LOW REGULARITY CLASSES AND ENTROPY NUMBERS

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In memory of Eduard Belinsky (1947 – 2004)

ABSTRACT. We note a sharp embedding of the Besov space $B^{\infty}_{0,q}(\mathbb{T})$ into exponential classes and prove entropy estimates for the compact embedding of subclasses with logarithmic smoothness, considered by Kashin and Temlyakov.

1. Introduction

We consider spaces of functions with low regularity and their embedding properties with respect to the exponential classes $\exp(L^\nu)$. For simplicity we work with functions on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (identified with 1-periodic functions on $\mathbb{R}$). We use the following characterization of the Luxemburg norm in $\exp L^\nu(\mathbb{T})$, found for example in [15]. For $\nu > 0$ set

(1) $$\|f\|_{\exp L^\nu(\mathbb{T})} = \sup_{1\le p<\infty} p^{-1/\nu} \|f\|_{L^p(\mathbb{T})};$$

this norm will be used in what follows.

We consider the Besov spaces $B^\infty_{0,q}$, defined via dyadic decompositions as follows. Let $\Phi \equiv \phi_0$ be an even C^∞ function on $\mathbb{R}$ with the property that $\Phi(s) = 1$ for $|s| \le 1$ and Φ is supported in $(-2, 2)$. For $k \ge 1$ set $\phi_k(s) = \Phi(2^{-k}s) - \Phi(2^{-k+1}s)$ and, for $k = 0, 1, 2, \dots$

$$L_k f(x) \equiv \phi_k(D) f(x) = \sum_n \phi_k(n) \widehat{f_n} e^{2\pi i n x}.$$

Then $B^\infty_{0,q}$ is defined as the space of distributions for which

$$\|f\|_{B^\infty_{0,q}} = \Big(\sum_{k=0}^\infty \big\|L_k f\big\|_\infty^q \Big)^{1/q}$$

is finite. It is well known that the class of functions defined in this way does not depend on the specific choice of Φ.

The space $B^\infty_{0,q}$ consists of locally integrable functions if and only if $q \le 2$ (see [6], p. 112) and it follows easily from the definition that it embeds into L^∞ if $q \le 1$. We shall show for the interesting range $1 < q \le 2$ a sharp embedding result involving the exponential classes.

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Theorem 1.1. *Let $1 < q \le 2$. Then the space $B^{\infty}_{0,q}$ is continuously embedded in $\exp L^{q'}$, $q' = q/(q-1)$.*

This can be read as a statement about the *growth envelope* of the space $B^{\infty}_{0,q}$, defined by

$$\mathcal{E}_q(t) = \sup\{f^*(t) : \|f\|_{B^{\infty}_{0,q}} \le 1\}; \tag{2}$$

here f^* is the nonincreasing rearrangement of f. It is shown in Corollary 2.3 of [3] that $\|f\|_{\exp L^{q'}} \approx \sup_{t>0} f^*(t) \log^{-1/q'}(e/t)$ so that Theorem 1.1 immediately implies an upper bound $C|\log t|^{1/q'}$ for $\mathcal{E}_q(t)$ when t is small. The corresponding lower bound is proved in [6], Prop. 8.24 (there also the nonoptimal upper bound $C|\log t|$ is derived). Thus we get

Corollary 1.2. *For $1 \le q \le 2$,*

$$\mathcal{E}_q(t) \approx |\log t|^{1/q'}, \quad 0 < t \le 1/2.$$

We shall now consider subclasses $LG^{\gamma}(\mathbb{T})$ of $B^{\infty}_{0,2}$ which are compactly embedded in Lebesgue and exponential classes; these were introduced by Kashin and Temlyakov [10]. For $\gamma > 1/2$ the class $LG^{\gamma}(\mathbb{T})$ is defined as the class of $L^1(\mathbb{T})$ functions for which $\|L_k f\|_\infty = O((1+k)^{-\gamma})$ and we set

$$\|f\|_{LG^{\gamma}(\mathbb{T})} = \sup_{k\ge 0} (1+k)^{\gamma} \|L_k f\|_\infty.$$

Clearly, for $\gamma > 1$ the class $LG^{\gamma}(\mathbb{T})$ is embedded in L^∞ and if $1/2 < \gamma \le 1$ then $LG^{\gamma}(\mathbb{T})$ is embedded in $\exp L^{\nu}(\mathbb{T})$ for $\nu < (1-\gamma)^{-1}$, by Theorem 1.1. We are interested in the compactness properties of this embedding and some related quantitative statements.

