RIESZ MEANS ASSOCIATED WITH CONVEX DOMAINS IN THE PLANE

Andreas Seeger Sarah Ziesler

1. Introduction

Let Ω be a bounded open convex set $\mathbb{R}^2$ which contains the origin. Let ρ be the associated Minkowski functional defined by

$$\rho(\xi) = \inf \{t > 0 : t^{-1}\xi \in \Omega\}.$$

We shall investigate the Riesz means of the inverse Fourier integral associated with Ω

$$\mathcal{R}_{\lambda,t} f(x) = \frac{1}{(2\pi)^2} \int_{\rho(\xi)\le t} \big(1 - \frac{\rho(\xi)}{t}\big)^{\lambda} \widehat{f}(\xi) e^{i\langle \xi, x\rangle} d\xi; \tag{1.1}$$

here our definition of the Fourier transform is $\widehat{f}(\xi) = \int f(y)e^{-i\langle y,\xi\rangle}dy$. For $t = 1$ we also set $\mathcal{R}_\lambda = \mathcal{R}_{\lambda,1}$ and refer to $\mathcal{R}_\lambda$ as the Bochner-Riesz operator associated with Ω. Note that for $\lambda = 0$ the Riesz means $\mathcal{R}_{0,t}$ are just the partial sum operators associated with the sets $t\Omega$, $t > 0$, while for $\lambda = 1$ one recovers the Féjer means, namely the averages $\mathcal{R}_{1,t} = t^{-1}\int_0^t \mathcal{R}_{0,s}ds$. The objective is to prove that $\mathcal{R}_{\lambda,t}f$ converges to f in $L^p(\mathbb{R}^2)$, for suitable $1 \le p < \infty$; for $p = \infty$ one has to replace L^∞ by the space $C^0(\mathbb{R}^2)$ of continuous functions with $\lim_{|x|\to\infty} |f(x)| = 0$. The main step is to establish the L^p boundedness of $\mathcal{R}_\lambda$ (and, equivalently, of $\mathcal{R}_{\lambda,t}$).

If Ω is the unit disc in $\mathbb{R}^2$ explicit calculations show that the convolution kernel of $\mathcal{R}_\lambda$ belongs to $L^p(\mathbb{R}^2)$ if and only if $\lambda > 2/p - 3/2$. In particular $\mathcal{R}_\lambda$ is bounded on L^p for $1 \le p \le \infty$ if $\lambda > 1/2$ (a result which goes back to Bochner), moreover $\mathcal{R}_\lambda$ is unbounded if $p \notin (p_\lambda, p'_\lambda)$ where $p_\lambda = 4/(3 + 2\lambda)$ and $p'_\lambda = p_\lambda/(p_\lambda - 1)$ is the conjugate exponent. Fefferman [6] showed that the partial sum operator $\mathcal{R}_0$ is bounded on L^p if and only if $p = 2$. The best possible result for all $\lambda \in (0, 1/2)$ was proved by Carleson and Sjölin [3] who obtained L^p boundedness for $p_\lambda < p < p'_\lambda$ (see also Fefferman [6], Córdoba [4] for different proofs).

Various generalizations of these results have been considered in the literature; in particular Sjölin [19] proved the analogous L^p inequalities for Bochner-Riesz multipliers associated to an arbitrary compact C^∞ curve in the plane.

In this paper we consider the case of Riesz means associated with *convex* domains, with no extra smoothness assumption on the boundary Ω. Only the Féjer means have been considered in this generality; see [12] where L^p boundedness for $1 \le p \le \infty$ is proved. We shall in fact show that all the above mentioned sufficient results for the unit disc remain true in our more general setting; these results are necessary and sufficient for convex sets with smooth boundary. Moreover we shall show how these results can be improved for some classes of convex domains with nonsmooth boundary.

In order to formulate this improvement we need to introduce a version of upper Minkowski dimension of the boundary $\partial\Omega$ with respect to a suitable families of caps or "balls". The use of

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these families is motivated by the estimates of Fourier transforms of measures carried by convex surfaces, see Bruna, Nagel and Wainger [2].

Let $P \in \partial\Omega$ and let ℓ be a line through P. Let $H_0(\ell)$ be the closed half-plane with boundary ℓ which contains the origin. We say that ℓ is a *supporting line* for Ω at P if $\Omega \subset H_0(\ell)$ and denote by $\mathcal{T}(\Omega, P)$ the set of supporting lines for Ω at P. Note that $\mathcal{T}(\Omega, P)$ consists precisely of the tangent line through P if Ω has a C^1 boundary.

Let $P \in \partial\Omega$. For any supporting line ℓ through P and $\delta > 0$ we define

$$B(P, \ell, \delta) = \{Y \in \partial\Omega : \operatorname{dist}(Y, \ell) < \delta\}. \tag{1.2}$$

Let $\mathcal{B}_\delta = \{B(P, \ell, \delta) : P \in \partial\Omega, \ell \in \mathcal{T}(\Omega, P)\}$ and let $N(\Omega, \delta)$ be the minimal number of balls $B \in \mathcal{B}_\delta$ needed to cover $\partial\Omega$. Let

$$\kappa_\Omega = \limsup_{\delta \to 0} \frac{\log N(\Omega, \delta)}{\log \delta^{-1}}; \tag{1.3}$$

this is our version of upper Minkowski dimension. It is not hard to see (*cf.* Lemma 2.3 below) that $0 \le \kappa_\Omega \le 1/2$.

In what follows Ω will always be an open *convex* set in $\mathbb{R}^2$ containing the origin. Our main result is:

Theorem 1.1. *Suppose that $1 \le p \le \infty$, $\lambda > 0$ and $\lambda > \kappa_\Omega(4|1/p - 1/2| - 1)$. Then $\mathcal{R}_{\lambda,t}$ is bounded on $L^p(\mathbb{R}^2)$.*

Suppose $f \in L^p(\mathbb{R}^2)$ if $p < \infty$ and $f \in C^0(\mathbb{R}^2)$ if $p = \infty$. Then

$$\lim_{t \to \infty} \|\mathcal{R}_{\lambda,t} f - f\|_p = 0.$$

The L^1 result of Theorem 1.1 has the following counterpart for pointwise convergence which follows from the weak type (1,1) bound for the appropriate maximal operator.

