-j}|w|}} |\varsigma_j(\alpha,w)|\,d\alpha\,dw\,;
\end{aligned}$$

here we have used $R\ge 5$. Let $m\le j+3$. Clearly

$$\int_{2^{m-1}\le|w|\le 2^m}\int_{|\alpha_1|\ge\frac{R-|4|}{2^{-j}|w|}} |\varsigma_j(\alpha,w)|\,d\alpha\,dw \lesssim (2^{j-m}R)^{-\varepsilon}\|\varsigma_j\|_{\mathcal{B}_{\varepsilon,1}} \lesssim 2^{m\varepsilon}(2^jR)^{-\varepsilon}\|\varsigma_j\|_{\mathcal{B}_\varepsilon}. \tag{12.6}$$

Also

$$\int_{2^{m-1}\le|w|\le 2^m}\int_{|\alpha_1|\ge\frac{R-|4|}{2^{-j}|w|}} |\varsigma_j(\alpha,w)|\,d\alpha\,dw \lesssim 2^{-m\varepsilon}\|\varsigma_j\|_{\mathcal{B}_{\varepsilon,4}} \lesssim 2^{-m\varepsilon}\|\varsigma_j\|_{\mathcal{B}_\varepsilon}. \tag{12.7}$$

We use (12.6) for $2^m<(2^jR)^{1/2}$ and (12.7) for $2^m\ge(2^jR)^{1/2}$, and sum. The assertion follows. ∎

Lemma 12.5. *For $l=1,2,n+1,n+2$, let $\{b_l^{j,k_1,k_2}: j,k_1,k_2\in\mathbb{N}\}\subset L^\infty(\mathbb{R}^d)$ be bounded sets with $\sup_{j,k_1,k_2}\|b_l^{j,k_1,k_2}\|_{L^\infty}=1$. Let $\beta>0$, $\delta>0$ and assume*

$$\operatorname{supp}(b_1^{j,k_1,k_2})\subseteq\big\{v:|v|\ge\max\{5,\beta 2^{k_1\delta+k_2\delta}\}\big\},\quad \forall j,k_1,k_2\in\mathbb{N} \tag{12.8}$$

and for $l=n+1,n+2$,

$$\operatorname{supp}(b_l^{j,k_1,k_2})\subseteq\{v:|v|\le 4\},\quad \forall j,k_1,k_2\in\mathbb{N}.$$

Then

$$\sum_{j,k_1,k_2\in\mathbb{N}} |\Lambda[\varsigma_j^{(2^j)}](b_1^{j,k_1,k_2},b_2^{j,k_1,k_2},b_3^j,\dots,b_n^j,b_{n+1}^{j,k_1,k_2},b_{n+2}^{j,k_1,k_2})| \lesssim \sup_j\|\varsigma_j\|_{L^1}\log^3(1+\beta^{-1}\Gamma_\varepsilon).$$

Here the implicit constant depends on δ, but not on β. The same result holds if instead of (12.8) *we have*

$$\operatorname{supp}(b_2^{j,k_1,k_2})\subseteq\big\{|v|\ge\max\{5,\beta 2^{k_1\delta+k_2\delta}\}\big\},\quad \forall j,k_1,k_2\in\mathbb{N}. \tag{12.9}$$

Proof. Because our definitions are symmetric in b_1 and b_2, the result with (12.9) in place of (12.8) follows from the result with (12.8). Thus, we may focus only on the proof with the assumption (12.8). Applying the previous lemma, we have

$$\begin{aligned}
&\sum_{j,k_1,k_2\in\mathbb{N}} \big|\Lambda[\varsigma_j^{(2^j)}](b_1^{j,k_1,k_2},b_2^{j,k_1,k_2},b_3^j,\dots,b_n^j,b_{n+1}^{j,k_1,k_2},b_{n+2}^{j,k_1,k_2})\big|\\
&\lesssim \sum_{j,k_1,k_2\in\mathbb{N}} \min\big\{2^{-j\varepsilon/2}\big(\max\{5,\beta 2^{k_1\delta+k_2\delta}\}\big)^{-\varepsilon/2}\sup_j\|\varsigma_j\|_{\mathcal{B}_\varepsilon},\,\sup_j\|\varsigma_j\|_{L^1}\big\}\\
&\lesssim \sup_j\|\varsigma_j\|_{L^1}\log^3(1+\beta^{-1}\Gamma_\varepsilon),
\end{aligned}$$

completing the proof. ∎

Proposition 12.6. *Let $b_1, b_2, b_{n+1}, b_{n+2} \in L^\infty(\mathbb{R}^d)$. Let $\mathfrak{S}_j$ be defined by*

$$\Lambda[\varsigma_j^{(2^j)}]((I-P_j)b_1,(I-P_j)b_2,b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2}) = \int b_{n+2}(x)\,\mathfrak{S}_j b_{n+1}(x)\,dx. \tag{12.10}$$

Consider $\mathfrak{S}_j$ as a bounded operator mapping L^∞ functions supported in $B^d(0,1)$ to $L^1(B^d(0,1))$. Then the sum $\sum \mathfrak{S}_j$ converges in the strong operator topology as bounded operators $L^\infty(B^d(0,1))$ to $L^1(B^d(0,1))$ and we have for $\mathrm{supp}(b_{n+1}), \mathrm{supp}(b_{n+2}) \subseteq \{y : |y| \le 1\}$,

$$\Big|\sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}]((I-P_j)b_1,(I-P_j)b_2,b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2})\Big|$$
$$\lesssim \sup_j \|\varsigma_j\|_{L^1}\log^3(1+n\Gamma_\varepsilon) \prod_{l\in\{1,2,n+1,n+2\}} \|b_l\|_\infty.$$

Proof. We may assume $\|b_l\|_{L^\infty} = 1$, $l = 1, 2, n+1, n+2$.

By Proposition 12.3 the required estimate holds for the sum over negative j and thus we only bound

$$\Big|\sum_{j\ge 0} \Lambda[\varsigma_j^{(2^j)}]((I-P_j)b_1,(I-P_j)b_2,b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2})\Big| \lesssim \sup_j \|\varsigma_j\|_{L^1}\log^3(1+n\Gamma_\varepsilon). \tag{12.11}$$

Let $0 < \beta \le 1$, $0 < \delta < 1$ be constants, to be chosen later (see (12.14)). Implicit constants below are allowed to depend on δ, but do not depend on β. For $l = 1, 2$ and $k_1, k_2 > 0$ define

$$b_{l,\infty}^{k_1,k_2}(v) := \begin{cases} b_l(v) & \text{if } |v| > \max\{10, \beta\cdot 2^{k_1\delta+k_2\delta+1}\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_{l,0}^{k_1,k_2}(v) := b_l(v) - b_{l,\infty}^{k_1,k_2}(v).$$

We have, by (6.2) and Remark 6.1,

$$\Big|\sum_{j\ge0} \Lambda[\varsigma_j^{(2^j)}](I-P_j)b_1,(I-P_j)b_2,b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2}\Big|$$
$$= \Big|\sum_{k_1,k_2>0}\sum_{j\ge0} \Lambda[\varsigma_j^{(2^j)}](Q_{j+k_1}b_1,Q_{j+k_2}b_2,b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2})\Big| \le (I)+(II)+(III)$$

where

$$(I) := \sum_{k_1,k_2>0}\sum_{j\ge0} \big|\Lambda[\varsigma_j^{(2^j)}](Q_{j+k_1}b_{1,\infty}^{k_1,k_2},Q_{j+k_2}b_2,b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2})\big|,$$
$$(II) := \sum_{k_1,k_2>0}\sum_{j\ge0} \big|\Lambda[\varsigma_j^{(2^j)}](Q_{j+k_1}b_{1,0}^{k_1,k_2},Q_{j+k_2}b_{2,\infty}^{k_1,k_2},b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2})\big|,$$
$$(III) := \sum_{k_1,k_2>0} \Big|\sum_{j\ge0} \Lambda[\varsigma_j^{(2^j)}](Q_{j+k_1}b_{1,0}^{k_1,k_2},Q_{j+k_2}b_{2,0}^{k_1,k_2},b_3^j,\dots,b_n^j,P_jb_{n+1},P_jb_{n+2})\Big|.$$

Because $j, k_1, k_2 \ge 0$, and by the supports of the functions in question, we have

$$\mathrm{supp}(Q_{j+k_1}b_{1,\infty}^{k_1,k_2}),\ \mathrm{supp}(Q_{j+k_2}b_{2,\infty}^{k_1,k_2}) \subseteq \big\{v : |v| > \max\{5, \beta\cdot 2^{k_1\delta+k_2\delta}\}\big\},$$

and

$$\mathrm{supp}(P_jb_{n+1}), \mathrm{supp}(P_jb_{n+2}) \subseteq \{v : |v| \le 4\}.$$

Lemma 12.5 applies to show

$$|(I)|+|(II)| \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+\beta^{-1}\Gamma_\varepsilon). \tag{12.12}$$

We now apply the L^2 result in Proposition 12.2. Let $\mathcal{T}_{k_1,k_2,j}$ be as in (12.4). Then $\sum_{j\ge 0}\mathcal{T}_{k_1,k_2,j}$ converges in the strong operator topology as operators $L^2\to L^2$, with equiconvergence with respect to bounded choices of $b_{n+1}, b_{n+2}\in L^\infty(B^d(0,1))$, moreover the operator norms involve some exponential deacy in k_1, k_2. If we apply this to $b_{1,0}^{k_1,k_2}, b_{2,0}^{k_1,k_2}$, we may replace the L^2 norms with L^∞-norms. Hence if we define operators $\mathfrak{S}_{k_1,k_2,j}$ by

$$\int b_{n+2}(x)\,\mathfrak{S}_{k_1,k_2,j}b_{n+1}(x)\,dx = \Lambda[\varsigma_j^{(2^j)}](Q_{j+k_1}b_1, Q_{j+k_2}b_2, b_3^j,\dots,b_n^j, P_jb_{n+1}, P_jb_{n+2})$$

we see that $\int\sum_j b_{n+2}(x)\mathfrak{S}_{k_1,k_2,j}b_{n+1}(x)dx$ converges with equiconvergence in the choice of b_{n+2} with $\|b_{n+2}\|_\infty\le 1$ and $\operatorname{supp}(b_{n+2})\subset B^d(0,1)$. Thus we get convergence of $\sum_{j=0}^\infty \mathfrak{S}_{k_1,k_2,j}$ in the strong operator topology as operators $L^\infty(B^d(0,1))\to L^1(B^d(0,1))$. For the quantitative estimates we apply the L^2 result in Proposition 12.2 and use the supports of $b_{1,0}^{k_1,k_2}, b_{2,0}^{k_1,k_2}$ to get for $\varepsilon'<c\varepsilon^2$

$$\begin{aligned}(III) &\lesssim \sum_{k_1,k_2>0} \max\big\{2^{-\varepsilon' k_1-\varepsilon' k_2} n \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \sup_j\|\varsigma_j\|_{L^1}\big\}\|b_{1,0}^{k_1,k_2}\|_2\|b_{2,0}^{k_1,k_2}\|_2\\ &\lesssim \sum_{k_1,k_2>0} \max\big\{2^{-\varepsilon' k_1-\varepsilon' k_2} n \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \sup_j\|\varsigma_j\|_{L^1}\big\}(\max\{5,\,\beta\cdot 2^{k_1\delta+k_2\delta}\})^{2d}.\end{aligned} \tag{12.13}$$

Set

$$\delta=\frac{\varepsilon'}{4d},\quad \beta=(n\Gamma_\varepsilon)^{-\frac{1}{2d}}. \tag{12.14}$$

Note that

$$(\beta\cdot 2^{k_1\delta+k_2\delta})^{2d}(2^{-\varepsilon' k_1-\varepsilon' k_2} n\sup_j\|\varsigma_j\|_{\mathcal{B}_\varepsilon}) = 2^{-\varepsilon' k_1/2-\varepsilon' k_2/2}\sup_j\|\varsigma_j\|_{L^1}.$$

Using this in (12.13), we obtain

$$\begin{aligned}(III) &\lesssim \sup\|\varsigma_j\|_{L^1}\sum_{k_1,k_2>0}\max\{2^{-\varepsilon' k_1-\varepsilon' k_2}n\Gamma_\varepsilon,\,1\}(1+\beta\cdot 2^{k_1\delta+k_2\delta})^{2d}\\ &\lesssim \sup_j\|\varsigma_j\|_{L^1}\log^2(1+n\Gamma_\varepsilon).\end{aligned}$$

Plugging the choice of β into (12.12) completes the proof of (12.11).