We recall that given a Banach space X and a subspace $Y \subset X$ one defines the *nth entropy number* $e_n(Y; X)$ as the infimum over all numbers $\varepsilon > 0$ for which there are 2^{n-1} balls of radius ε in X which cover the unit ball $\{y \in Y : \|y\|_Y \le 1\}$ embedded in X. It is easy to see that the embedding of Y in X is a compact operator if and only if $\lim_{n\to\infty} e_n(Y; X) = 0$.

For $\gamma > 1$ the embedding of $LG^{\gamma}(\mathbb{T})$ into L^∞ is compact and Kashin and Temlyakov [10] determined sharp bounds for the entropy numbers for the embedding into L^∞ and L^p, $p < \infty$; they showed that for $n \ge 2$ and $\gamma > 1$

$$e_n(LG^{\gamma}, L^p) \approx \begin{cases} (\log n)^{1/2-\gamma}, & 1 \le p < \infty, \\ (\log n)^{1-\gamma}, & p = \infty. \end{cases} \tag{3}$$

We note that the restriction $\gamma > 1$ in [10] is only used to ensure the imbedding into L^∞; indeed it is implicitly in [10] that for $p < \infty$ the L^p result (3) holds for all $\gamma > 1/2$. The hard part in the Kashin-Temlyakov result are the lower bounds. The L^p lower bound is derived using Littlewood-Paley theory from lower bounds for classes of trigonometric polynomials in [9]. The L^∞ bounds require fine estimates for certain Riesz products (*cf.* Theorem 2.3 in [10]).

It is desirable to explain the jump in the exponent that occurs in (3) when $p \to \infty$. To achieve this Belinsky and Trebels ([1], Theorem 5.3) studied the entropy numbers $e_n(LG^\gamma, \exp L^\nu)$ for the natural embedding into the exponential classes; they obtained the equivalence $e_n \approx (\log n)^{1/2-\gamma}$ for $\nu \le 1$. For $\nu \ge 2$ they obtained an almost sharp result, namely that e_n is essentially $(\log n)^{1-\gamma-1/\nu}$, albeit with a loss of $(\log\log n)^{1/\nu}$ for the upper bound. A more substantial gap between lower and upper bounds remained for $1 \le \nu < 2$. In [1] it was also noticed that this gap could be closed if Pichorides conjecture [13] on the constant in the reverse Littlewood-Paley inequality were proved; this however is still an open problem. Nevertheless we shall use this insight to close the gap in [1].

Theorem 1.3. *The embedding $LG^\gamma(\mathbb{T}) \to \exp L^\nu(\mathbb{T})$ is compact if either $\gamma > 1/2$, $\nu < 2$, or $\nu \ge 2$, $\gamma > 1 - \nu^{-1}$, and there are the following upper and lower bounds for the entropy numbers.*

(i) For $\gamma > 1/2$, and $\nu < 2$,

$$e_n(LG^\gamma, \exp L^\nu) \approx (\log n)^{1/2-\gamma}. \tag{4}$$

(ii) For $\nu \ge 2$ and $\gamma > 1 - \nu^{-1}$,

$$e_n(LG^\gamma, \exp L^\nu) \approx (\log n)^{1-\gamma-1/\nu}. \tag{5}$$

The lower bounds are known; for $\nu \le 2$ they follow immediately from (3). It was pointed out in [1] that for $\nu > 2$ the lower bounds follow from the L^∞ lower bound in (3) and $L^\infty \to \exp(L^\nu)$ Nikolskii inequalities for trigonometric polynomials.

We thus are left to establish the upper bounds for the entropy numbers. The idea here is to embed the classes LG^γ into slightly larger classes LG^γ_{dyad} which contain discontinuous functions but satisfy the same entropy estimates with respect to the exponential classes. Instead of the Pichorides conjecture we shall then use the well known bounds for a martingale analogue, due to Chang, Wilson and Wolff [2]. This philosophy also applies to the proof of Theorem 1.1; it has been used in other papers, among them [7], [8], [5] (see also references contained in these papers).