Theorem 1.2. *Suppose that $f \in L^1(\mathbb{R}^2)$, $\lambda > \kappa_\Omega$. Then $\lim_{t\to\infty} \mathcal{R}_{\lambda,t} f(x) = f(x)$ almost everywhere.*

It is well known that Theorem 1.1 is sharp if the boundary of Ω is smooth; then $\mathcal{R}_{\lambda,t}$ is bounded on L^p if and only if $\lambda > \max\{0, 2|1/p - 1/2| - 1/2\}$; in fact the necessity of this condition follows from Theorem 3 in [11] if Ω has merely C^2 boundary. This is the minimal smoothness assumption to ensure $\kappa_\Omega = 1/2$ since for every $\alpha \in (0,1)$ there exist convex domains Ω with $C^{1,\alpha}$ boundary and $\kappa_\Omega = \alpha/(\alpha+1) < 1/2$. In §4 we construct domains with this property for which Theorem 1.1 is sharp:

Theorem 1.3. *Let $0 < \kappa < 1/2$. Then there exists a convex domain Ω_κ with $C^{1,\frac{\kappa}{1-\kappa}}$ boundary satisfying $\kappa_\Omega = \kappa$ so that for $1 \le p < 4/3$ the operators $\mathcal{R}_{\lambda,t}$ associated to Ω are bounded on $L^p(\mathbb{R}^2)$ if and only if $\lambda > \kappa(4/p - 3)$.*

Remarks. 1. If Ω has a C^∞ boundary then Theorem 1.1 is a special case of Sjölin's theorem [19] (the previously proved Carleson-Sjölin theorem [3] covered the case of domains with nonvanishing curvature; see also [9] for the case of finite type curves). If Ω has C^2 boundary then there is a point on $\partial\Omega$ where the curvature does not vanish and working near this point one easily checks that $\kappa_\Omega = 1/2$ in this case. We remark that Sjölin's proof relies on the assumption $\gamma \in C^\infty$ since it uses approximation to reduce the general case of smooth functions to the case of polynomials. It would be interesting to investigate whether the smoothness assumption is needed in the nonconvex case.

2. Podkorytov [12] showed L^1 convergence of the Féjer means (for $\lambda = 1$) associated to arbitrary convex domains; this can be improved to the condition $\lambda > 1/2$ (since $\kappa_\Omega \le 1/2$ in Theorem 1.1). A different extreme case occurs when $\kappa_\Omega = 0$, in particular if Ω is a convex polygon. This particular case was considered before; Podkorytov [13] proved L^1 convergence for Riesz means $\mathcal{R}_{\lambda,t}$ associated to arbitrary (not necessarily convex) polyhedra in $\mathbb{R}^n$, for every $\lambda > 0$. The corresponding result for pointwise convergence was worked out by one of the authors in [17] using the method in [13] and a result by E. Stein and N. Weiss [22] on adding weak type functions; *cf.* also the more recent paper [1].

3. Intermediate rates for Lebesgue constants of trigonometric series, for certain polygonal domains in the plane with infinitely many vertices, were found by Podkorytov [14], and by A. and V. Yudin [24]; their examples are related to the example discussed in §4.

4. Let $1 < p_0 < 4/3$. It would be interesting to find convex domains for which $\mathcal{R}_\lambda$ is bounded in L^p for all $\lambda > 0$ if and only if $p_0 < p < p_0'$.

5. Our method also applies to multipliers of the form $\chi(\xi)(\xi_2 - \gamma(\xi_1))_+^\lambda$ if γ is a convex function or $\chi(\xi)\text{dist}(\xi,\Gamma)^\lambda$ if Γ is a convex curve; here $\chi \in C_0^\infty$. Moreover the subordination formula (4.19) below allows the extension to more general multipliers of the form $m \circ \rho$.

Notation: The Fourier transform of f is denoted by $\widehat{f}$, the inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}[f]$. By C^0 we denote the space of continuous functions with $\lim_{|x|\to\infty} |f(x)| = 0$. By C_0^∞ we denote the space of smooth functions with compact support. By $C^{1,\alpha}$ we denote a space of differentiable functions whose derivatives are Hölder continuous with exponent α. Given two quantities A and B we write $A \lesssim B$ if there is a positive constant C, such that $A \le CB$. Such constants may depend on the number M in (2.1) below. We write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

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2. Convex sets and plane geometry

Let Ω be an open convex domain in $\mathbb{R}^2$ containing the origin. Since the statements of Theorems 1.1 and 1.2 are invariant under dilations and since ρ is homogeneous of degree 1 it is no loss of generality to assume that the closed ball of radius 4 centered at the origin is contained in Ω. Then there is an integer $M \ge 3$ so that

$$\{\xi : |\xi| \le 4\} \subset \Omega \subset \overline{\Omega} \subset \{\xi : |\xi| < 2^M\}; \tag{2.1}$$

this is henceforth assumed.

Let $u_\perp$, u be orthonormal unit vectors, so that $\det(u_\perp, u) = 1$ and define the half strip

$$\mathfrak{S}_u = \{\xi : |\langle \xi, u_\perp\rangle| \le 2, \langle \xi, u\rangle \le 0\}. \tag{2.2}$$

We now give some properties of the boundary $\partial\Omega$ relying on elementary facts on convex functions (see *e.g.* [10, §1.1]).

Lemma 2.1. *$\partial\Omega \cap \mathfrak{S}_u$ can be parametrized by*

$$t \mapsto tu_\perp + \gamma(t)u, \quad -2 \le t \le 2 \tag{2.3}$$

where (i)

$$-2^M < \gamma(t) < -2, \quad -2 \le t \le 2. \tag{2.4}$$

(ii) γ is a convex function on $[-2,2]$, so that the left and right derivatives γ_L' and γ_R' exist everywhere in $(-2,2)$ and

$$-2^{M-1} \le \gamma_L'(t) \le \gamma_R'(t) \le 2^{M-1} \tag{2.5}$$

for $t \in [-2,2]$. The functions γ_L' and γ_R' are increasing functions; γ_L' is left continuous and γ_R' is right continuous in $[-2,2]$.

(iii) Let ℓ be a supporting line through $\xi \in \partial\Omega$ and let n be an outward normal vector (e.g. normal to ℓ). Then

$$\langle \xi, n\rangle \ge 2^{-M}|\xi|. \tag{2.6}$$

Proof. By assumption (2.1) the line segment $\{su_\perp : |s| \le 4\}$ is contained in Ω. Now fix s with $|s| \le 2$ and consider the ray $\{su_\perp - Ru : R > 0\}$. For any point $P \in \Omega$ which is on this ray the line segment connecting P to $su_\perp$ also belongs to Ω. Hence there is exactly one point on this ray, which is also a boundary point. Therefore there is a function $t \mapsto \gamma(t)$ on $[-2,2]$ so that $\partial\Omega \cap \mathfrak{S}_u$ can be parametrized by (2.3) and (2.4) is satisfied. Then γ is a convex function; for the existence and continuity properties of left and right derivatives see [10, §1.1].