Finally, we reexamine the proof to get the asserted convergence in the strong operator topology. This is immediate for the sums corresponding to the terms (I), (II) in view of the decay estimates in the proof of Lemma 12.5. For (III) we easily get the assertion from the above statements about convergence of $\sum_{j\ge 0}\mathfrak{S}_{k_1,k_2,j}$ and the exponential decay estimates in k_1,k_2. □

Proof of Theorem 12.1, conclusion. We shall apply Theorem 8.23. We need to verify that for every ball $B^d(x_0,r)$, $b_{n+1}\in L^\infty(B^d(x_0,r))$, $\|b_{n+1}\|_\infty=1$,

$$\int_{B_d(x_0,r)}\Big|\sum_{|j|>N} P_jS_j[b_1,b_2]P_jb_{n+1}(x)\Big|dx\to 0$$

as $N \to \infty$ and

$$\sup_N r^{-d} \int_{B_d(x_0,r)} \Big| \sum_{|j|\le N} P_j S_j[b_1,b_2] P_j b_{n+1}(x) \Big| \, dx \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \|b_1\|_\infty \|b_2\|_\infty . \tag{12.15}$$

For $x_0 = 0$ and $r = 1$ these statements follow from Proposition 12.6. We argue by rescaling to obtain the same statement for other balls. Let ℓ be such that $2^{\ell-1} \le r \le 2^\ell$. Let $\widetilde{b}_i(x) = b_i(x_0 + 2^\ell x)$, $i = 1, 2, n+1, n+2$ and $\widetilde{b}_i^j(x) = b_i^{j-\ell}(x_0 + 2^\ell x)$, $3 \le i \le n$. Then by changes of variables

$$\int b_{n+2}(x)\, S_j[b_1,b_2] b_{n+1}(x)\, dx = 2^{\ell d} \Lambda[\varsigma^{(2^{j+\ell})}]\big((I - P_{j+\ell})\widetilde{b}_1, (I-P_{j+\ell})\widetilde{b}_2, \widetilde{b}_3^{j+\ell}, \dots, \widetilde{b}_n^{j+\ell}, \widetilde{b}_{n+1}, \widetilde{b}_{n+2}\big).$$

We use the fact that the functions $\widetilde{b}_{n+1}$, $\widetilde{b}_{n+2}$ are supported in the unit ball centered at the origin. Then the result follows immediately from the statement for $x_0 = 0$, $r = 1$.

In order to verify the Op_ε-assumptions in Theorem 8.23 we use Lemma 8.24 with $C_0 \lesssim \sup_j \|\varsigma_j\|_{L^1}$ and $C_\varepsilon \lesssim \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}$. Now Theorem 8.23 yields

$$\|S[b_1,b_2]\|_{L^2\to L^2} \lesssim \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} (\sup_j \|\varsigma_j\|_{L^1}) \log^3(1+n\Gamma_\varepsilon).$$

Finally we combine this inequality with Corollary 8.10, with the choices $A \lesssim \sup_j \|\varsigma_j\|_{L^1}$ and $B \lesssim \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}$. This yields the asserted L^p bound. □

13. Proof of Theorem 5.1: Part IV

Let $1 \le l \le n+2$. In this section, we consider the multilinear form

$$\Lambda_l^2(b_1,\dots,b_{n+2}) := \sum_{j\in\mathcal{J}} \Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{l-1}, (I-P_j) b_l, P_j b_{l+1}, \dots, P_j b_{n+2}),$$

where $\mathcal{J} \subset \mathbb{Z}$ is a finite set, and, given some fixed $\epsilon > 0$, $\vec{\varsigma} = \{\varsigma_j : j \in \mathbb{Z}\} \subset \mathcal{B}_\epsilon(\mathbb{R}^n \times \mathbb{R}^d)$ is a bounded set with $\int \varsigma_j(\alpha, v)\, dv = 0$, $\forall \alpha, j$. Our task is to show that for $p \in (1,2]$,

$$|\Lambda_l^2(b_1,\dots,b_{n+2})| \le C_{d,p,\epsilon} n \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \big[\prod_{i=1}^n \|b_i\|_\infty\big] \|b_{n+1}\|_p \|b_{n+2}\|_{p'} \tag{13.1}$$

where the implicit constant is independent of $\mathcal{J}$. Moreover we wish to show that the sum defining the operator T_l^2 associated to Λ_l^2 via (5.12) converges in the strong operator topology as operators bounded on L^p. The heart of the proof lies in the next theorem which we shall prove first. Let $\Gamma_\varepsilon \equiv \Gamma_\varepsilon(\vec{\varsigma})$ be as in (5.6).

Theorem 13.1. *Let $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$, $b_{n+1}, b_{n+2} \in L^2(\mathbb{R}^d)$. Then,*

$$\lim_{N\to\infty} \sum_{j=-N}^{N} \Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{l-1}, (I-P_j) b_l, P_j b_{l+1}, \dots, P_j b_{n+2}) = \Lambda_{n+2}^2(b_1,\dots,b_{n+2})$$

and Λ_{n+2}^2 satisfies

$$|\Lambda_{n+2}^2(b_1,\dots,b_{n+2})| \le C_{d,\epsilon} n (\sup_j \|\varsigma_j\|_{L^1}) \log^3(1+n\Gamma_\varepsilon) \big[\prod_{m=1}^n \|b_m\|_{L^\infty}\big] \|b_{n+1}\|_{L^2} \|b_{n+2}\|_{L^2}.$$

Moreover the sums defining the operator T^2_{n+2} associated with Λ^2_{n+2} converge in the strong operator topology as operators $L^2 \to L^2$.

The full proof of (13.1) will be given in §13.3 below.

13.1. **Outline of the proof of Theorem 13.1.** We give an outline of the steps and refer to §13.2 for some technical details.

We first describe the basic decomposition of $\Lambda^2_{n+2}(b_1, \dots, b_{n+2})$ which is derived from a decomposition of $\Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+1}, (I-P_j)b_{n+2}$, for fixed j. Write

$$\begin{aligned}
&\Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+1}, (I-P_j)b_{n+2}) \\
&= \lim_{M\to\infty} \Big(\Lambda[\varsigma_j^{(2^j)}](P_{j+M}P_j b_1, \dots, P_{j+M}P_j b_{n+1}, (I-P_j)b_{n+2}) \\
&\qquad\qquad - \Lambda[\varsigma_j^{(2^j)}](P_{j-M}P_j b_1, \dots, P_{j-M}P_j b_{n+1}, (I-P_j)b_{n+2}) \Big) \\
&= \lim_{M\to\infty} \sum_{m=-M+1}^{M} \Big(\Lambda[\varsigma_j^{(2^j)}](P_{j+m}P_j b_1, \dots, P_{j+m}P_j b_{n+1}, (I-P_j)b_{n+2}) \\
&\qquad\qquad - \Lambda[\varsigma_j^{(2^j)}](P_{j+m-1}P_j b_1, \dots, P_{j+m-1}P_j b_{n+1}, (I-P_j)b_{n+2}) \Big)
\end{aligned}$$

and use the multilinearity to obtain the decomposition

$$\begin{aligned}
&\Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+1}, (I-P_j)b_{n+2}) \\
&= \sum_{l=1}^{n+1} \sum_{m=-\infty}^{\infty} \Lambda[\varsigma_j^{(2^j)}]\big(P_{j+m-1}P_j b_1, \dots, P_{j+m-1}P_j b_{l-1}, Q_{j+m}P_j b_l, \\
&\qquad\qquad P_{j+m}P_j b_{l+1}, \dots, P_{j+m}P_j b_{n+1}, (I-P_j)b_{n+2}\big).
\end{aligned} \tag{13.2}$$

The terms for $l = 1, \dots, n$ are handled in a similar fashion, in fact the estimates can be reduced to the case $l = 1$ by using Theorem 2.9, permuting the first and the l^{th} entry, and accordingly changing the family $\{\varsigma_j\}$.

Now let

$$X^i_k \in \{P_k, P_{k-1}\}. \tag{13.3}$$

Then we need to show

$$\begin{aligned}
&\Big| \sum_{j=-N}^{N} \sum_{m=-\infty}^{\infty} \Lambda[\varsigma_j^{(2^j)}](X^1_{j+m}P_j b_1, X^2_{j+m}P_j b_2, \dots, Q_{j+m}P_j b_{n+1}, (I-P_j)b_{n+2}) \Big| \\
&\qquad \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^2(1+n\Gamma_\varepsilon) \big(\prod_{i=1}^{n} \|b_i\|_\infty \big) \|b_{n+1}\|_2 \|b_{n+2}\|_2
\end{aligned} \tag{13.4}$$

and

$$\begin{aligned}
&\Big| \sum_{j=-N}^{N} \sum_{m=-\infty}^{\infty} \Lambda[\varsigma_j^{(2^j)}](Q_{j+m}P_j b_1, X^2_{j+m}P_j b_2, \dots, X^{n+1}_{j+m}P_j b_{n+1}, (I-P_j)b_{n+2}) \Big| \\
&\qquad \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^2(1+n\Gamma_\varepsilon) \big(\prod_{i=1}^{n} \|b_i\|_\infty \big) \|b_{n+1}\|_2 \|b_{n+2}\|_2
\end{aligned} \tag{13.5}$$

with implicit constants uniform in N; moreover we need to show the existence of the limits as $N \to \infty$, for the corresponding operator sums in the strong operator topology. By another application of Theorem 2.9 (this time permuting the entries $(1,n+1)$), with the corresponding change of the family $\{\varsigma_j\}$), we see that (13.4) can be deduced from

$$\Big|\sum_{j=-N}^{N}\sum_{m=-\infty}^{\infty}\Lambda[\varsigma_j^{(2^j)}](Q_{j+m}P_jb_1, X_{j+m}^2P_jb_2,\dots,X_{j+m}^{n+1}P_jb_{n+1},(I-P_j)b_{n+2})\Big| \\ \lesssim \sup_j\|\varsigma\|_{L^1}\log^2(1+n\Gamma_\varepsilon)\Big(\prod_{i=2}^{n+1}\|b_i\|_\infty\Big)\|b_1\|_2\|b_{n+2}\|_2. \tag{13.6}$$

It remains to prove (13.5), (13.6). We shall also decompose further using $(I-P_j)b_{n+2}=\sum_{m_2\in\mathbb{N}}Q_{j+m_2}b_{n+2}$. This leads to the following definition.

Definition 13.2. Let $m,m_1\in\mathbb{Z}$, $m_2>0$.

For $b_{n+1}\in L^\infty(\mathbb{R}^d)$ the operators $S_j^{m_1,m_2}[b_{n+1}]$ are defined by

$$\int g(x)S_j^{m_1,m_2}[b_{n+1}]f(x)\,dx \\ := \Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_jg, X_{j+m_1}^2P_jb_2,\cdots,X_{j+m_1}^nP_jb_n, X_{j+m_1}^{n+1}P_jb_{n+1},Q_{j+m_2}f). \tag{13.7}$$

For $b_1\in L^\infty(\mathbb{R}^d)$ the operators $T_j^{m_1,m_2}[b_1]$ are defined by

$$\int g(x)T_j^{m_1,m_2}[b_1]f(x)\,dx \\ := \Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_jb_1, X_{j+m_1}^2P_jb_2,\cdots,X_{j+m_1}^nP_jb_n, X_{j+m_1}^{n+1}P_jg,Q_{j+m_2}f). \tag{13.8}$$

We formulate an auxiliary result. It gives bounds in the $\mathrm{Op}(\varepsilon)$-classes defined in (8.36) for suitable normalizing dilates of the operators $S_j^{m_1,m_2}[b_{n+1}]$, $T_j^{m_1,m_2}[b_1]$. We use the same notation for these operators and their Schwartz kernels.