Notation. If X, Y are normed linear spaces we use the notation $Y \hookrightarrow X$ to indicate that $Y \subset X$ and the embedding is continuous.

This paper. The proof of Theorem 1.1 is given in §2, and the proof of Theorem 1.3 in §3.

2. Embedding into the exponential classes

We shall work with dyadic versions of the Besov spaces where the Littlewood-Paley operators L_k are replaced by martingale difference operators. Let k be a nonnegative integer. For a function on $[0, 1]$ we define the conditional expectation operator

$$\mathbb{E}_k f(x) = 2^k \int_{(m-1)2^{-k}}^{m2^{-k}} f(t)dt, \quad (m-1)2^{-k} \le x < m2^{-k}, \quad m = 1, \dots, 2^k,$$

and define

$$\mathbb{D}_k f(x) = \mathbb{E}_k f(x) - \mathbb{E}_{k-1} f(x), \quad k \ge 1,$$
$$\mathbb{D}_0 f(x) = \mathbb{E}_0 f(x);$$

clearly both $\mathbb{E}_k f$ and $\mathbb{D}_k f$ define 1-periodic functions and can be viewed as functions on $\mathbb{T}$. Note that the functions $\mathbb{D}_k f$ are piecewise constant and (typically) discontinuous at $m2^{-k}$, $m = 0, \dots, 2^k - 1$. We also observe that $f = \sum_{k \ge 0} \mathbb{D}_k f$ almost everywhere for $f \in L^1$.

Definition 2.1. Let $1 \le q \le 2$. The dyadic Besov-type spaces $\ell^q(B^\infty_{\text{dyad}})$ consists of all $f \in L^1(\mathbb{T})$ for which the sequence $\{\|\mathbb{D}_k f\|_\infty\}_{k=0}^\infty$ belongs to ℓ^q; the norm is given by

$$\|f\|_{\ell^q(B^\infty_{\text{dyad}})} = \Big(\sum_{k=0}^\infty \|\mathbb{D}_k f\|_\infty^q \Big)^{1/q}.$$

Proposition 2.2. *Let $1 \le q \le 2$. Then*

$$B^\infty_{0,q} \hookrightarrow \ell^q(B^\infty_{dyad}) \,.$$

This is easily reduced to the following estimate on compositions of the difference operators with the convolutions $\phi(D/\lambda)$ for large λ.

Lemma 2.3. *Let $\lambda \ge 1$ and $k \ge 0$. Let $\psi \in C^\infty$ be even, with support in $(-2,-1/2) \cup (1/2,2)$ and let $\mathcal{L}_\lambda = \psi(\lambda^{-1}D)$. Then*

$$\big\|\mathbb{E}_k \mathcal{L}_\lambda\big\|_{L^\infty \to L^\infty} \le C \min\{\lambda^{-1}2^k, 1\}, \quad k \ge 0, \tag{6}$$

$$\big\|\mathbb{D}_k \mathcal{L}_\lambda\big\|_{L^\infty \to L^\infty} \le C \min\{\lambda^{-1}2^k, \lambda 2^{-k}\}, \quad k \ge 1. \tag{7}$$

Proof. Use the notation $\psi_{-1}(s) = (2\pi i s)^{-1}\psi(s), \psi_1(s) = s\,\psi(s)$ and observe that ψ, ψ_{-1}, ψ_1 are C^∞–functions with compact support away from the origin so that by standard $\widehat{L^1}$-theory the sequences $\ell \mapsto \psi(\lambda^{-1}\ell)$, $\psi_{-1}(\lambda^{-1}\ell)$, $\psi_1(\lambda^{-1}\ell)$ define the Fourier coefficients of $L^1(\mathbb{T})$ functions, with L^1 norms uniformly in λ. Therefore,

$$\|\psi(\lambda^{-1}D)f\|_\infty + \|\psi_{-1}(\lambda^{-1}D)f\|_\infty + \|\psi_1(\lambda^{-1}D)f\|_\infty \le C\|f\|_\infty \,. \tag{8}$$

In particular it is clear that $\|\mathbb{E}_k \mathcal{L}_\lambda\|_{L^\infty \to L^\infty} = O(1)$.