In order to obtain the bounds on the derivatives fix $t_0 \in [-2,2]$. One notes that the intersection of $\overline{\Omega}$ and the ray through $t_0u_\perp + \gamma(t_0)u$ starting at $4u_\perp$ is precisely the line segment connecting those two points; an analogous statement holds with $4u_\perp$ replaced by $-4u_\perp$. This implies for $t, t_0 \in [-2,2]$ that

$$\begin{aligned}
\frac{\gamma(t_0)}{4+t_0}(t-t_0) &\le \gamma(t)-\gamma(t_0) \le \frac{-\gamma(t_0)}{4-t_0}(t-t_0), \qquad t_0 \le t,\\
\frac{\gamma(t_0)}{4-t_0}(t_0-t) &\le \gamma(t)-\gamma(t_0) \le \frac{-\gamma(t_0)}{4+t_0}(t_0-t), \qquad t \le t_0,
\end{aligned}$$

and (2.5) follows from (2.4).

In order to see (2.6) we choose $u = \xi/|\xi|$ and parametrize $\partial\Omega$ near ξ by (2.3) (the function γ depends of course on u). The vector n is given by

$$n = \frac{1}{\sqrt{1+\sigma^2}}(\sigma u_\perp - u) \quad \text{where } \gamma_L'(0) \le \sigma \le \gamma_R'(0).$$

Since $\langle \xi, n\rangle = |\xi|(1+\sigma^2)^{-1/2}$ the assertion (2.6) is an immediate consequence. □

It will be useful to approximate convex domains by smooth ones.

Lemma 2.2. *Suppose that Ω satisfies (2.1). There is a sequence of convex domains Ω_n containing the origin, with Minkowski-functionals $\rho_n(\xi) = \inf\{t : \xi/t \in \Omega_n\}$, so that the following holds:*

(i) $\Omega_n \subset \Omega_{n+1} \subset \Omega$ and $\cup_n \Omega_n = \Omega$.

(ii) $\rho_n(\xi) \ge \rho_{n+1}(\xi) \ge \rho(\xi)$ and

$$\frac{\rho_n(\xi)-\rho(\xi)}{\rho(\xi)} \le 2^{-n-1};$$

in particular $\lim_{n\to\infty}\rho_n(\xi) = \rho(\xi)$, with uniform convergence on compact sets.

(iii) Ω_n has C^∞ boundary.

(iv) If $\delta \geq 2^{-n+2}$ then

$$N(\Omega_n, 2\delta) \lesssim N(\Omega, \delta) \tag{2.7}$$

where $N(\Omega,\delta)$ denotes the covering number for the boundary as defined in the introduction.

Proof. We first approximate the boundary of $(1-4^{-n})\Omega = \{\xi : \rho(\xi) \leq 1-4^{-n}\}$ by a convex polygon. Let $\theta_\nu = 2\pi\nu 4^{-n-M}$ and let $u^\nu = (\cos\theta_\nu, \sin\theta_\nu)$, $u^\nu_\perp = (-\sin\theta_\nu, \cos\theta_\nu)$. Let $P_\nu = -R_\nu u^\nu$ where R_ν is such that $\rho(P_\nu) = 1-4^{-n}$. Let $\widetilde{\Omega}_n$ be the polygon with vertices P_ν, $\nu = 0, \dots, 4^{n+M}-1$. We wish to smooth out the boundary near the vertices and therefore modify this boundary only on a small part in the narrow half strips

$$\mathfrak{S}^\nu = \{\xi : |\langle \xi, u^\nu_\perp\rangle| \leq 16^{-n-M}, \langle \xi, u^\nu\rangle \leq 0\}.$$

Define $\tilde\gamma_\nu(t) = -R_\nu + c^+_\nu t$ if $t \geq 0$ and $\tilde\gamma_\nu(t) = -R_\nu - c^-_\nu t$ if $t \leq 0$ where the slopes $c^\pm_\nu$ are chosen so that the portion of the boundary which is in $\mathfrak{S}^\nu$ is parametrized by $t \mapsto tu^\nu_\perp + \tilde\gamma_\nu(t) u^\nu$, $|t| \leq 16^{-n-M}$.

Now let $\eta \in C^\infty_0(\mathbb{R})$ be an even nonnegative function supported in $(-1/2, 1/2)$ so that $\int \eta(t)dt = 1$. We define

$$\gamma_\nu(t) = \int 64^{n+M}\eta(64^{n+M}s)\tilde\gamma_\nu(t-s)ds.$$

Since η is even it is straightforward to check that $\gamma_\nu(t) = \tilde\gamma_\nu(t)$ when $32^{-n-M} \leq |t| \leq 16^{-n-M}$. Replacing $\partial\Omega \cap \mathfrak{S}_\nu$ parametrized by $\tilde\gamma_\nu$ by the curve parametrized by γ_ν yields a convex domain Ω_n with the required properties. If $2^{-n+2} \leq \delta$ and $\{B_j\}$ denotes a cover of Ω with "balls" of the form (1.2) then the balls with double height cover the boundary of Ω_n. This yields (2.7). □

A decomposition of the boundary. Let $\mathfrak{S}_u$ be as in (2.2). We introduce a decomposition of $\partial\Omega \cap \mathfrak{S}_u$ in order to use the geometric properties of $\partial\Omega$ in terms of the covering numbers $N(\Omega,\delta)$; we assume that $\delta \leq 2^{-100-M}$. Consider the parametrization of $\partial\Omega \cap \mathfrak{S}_u$ by (2.3). We define a finite sequence of increasing numbers

$$\mathfrak{A}_u(\delta) = \{a_0, \dots, a_Q\}$$

inductively as follows. Let $a_0 = -1$. Suppose $a_0, \dots, a_{j-1}$ are already defined. If

$$(t - a_{j-1})(\gamma_L'(t) - \gamma_R'(a_{j-1})) \leq \delta \text{ for all } t \in (a_{j-1}, 1])$$

then let $a_j = 1$ if $a_{j-1} \leq 1 - 2^{-M}\delta$ and $a_j = a_{j-1} + 2^{-M}\delta$ if $a_{j-1} > 1 - 2^{-M}\delta$. Otherwise define

$$a_j = \inf\{t \in (a_{j-1}, 1] : (t - a_{j-1})(\gamma_L'(t) - \gamma_R'(a_{j-1})) > \delta\}.$$