Proposition 13.3. *Let*

$$\sigma_j^{m_1,m_2}=\begin{cases}\mathrm{Dil}_{2^{-j}}(S_j^{m_1,m_2}[b_{n+1}]) & \text{if } m_1\ge 0,\\ \mathrm{Dil}_{2^{-j-m_1}}(S_j^{m_1,m_2}[b_{n+1}]) & \text{if } m_1<0,\end{cases} \tag{13.9}$$

and

$$\tau_j^{m_1,m_2}=\begin{cases}\mathrm{Dil}_{2^{-j}}(T_j^{m_1,m_2}[b_1]) & \text{if } m_1\ge 0,\\ \mathrm{Dil}_{2^{-j-m_1}}(T_j^{m_1,m_2}[b_1]) & \text{if } m_1<0.\end{cases} \tag{13.10}$$

There exists $\varepsilon'>c(\varepsilon)$ (independent of n) such that, for $m_2>0$,

$$\begin{aligned}\big\|\sigma_j^{m_1,m_2}\big\|_{\mathrm{Op}_\varepsilon}&\lesssim 2^{-\varepsilon'(|m_1|+m_2)}n^2\|\varsigma_j\|_{\mathcal{B}_\varepsilon}\|b_{n+1}\|_\infty,\\ \big\|\sigma_j^{m_1,m_2}\big\|_{\mathrm{Op}_0}&\lesssim\|\varsigma_j\|_{L^1}\|b_{n+1}\|_\infty,\end{aligned} \tag{13.11}$$

and

$$\begin{aligned}\big\|\tau_j^{m_1,m_2}\big\|_{\mathrm{Op}_\varepsilon}&\lesssim 2^{-\varepsilon'(|m_1|+m_2)}n^2\|\varsigma_j\|_{\mathcal{B}_\varepsilon}\|b_1\|_\infty,\\ \big\|\tau_j^{m_1,m_2}\big\|_{\mathrm{Op}_0}&\lesssim\|\varsigma_j\|_{L^1}\|b_1\|_\infty\,.\end{aligned} \tag{13.12}$$

The proof will be given in §13.2 below. Note that we have the trivial estimate $\|\cdot\|_{\mathrm{Op}_0}\le\|\cdot\|_{\mathrm{Op}_\varepsilon}$, and therefore the Op_0 bounds stated in Proposition 13.3 will only be used for $2^{\varepsilon(|m_1|+m_2)}\lesssim n^2\Gamma_\varepsilon$.

The estimates (13.5), (13.6) and the asserted existence of the limits follow easily from the following Proposition.

Proposition 13.4. *Let $b_2,\dots,b_n \in L^\infty(\mathbb{R}^d)$, with $\|b_i\|_\infty \le 1$, $i=2,\dots,n$. Let $\vec\varsigma = \{\varsigma_j\}$ be a bounded family in $\mathcal{B}_\varepsilon$, $\mathcal{J} \subset \mathbb{Z}^d$ with $\#\mathcal{J} < \infty$ and let $m_1 \in \mathbb{Z}$, $m_2 \in \mathbb{N}$.*

Then there exist $\varepsilon' > 0$ so that the following estimates hold, uniformly in $\mathcal{J}$.

(i) If $b_{n+1} \in L^\infty(\mathbb{R}^d)$,

$$\Big\| \sum_{j\in\mathcal{J}} S_j^{m_1,m_2}[b_{n+1}] \Big\|_{L^2\to L^2} \lesssim \min\big\{2^{-\varepsilon'(|m_1|+m_2)} n^2 \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \sup_j \|\varsigma_j\|_{L^1}\big\} \|b_{n+1}\|_\infty. \tag{13.13}$$

(ii) We have $\lim_{N\to\infty} \sum_{j=-N}^N S_j^{m_1,m_2}[b_{n+1}] = S^{m_1,m_2}[b_{n+1}]$ in the strong operator topology (as operators $L^2 \to L^2$) and the bound (13.13) *remains true for the limit S^{m_1,m_2}.*

(iii) We have $\sum_{m_1\in\mathbb{Z}} \sum_{m_2>0} S^{m_1,m_2}[b_{n+1}] \to S[b_{n+1}]$ with absolute convergence in $\mathcal{L}(L^2,L^2)$. Also $\sum_{j=-N}^N S_j[b_{n+1}]$ converges to an operator $S[b_{n+1}]$ in the strong operator topology as operators $L^2 \to L^2$ and

$$\|S[b_{n+1}]\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^2(1+n\Gamma_\varepsilon)\, \|b_{n+1}\|_\infty.$$

(iv) In (ii), (iii) the convergence in the strong operator topology is equicontinuous with respect to $\{b_{n+1} : \|b_{n+1}\|_\infty \le 1\}$.

Proof of Proposition 13.4, given Proposition 13.3. For the proof of (i) we apply the almost orthogonality Lemma 9.1. To this end we need to derive the estimate

$$\begin{aligned} &\big\| \mathcal{Q}_{k_1} S_{j+k_1}^{m_1,m_2}[b_{n+1}] \mathcal{Q}_{j+k_1+k_2} \big\|_{L^2\to L^2} \\ &\lesssim A_{j,k_2}^{m_1,m_2} := \min \|b_{n+1}\|_\infty \big\{ 2^{-\varepsilon_1(|m_1|+m_2)} n^2 \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, 2^{-|j+m_1|-|m_2+k_2|} \sup_j \|\varsigma_j\|_{L^1} \big\} \end{aligned} \tag{13.14}$$

for some $\varepsilon_1 > 0$. To see this we note that the bound

$$\big\| S_{j+k_1}^{m_1,m_2}[b_{n+1}] \big\|_{L^2\to L^2} \lesssim \min \|b_{n+1}\|_\infty \big\{ 2^{-\varepsilon_1(|m_1|+m_2)} n^2 \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} \big\}$$

(and hence the corresponding estimate for $\mathcal{Q}_{k_1} S_{j+k_1}^{m_1,m_2}[b_{n+1}] \mathcal{Q}_{j+k_1+k_2}$) follows from Proposition 13.3. The bound

$$\big\| \mathcal{Q}_{k_1} S_{j+k_1}^{m_1,m_2}[b_{n+1}] \mathcal{Q}_{j+k_1+k_2} \big\|_{L^2\to L^2} \lesssim 2^{-|j+m_1|-|m_2+k_2|} \sup_j \|\varsigma_j\|_{L^1}$$

follows from the fact that $\|\mathcal{Q}_k Q_l\|_{L^2\to L^2}, \|Q_l \mathcal{Q}_k\|_{L^2\to L^2} \lesssim 2^{-|k-l|}$, the definition of $S_{j+k}^{m_1,m_2}$, and Lemma 2.7.

We now observe that for $A_{j,k_2}^{m_1,m_2}$ as in (13.14) we have

$$\sum_{j,k_2} A_{j,k_2}^{m_1,m_2} \lesssim \|b_{n+1}\|_\infty \min\big\{ \sup_j \|\varsigma_j\|_{L^1},\, 2^{-\varepsilon_1(|m_1|+m_2)} (|m_1|+m_2)^2 n^2 \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} \big\}.$$

By an application of Lemma 9.1 this yields (13.13) and the convergence result in (ii), with equiconvergence with respect to b_{n+1} in the unit ball of $L^\infty(\mathbb{R}^d)$. Summing in m_1, m_2 yields (iii). □

Proposition 13.5. *Let $b_2,\dots,b_n \in L^\infty(\mathbb{R}^d)$, with $\|b_i\|_\infty \le 1$, $i=2,\dots,n$. Let $\vec\varsigma = \{\varsigma_j\}$ be a bounded family in $\mathcal{B}_\varepsilon$, $\mathcal{J} \subset \mathbb{Z}^d$ with $\#\mathcal{J} < \infty$ and let $m_1 \in \mathbb{Z}$, $m_2 \in \mathbb{N}$.*

(i) If $b_1 \in L^\infty(\mathbb{R}^d)$,

$$\Big\| \sum_{j\in\mathcal{J}} T_j^{m_1,m_2}[b_1] \Big\|_{L^2\to L^2} \lesssim \min\big\{ 2^{-\varepsilon'|m_1|-\varepsilon' m_2} n^2 \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon},\, \sup_j \|\varsigma_j\|_{L^1} \log(1+n^2\Gamma_\varepsilon)\big\} \|b_1\|_\infty. \tag{13.15}$$

(ii) We have $\lim_{N\to\infty}\sum_{j=-N}^{N} T_j^{m_1,m_2}[b_1] = T^{m_1,m_2}[b_1]$ in the strong operator topology (as operators $L^2 \to L^2$) and the bound (13.15) remains true for the limit T^{m_1,m_2}.

(iii) We have $\sum_{m_1\in\mathbb{Z}}\sum_{m_2>0} T^{m_1,m_2}[b_1] \to T[b_1]$ with absolute convergence in $\mathcal{L}(L^2,L^2)$. Moreover $\sum_{j=-N}^{N} T_j[b_1]$ converges to an operator $T[b_1]$ in the strong operator topology as operators $L^2 \to L^2$ and

$$\|T[b_1]\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon)\, \|b_1\|_\infty.$$

Proof. Use Propositions 13.4 and 13.3, together with Theorem 8.22 to deduce that $S^{m_1,m_2}[b_{n+1}] = \sum_j S_j^{m_1,m_2}[b_{n+1}]$ converges in the strong operator topology as operators $H^1 \to L^1$, with uniformity in b_{n+1}, $\|b_{n+1}\|_\infty \le 1$, and we get the estimate

$$\big\|S^{m_1,m_2}[b_{n+1}]\big\|_{H^1\to L^1} \lesssim \sup \|\varsigma_j\|_{L^1} \min\big\{ \log(1+n^2\Gamma_\varepsilon),\, 2^{-\varepsilon'(|m_1|+m_2)} n^2\Gamma_\varepsilon\big\} \|b_{n+1}\|_\infty$$

Now for $b_1 \in L^\infty$, $b_{n+1} \in L^\infty$ we have by (13.7), (13.8)

$$\int b_1(x)\, S_j^{m_1,m_2}[b_{n+1}] f(x)\, dx = \int b_{n+1}(x)\, T_j^{m_1,m_2}[b_1] f(x)\, dx\,.$$

The uniformity with respect to b_{n+1} in the strong operator convergence of $\sum_j S_j^{m_1,m_2}[b_{n+1}]$ now implies that $T^{m_1,m_2}[b_1] = \sum_j T_j^{m_1,m_2}[b_1]$ converges in the strong operator topology as operators $H^1 \to L^1$ and we have the estimate

$$\big\|T^{m_1,m_2}[b_1]\big\|_{H^1\to L^1} \lesssim \|b_1\|_\infty \sup \|\varsigma_j\|_{L^1} \min\big\{ \log(1+n^2\Gamma_\varepsilon),\, 2^{-\varepsilon'(|m_1|+m_2)} n^2\Gamma_\varepsilon\big\}\,.$$