Now fix k so that $2^k < \lambda$ and let $x_{m,k} = m2^{-k}$. Then for $x \in [x_{m,k}, x_{m+1,k})$,

$$\begin{aligned}
\mathbb{E}_k \mathcal{L}_\lambda f(x) &= 2^k \int_{x_{m,k}}^{x_{m+1,k}} \bigg(\sum_{\ell \in \mathbb{Z}} \psi(\lambda^{-1}\ell) \int_0^1 e^{-2\pi i \ell y} f(y)\,dy\, e^{2\pi i \ell x} \bigg) dx \\
&= 2^k \sum_{\ell \in \mathbb{Z}} \psi(\lambda^{-1}\ell) \int_0^1 \frac{e^{2\pi i \ell (x_{m+1,k}-y)} - e^{2\pi i \ell (x_{m,k}-y)}}{2\pi i\, \ell} f(y)\,dy \\
&= 2^k \lambda^{-1} \Big(\psi_{-1}(D/\lambda) f(x_{m+1,k}) - \psi_{-1}(D/\lambda) f(x_{m,k}) \Big)
\end{aligned}$$

and (6) follows by (8).

Inequality (7) for $2^k < \lambda$ is an immediate consequence and it remains to consider the case $2^k \geq \lambda$. Fix x, then $\mathbb{E}_k \mathcal{L}_\lambda f(x)$ is the average of $\mathcal{L}_\lambda f$ over an interval of length 2^{-k} containing x. Thus, by the mean value theorem applied to $\mathbb{E}_k \mathcal{L}_\lambda f(x)$ and $\mathbb{E}_{k-1} \mathcal{L}_\lambda f(x)$, we can write for $k \geq 1$

$$\mathbb{D}_k \mathcal{L}_\lambda f(x) = \mathcal{L}_\lambda f(x') - \mathcal{L}_\lambda f(x'') = (\mathcal{L}_\lambda f)'(\tilde{x})(x' - x'')$$

where $x', x'', \tilde{x}$ have distance at most 2^{-k+1} from x. Now $(\mathcal{L}_\lambda f)' = \lambda \psi_1(D/\lambda) f$ and thus

$$\|\mathbb{D}_k \mathcal{L}_\lambda f\|_\infty \leq 2^{1-k} \|(\mathcal{L}_\lambda f)'\|_\infty \leq C\lambda 2^{-k} \|f\|_\infty .$$

□

Proof of Proposition 2.2. Let Ψ_0 be a C^∞ function supported in $(-4, 4)$ which satisfies $\Psi_0(s) = 1$ in $(-2, 2)$ and let $\Psi_n = \Psi(2^{-n}\cdot)$ where Ψ is supported in $(-8, -1/8) \cup (1/8, 8)$ so that $\Psi(s) = 1$ for $|s| \in (1/2, 4)$. Then $\Psi_n \phi_n = \phi_n$ for all n, so that $\Psi_n(D) L_n = L_n$, and we can write

$$\begin{aligned} \big\|\mathbb{D}_k f\big\|_\infty &= \Big\|\mathbb{D}_k \sum_{n=0}^\infty \Psi_n(D) L_n f\Big\|_\infty \leq \sum_{n=0}^\infty \big\|\mathbb{D}_k \Psi_n(D)\big\|_{L^\infty \to L^\infty} \big\|L_n f\big\|_\infty \\ &\leq C \sum_{n=0}^\infty 2^{-|k-n|} \|L_n f\|_\infty \end{aligned}$$

and therefore

$$\|f\|_{\ell^q(B^\infty_{\mathrm{dyad}})} \leq C \sum_{m=0}^\infty 2^{-m} \Big\| \big\{ \|L_{k+m} f\|_\infty \big\}_{k=-m}^\infty \Big\|_{\ell^q} \leq C' \|f\|_{B^\infty_{0,q}} .$$