Since $|\gamma_L'|$, $|\gamma_R'|$ are bounded by 2^{M-1} we see that $|t-s||\gamma_L'(t) - \gamma_R'(s)| < \delta$ if $|t-s| < \delta 2^{-M}$; therefore the first case occurs after a finite number of steps. We obtain a sequence $a_0 < a_1 < \cdots < a_Q$ with so that for $j = 0, \dots, Q-1$

$$(a_{j+1} - a_j)(\gamma_L'(a_{j+1}) - \gamma_R'(a_j)) \leq \delta, \tag{2.8.1}$$

and for $0 \leq j < Q-1$

$$(t - a_j)(\gamma_L'(t) - \gamma_R'(a_j)) \leq \delta, \quad \text{if} \quad t > a_{j+1}. \tag{2.8.2}$$

Condition (2.8.1) is satisfied since γ_L' is left continuous.

It will also be convenient to define

$$\mathfrak{A}_u(\delta, r) = \{a_j \in \mathfrak{A}_u(\delta) : 2^{-r} \leq a_{j+1} - a_j < 2^{-r+1}\}. \tag{2.9}$$

where $r \in \mathbb{N}$ so that $2^{-M}\delta \leq 2^{-r} \leq 1$. Note that $\mathfrak{A}_u(\delta) = \cup_r \mathfrak{A}_u(\delta, r)$.

The number Q in (2.8.1/2) is also denoted by $Q(\delta)$ or $Q_u(\delta)$ if it becomes necessary to indicate the dependence of δ or u.

The following Lemma relates the numbers $Q_u(\delta)$ to the covering numbers $N(\Omega,\delta)$.

Lemma 2.3.

There exist a positive constant C_M, so that the following statements hold.

(i) $Q_u(\delta) \le C_M \delta^{-1/2}$.

(ii) $0 \le \kappa_\Omega \le 1/2$.

(iii) $Q_u(\delta) \le C_M N(\Omega, \delta) \log(2+\delta^{-1})$.

(iv) For $\nu = 1, \dots, 2^{2M}$ *let* $u_\nu = (\cos(2\pi\nu 2^{-2M}), \sin(2\pi\nu 2^{-2M}))$. *Then*

$$C_M^{-1} N(\Omega, \delta) \le \sum_\nu Q_{u_\nu}(\delta) \le C_M N(\Omega, \delta) \log(2+\delta^{-1}).$$

Proof. For (i) apply the Cauchy-Schwarz inequality and (2.8) to obtain

$$\begin{aligned} Q-1 &\le \delta^{-1/2} \sum_{j=1}^{Q-1} (a_j - a_{j-1})^{1/2} (\gamma_L'(a_j) - \gamma_R'(a_{j-1}))^{1/2} \\ &\le \delta^{-1/2} \Big(\sum_{j=1}^{Q-1} a_j - a_{j-1} \Big)^{1/2} \Big(\sum_{j=1}^{Q-1} \gamma_L'(a_j) - \gamma_R'(a_{j-1}) \Big)^{1/2} \\ &\le 4 \cdot 2^{M/2} \delta^{-1/2}. \end{aligned}$$

(ii) is an immediate consequence of (i).

The left inequality in (iv) follows easily from the definitions once one observes that for a slope $\sigma \ge \gamma_R'(a_j)$ we have

$$\begin{aligned} \gamma(a_{j+1}) - \gamma(a_j) - \sigma(a_{j+1} - a_j) &= \int_{a_j}^{a_{j+1}} (\gamma'(s) - \sigma) ds \\ &\le (\gamma_L'(a_{j+1}) - \gamma_R'(a_j))(a_{j+1} - a_j). \end{aligned} \tag{2.10}$$

The other inequality in (iv) follows from (iii). For the proof of (iii) pick r_0 (with $2^{r_0} \le 2^M \delta^{-1}$) so that among the sets $\mathfrak{A}_u(\delta, r)$ the set $\mathfrak{A}_u(\delta, r_0)$ has maximal cardinality Q_0. Note that

$$Q_0 \ge \frac{Q_u(\delta)}{2(M + \log(2+\delta^{-1}))}.$$

We may assume that $Q_0 \ge 2^{M+100}$ (otherwise (iii) follows easily). We may split $\mathfrak{A}_u(\delta, r_0)$ into not more than 2^{40+M} families $\mathfrak{A}_u^m(\delta, r_0)$ with the property that for every choice of different $a_j, a_k \in \mathfrak{A}_u^m(\delta, r_0)$ there are at least 2^{20+M} numbers $a_i \in \mathfrak{A}_u(\delta, r_0)$ between a_j and a_k.

In order to verify (iii) we have to show that

$$Q_0 \le c_M N(\Omega, \delta). \tag{2.11}$$

We now fix $a_j, a_k \in \mathfrak{A}_u^m(\delta, r_0)$, $a_j < a_k$. Let $t_0 \in [a_j, a_{j+1}]$ and let ℓ be a supporting line for Ω at $P_0 = t_0 u_\perp + \gamma(t_0) u$. Pick any $t_1 \in [a_k, a_{k+1}]$ and let $P_1 = t_1 u_\perp + \gamma(t_1) u$. Observe that (2.11) follows if one can show that the distance of P_1 to ℓ is greater than δ. This we now verify. If σ is the slope of ℓ then $\sigma \le \gamma_R'(t_0)$ and $\sigma u_\perp - u$ is the normal to ℓ which is outward with respect to Ω. Then

$$\begin{aligned} \operatorname{dist}(P_1, \ell) &= \frac{1}{\sqrt{1+\sigma^2}} \big(\gamma(t_1) - \gamma(t_0) - \sigma(t_1 - t_0) \big) \\ &\ge \frac{1}{\sqrt{1+\sigma^2}} \big(\gamma(t_1) - \gamma(t_0) - \gamma_R'(t_0)(t_1 - t_0) \big) \end{aligned} \tag{2.12}$$