From Theorem 8.22 we then get

$$\big\|T^{m_1,m_2}[b_1]\big\|_{L^2\to L^2} \lesssim \|b_1\|_\infty \sup \|\varsigma_j\|_{L^1} \min\big\{ \log(1+n^2\Gamma_\varepsilon),\, 2^{-\varepsilon'(|m_1|+m_2)} n^2\Gamma_\varepsilon\big\}$$

which is (ii). Statement (iii) follows after summing in m_1, m_2. □

13.2. Op_ε-bounds and the proof of Proposition 13.3.

Lemma 13.6. *Let $\varepsilon > 0$, $\phi_0 \in C^1$, supported in $\{y : |y| \le 10\}$, $\varsigma \in \mathcal{B}_\epsilon(\mathbb{R}^n \times \mathbb{R}^d)$. For $\ell \ge 0$ define*

$$F_\ell(x,y) := \iiint_{|x-v-y|\le 100} |\varsigma^{(2^\ell)}(\alpha,v)| |\phi_0(y-\alpha_1 v - y') - \phi_0(y-y')|\, dv\, d\alpha\, dy'.$$

Then,

$$\sup_x \int (1+|x-y|)^{\varepsilon/2} |F_\ell(x,y)|\, dy + \sup_y \int (1+|x-y|)^{\varepsilon/2} |F_\ell(x,y)|\, dx \lesssim 2^{-\ell\varepsilon/2} \|\phi_0\|_{C^1} \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Proof. We may assume $\|\phi\|_{C^1} = 1$. We estimate, for each y,

$$\begin{aligned}
&\int (1+|x-y|)^{\varepsilon/2} |F_\ell(x,y)|\, dx \\
&= \iiiint\limits_{|x-v-y|\le 100} (1+|x-y|)^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)||\phi_0(y-\alpha_1 v - y') - \phi_0(y-y')|\, dv\, d\alpha\, dy'\, dx \\
&\lesssim \iiint (1+|v|)^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)||\phi_0(y-\alpha_1 v - y') - \phi_0(y-y')|\, dv\, d\alpha\, dy' \\
&\lesssim \iint (1+|v|)^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)| \min\{1, |\alpha_1 v|^{\varepsilon/2}\}\, dv\, d\alpha \\
&\lesssim \iint |v|^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)|\, dv\, d\alpha + \iint |\alpha_1 v|^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)|\, dv\, d\alpha\,.
\end{aligned}$$

Now

$$\iint |v|^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)|\, d\alpha\, dv = 2^{-\ell\varepsilon/2} \iint |v|^{\varepsilon/2} |\varsigma(\alpha,v)|\, d\alpha\, dv \lesssim 2^{-\ell\varepsilon/2} \|\varsigma\|_{\mathcal{B}_{\varepsilon/2}} \lesssim 2^{-\ell\varepsilon/2} \|\varsigma\|_{\mathcal{B}_\varepsilon},$$

and

$$\begin{aligned}
&\iint |\alpha_1 v|^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)|\, d\alpha\, dv = 2^{-\ell\varepsilon/2} \iint |\alpha_1 v|^{\varepsilon/2} |\varsigma(\alpha,v)|\, d\alpha\, dv \\
&\le 2^{-\ell\varepsilon/2} \iint (|\alpha_1| + |v|)^{\varepsilon} |\varsigma(\alpha,v)|\, d\alpha\, dv \lesssim 2^{-\ell\varepsilon/2} \|\varsigma\|_{\mathcal{B}_\epsilon}.
\end{aligned}$$

This completes the proof that $\sup_y \int (1+|x-y|)^{\varepsilon/2} |F_\ell(x,y)|\, dx \lesssim 2^{-\ell\varepsilon/2} \|\varsigma\|_{\mathcal{B}_\epsilon}$.

Next we estimate for $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\int (1+|x-y|)^{\varepsilon/2} |F_\ell(x,y)|\, dy \\
&= \iiiint\limits_{|x-v-y|\le 100} (1+|x-y|)^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)||\phi_0(y-\alpha_1 v - y') - \phi_0(y-y')|\, dv\, d\alpha\, dy'\, dy \\
&\lesssim \iiiint\limits_{|x-v-y|\le 100} (1+|v|)^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)| \min\{1, |\alpha_1 v|^{\varepsilon/2}\} \mathbb{1}_{\{|y-\alpha_1 v - y'|\le 10 \text{ or } |y-y'|\le 10\}}\, dv\, d\alpha\, dy'\, dy \\
&\lesssim \iiint\limits_{|x-v-y|\le 100} (1+|v|)^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)| \min\{1, |\alpha_1 v|^{\varepsilon/2}\}\, dv\, d\alpha\, dx \\
&\lesssim \iint (1+|v|)^{\varepsilon/2} |\varsigma^{(2^\ell)}(\alpha,v)| \min\{1, |\alpha_1 v|^{\varepsilon/2}\}\, dv\, d\alpha
\end{aligned}$$

and above the last quantity has already been shown to be $\lesssim 2^{-\ell\varepsilon/2} \|\varsigma\|_{\mathcal{B}_\epsilon}$. This completes the proof of the lemma. □

Lemma 13.7. *Let $\epsilon > 0$. For $\phi \in C^1$, supported in $\{y : |y| \le 10\}$, $\varsigma \in \mathcal{B}_\varepsilon(\mathbb{R}^d \times \mathbb{R}^n)$, $j \ge 0$, let*

$$g_j(x,y) = \int |\varsigma^{(2^j)}(\alpha,v)||\phi(x-v-y) - \phi(x-y)|\, d\alpha\, dv.$$

Then

$$\sup_x \int g_j(x,y)\, dy + \sup_y \int g_j(x,y)\, dx \lesssim 2^{-\varepsilon j} \|\varsigma\|_{\mathcal{B}_\varepsilon} \|\phi\|_{C^1}\,.$$

Proof. We may assume $\|\phi\|_{C^1} = 1$. For any x, we have

$$\begin{aligned}
&\iiint (1+|x-y|)^{\varepsilon} |\varsigma^{(2^j)}(\alpha,v)| |\phi(x-v-y)-\phi(x-y)|\, d\alpha\, dv\, dy \\
&\lesssim \iiint (1+|x-y|)^{\varepsilon} |\varsigma^{(2^j)}(\alpha,v)| \min\{1,|v|^{\varepsilon}\} \chi_{\{|x-v-y|\le 10 \text{ or } |x-y|\le 10\}}\, d\alpha\, dv\, dy \\
&\lesssim \iint (1+|v|)^{\varepsilon} |\varsigma^{(2^j)}(\alpha,v)| \min\{1,|v|^{\varepsilon}\}\, d\alpha\, dv \\
&\lesssim \iint |v|^{\varepsilon} |\varsigma^{(2^j)}(\alpha,v)|\, d\alpha\, dv \lesssim 2^{-j\epsilon} \|\varsigma\|_{\mathcal{B}_{\varepsilon}},
\end{aligned}$$

where the last inequality has already been used in the proof of Lemma 13.6. By symmetry we also get the corresponding second inequality with the roles of x and y reversed. □

Lemma 13.8. *For $\epsilon > 0$ there is $\varepsilon' > 0$ such that the following holds. Let $\phi_1, \dots, \phi_{n+1} \in C^2$ supported in $\{y : |y| \le 10\}$ and such that for all but at most two l, $\phi_l \ge 0$ and $\int \phi_l = 1$. For $k \in \mathbb{Z}$ set $Y_k^l f = f * \phi_l^{(2^k)}$. For $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$, $\varsigma \in \mathcal{B}_\epsilon(\mathbb{R}^n \times \mathbb{R}^d)$ with*

$$\int \varsigma(\alpha, v)\, dv = 0, \tag{13.16}$$

and define a kernel $K_{j,k} \equiv K_{j,k}[b_1, \dots, b_n]$ by

$$\int g(x) \int K_{j,k}(x,y) f(y)\, dy\, dx = \Lambda[\varsigma^{(2^j)}](Y_k^1 b_1, \dots, Y_k^n b_n, Y_k^{n+1} g, f).$$

Then, for $j \ge k$,

$$\|\mathrm{Dil}_{2^{-k}} K_{j,k}\|_{\mathrm{Op}_{\varepsilon'}} \lesssim 2^{-\epsilon'(j-k)} n \|\varsigma\|_{\mathcal{B}_\epsilon} \prod_{i=1}^n \|b_i\|_\infty,$$

$$\|\mathrm{Dil}_{2^{-k}} K_{j,k}\|_{\mathrm{Op}_0} \lesssim \|\varsigma\|_{L^1} \prod_{i=1}^n \|b_i\|_\infty.$$

Here, the implicit constants may depend on $\max\limits_{i_1,i_2,i_3,i_4 \in \{1,\dots,n+1\}} \|\phi_{i_1}\|_{C^2} \|\phi_{i_2}\|_{C^2} \|\phi_{i_3}\|_{C^2} \|\phi_{l_4}\|_{C^2}$.

Proof. The bound for the Op_0 norm is immediate so we focus only on the bound for the Op_ε-norms. Note that by scaling (see Lemma 4.16)

$$\Lambda[\varsigma^{(2^j)}](Y_k^1 b_1, \dots, Y_k^n b_n, Y_k^{n+1} g, f) = 2^{-kd} \Lambda[\varsigma^{(2^{j-k})}](Y_0 b_1^k, \dots, Y_0 b_n^k, Y_0 g^k, f^k)$$

where $b_i^k = b_i(2^{-k}\cdot)$, $f^k = f(2^{-k}\cdot)$, $g^k = g(2^{-k}\cdot)$. This leads to

$$K_{j,k}[b_1, \dots, b_n](x,y) = 2^{kd} K_{j-k,0}[b_1^k, \dots, b_n^k](2^k x, 2^k y).$$

Now $\|b_i^k\|_\infty = \|b_i\|_\infty$, $i = 1, \dots, n$, and hence after replacing the functions b_i by b_i^k, $i = 1, \dots, n$, it suffices to check the case $k = 0$. That is, we need to prove, for $\ell \ge 0$,

$$\|K_{\ell,0}[b_1, \dots, b_n]\|_{\mathrm{Op}_\varepsilon} \lesssim 2^{-\varepsilon' \ell} n \|\varsigma\|_{\mathcal{B}_\varepsilon} \prod_{i=1}^n \|b_i\|_\infty. \tag{13.17}$$

In what follows we may assume $\|b_i\|_{L^\infty} = 1$, $i = 1, \dots, n$. We will prove, under the assumption that all but at most *three* of the ϕ_i satisfy $\phi_i \ge 0$, $\int \phi_i = 1$ we have

$$\sup_x \int (1+|x-y|)^{\varepsilon'} |K_{\ell,0}(x,y)|\, dy + \sup_y \int (1+|x-y|)^{\varepsilon'} |K_{\ell,0}(x,y)|\, dx \lesssim 2^{-\varepsilon' \ell} n \|\varsigma\|_{\mathcal{B}_\varepsilon}, \tag{13.18}$$

where the implicit constant is allowed to depend on the C^1 norms of up to three of ϕ_i (instead of the C^2 norms).

First we see why (13.18) yields the result. The explicit formula for the kernel is

$$K_{\ell,0}(x,y) = \int \phi_{n+1}(y-v-x) \int \varsigma^{(2^\ell)}(\alpha, v) \prod_{i=1}^{n} Y_0^i b_i(y-\alpha_i v)\, d\alpha\, dv. \tag{13.19}$$

It implies that $\partial_{x_m} K_{\ell,0}(x,y)$ is a term of the form covered by (13.18) (with ϕ_{n+1} replaced by $-\partial_{x_m}\phi_{n+1}$). Moreover, $\partial_{y_m} K_{\ell,0}(x,y)$ is a sum of $n+1$ terms of the form covered by (13.18), indeed differentiating (13.19) yields (setting $b_{n+1} := g$)

$$\int b_{n+1}(x) \int \partial_{y_m} K_{\ell,0}(x,y) f(y)\, dy\, dx$$
$$= \sum_{i=1}^{n+1} \Lambda[\varsigma^{(2^\ell)}](Y_0^1 b_1, \dots, Y_0^{i-1} b_{i-1}, \partial_{x_m} Y_0^l b_i, Y_0^{i+1} b_{i+1}, \dots, Y_0^{n+1} b_{n+1}, f).$$

Thus, $\partial_{x_m} K_{\ell,0}(x,y)$ is a sum of $n+1$ terms of the form covered by (13.18). From these remarks, it follows, given (13.18), that the expressions

$$\sup_{\substack{y\\0<|h|\le 1}} |h|^{-1} \int |K_{\ell,0}(x,y+h) - K_{\ell,0}(x,y)|\, dx,$$
$$\sup_{\substack{x\\0<|h|\le 1}} |h|^{-1} \int |K_{\ell,0}(x,y+h) - K_{\ell,0}(x,y)|\, dy,$$
$$\sup_{\substack{y\\0<|h|\le 1}} |h|^{-1} \int |K_{\ell,0}(x+h,y) - K_{\ell,0}(x,y)|\, dx,$$
$$\sup_{\substack{x\\0<|h|\le 1}} |h|^{-1} \int |K_{\ell,0}(x+h,y) - K_{\ell,0}(x,y)|\, dy$$

are all bounded by a constant times $2^{-\ell\varepsilon'} n \|\varsigma\|_{\mathcal{B}_\epsilon}$.