□

We now introduce the square-function and the maximal function

$$\mathfrak{S}(f) := \Big(\sum_{k \geq 0} |\mathbb{D}_k f(x)|^2 \Big)^{1/2}, \qquad \mathfrak{M}_0(f) := \sup_{k \geq 0} |\mathbb{E}_k f(x) - \mathbb{E}_0 f(x)| ,$$

resp., and recall the following deep "good λ inequality" due to Chang, Wilson and Wolff (Corollary 3.1 in [2]): There are absolute constants c and C so that for all $\lambda > 0$, $0 < \varepsilon < 1$,

$$\begin{aligned} \text{(9)} \quad & \mathrm{meas}\big(\{x : \mathfrak{M}_0(f)(x) > 2\lambda,\, \mathfrak{S}(f) < \varepsilon\lambda\}\big) \\ & \qquad \leq C \exp(-\frac{c}{\varepsilon^2}) \mathrm{meas}\big(\{x : \sup_{k \geq 0} |\mathbb{E}_k f(x)| > \lambda\}\big). \end{aligned}$$

It is standard that this implies the inequality

$$\|f\|_p \leq C\sqrt{p}\, \|\mathfrak{S}(f)\|_p \tag{10}$$

for all $p \geq 2$, and some absolute constant $C \geq 1$. Indeed, if we integrate out the L^p norms using the distribution function, where we observe that

$$\{x : \mathfrak{M}_0(f)(x) > 2\lambda\} \subset \{x : \mathfrak{M}_0(f) > 2\lambda,\, \mathfrak{S}(f) < \varepsilon\lambda\} \cup \{x : \mathfrak{S}(f) \geq \varepsilon\lambda\} ,$$

we obtain

$$\big\| \sup_k |\mathbb{E}_k f| \big\|_p \le \|\mathbb{E}_0 f\|_p + 2C^{1/p} e^{-c\varepsilon^{-2}p^{-1}} \| \sup_k |\mathbb{E}_k f| \big\|_p + 2\varepsilon^{-1} \big\| \mathfrak{S}(f) \big\|_p.$$

Now we choose $\varepsilon = ap^{-1/2}$ with a so small that $2Ce^{-ca^{-2}} = 1/2$. Since $\mathbb{D}_0 = \mathbb{E}_0$ is incorporated in the definition of the square-function, $|f(x)| \le \sup_k |\mathbb{E}_k g|(x)$ a.e., the asserted bound (10) follows.

The following interpolation result is a quick consequence of (10).

Lemma 2.4. *There is a constant C so that for $1 \le s \le 2$, $s' = s/(s-1)$, $2 \le p < \infty$, and all sequences $\{f_k\}$ of $L^p(\mathbb{T})$ functions,*

$$\Big\| \sum_{k=0}^\infty \mathbb{D}_k f_k \Big\|_{L^p(\mathbb{T})} \le C p^{1/s'} \Big(\sum_{k=0}^\infty \|f_k\|_{L^p(\mathbb{T})}^s \Big)^{1/s}.$$

Proof. The statement is trivial for $s = 1$, because of the uniform L^p bounds for the operators $\mathbb{D}_k$. We thus only need to prove the statement for $s = 2$ since then the general case follows by complex interpolation. By a straightforward limiting argument we may assume that $f_k = 0$ for all but finitely many k.

We use that $\mathbb{D}_k \mathbb{D}_l = 0$ if $k \ne l$, and define $g = \sum \mathbb{D}_k f_k$. Then by (10)

$$\|g\|_p = \Big\| \sum_l \mathbb{D}_l g \Big\|_p \le C\sqrt{p} \Big\| \Big(\sum_l |\mathbb{D}_l g|^2 \Big)^{1/2} \Big\|_p,$$

and since $p \ge 2$ we can use Minkowski's inequality to bound this by

$$C\sqrt{p} \Big(\sum_l \|\mathbb{D}_l g\|_p^2 \Big)^{1/2} = C\sqrt{p} \Big(\sum_l \|\mathbb{D}_l f_l\|_p^2 \Big)^{1/2} \le C'\sqrt{p} \Big(\sum_l \|f_l\|_p^2 \Big)^{1/2}.$$

□

Theorem 1.1 is an immediate consequence of Proposition 2.2 and the following imbedding result which is based on (10) (or rather the case $s = 2$ of Lemma 2.4).