By definition of $\mathfrak{A}_u^m(\delta, r_0)$ we may pick $L \equiv 2^{M+5}$ intervals $[c_i, d_i]$ with $a_{j+1} < c_1$, $d_L < a_k$, $2^{-r} \le d_i - c_i \le 2^{-r+2}$, $(\gamma_L'(d_i) - \gamma_R'(c_i))(d_i - c_i) > \delta$. We set $d_0 = a_{j+1}$, $c_{L+1} = a_k$. Then

$$\begin{aligned}
&\gamma(t_1) - \gamma(t_0) - \gamma_R'(t_0)(t_1 - t_0) \\
&= \gamma(t_1) - \gamma(c_{L+1}) - \gamma_R'(t_0)(t_1 - c_{L+1}) + \sum_{i=0}^{L} \Big[\gamma(c_{i+1}) - \gamma(d_i) - \gamma_R'(t_0)(c_{i+1} - d_i)\Big] \\
&\quad + \sum_{i=1}^{L} \Big[\gamma(d_i) - \gamma(c_i) - \gamma_R'(t_0)(d_i - c_i)\Big] + \gamma(d_0) - \gamma(t_0) - \gamma_R'(t_0)(d_0 - t_0) \\
&\ge \sum_{i=2}^{L} \Big[\gamma(d_i) - \gamma(c_i) - \gamma_R'(t_0)(d_i - c_i)\Big] \ge \sum_{i=2}^{L} \Big[(\gamma_L'(d_i) - \gamma_R'(c_i))(d_i - c_i)\Big] \ge (L-1)\delta
\end{aligned}$$

and thus, since $L = 2^{M+5}$,

$$\operatorname{dist}(P_1, \ell) \ge \frac{L-1}{\sqrt{1 + 2^{2M}}}\delta > \delta. \quad \square$$

The following Lemma is concerned with a disjointness property for algebraic sums of balls of the form (1.2). This will be used in the proof of the L^4 estimate in §3, using an orthogonality argument due to Fefferman [7].

Lemma 2.4. *Let $B \ge 1$ and $\mathfrak{A}_u(\delta, r)$ as in (2.9). Let $\mathfrak{a}$ be a subset of $\mathfrak{A}_u(\delta, r)$ with the property that*

$$a_j \in \mathfrak{a}, \quad a_k \in \mathfrak{a}, \quad a_j < a_k \quad \Longrightarrow \quad k - j > 2^{10}B. \tag{2.13}$$

Let $I_j = [a_j - \delta 2^{-M}, a_{j+1} + \delta 2^{-M}]$ and

$$G_j = \{\xi : \langle \xi, u_\perp \rangle \in I_j, |\langle \xi, u \rangle - \gamma(\langle \xi, u_\perp \rangle)| \le B\delta\} \tag{2.14}$$

Then for any $\xi \in \mathbb{R}^2$ there are at most two pairs (j,k) with $a_j, a_k \in \mathfrak{a}$ so that ξ belongs to $G_j + G_k$.

Proof. We may assume $u_\perp = (1,0)$, $u = (0,1)$. Suppose without loss of generality that

$$\xi \in (G_j + G_k) \cap (G_m + G_n), \qquad j \le k, \quad m \le n; \tag{2.15}$$

we shall then show that $j = m$, $k = n$. Now $\xi = (\xi_1, \xi_2)$ with

$$\xi_1 = \alpha_j + \alpha_k = \alpha_m + \alpha_n, \qquad \alpha_i \in I_i, \ i = j, k, m, n, \tag{2.16}$$

and

$$\xi_2 = \gamma(\alpha_j) + \gamma(\alpha_k) + (t_j + t_k)\delta, \tag{2.17}$$

$$\xi_2 = \gamma(\alpha_m) + \gamma(\alpha_n) + (t_m + t_n)\delta, \tag{2.18}$$

with

$$|t_i| \le B \quad \text{for } i = j, k, m, n.$$

Recall that $2^{-r} > \delta 2^{-M}$ and therefore

$$|\alpha_i - a_i| \le \delta 2^{-M+1} + 2^{-r+1} \le 2^{-r+2} \text{ for } \alpha_i \in I_i.$$

We shall distinguish three cases.

Case I: $a_j = a_k$, $a_m = a_n$.

In this case by (2.16) $|a_j - a_m| \leq 2^{-r+4}$ and the condition (2.13) implies $a_j = a_m$ and therefore $a_j = a_k = a_m = a_n$.

Case II: Suppose $a_j = a_k$, $a_m < a_n$. We show that this case does not occur if (2.13) and (2.15) hold. Without loss of generality we may assume that $\alpha_j \leq \alpha_k$.

The interval $[a_{m+1}, a_n]$ contains at least 2^{10} intervals of length 2^{-r}, in particular

$$\begin{aligned} 2a_j &\geq \alpha_j + \alpha_k - 2^{-r+3} = \alpha_m + \alpha_n - 2^{-r+3} \geq a_m + a_n - 2^{-r+4} \\ &\geq 2a_m - 2^{-r+4} + B2^{10-r} \geq 2a_{m+1} - 2^{-r+5} + B2^{10-r} \end{aligned}$$

hence $a_j > a_{m+1} + 2^{-r}$. A similar argument shows that $a_{j+1} < a_n - 2^{-r}$. Thus $I_j \subset [a_{m+1}, a_n]$ and in particular $(\alpha_j + \alpha_k)/2 \in [a_{m+1}, a_n]$. By (2.17) and (2.18)

$$|\gamma(\alpha_n) + \gamma(\alpha_m) - \gamma(\alpha_j) - \gamma(\alpha_k)| \leq 4B\delta.$$

Choose $\beta_j, \beta_k \in [a_j, a_{j+1}]$ so that $|\beta_j - \alpha_j| \leq \delta 2^{-M}$ and $|\beta_k - \alpha_k| \leq \delta 2^{-M}$. From

$$\begin{aligned} |\gamma(\beta_j) - 2\gamma(\frac{\beta_j + \beta_k}{2}) + \gamma(\beta_k)| &\leq (\gamma'_L(\beta_k) - \gamma'_R(\beta_j)(\beta_k - \beta_j) \\ &\leq (\gamma'_L(a_{j+1}) - \gamma'_R(a_j))(a_{j+1} - a_j) \leq \delta \end{aligned}$$

we see that also

$$|\gamma(\alpha_j) - 2\gamma(\frac{\alpha_j + \alpha_k}{2}) + \gamma(\alpha_k)| \leq \delta + 4 \cdot 2^{-M}\delta\|\gamma'\|_\infty \leq 3\delta$$

and therefore

$$|\gamma(\alpha_n) - 2\gamma(\frac{\alpha_j + \alpha_k}{2}) + \gamma(\alpha_m)| \leq (4B+3)\delta. \tag{2.19}$$

Now

$$\gamma(\alpha_n) - 2\gamma(\frac{\alpha_j + \alpha_k}{2}) + \gamma(\alpha_m) = \int_{\frac{\alpha_j+\alpha_k}{2}}^{\alpha_n} (\alpha_n - t) d\gamma'_R(t) + \int_{\alpha_m}^{\frac{\alpha_j+\alpha_k}{2}} (t - \alpha_m) d\gamma'_R(t)$$

plus a remainder term $(\alpha_j + \alpha_k - \alpha_n - \alpha_m)\gamma'_R(\frac{\alpha_j+\alpha_k}{2})$ which vanishes in view of (2.16).