It remains to prove (13.18). We first compute, with $\tilde\varsigma(\alpha,v) = \varsigma(1-\alpha_1,\dots,1-\alpha_n,v)$,

$$\Lambda[\varsigma^{(2^\ell)}](Y_0^1 b_1, \dots, Y_0^n, Y_0^{n+1} g, f) = \Lambda[\varsigma^{(2^\ell)}](Y_0^1 b_1, \dots, Y_0^n, f, Y_0^{n+1} g)$$
$$= \iiint \tilde\varsigma^{(2^\ell)}(\alpha, w-y) f(y) \int \phi_{n+1}(w-x) g(x) dx \prod_{i=1}^{n} Y_0^i b_i(w(1-\alpha_i)+\alpha_i y)\, d\alpha\, dw\, dy$$
$$= \iint g(x) f(y) \iint \tilde\varsigma^{(2^\ell)}(\alpha, v) \phi_{n+1}(y+v-x) \prod_{i=1}^{n} Y_0^i b_i(y+(1-\alpha_i)v)\, dv\, d\alpha\, dx\, dy$$

and changing variable in α again we get

$$K_{\ell,0}(x,y) = \iint \varsigma^{(2^\ell)}(\alpha, v) \phi_{n+1}(y+v-x) \prod_{i=1}^{n} Y_0^i b_i(y+\alpha_i v)\, dv\, d\alpha$$
$$= \iint \varsigma^{(2^\ell)}(\alpha, v) \Big[\phi_{n+1}(y+v-x) \prod_{i=1}^{n} Y_0^i b_i(y+\alpha_i v) - \phi_{n+1}(y-x) \prod_{i=1}^{n} Y_0^i b_i(y)\Big]\, dv\, d\alpha;$$

here we have used the cancellation condition (13.16). Now

$$|K_{\ell,0}(x,y)| \le I(x,y) + \sum_{i=1}^{n} II_i(x,y)$$

where

$$I(x,y) = \iint |\varsigma^{(2^\ell)}(\alpha,v)||\phi_{n+1}(y+v-x)-\phi_{n+1}(y-x)|\,dv\,d\alpha\,,$$

$$II_i(x,y) = \iint |\varsigma^{(2^\ell)}(\alpha,v)||\phi_{n+1}(y-x)|\int |\phi_i(y+\alpha_i v-w)-\phi_i(y-w)|dw\,dv\,d\alpha\,.$$

Now apply Lemma 13.6 to the expressions II_i and Lemma 13.7 to I, and (13.18) follows. This completes the proof. □

Proof of Proposition 13.3, conclusion. We focus on the estimates for $S_j^{m_1,m_2}[b_{n+1}]$ as the estimates for $T_j^{m_1,m_2}[b_1]$ are analogous (switch the roles of b_1 and b_{n+1}). We may assume $\|b_{n+1}\|_\infty = 1$.

In what follows we identify operators with their Schwartz kernels. For an operator R we denote by $\partial_{x_\mu} R$ the operator with Schwartz kernel $\partial_{x_\mu} R(x,y)$.

We use Lemma 6.8 to write $Q_{j+m_2} = \sum_{\mu=1}^d 2^{-(j+m_2)}\partial_{x_\mu} R^\mu_{j+m_2}$, where $R^\mu_{j+m_2} = f * \tilde\phi_\mu^{(2^{j+m_2})}$, and $\tilde\phi_l \in C_0^\infty$ supported in $\{x : |x| \le 2\}$. Now

$$\begin{aligned}
&\Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_j b_1, X^2_{j+m_1}P_j b_2, \dots, X^{n+1}_{j+m_1}P_j b_{n+1}, Q_{j+m_2} f)\\
&= 2^{-(j+m_2)}\sum_{\mu=1}^d \iint \varsigma_j^{(2^j)}(\alpha,v)\int \partial_{x_\mu} R^\mu_{j+m_2} f(x) X^{n+1}_{j+m_1}P_j b_{n+1}(x-v)\,\times\\
&\qquad\qquad Q_{j+m_1}P_j b_1(x-\alpha_1 v)\prod_{i=2}^n Q_{j+m_1}P_j b_i(x-\alpha_i v)\,dx\ dv\,d\alpha\,.
\end{aligned}$$

Integrating by parts we see that this expression equals

$$\begin{aligned}
(13.20)\quad & -2^{-(j+m_2)}\sum_{\mu=1}^d \Big(\Lambda[\varsigma_j^{(2^j)}](\partial_{x_\mu} Q_{j+m_1}P_j b_1, X^2_{j+m_1}P_j b_2,\dots, X^{n+1}_{j+m_1}P_j b_{n+1}, R^\mu_{j+m_2} f)\\
&+\sum_{\nu=2}^{n+1}\Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_j b_1, X^2_{j+m_1}P_j b_2,\dots,\partial_{x_\mu} X^\nu_{j+m_1}P_j b_\nu \dots, X^{n+1}_{j+m_1}P_j b_{n+1}, R^\mu_{j+m_2} f)\Big).
\end{aligned}$$

We distinguish the cases $m_1 \le 0$ and $m_1 \ge 0$.

For $m_1 \le 0$ we write (13.20) as

$$\begin{aligned}
&\Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_j b_1, X^2_{j+m_1}P_j b_2,\dots, X^{n+1}_{j+m_1}P_j b_{n+1}, Q_{j+m_2} f)\\
&= -2^{-m_2+m_1}\sum_{\mu=1}^d\sum_{\nu=1}^{n+1}\Lambda[\varsigma_j^{(2^j)}](Y^{1,\mu,\nu}_{j+m_1,j} b_1,\dots, Y^{n+1,\mu,\nu}_{j+m_1,j} b_{n+1}, R^\mu_{j+m_2} f)
\end{aligned}$$

where, for $m_1 \le 0$, the operators $Y^{i,\mu,\nu}_{j+m_1,j}$ are given by

$$Y^{1,\mu,\nu}_{j+m_1,j} = \begin{cases} 2^{-j-m_1}\partial_{x_\mu}(Q_{j+m_1}P_j) & \text{if } \nu = 1,\\ Q_{j+m_1}P_j & \text{if } \nu \in \{2,\dots,n+1\}\end{cases}$$

if $i = 1$, and by

$$Y^{i,\mu,\nu}_{j+m_1,j} = \begin{cases} 2^{-j-m_1}\partial_{x_\mu}(P_{j+m_1}P_j) & \text{if } \nu = i,\\ P_{j+m_1}P_j & \text{if } \nu \in \{1,\dots,n+1\}\setminus\{i\}\end{cases}$$

if $2 \le i \le n+1$.

Hence for $m_1 \le 0$

$$\begin{aligned}&\Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_jb_1, X^2_{j+m_1}P_jb_2, \dots, X^{n+1}_{j+m_1}P_jb_{n+1}, Q_{j+m_2}f)\\&= 2^{-m_2+m_1}\sum_{\mu=1}^{d}\sum_{\nu=1}^{n}\int b_{n+1}(x)K^{\mu,\nu}_{j+m_1,j}(x,y)R^{\mu}_{j+m_2}f(y)dy\end{aligned}$$

and by Lemma 13.8

$$\|\mathrm{Dil}_{2^{-j-m_1}}K^{\mu,\nu}_{j+m_1,j}\|_{Op_{\varepsilon'}} \lesssim \|\varsigma_j\|_{\mathcal{B}_\varepsilon}$$

for some $\varepsilon' \le \varepsilon$. This, together with Lemma 8.25, implies the asserted bound (13.11), for $m_1 \le 0$.

We now consider the case $m_1 > 0$. Now use the cancellation and support properties of Q_{j+m_1} to write

$$Q_{j+m_1}P_j = 2^{-m_1}Z_{j,m_1}$$

where $Z_{j,m_1} = f * \upsilon^{(2^j)}_{j,m}$ and $\{\upsilon_{j,m} : j \in \mathbb{Z}, m_1 \in \mathbb{N}\}$ is a bounded family of C^∞_c functions supported in $\{y : |y| \le 2\}$.

We now write (13.20) as

$$\begin{aligned}&\Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_jb_1, X^2_{j+m_1}P_jb_2, \dots, X^{n+1}_{j+m_1}P_jb_{n+1}, Q_{j+m_2}f)\\&= -\,2^{-m_2-m_1}\sum_{\mu=1}^{d}\sum_{\nu=1}^{n+1}\Lambda[\varsigma_j^{(2^j)}](Y^{1,\mu,\nu}_{j+m_1,j}b_1, \dots, Y^{n+1,\mu,\nu}_{j+m_1,j}b_{n+1}, R^{\mu}_{j+m_2}f)\end{aligned}$$

where (now for $m_1 > 0$)

$$Y^{1,\mu,\nu}_{j+m_1,j} = \begin{cases} 2^{-j}\partial_{x_\mu}Z_{j,m_1} & \text{if } \nu = 1,\\ Z_{j,m_1} & \text{if } \nu \in \{2,\dots,n+1\},\end{cases}$$

and for $2 \le i \le n+1$

$$Y^{i,\mu,\nu}_{j+m_1,j} = \begin{cases} 2^{-j}\partial_{x_\mu}(P_{j+m_1}P_j) & \text{if } \nu = i,\\ P_{j+m_1}P_j & \text{if } \nu \in \{1,\dots,n+1\}\setminus\{i\}\,.\end{cases}$$

We see, using Lemma 13.8, that for $m_1 > 0$

$$\begin{aligned}&\Lambda[\varsigma_j^{(2^j)}](Q_{j+m_1}P_jb_1, X^2_{j+m_1}P_jb_2, \dots, X^{n+1}_{j+m_1}P_jb_{n+1}, Q_{j+m_2}f)\\&= 2^{-m_2-m_1}\sum_{\mu=1}^{d}\sum_{\nu=1}^{n}\int b_{n+1}(x)K^{\mu,\nu,m_1}_{j}(x,y)R^{\mu}_{j+m_2}f(y)\,dy\end{aligned}$$

with

$$\big\|\mathrm{Dil}_{2^{-j}}K^{\mu,\nu,m_1}_j\big\|_{\mathrm{Op}_\varepsilon} \lesssim \|\varsigma_j\|_{\mathcal{B}_\varepsilon}\,.$$

Using also Lemma 8.25 we obtain the asserted bound (13.11), for $m_1 > 0$. □

13.3. **Proof of the bound** (13.1)**, concluded.** The following proposition will conclude the proof of part IV in Theorem 5.1.