Proposition 2.5. *Let $1 \le q \le 2$. Then*

$$\ell^q(B^\infty_{dyad}) \hookrightarrow \exp L^{q'}.$$

Proof. We modify an argument from [1] (which was based there on the Pichorides conjecture). Fix $f \in \ell^q(B^\infty_{\text{dyad}})$ and let $n \to k(n,f)$ be a bijection of $\mathbb{N} \cup \{0\}$ so that the sequence $n \to \|\mathbb{D}_{k(n,f)} f\|_\infty$ is nonincreasing (in other words, we form the nonincreasing rearrangement of the sequence $\{\|\mathbb{D}_k f\|\}$).

For $p \ge 2$ we need to estimate $p^{-1/q'} \|f\|_p$. Thus fix $p > 2$ and let $N \in \mathbb{N}$ so that $p \le N < p+1$. We then split

$$f = \sum_{n=0}^N \mathbb{D}_{k(n,f)} f + \sum_{n=N+1}^\infty \mathbb{D}_{k(n,f)} f := I_N f + II_N f.$$

By Hölder's inequality

$$\|I_N f\|_p \le \sum_{n=0}^{N} \|\mathbb{D}_{k(n,f)} f\|_p \le \sum_{n=0}^{N} \|\mathbb{D}_{k(n,f)} f\|_\infty$$

$$\le (N+1)^{1/q'} \Big(\sum_{n=0}^{N} \|\mathbb{D}_{k(n,f)} f\|_\infty^q \Big)^{1/q} \le C p^{1/q'} \|f\|_{\ell^q(B^\infty_{\text{dyad}})}. \tag{11}$$

For the second term we get a bound in terms of the Lorentz-Besov type space $\ell^{q,2}(B^\infty_{\text{dyad}})$ defined similarly as $\ell^q(B^\infty_{\text{dyad}})$, but with the sequence space ℓ^q replaced by the Lorentz variant $\ell^{q,2}$. Since $\ell^q \hookrightarrow \ell^{q,2}$ for $q \le 2$; this is a better estimate. Note that

$$\Big\| \{\mathbb{D}_k f\}_{k=0}^\infty \Big\|_{\ell^{q,2}} \approx \Big(\sum_{n=0}^{\infty} \big[n^{1/q} \|\mathbb{D}_{k(n,f)} f\|_\infty \big]^2 n^{-1} \Big)^{1/2}. \tag{12}$$

We now use the case $s = 2$ of Lemma 2.4 to obtain

$$\|II_N f\|_p \le C p^{1/2} \Big(\sum_{n=N+1}^{\infty} \big\|\mathbb{D}_{k(n,f)} f\big\|_p^2 \Big)^{1/2} \le C p^{1/2} \Big(\sum_{n=N+1}^{\infty} \big\|\mathbb{D}_{k(n,f)} f\big\|_\infty^2 \Big)^{1/2}$$

$$\le C p^{1/2} N^{-1/2+1/q'} \Big(\sum_{n=N+1}^{\infty} n^{1-2/q'} \big\|\mathbb{D}_{k(n,f)} f\big\|_\infty^2 \Big)^{1/2},$$

and, since $1 - 2/q' = 2/q - 1$ and $p \approx N$, we get from (12)

$$p^{-1/q'} \|II_N f\|_p \le C \|f\|_{\ell^{q,2}(B^\infty_{\text{dyad}})} \le C' \|f\|_{\ell^q(B^\infty_{\text{dyad}})}. \tag{13}$$

Estimates (11) and (13) yield

$$\|f\|_{\exp L^{q'}} \lesssim \|f\|_{\ell^q(B^\infty_{\text{dyad}})}$$

and thus the assertion. □

3. Entropy numbers for the Kashin-Temlyakov classes

We now give a proof of Theorem 1.3. As discussed in the introduction only the upper bounds have to be proved. It will be advantageous to define larger "dyadic" analogues of the LG classes.