By the assumption on $\mathfrak{a}$ at least one of the intervals $[\alpha_m, \frac{\alpha_j+\alpha_k}{2}]$, $[\frac{\alpha_j+\alpha_k}{2}, \alpha_n]$ contains an interval $[a_i - \epsilon, a_{i+1} + \epsilon]$ with $2^{-r} \leq a_{i+1} - a_i \leq 2^{-r+1}$ and $0 < \epsilon \leq 2^{-r-1}$ so that

$$a_m + 2^{8-r}B \leq a_i < a_{i+1} < a_n - 2^{8-r}B \tag{2.20}$$

Suppose first that $[a_i - \epsilon, a_{i+1} + \epsilon] \subset [\alpha_m, \frac{\alpha_j+\alpha_k}{2}]$. Then by (2.19)

$$\begin{aligned} (4B+3)\delta &\geq \int_{\alpha_m}^{\frac{\alpha_j+\alpha_k}{2}} (t - \alpha_m) d\gamma'_R(t) \\ &\geq \int_{a_i-\epsilon}^{a_{i+1}+\epsilon} (a_i - a_{m+1} - 2^{-M}\delta) d\gamma'_R(t) \\ &= (a_i - a_{m+1} - 2^{-M}\delta)(\gamma'_R(a_{i+1}+\epsilon) - \gamma'_R(a_i - \epsilon)) \\ &\geq (a_i - a_{m+1})(\gamma'_L(a_{i+1}+\epsilon) - \gamma'_R(a_i-\epsilon)) - \delta \geq \frac{a_i - a_{m+1}}{a_{i+1} - a_i + 2\epsilon}\delta - \delta \end{aligned}$$

Therefore $a_i - a_m \le a_i - a_{m+1} + 2^{-r+1} \le (4B+4)(2^{-r+1} + 2\epsilon) \le B2^{-r+5}$, in contradiction to (2.20).

Similarly if $[a_i - \epsilon, a_{i+1} + \epsilon] \subset [\frac{\alpha_j + \alpha_k}{2}, \alpha_n]$ we deduce that $a_n - a_{i+1} \le B2^{-r+5}$, again in contradiction to (2.20).

Case III. We now suppose that $a_j < a_k$, $a_m < a_n$ and show again that this case does not occur. Without loss of generality $m \le j$ which then implies by (2.13) that $\alpha_m < \alpha_j < \alpha_k < \alpha_n$.

Since $\alpha_n = \alpha_j + \alpha_k - \alpha_m$ we obtain

$$0 \le \int_{\alpha_m}^{\alpha_j} \big(\gamma_R'(\alpha_j + \alpha_k - u) - \gamma_R'(u)\big) du = \gamma(\alpha_m) + \gamma(\alpha_n) - \gamma(\alpha_j) - \gamma(\alpha_k) \le 4B\delta$$

where the last inequality follows from (2.17-18).

If $u \in [\alpha_m, \alpha_j]$ then $[\alpha_j, \alpha_k] \subset [u, \xi_1 - u]$ and, by our assumption on $\mathfrak{a}$, $[\alpha_j, \alpha_k]$ contains an interval $[a_i - \epsilon, a_{i+1} + \epsilon]$ with $2^{-r} \le a_{i+1} - a_i \le 2^{-r+1}$ and $0 < \epsilon \le 2^{-r-1}$. We conclude that

$$\begin{aligned} 4B\delta &\ge \int_{\alpha_m}^{\alpha_j} \big(\gamma_R'(\alpha_j + \alpha_k - u) - \gamma_R'(u)\big) du \\ &\ge \int_{\alpha_m}^{\alpha_j} (\gamma_L'(a_{i+1} + \epsilon) - \gamma_R'(a_i - \epsilon)) du \ge \frac{\alpha_j - \alpha_m}{a_{i+1} - a_i + 2\epsilon} \delta \end{aligned}$$

and therefore $a_j - a_m \le \alpha_j - \alpha_m + 2^{-r+2} \le 4B(2^{-r+1} + 2\epsilon) + 2^{-r+2} \le 2^{5-r}B$ in contradiction to (2.13).

Thus only Case I can occur and the Lemma is proved. □

3. Estimates

Our first Lemma in this section is used to prove estimates for $h \circ \rho$ where h is sufficiently regular. The bounds (3.2) are not best possible; they will be used later to consider error terms in the proof of Proposition 3.2 below.

Lemma 3.1. *Let h be an absolutely continuous function on $[0, \infty)$ and suppose that $\lim_{t\to\infty} h(t) = 0$. Suppose that $s \mapsto sh'(s)$ defines an L^1 function on $[0, \infty)$ and let*

$$F(\tau) = \int_0^\infty h'(s) e^{is\tau} ds.$$

Suppose that $\mu > 0$ and that

$$|F(\tau)| + |F'(\tau)| \le B(1 + |\tau|)^{-\mu}. \tag{3.1}$$

Let $B(0, R)$ be the ball with radius R and center 0, and define $\mathcal{A}_k = B(0, 2^k) \setminus B(0, 2^{k-1})$, for $k > 0$, and $\mathcal{A}_0 = B(0, 1)$.

Then

$$\int_{\mathcal{A}_k} \sup_{1/2 \le t \le 2} |\mathcal{F}^{-1}[h \circ \frac{\rho}{t}](x)| dx \lesssim B\big[2^{-k(\mu-1)} + k2^{-k}\big]. \tag{3.2}$$

Proof. Denote by $d\sigma$ surface measure on $\Sigma_\rho = \partial\Omega$ and for $x \in \partial\Omega$ by n the outward unit normal vector. We shall first assume that Σ_ρ is a C^2 surface, but the bounds will depend only on the Lip(1) norm of parametrizations.