Proposition 13.9. *Let $1 \le l_1 \ne l_2 \le n+2$. Then, for $p \in (1,2]$ and $p' = p/(p-1)$*

$$\begin{aligned}&\Big|\sum_{j\in\mathbb{Z}}\Lambda[\varsigma_j^{(2^j)}](P_jb_1, \dots, P_jb_{n+1}, (I-P_j)b_{n+2})\Big|\\&\qquad\le C_{d,p,\epsilon}\,n(\sup_j\|\varsigma_j\|_{L^1})\log^3(1+n\Gamma_\varepsilon)\Big(\prod_{l\ne l_1,l_2}\|b_l\|_\infty\Big)\|b_{l_1}\|_p\|b_{l_2}\|_{p'}\,.\end{aligned}$$

Proof. By symmetry of the roles of $b_1, \dots, b_{n+1}$, via Theorem 2.9, it suffices to prove the result for three cases: $(l_1, l_2) = (n+1, n+2)$, $(l_1, l_2) = (n+2, n+1)$, and $(l_1, l_2) = (1, n+1)$.

We begin with the case $(l_1, l_2) = (n+1, n+2)$. For this we define an operator $S_{1,j} \equiv S_{1,j}[b_1, \dots, b_n]$ by

$$\int g(x)(S_{1,j}[b_1, \dots, b_n]f)(x)\, dx := \Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+1}, (I - P_j)b_{n+2}).$$

It is straightforward to verify the inequalities

$$\|\mathrm{Dil}_{2^{-j}} S_{1,j}\|_{\mathrm{Op}_\varepsilon} \lesssim n(\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{\mathcal{B}_\varepsilon} \prod_{i=1}^{n} \|b_i\|_\infty,$$

$$\|\mathrm{Dil}_{2^{-j}} S_{1,j}\|_{\mathrm{Op}_0} \lesssim (\sup_{j\in\mathbb{Z}} \|\varsigma_j\|_{L^1}) \prod_{i=1}^{n} \|b_i\|_\infty;$$

here $\varepsilon \le 1$ and the Op_ε, Op_0 norms are as in (8.36), (8.37).

Theorem 13.1 shows

$$\Big\| \sum_{j\in\mathbb{Z}} S_{1,j}[b_1, \dots, b_n] \Big\|_{L^2\to L^2} \lesssim n(\sup_j \|\varsigma_j\|_{L^1}) \log^3(1 + n\Gamma_\varepsilon) \prod_{i=1}^{n} \|b_i\|_\infty.$$

with convergence in the strong operator topology. By Proposition 8.9 we get, for $1 < p \le 2$,

$$\Big\| \sum_{j\in\mathbb{Z}} S_{1,j}[b_1, \dots, b_n] \Big\|_{L^p\to L^p} \le C_{d,p,\epsilon} n(\sup_j \|\varsigma_j\|_{L^1}) \log^3(1 + n\Gamma_\varepsilon) \prod_{i=1}^{n} \|b_i\|_\infty,$$

and

$$\Big\| \sum_{j\in\mathbb{Z}} {}^tS_{1,j}[b_1, \dots, b_n] \Big\|_{L^p\to L^p} \le C_{d,p,\epsilon} n(\sup_j \|\varsigma_j\|_{L^1}) \log^3(1 + n\Gamma_\varepsilon) \prod_{i=1}^{n} \|b_i\|_\infty,$$

which are equivalent to the statement of the proposition in the cases $(l_1, l_2) = (n+1, n+2)$ and $(l_1, l_2) = (n+2, n+1)$, respectively. The convergence is in the sense of the strong operator topology (as operators bounded on L^p).

We now turn to the case $(l_1, l_2) = (1, n+1)$. If we apply Theorem 8.22 to $\sum {}^tS_{1,j}$ we also get an $H^1 \to L^1$ bound

$$\Big\| \sum_{j\in\mathbb{Z}} {}^tS_{1,j}[b_1, \dots, b_n] \Big\|_{H^1\to L^1} \lesssim n(\sup_j \|\varsigma_j\|_{L^1}) \log^3(1 + n\Gamma_\varepsilon) \prod_{i=1}^{n} \|b_i\|_{L^\infty}.$$

This means that for $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$, $b_{n+2} \in L^\infty(\mathbb{R}^d)$, $b_{n+1} \in H^1(\mathbb{R}^d)$, we have

$$\Big| \sum_{j\in\mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+1}, (I - P_j)b_{n+2}) \Big| \lesssim n(\sup_j \|\varsigma_j\|_{L^1}) \log^3(1 + n\Gamma_\varepsilon) \Big(\prod_{i=1}^{n} \|b_i\|_\infty \Big) \|b_{n+1}\|_{H^1} \|b_{n+1}\|_\infty. \tag{13.21}$$

For $j \in \mathbb{Z}$, define an operator $S_{2,j}[b_2, \dots, b_n, b_{n+2}]$ by

$$\int g(x)(S_{2,j}[b_2, \dots, b_n, b_{n+2}]f)(x)\, dx := \Lambda[\varsigma_j^{(2^j)}](g, P_j b_2, \dots, P_j b_n, f, (I - P_j)b_{n+2}).$$

Since ${}^tP_j = P_j$ the case $(l_1, l_2) = (1, n+1)$ is equivalent to the inequality

$$\Big\| \sum_{j\in\mathbb{Z}} P_j S_{2,j}[b_2, \dots, b_n, b_{n+2}] P_j \Big\|_{L^p\to L^p} \lesssim n \sup_j \|\varsigma\|_{L^1} (1 + n\Gamma_\varepsilon) \prod_{l\in\{2,\dots,n,n+2\}} \|b_l\|_\infty. \tag{13.22}$$

To show (13.22) we first observe that by Theorem 2.9, there is a $c > 0$ (independent of n) such that for $\varepsilon' < c\varepsilon$ there are $\tilde{\varsigma}_j \in \mathcal{B}_{\varepsilon'}(\mathbb{R}^n \times \mathbb{R}^d)$ with $\|\tilde{\varsigma}_j\|_{\mathcal{B}_{\varepsilon'}} \lesssim n\|\varsigma\|_{\mathcal{B}_\varepsilon}$ and $\|\tilde{\varsigma}_j\|_{L^1} = \|\varsigma_j\|_{L^1}$ such that

$$\int b_1(x)(S_{2,j}[b_2,\dots,b_n,b_{n+2}]b_{n+1})(x)\,dx = \Lambda[\tilde{\varsigma}_j^{(2^j)}](P_jb_2,\dots,P_jb_n,(I-P_j)b_{n+2},b_1,b_{n+1}).$$

If we apply (13.21) with the family $\{\tilde{\varsigma}_j\}$ in place of $\{\varsigma_j\}$ and ε' in place of ε) we get

$$\begin{aligned}
&\Big|\sum_{j\in\mathbb{Z}}\Lambda[\tilde{\varsigma}_j^{(2^j)}](P_jb_1,\dots,P_jb_{n+1},(I-P_j)b_{n+2})\Big| \\
&\quad\lesssim n(\sup_j\|\varsigma_j\|_{L^1})\log^3\big(1+n\frac{\sup_j\|\tilde{\varsigma}_j\|_{\mathcal{B}_\varepsilon}}{\sup_j\|\tilde{\varsigma}_j\|_{L^1}}\big)\big(\prod_{i=1}^n\|b_i\|_\infty\big)\|b_{n+1}\|_{H^1}\|b_{n+1}\|_\infty \\
&\quad\lesssim n(\sup_j\|\varsigma_j\|_{L^1})\log^3(1+n\Gamma_\varepsilon)\big(\prod_{i=1}^n\|b_i\|_\infty\big)\|b_{n+1}\|_{H^1}\|b_{n+1}\|_\infty
\end{aligned}$$

which (in view of ${}^tP_j = P_j$) can be rephrased as

$$\Big\|\sum_j P_jS_{2,j}[b_2,\dots,b_n,b_{n+2}]P_j\Big\|_{H^1\to L^1} \lesssim n(\sup_j\|\varsigma_j\|_{L^1})\log^3(1+n\Gamma_\varepsilon)\prod_{l\in\{2,\dots,n,n+2\}}\|b_l\|_\infty.$$

We wish to apply Lemma 8.24 to the kernels $\sigma_j = \mathrm{Dil}_{2^{-j}}S_{2,j}$. Observe that the Schur integrability norms for these kernels satisfy the uniform estimates

$$\mathrm{Int}^1_\varepsilon[\sigma_j] + \mathrm{Int}^\infty_\varepsilon[\sigma_j] \lesssim \|\tilde{\varsigma}_j\|_{\mathcal{B}_{\epsilon'}}\prod_{l\in\{2,\dots,n,n+2\}}\|b_l\|_{L^\infty} \lesssim n\sup_j\|\varsigma_j\|_{\mathcal{B}_\epsilon}\prod_{l\in\{2,\dots,n,n+2\}}\|b_l\|_\infty,$$

and

$$\mathrm{Int}^1_\varepsilon[\sigma_j] + \mathrm{Int}^\infty_\varepsilon[\sigma_j] \lesssim \|\tilde{\varsigma}_j\|_{L^1}\prod_{l\in\{2,\dots,n,n+2\}}\|b_l\|_\infty \le \sup_j\|\varsigma_j\|_{L^1}\prod_{l\in\{2,\dots,n,n+2\}}\|b_l\|_\infty.$$

Now Theorem 8.22 in conjunction with Lemma 8.24 applies to show

$$\Big\|\sum_j P_jS_{2,j}[b_2,\dots,b_n,b_{n+2}]P_j\Big\|_{L^2\to L^2} \lesssim n(\sup_j\|\varsigma_j\|_{L^1})\log^3(1+n\Gamma_\varepsilon)\prod_{l\in\{2,\dots,n,n+2\}}\|b_l\|_\infty,$$

with convergence in the strong operator topology. Finally (13.22) follows by interpolation (see Corollary 8.10). This completes the proof. □

14. Proof of Theorem 5.1: Part V

In this section, we consider the multilinear form

$$\Lambda^3(b_1,\dots,b_{n+2}) := \sum_j \Lambda[\varsigma_j^{(2^j)}](P_jb_1,\dots,P_jb_{n+2}),$$

where the summation is a priori extended over a finite subset of $\mathbb{Z}$, and where, for some fixed $\epsilon > 0$, $\{\varsigma_j : j\in\mathbb{Z}\} \subset \mathcal{B}_\epsilon(\mathbb{R}^n\times\mathbb{R}^d)$ is a bounded set with $\int\varsigma_j(\alpha,v)\,dv = 0$, for all j and almost every α. To prove part V of Theorem 5.1 we need to establish for $1 < p \le 2$ the inequality

$$|\Lambda^3(b_1,\dots,b_{n+2})| \le C_{d,p,\epsilon}n^2(\sup_j\|\varsigma_j\|_{L^1})\log^3(1+n\Gamma_\varepsilon)\big(\prod_{i=1}^n\|b_i\|_\infty\big)\|b_{n+1}\|_{p'}\|b_{n+2}\|_p. \tag{14.1}$$

As in the previous section the heart of the proof lies in the case $p = 2$ which we state as a theorem.

Theorem 14.1. *Let $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$ and $b_{n+1}, b_{n+2} \in L^2(\mathbb{R}^d)$. Then,*

$$\lim_{N\to\infty} \sum_{j=-N}^{N} \Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+2}) = \Lambda^3(b_1, \dots, b_{n+2})$$

and Λ^3 satisfies

$$|\Lambda^3(b_1, \dots, b_{n+2})| \le C_{d,\epsilon} n^2 \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon)\big(\prod_{i=1}^n \|b_i\|_\infty\big)\|b_{n+1}\|_2 \|b_{n+1}\|_2.$$

The sum defining the operator $T^3[n_1, \dots, b_n]$ associated to Λ^3 converges in the strong operator topology as bounded operators $L^2 \to L^2$.