Definition 3.1. Let $\gamma > 1/2$ and let $LG^\gamma_{\text{dyad}}(\mathbb{T})$ denote the class of $L^1(\mathbb{T})$ functions for which $\|\mathbb{D}_k f\|_\infty = O(k^{-\gamma})$ as $k \to \infty$. We set

$$\|f\|_{LG^\gamma_{\text{dyad}}} = \sup_{k \ge 0} (k+1)^\gamma \|\mathbb{D}_k f\|_\infty.$$

We note that the classes $LG^\gamma(\mathbb{T})$ consist of continuous functions provided that $\gamma > 1$. This is not the case for the dyadic analogue $LG^\gamma_{\text{dyad}}(\mathbb{T})$ as even the building blocks $\mathbb{D}_k f$ are piecewise constant and typically discontinuous at $m2^{-k}$, $m = 0, \dots, 2^k - 1$. We prove the following embedding result.

Lemma 3.2. *For* $\gamma > 1/2$

$$LG^{\gamma}(\mathbb{T}) \hookrightarrow LG^{\gamma}_{dyad}(\mathbb{T})\,.$$

Proof. This follows easily from Lemma 2.3. Indeed let $f \in LG^{\gamma}(\mathbb{T})$, so that $\|L_n f\|_\infty \lesssim \|f\|_{LG^\gamma}(1+n)^{-\gamma}$. As in §2 we can write $L_n = \Psi_n(D)L_n$ where the operator $\mathbb{D}_k\Psi_n(D)$ has $L^\infty \to L^\infty$ operator norm $O(2^{-|k-n|})$. Thus

$$\begin{aligned}\big\|\mathbb{D}_k f\big\|_\infty &= \Big\|\mathbb{D}_k \sum_{n=0}^{\infty} \Psi_n(D) L_n f\Big\|_\infty \le C\sum_{n=0}^{\infty} 2^{-|k-n|}\big\|L_n f\big\|_\infty \\ &\le C_0 \sum_{n=0}^{\infty} 2^{-|k-n|}(1+n)^{-\gamma}\|f\|_{LG^\gamma} \le C'(1+k)^{-\gamma}\|f\|_{LG^\gamma}.\end{aligned}$$

This proves the assertion. □

We now state a crucial approximation result which will be derived as a quick consequence of Lemma 2.4.

Lemma 3.3. *Let* $1/2 < \gamma < 1$ *and* $0 < \nu < (1-\gamma)^{-1}$ *or* $\gamma \ge 1$ *and* $0 < \nu < \infty$. *There is a constant* $C = C(\gamma, \nu)$ *so that for* $M = 1, 2, \dots$

$$\sup_{\|f\|_{LG^\gamma_{dyad}} \le 1} \|f - \mathbb{E}_M f\|_{\exp L^\nu} \le C \begin{cases} M^{1/2-\gamma}, & \nu \le 2,\ \gamma > 1/2, \\ M^{1-1/\nu-\gamma}, & \nu \ge 2,\ \gamma > 1-\nu^{-1}. \end{cases}$$

Proof. Consider $f \in LG^\gamma_{\text{dyad}}$, $\|f\|_{LG^\gamma_{\text{dyad}}} \le 1$, and write

$$f - \mathbb{E}_M f = \sum_{k=M+1}^{\infty} \mathbb{D}_k \mathbb{D}_k f\,.$$

By Lemma 2.4 we have for $2 \le p < \infty$, and $s\gamma > 1$

$$\begin{aligned} p^{-1/\nu}\|f - \mathbb{E}_M f\|_p &\le C p^{1/s'-1/\nu}\Big(\sum_{k=M+1}^{\infty} \big\|\mathbb{D}_k f\big\|_p^s\Big)^{1/s} \\ &\le C p^{1/s'-1/\nu}\Big(\sum_{k=M+1}^{\infty} \big\|\mathbb{D}_k f\big\|_\infty^s\Big)^{1/s} \le C p^{1/s'-1/\nu}\Big(\sum_{k=M+1}^{\infty}(1+k)^{-s\gamma}\Big)^{1/s} \\ &\le \mathcal{C}(s,\gamma)p^{1-1/\nu-1/s}M^{1/s-\gamma}. \end{aligned}$$