We begin following Hlawka [8] and Randol [15]. Using integration by parts and the divergence theorem applied to the vector field $\xi \mapsto (is)^{-1}|x|^{-2}xe^{is\langle x,\xi\rangle}$ we obtain

$$\begin{aligned}(2\pi)^2\mathcal{F}^{-1}[h(\frac{\rho(\cdot)}{t})](x) &= \int_\Omega h(\rho(\xi)/t)e^{i\langle x,\xi\rangle}d\xi \\ &= -\int_\Omega e^{i\langle x,\xi\rangle}\int_{\rho(\xi)}^\infty t^{-1}h'(s/t)ds\,d\xi \\ &= -t^{-1}\int_0^\infty h'(s/t)\int_{s\Omega}e^{i\langle x,\xi\rangle}d\xi\,ds \\ &= -t^{-1}\int_0^\infty h'(s/t)s^2\int_\Omega e^{i\langle x,s\xi\rangle}d\xi\,ds \\ &= it^{-1}|x|^{-2}\int_0^\infty sh'(s/t)\int_{\partial\Omega}e^{is\langle x,\xi\rangle}\langle x,n(\xi)\rangle d\sigma(\xi)\,ds. \end{aligned} \tag{3.3}$$

Let $\widetilde{\chi}\in C^\infty(\mathbb{R}^2\setminus\{0\})$ be supported in a sector

$$S=\{\xi:|\langle\frac{\xi}{|\xi|},u\rangle+1|\le\varepsilon\}$$

where u is a unit vector and ϵ is small; the choice

$$\varepsilon\le 2^{-10-2M} \tag{3.4}$$

will certainly suffice. Let $u_\perp$ be the unit vector perpendicular to u, so that $\det(u_\perp,u)=1$. Then $\Sigma_\rho\cap S$ can be parametrized by $\alpha\mapsto u_\perp\alpha+u\gamma(\alpha)$ so that (2.3)-(2.6) holds. Set $\chi(\alpha)=\widetilde{\chi}(u_\perp\alpha+u\gamma(\alpha))$; then $\chi(\alpha)=0$ for $|\alpha|\ge 2^{-10-M}$. In S we introduce homogeneous coordinates (*i.e.* polar coordinates associated to $\partial\Omega$) given by

$$(s,\alpha)\mapsto\xi(s,\alpha)=s(u_\perp\alpha+u\gamma(\alpha)) \tag{3.5}$$

with $\xi_0=u\gamma(0)$ with $4\le-\gamma(0)\le 2^M$; then

$$\rho(u_\perp\alpha+u\gamma(\alpha))=1.$$

Note that for the Jacobian of the map (3.5) we have

$$\det\big(\frac{\partial\xi}{\partial(s,\alpha)}\big)=s(\alpha\gamma'(\alpha)-\gamma(\alpha))$$

which is bounded below by $2s$ on the support of χ (since $-\gamma(\alpha)\ge 4-2^M\varepsilon\ge 3$ and $|\alpha\gamma'(\alpha)|\le 2^{2M}\varepsilon\le 2^{-10}$ where ε is as in (3.4)). Let

$$K_t(x)=it^{-1}|x|^{-2}\int_0^\infty sh'(s/t)\int_{\partial\Omega}e^{is\langle x,\xi\rangle}\langle x,n(\xi)\rangle\tilde{\chi}(\xi)d\sigma(\xi)\,ds$$

and let R_u be the rotation with $R_ue_1=u_\perp$, $R_ue_2=u$. Using a partition of unity we see that it suffices to estimate

$$K_t(R_ux)=it^{-1}|x|^{-2}\int_0^\infty sh'(s/t)\int e^{is(x_1\alpha+x_2\gamma(\alpha))}(x_2-x_1\gamma'(\alpha))\chi(\alpha)(\alpha\gamma'(\alpha)-\gamma(\alpha))\,d\alpha\,ds. \tag{3.6}$$

Let $\eta \in C^\infty(\mathbb{R})$ be an even function supported in $[-\varepsilon, \varepsilon]$ so that $\eta(\alpha) = 1$ for $|\alpha| \le \varepsilon/2$. We split

$$K_t(R_u x) = K_{t,1}(x) + K_{t,2}(x)$$

where

$$(3.7)\quad K_{t,1}(x) = \\ it^{-1}|x|^{-2} \int_0^\infty s h'(s/t) \int_{\partial\Omega} e^{is(x_1\alpha + x_2\gamma(\alpha))}(x_2 - x_1\gamma'(\alpha))\eta(\tfrac{x_1 + x_2\gamma'(\alpha)}{|x|})\chi(\alpha)(\alpha\gamma'(\alpha) - \gamma(\alpha))\, d\alpha\, ds$$

and $K_{t,2}$ is defined in the same way, with $\eta(\cdots)$ replaced by $1 - \eta(\cdots)$.

In (3.7) we interchange the order of integration and see that

$$|K_{t,1}(x)| \lesssim Bt|x|^{-2} \int \big|\eta(\tfrac{x_1 + x_2\gamma'(\alpha)}{|x|}) F'(tx_1\alpha + tx_2\gamma(\alpha))(x_2 - x_1\gamma'(\alpha))\chi(\alpha)\big|\, d\alpha. \tag{3.8}$$

If $\eta(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}) \ne 0$ then $|x_2| \ge |x_1| 2^{-M-1}$ and, since $|\alpha| \le 2\varepsilon$,

$$|x_1\alpha + x_2\gamma(\alpha)| \ge |x_2||\gamma(\alpha) - \alpha\gamma'(\alpha)| - 2^{M+2}\varepsilon|x| \ge 2^{-M-1}|x| \tag{3.9}$$

and it follows from (3.1) and (3.8), (3.9) that

$$\sup_{1/2 \le t \le 2} |K_{t,1}(x)| \lesssim B|x|^{-1-\mu}$$

and hence

$$\int_{\mathcal{A}_k} \sup_{1/2 \le t \le 2} |K_{t,1}(x)| dx \lesssim B 2^{k(1-\mu)}. \tag{3.10}$$

In order to estimate $K_{t,2}(x)$ we integrate by parts with respect to α. This yields

$$\begin{aligned} K_{t,2}(x) &= |x|^{-2} \int_0^1 t^{-1} h'(s/t) \int e^{is(x_1\alpha + x_2\gamma(\alpha))} \partial_\alpha g(x, \alpha) d\alpha\, ds \\ &= t|x|^{-2} \int F(tx_1\alpha + tx_2\gamma(\alpha)) \partial_\alpha g(x, \alpha) d\alpha \end{aligned}$$

with

$$g(x, \alpha) = \frac{(x_2 - x_1\gamma'(\alpha))(1 - \eta(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}))\chi(\alpha)(\alpha\gamma'(\alpha) - \gamma(\alpha))}{x_1 + x_2\gamma'(\alpha)}$$