Proof of (14.1) *given Theorem 14.1.* We may assume $\|b_l\|_{L^\infty} = 1$, $l = 1, \dots, n$. For $j \in \mathbb{Z}$ define the operator T_j by

$$\int g(x)\, T_j f(x)\, dx := \Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_n, P_j g, f).$$

Theorem 14.1 is equivalent to

$$\Big\|\sum_{j\in\mathbb{Z}} T_j P_j\Big\|_{L^2\to L^2} \lesssim n^2 \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \prod_{i=1}^n \|b_i\|_p$$

Corollary 8.10 applies since $\sup_j \mathrm{Int}^1_\varepsilon[\mathrm{Dil}_{2^{-j}} T_j] \lesssim \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}$, $\sup_j \mathrm{Int}^1_0[\mathrm{Dil}_{2^{-j}} T_j] \lesssim \sup_j \|\varsigma_j\|_{L^1}$. This completes the proof. □

We now turn to the proof of Theorem 14.1. The argument is analogous to the arguments in the previous section and therefore we shall be brief.

14.1. **Basic decompositions.** We argue as in §13.1 and decompose

$$\begin{aligned}
&\Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+2}) \\
&= \lim_{M\to\infty} \Big(\Lambda[\varsigma_j^{(2^j)}](P_{j+M} P_j b_1, \dots, P_{j+M} P_j b_{n+1}, P_{j+M} P_j b_{n+2}) \\
&\qquad\qquad - \Lambda[\varsigma_j^{(2^j)}](P_{j-M} P_j b_1, \dots, P_{j-M} P_j b_{n+1}, P_{j-M} P_j b_{n+2})\Big) \\
&= \lim_{M\to\infty} \sum_{m=-M+1}^{M} \Big(\Lambda[\varsigma_j^{(2^j)}](P_{j+m} P_j b_1, \dots, P_{j+m} P_j b_{n+2}) \\
&\qquad\qquad - \Lambda[\varsigma_j^{(2^j)}](P_{j+m-1} P_j b_1, \dots, P_{j+m-1} P_j b_{n+1}, P_{j+m-1} P_j b_{n+2})\Big)
\end{aligned}$$

and thus

$$\begin{aligned}
&\Lambda[\varsigma_j^{(2^j)}](P_j b_1, \dots, P_j b_{n+2}) = \\
&\sum_{l=1}^{n+1} \sum_{m=-\infty}^{\infty} \Lambda[\varsigma_j^{(2^j)}](P_{j+m-1} P_j b_1, \dots, P_{j+m-1} P_j b_{l-1}, Q_{j+m} P_j b_l, P_{j+m} P_j b_{l+1}, \dots, P_{j+m} P_j b_{n+2}).
\end{aligned}$$

We repeat the same procedure to each term and write, for fixed $m \in \mathbb{Z}$

$$\Lambda[\varsigma_j^{(2^j)}](P_{j+m-1} P_j b_1, \dots, P_{j+m-1} P_j b_{l-1}, Q_{j+m} P_j b_l, P_{j+m} P_j b_{l+1}, \dots, P_{j+m} P_j b_{n+2})$$

as the limit (as $M \to \infty$) of the differences

$$\begin{aligned}\Lambda[\varsigma_j^{(2^j)}]\big(&P_{j+M}P_{j+m-1}P_jb_1, \dots, P_{j+M}P_{j+m-1}P_jb_{l-1},\\ &P_{j+M}Q_{j+m}P_jb_l, P_{j+M}P_{j+m}P_jb_{l+1}, \dots, P_{j+M}P_{j+m}P_jb_{n+2}\big)\\ -\Lambda[\varsigma_j^{(2^j)}](&P_{j-M}P_{j+m-1}P_jb_1, \dots, P_{j-M}P_{j+m-1}P_jb_{l-1},\\ &P_{j-M}Q_{j+m}P_jb_l, P_{j-M}P_{j+m}P_jb_{l+1}, \dots, P_{j-M}P_{j+m}P_jb_{n+2}).\end{aligned}$$

We continue as above, writing each difference as a collapsing sum, and than expanding each summand using the multilinearity of the functionals. The limit of the expressions in the last display becomes

$$\Lambda[\varsigma_j^{(2^j)}](P_jb_1, \dots, P_jb_{n+2}) = \sum_{\substack{(l_1,l_2)\\ 1\le l_1\ne l_2\le n+2}} \sum_{(m_1,m_2)\in\mathbb{Z}^2} \lambda_{j,l_1,l_2}^{m_1,m_2}(b_1, \dots, b_{n+2})$$

where, for $l_1 < l_2$,

$$\begin{aligned}&\lambda_{j,l_1,l_2}^{m_1,m_2}(b_1, \dots, b_{n+2}) :=\\ &\Lambda[\varsigma_j^{(2^j)}]\big(P_{j+m_2-1}P_{j+m_1-1}P_jb_1, \dots, P_{j+m_2-1}P_{j+m_1-1}P_jb_{l_1-1},\\ &\qquad P_{j+m_2-1}Q_{j+m_1}P_jb_{l_1}, P_{j+m_2-1}P_{j+m_1}P_jb_{l_1+1}, \dots, P_{j+m_2-1}P_{j+m_1}P_jb_{l_2-1},\\ &\qquad Q_{j+m_2}P_{j+m_1}P_jb_{l_2}, P_{j+m_2}P_{j+m_1}P_jb_{l_2+1}, \dots, P_{j+m_2}P_{j+m_1}P_jb_{n+2}\big).\end{aligned}$$

For $l_1 > l_2$ there is an obvious modification.

There are $(n+2)(n+1) = O(n^2)$ terms in the sum $\sum_{1\le l_1\ne l_2\le n+2}$. It is therefore our task to show that

$$\Big|\sum_{m_1,m_2}\sum_j \lambda_{j,l_1,l_2}^{m_1,m_2}(b_1, \dots, b_{n+2})\Big| \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon)\Big(\prod_{i=1}^n \|b_i\|_\infty\Big)\|b_{n+1}\|_2\|b_{n+2}\|_2; \tag{14.2}$$

then summing the $O(n^2)$ terms will complete the proof.

14.2. **Proof of the bound** (14.2)**.** For $k \in \mathbb{Z}$, $1 \le l \le n+2$, let

$$X_k^{1,l}, X_k^{2,l} \in \{P_k, P_{k-1}\}.$$

For $1 \le l_1, l_2 \le n+2$, $j, k_1, k_2 \in \mathbb{Z}$, define the operator

$$\begin{aligned}&\int b_{l_1}(x)\, T_{j,l_1,l_2}^{m_1,m_2} b_{l_2}(x)\, dx\\ &:= \Lambda[\varsigma_j^{(2^j)}](X_{j+m_1}^{1,1}X_{j+m_2}^{2,1}P_jb_1, \dots, X_{j+m_1}^{1,n}X_{j+m_2}^{2,n}P_jb_n, X_{j+m_1}^{1,n+1}Q_{j+m_2}P_jb_{n+1}, Q_{j+m_1}P_jb_{n+2}),\end{aligned}$$

where we have suppressed the dependance of $T_{j,l_1,l_2}^{m_1,m_2}$ on b_l, $l \ne l_1, l_2$.

Lemma 14.2. *Let $\rho_{j,m_1,m_2} = \min\{2^j, 2^{j+m_1}, 2^{j+m_2}\}$. There is a $c > 0$ (independent of n so that for $\varepsilon' > c\varepsilon$*

$$\big\|\mathrm{Dil}_{\rho_{m_1,m_2,j}^{-1}} T_{j,l_1,l_2}^{m_1,m_2}\big\|_{\mathrm{Op}_{\varepsilon'}} \lesssim \min\{2^{-\varepsilon'|m_1|}, 2^{-\varepsilon'|m_2|}\} n^2 \|\varsigma_j\|_{\mathcal{B}_\epsilon} \prod_{l\ne l_1,l_2} \|b_l\|_{L^\infty}, \tag{14.3}$$

and

$$\|\mathrm{Dil}_{\rho_{m_1,m_2,j}^{-1}} T_{j,l_1,l_2}^{m_1,m_2}\|_{\mathrm{Op}_0} \lesssim \|\varsigma_j\|_{L^1} \prod_{l\ne l_1,l_2} \|b_l\|_\infty .$$

Proof. The bound for $\|\mathrm{Dil}_{\rho^{-1}_{m_1,m_2,j}} T^{m_1,m_2}_{j,l_1,l_2}\|_{\mathrm{Op}_0}$, and, equivalently, for $\|T^{m_1,m_2}_{j,l_1,l_2}\|_{\mathrm{Op}_0}$ is immediate, so we focus only on the bound for $\|\mathrm{Dil}_{\rho^{-1}_{m_1,m_2,j}} T^{m_1,m_2}_{j,l_1,l_2}\|_{\mathrm{Op}_{\varepsilon'}}$. Fix l_1, l_2. We may assume $\|b_l\|_{L^\infty} = 1$, $l \neq l_1, l_2$. We distinguish the cases (i) $m_1, m_2 \geq 0$, (ii) $m_1 \leq \min\{0, m_2\}$, (iii) $m_2 \leq \min\{0, m_1\}$.

(i) The case $m_1, m_2 \geq 0$. Now $\rho_{j,m_1,m_2} = 2^j$. One uses that, for $m \geq 0$, $Q_{j+m} P_j = 2^{-m} X_{m,j}$, where $X_{m,j} f = f * \phi^{(2^j)}_{m,j}$ and $\{\phi_{m,j} : m \geq 0\}$ is a bounded subset of C^∞ functions supported in $\{|y| \leq 2\}$. Then the bound

$$\big\|\mathrm{Dil}_{2^{-j}} T^{m_1,m_2}_{j,l_1,l_2}\big\|_{\mathrm{Op}_{\varepsilon_1}} \lesssim 2^{-m_1-m_2} \|\varsigma_j\|_{\mathcal{B}_\epsilon}$$

follows quickly. (14.3) follows in this case.

(ii) The case $m_1 \leq \min\{0, m_2\}$, *that is,* $\rho_{j,m_1,m_2} = 2^{j+m_1}$. Lemma 13.8 (combined with Theorem 2.9) shows that we have

$$\big\|\mathrm{Dil}_{2^{-j-m_1}} T^{m_1,m_2}_{j,l_1,l_2}\big\|_{\mathrm{Op}_{\varepsilon_2}} \lesssim 2^{-\epsilon_2 m_1} n^2 \|\varsigma_j\|_{\mathcal{B}_\epsilon}.$$

Using that $X^{1,n+1}_{j+m_1} Q_{j+m_2} = 2^{-(m_2-m_1)} X_{j,m_1,m_2} f$, where $X_{j,m_1,m_2} f = f * \phi^{(2^{j+m_1})}_{j,m_1,m_2}$ and $\{\phi_{j,m_1,m_2} : m_2 \geq m_1\} \subset C^\infty_0(B^d(2))$ is a bounded set, the bound

$$\big\|\mathrm{Dil}_{2^{-j-m_1}} T^{m_1,m_2}_{j,l_1,l_2}\big\|_{\mathrm{Op}_{\varepsilon_3}} \lesssim 2^{-(m_2-m_1)} \|\varsigma_j\|_{\mathcal{B}_\epsilon}$$

follows easily. Combining these two estimates, (14.3) follows.