If $\nu \le 2$ then we may apply this bound for $s = 2$, $\gamma > 1/2$ and get the bound $\|f - \mathbb{E}_M f\|_{\exp L^\nu} = O(M^{1/2-\gamma})$. If $\nu > 2$ we may apply it with $s = \nu/(\nu-1) \in (1,2)$, indeed we have $s\gamma > 1$ in view of our assumption $\gamma > 1-\nu^{-1}$; the result is the asserted bound $\|f - \mathbb{E}_M f\|_{\exp L^\nu} = O(M^{1-1/\nu-\gamma})$. □

We apply a result of Lorentz [11], *cf.* Theorem 3.1 in [12], p. 492. Here one considers a Banach space X of functions, a sequence $\mathcal{G} = \{g_1, g_2, \dots\}$ of linearly independent functions whose linear span is dense in X. Set $X_0 = 0$, and let, for $n \ge 1$, X_n be the linear span of $g_1, \dots, g_n$. Let

$$D_n(x) = \inf\{\|x - y\| : y \in X_n\}$$

and let $\mathfrak{d} = (\delta_0, \delta_1, \dots)$ be a nonincreasing sequence of positive numbers with $\lim_{n\to\infty} \delta_n = 0$. Let

$$A(\mathfrak{d}) = \{x \in X : D_n(x) \le \delta_n,\ n = 0, 1, 2, \dots\}$$

be the approximation set associated with $\mathfrak{d}$, $\mathcal{G}$.

Next let $\mathcal{N}_\varepsilon(A(\mathfrak{d}))$ denotes the minimal number of balls of radius ε needed to cover $A(\mathfrak{d})$. The following inequality for the natural logarithm of $\mathcal{N}_\varepsilon(A(\mathfrak{d}))$ is a special case of Lorentz' result.

$$\log N_\varepsilon(A(\mathfrak{d})) \le 2n \log\left(\tfrac{18\delta_0}{\varepsilon}\right), \qquad \text{if } \varepsilon \ge \delta_n\,. \tag{14}$$

We apply (14) to prove the dyadic analogue of the upper bound in Theorem 1.3.

Proposition 3.4. *The embedding $LG^\gamma_{dyad}(\mathbb{T}) \to \exp L^\nu(\mathbb{T})$ is compact if $\gamma > 1/2$, $\nu < 2$ or $\nu \ge 2$, $\gamma > 1 - \nu^{-1}$ and we have*

$$e_n(LG^\gamma_{dyad}, \exp L^\nu) \le C(\log n)^{1/2-\gamma}, \quad \gamma > 1/2, \quad \nu \le 2, \tag{15}$$

$$e_n(LG^\gamma_{dyad}, \exp L^\nu) \le C(\log n)^{1-\gamma-1/\nu}, \quad \gamma > 1 - 1/\nu, \quad \nu \ge 2. \tag{16}$$

Proof. We set $X = \exp L^\nu$, and, for $n = 2^M + j$, $j = 0, 1, \dots, 2^M - 1$, let g_n be the characteristic function of the interval $[j2^{-M}, (j+1)2^{-M})$. If X_n, $D_n(x)$ are defined as above then we note that Lemma 3.3 says that for f in the unit ball of LG^γ_{dyad} we have

$$D_n(f) \le C_0(\log(n+2))^{-a}$$

where $a = \gamma - 1/2$ if $\gamma > 1/2$ and $\nu \le 2$, and $a = \gamma + \nu^{-1} - 1$ if $\nu \ge 2$ and $\gamma > 1 - 1/\nu$. We now note that (14) implies that

$$e_{\widetilde{n}}(LG^\gamma_{\text{dyad}}, \exp L^\nu) \le (\log(n+1))^{-a}$$

if $\widetilde{n} > Cn \log\log n$. As $\log \widetilde{n} \approx \log n$ the asserted inequalities follow. □

Conclusion of the proof. By Lemma 3.2 we have

$$e_n(LG^\gamma, \exp L^\nu) \le Ce_n(LG^\gamma_{\text{dyad}}, \exp L^\nu) \tag{17}$$

and the assertion of the Theorem 1.3 follows from Proposition 3.4. □

Remark: We note that in the dyadic case, there are also similar lower bounds matching (15), (16) for the entropy numbers $e_n(LG^\gamma_{\text{dyad}}, \exp L^\nu)$. These follow from (17) and the known lower bounds for the entropy numbers for LG^γ.

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