(iii) The case $m_2 \leq \min\{0, m_1\}$, *that is* $\rho_{j,m_1,m_2} = 2^{j+m_2}$. Now we use an integration by parts argument as in the proof of Proposition 13.3 to obtain

$$\big\|\mathrm{Dil}_{2^{-j-m_2}} T^{m_1,m_2}_{j,l_1,l_2}\big\|_{\mathrm{Op}_{\varepsilon_4}} \lesssim 2^{-(m_1-m_2)} \|\varsigma_j\|_{\mathcal{B}_\epsilon}.$$

Using Lemma 13.8 (combined with Theorem 2.9), as above, we have

$$\big\|\mathrm{Dil}_{2^{-j-m_2}} T^{m_1,m_2}_{j,l_1,l_2}\big\|_{\mathrm{Op}_{\varepsilon_5}} \lesssim 2^{-\epsilon' m_1} n^2 \|\varsigma_j\|_{\mathcal{B}_\epsilon}.$$

Combining these two estimates yields (14.3) in this last case and the proof is complete. □

Proposition 14.3. *For each* m_1, m_2, $\sum_{j \in \mathbb{Z}} T^{m_1,m_2}_{j,n+1,n+2}$ *converges in the strong operator topology as operators* $L^2 \to L^2$ *(with equiconvergence with respect to the* $\{(b_1, \dots, b_n) : \|\|b_i\|_\infty \leq C\}$*) and the estimates*

$$\Big\|\sum_{j \in \mathbb{Z}} T^{m_1,m_2}_{j,n+1,n+2}\Big\|_{L^2 \to L^2} \lesssim \min\big\{2^{-\varepsilon'(|m_1|+|m_2|)} n^M \sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}, \sup_j \|\varsigma_j\|_{L^1}\big\} \prod_{i=1}^n \|b_i\|_\infty, \tag{14.4}$$

for suitable $M \lesssim 1$, *and*

$$\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \Big\|\sum_{j \in \mathbb{Z}} T^{m_1,m_2}_{j,n+1,n+2}\Big\|_{L^2 \to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^2(1 + n\Gamma_\varepsilon) \prod_{i=1}^n \|b_i\|_\infty. \tag{14.5}$$

hold.

Proof. With Lemma 14.2 in hand, (14.4) is based on almost orthogonality (Lemma 9.1) and follows just as in the proof of Proposition 13.4. (14.5) follows after summing in m_1, m_2. □

We combine the above results with several applications of Theorem 8.22 to prove our last proposition.

Proposition 14.4. *For $1 \le l_1, l_2 \le n+2$,*

$$\Big\| \sum_{j,m_1,m_2} T^{m_1,m_2}_{j,l_1,l_2} \Big\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \prod_{i=1}^n \|b_i\|_\infty.$$

The sum converges in the strong operator topology, with equiconvergence with respect to $\{(b_1,\dots,b_n) : \|b_i\|_\infty \le C\}$.

Proof. For $r \in \mathbb{Z}$ define

$$S_{r,l_1,l_2} := \sum_{\substack{j,m_1,m_2:\\ \min\{j,j+m_1,j+m_2\}=r}} T^{m_1,m_2}_{j,l_1,l_2}.$$

Note that $\sum_{r\in\mathbb{Z}} S_{r,l_1,l_2} = \sum_{j,m_1,m_2\in\mathbb{Z}} T^{m_1,m_2}_{j,l_1,l_2}$, and Lemma 14.2 shows

$$\big\|\mathrm{Dil}_{2^{-r}} S_{r,l_1,l_2}\big\|_{\mathrm{Op}'_\varepsilon} \lesssim n^M \sup_j \|\varsigma_j\|_{\mathcal{B}_\epsilon} \prod_{l\ne l_1,l_2} \|b_l\|_\infty, \tag{14.6}$$

and

$$\big\|\mathrm{Dil}_{2^{-r}} S_{r,l_1,l_2}\big\|_{\mathrm{Op}_0} \lesssim \log^2(1+n\Gamma_\varepsilon) \sup_j \|\varsigma_j\|_{L^1} \prod_{l\ne l_1,l_2} \|b_l\|_\infty. \tag{14.7}$$

By Proposition 14.3,

$$\Big\| \sum_{r\in\mathbb{Z}} S_{r,n+1,n+2} \Big\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^2(1+n\Gamma_\varepsilon) \prod_{i=1}^n \|b_i\|_\infty$$

and using (14.6), (14.7), Theorem 8.22 shows

$$\Big\| \sum_{r\in\mathbb{Z}} S_{r,n+1,n+2} \Big\|_{H^1\to L^1} \lesssim (\sup_j \|\varsigma_j\|_{L^1}) \log^3(1+n\Gamma_\varepsilon) \prod_{i=1}^n \|b_i\|_\infty.$$

Here we have convergence in the strong operator topology (as operators $H^1 \to L^1$), with equicontinuity with respect to $b_1,\dots,b_n$ in bounded subsets of $L^\infty(\mathbb{R}^d)$. Using the definition of S_{r,l_1,l_2}, this is equivalent to

$$\Big\| \sum_{r\in\mathbb{Z}} S_{r,l_2,n+2} \Big\|_{H^1\to L^1} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \prod_{l\ne l_2,n+2} \|b_l\|_\infty,$$

with convergence in the strong operator topology (as operators $H^1 \to L^1$) with equicontinuity with respect to b_l, $l \notin \{l_2, n+2\}$, in bounded subsets of $L^\infty(\mathbb{R}^d)$. This argument will now be used repeatedly. Using this $L^1 \to L^1$ result together with (14.6) and (14.7), Theorem 8.22 shows

$$\Big\| \sum_{r\in\mathbb{Z}} S_{r,l_2,n+2} \Big\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \prod_{l\ne l_2,n+2} \|b_l\|_\infty.$$

Taking transposes, this shows

$$\Big\| \sum_{r\in\mathbb{Z}} S_{r,n+2,l_2} \Big\|_{L^2\to L^2} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \prod_{l\ne l_2,n+2} \|b_l\|_\infty.$$

Using this, (14.6) and (14.7), Theorem 8.22 shows

$$\Big\| \sum_{r\in\mathbb{Z}} S_{r,n+2,l_2} \Big\|_{H^1\to L^1} \lesssim \sup_j \|\varsigma_j\|_{L^1} \log^3(1+n\Gamma_\varepsilon) \prod_{l\ne l_2,n+2} \|b_l\|_\infty.$$

Using the definition of S_{r,l_1,l_2}, this is equivalent to

$$\Big\|\sum_{r\in\mathbb{Z}} S_{r,l_1,l_2}\Big\|_{H^1\to L^1} \lesssim \sup_j \|\varsigma_j\|_{L^1}\log^3(1+n\Gamma_\varepsilon)\prod_{l\neq l_1,l_2}\|b_l\|_\infty.$$

Finally, using this again with (14.6) and (14.7), one last application of Theorem 8.22 completes the proof of the proposition. □

15. Interpolation

We use complex interpolation to show that the $L^{p_1}\times\cdots\times L^{p_{n+2}}$ estimates in Theorem 2.8 follow from the special case in Theorem 2.10, together with Theorem 2.9.

Let $K=\sum_j \varsigma_j^{(2^j)}$ be as in the assumption of Theorem 2.8 with $\sup\|\varsigma_j\|_{\mathcal{B}_\varepsilon}<\infty$. Define for a permutation ϖ of $\{1,\dots,n+2\}$

$$\Lambda^\varpi[K](b_1,\dots,b_{n+2})=\Lambda[K](b_{\varpi(1)},\dots,b_{\varpi(n+2)})$$

so that $\Lambda^\varpi[K]=\Lambda[K^\varpi]$ with

$$K^\varpi=\sum_j(\ell_\varpi\varsigma_j)^{(2^j)}$$

where ℓ_ϖ is as in Theorem 2.9. There is $\varepsilon'>c(\varepsilon)$, $B\geq 1$, both independent of n, such that for all permutations $\|\ell_\varpi\sigma\|_{\mathcal{B}_{\varepsilon'}}\leq Bn^2\|\varsigma\|_{\mathcal{B}_\varepsilon}$ and $\|\ell_\varpi\sigma\|_{L^1}=\|\varsigma\|_{L^1}$. As a consequence we get for any pair $l_1,l_2\in\{1,\dots,n+2\}$, $l_1\neq l_2$ the estimate

$$\begin{aligned}
&|\Lambda[K](b_1,\dots,b_{n+2})|\\
&\leq C_{\varepsilon',d,\delta}n^2\sup_{j\in\mathbb{Z}}\|\varsigma_j\|_{L^1}\ \log^3\big(2+n\frac{Bn^2\sup_{j\in\mathbb{Z}}\|\varsigma_j\|_{\mathcal{B}_\epsilon}}{\sup_{j\in\mathbb{Z}}\|\varsigma_j\|_{L^1}}\big)\Big(\prod_{l\notin\{l_1,l_2\}}\|b_l\|_\infty\Big)\|b_{l_1}\|_p\|b_{l_2}\|_{p'}\\
&\leq A\Big(\prod_{l\notin\{l_1,l_2\}}\|b_l\|_\infty\Big)\|b_{l_1}\|_p\|b_{l_2}\|_{p'} \qquad (15.1)
\end{aligned}$$

where $1+\delta\leq p\leq 2$ and

$$A:=3^3BC_{\epsilon',d,\delta}n^2\sup_{j\in\mathbb{Z}}\|\varsigma_j\|_{L^1}\ \log^3\big(2+n\frac{\sup_{j\in\mathbb{Z}}\|\varsigma_j\|_{\mathcal{B}_\epsilon}}{\sup_{j\in\mathbb{Z}}\|\varsigma_j\|_{L^1}}\big).$$

Let $\mathcal{R}$ be the set of points $(p_1^{-1},\dots,p_{n+2}^{-1})\in[0,1]^{n+2}$ for which the inequality

$$|\Lambda[K](b_1,\dots,b_{n+2})|\leq A\prod_{i=1}^{n+2}\|b_i\|_{p_i} \qquad (15.2)$$

holds for all $(b_1,\dots,b_{n+2})\in L^{p_1}(\mathbb{R}^d)\times\cdots\times L^{p_{n+2}}(\mathbb{R}^d)$.

We note that if $P_0=(p_{1,0}^{-1},\dots,p_{n+2,0}^{-1})$ and $P_1=(p_{1,1}^{-1},\dots,p_{n+2,1}^{-1})$ both belong to $\mathcal{R}$ then, by complex interpolation for multilinear functionals, we also have for $0\leq\vartheta\leq 1$

$$|\Lambda[K](b_1,\dots,b_{n+2})|\leq A\prod_{i=1}^{n+2}\|b_i\|_{[L^{p_{i,0}},L^{p_{i,1}}]_\vartheta}$$

where $[\cdot,\cdot]_\theta$ denotes Calderón's complex interpolation method, see Theorem 4.4.1 in [1]. By Theorem 5.1.1 in [1] (a version of the Riesz-Thorin theorem) we have the identification of the complex interpolation norm with the standard L^p norm:

$$\|f\|_{[L^{p_{i,0}},L^{p_{i,1}}]_\vartheta}=\|f\|_{L^p},\quad p^{-1}=(1-\vartheta)p_{i,0}^{-1}+\vartheta p_{i,1}^{-1}.$$

We conclude that the set $\mathcal{R}$ is convex. Denote by e_i, $i = 1, \dots, n+2$, the standard basis in $\mathbb{R}^{n+2}$. By (15.1), $\mathcal{R}$ contains all points in $\mathbb{R}^{n+2}$ of the form

$$P_{i,j}(\delta) = \frac{\delta}{1+\delta} e_i + \frac{1}{1+\delta} e_j, \quad i \neq j.$$

Let

$$\mathfrak{P}_\delta = \Big\{ x \in \mathbb{R}^{n+2} : \sum_{i=1}^{n+2} x_i = 1, \quad 0 \le x_j \le (\delta+1)^{-1}, \, j = 1, \dots, n+2 \Big\}.$$

$\mathfrak{P}_\delta$ is a compact convex subset of $\mathbb{R}^{n+2}$, of dimension $n+1$. It is easy to see that $\{P_{i,j}(\delta) : i \neq j\}$ is the set of the extreme points of $\mathfrak{P}_\delta$. By Minkowski's theorem (see e.g. Theorem 2.1.9 in [24]) every point in $\mathfrak{P}_\delta$ is a convex combination of (at most $n+2$ of) the extreme points $P_{i,j}(\delta)$. Thus we can conclude

$$\mathfrak{P}_\delta \subset \mathcal{R},$$

and we have verified (15.2) for all $(n+2)$-tuples of exponents p_i, with $\sum_{i=1}^{n+2} p_i^{-1} = 1$ and $1 + \delta \le p_i \le \infty$. This completes the proof. □

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