Notice that $|x_1 + x_2\gamma'(\alpha)| \ge c|x|$ in the support of $1 - \eta(\cdots)$. This yields

$$|\partial_\alpha g(x, \alpha)| \le C_M(1 + |\gamma''(\alpha)|)$$

and consequently, using also (3.1), we obtain for $k \ge 1$

$$\int_{\mathcal{A}_k} \sup_{1/2 \le t \le 2} |K_{t,2}(x)| dx \lesssim \int (1 + |\gamma''(\alpha)|) \int_{\mathcal{A}_k} |x|^{-2}(1 + |x_1\alpha + x_2\gamma(\alpha)|)^{-\mu} dx\, d\alpha.$$

The inner integral is $O(2^{-k\mu})$ if $\mu < 1$, $O(2^{-k}k)$ if $\mu = 1$ and $O(2^{-k})$ if $\mu > 1$. By convexity $\int |\gamma''(\alpha)| d\alpha \lesssim 2^M$ and hence

$$\int_{\mathcal{A}_k} \sup_{1/2 \le t \le 2} |K_{t,2}(x)| dx \lesssim B \max\{2^{-k\mu}, k2^{-k}\}. \tag{3.11}$$

The estimate follows from (3.10) and (3.11).

Finally in order to remove the assumption of a C^2 boundary we can approximate the domain Ω by an increasing sequence of convex sets Ω_n with smooth boundaries, so that the C^1 bounds for parametrizations of the boundary are uniform in n. The estimate (3.2) holds then for ρ_n with bounds uniformly in n. The Minkowski functionals ρ_n associated with Ω_n converge to ρ, uniformly on compact sets and therefore the estimate (3.2) holds for ρ as well. □

We shall now investigate the multiplier $(1-\rho)_+^\lambda$ near the boundary of Ω; it suffices to consider $(1-\rho)_+^\lambda b$ where b is supported in a narrow sector. Since the number κ_Ω is invariant under rotations, there is no loss of generality to assume that this sector contains $u = (0, 1)$.

Proposition 3.2. *Let Ω be a convex domain as in (2.1) and let $b \in C_0^\infty$ be supported in the sector $S = \{\xi : |\xi_1| \le 2^{-10M}|\xi_2|; \xi_2 < 0\}$. Let $\alpha \mapsto (\alpha, \gamma(\alpha))$ be the parametrization of $\partial\Omega \cap S$ as a graph, as in Lemma 2.1. For any subinterval I of $[-1/2, 1/2]$ denote by I^* the interval with same center and with length $4/3|I|$. For $\ell > 1$ let $\mathfrak{I}_\ell$ be the set of subintervals I of $[-1, 1]$ with the property that $|I| \ge 2^{-\ell-5M}$ and*

$$(t-s)(\gamma_L'(t) - \gamma_R'(s)) \le 2^{-\ell+5} \text{ for } s < t, \ s, t \in I^*. \tag{3.12}$$

Let $\mathfrak{B}$ be the set of C^2 functions β supported on $(-1/2, 1/2)$ so that

$$|\beta^{(k)}(t)| \le 1, \quad k = 0, \dots, 4. \tag{3.13}$$

(i) Suppose $I = (c_I - |I|/2, c_I + |I|/2) \in \mathfrak{I}_\ell$. Let

$$m(\xi) = b(\xi)\beta_1(2^{\ell-1}(1-\rho(\xi))\beta_2(|I|^{-1}(\xi_1 - c_I)) \tag{3.14}$$

where $\beta_1, \beta_2 \in \mathfrak{B}$. Then

$$\int_{|x| \ge 2^{10\ell}} \sup_{1/2 \le t \le 2} |t^2 \mathcal{F}^{-1}[m](tx)| dx \lesssim 2^{-5\ell}. \tag{3.15}$$

Moreover

$$\int \sup_{1/2 \le t \le 2} t^2 \big[|\mathcal{F}^{-1}[m](tx)| + |\nabla \mathcal{F}^{-1}[m](tx)|\big] dx \lesssim 1 + \ell. \tag{3.16}$$

and

$$\int_{|x| > 2|y|} \sup_{0 < t < \infty} t^2 \big|\mathcal{F}^{-1}[m](tx - ty) - \mathcal{F}^{-1}[m](tx)\big| dx \lesssim (1+\ell)^2. \tag{3.17}$$

(ii) Denote by $m_{\beta_1,\beta_2,I}$ the right hand side of (3.14) and let

$$\mathfrak{M}_\ell f(x) = \sup_{\beta_1,\beta_2 \in \mathfrak{B}} \sup_{I \in \mathfrak{I}_\ell} |\mathcal{F}^{-1}[m_{\beta_1,\beta_2,I} \widehat{f}](x)|. \tag{3.18}$$

Then

$$\|\mathfrak{M}_\ell f\|_2 \lesssim (1+\ell)^4 \|f\|_2$$

for all $f \in L^2(\mathbb{R}^2)$.

Proof. Again we first assume that $\gamma \in C^2$ but our bounds will only depend on the L^∞ norms of γ and γ'. This restriction can then be removed by using Lemma 2.2.

We shall now fix β_1 and β_2 and set $h_\ell(s) = \beta_1(2^\ell(1-s))$. Let

$$\Lambda_\ell(\tau) = \int_0^\infty h_\ell(s) e^{is\tau} ds \qquad \text{and} \qquad \widetilde{\Lambda}_\ell(\tau) = \int_0^\infty h'_\ell(s) e^{is\tau} ds.$$

Then

$$|\Lambda_\ell(\tau)| + |\Lambda'_\ell(\tau)| \lesssim 2^{-\ell}(1+2^{-\ell}|\tau|)^{-4} \tag{3.19.1}$$

$$|\widetilde{\Lambda}_\ell(\tau)| + |\widetilde{\Lambda}'_\ell(\tau)| \lesssim (1+2^{-\ell}|\tau|)^{-3} \tag{3.19.2}$$

by an integration by parts.

We apply Lemma 3.1 to $h = h_\ell$ and $\mu = 2$ (so $F = \widetilde{\Lambda}_\ell$ in (3.1)). Since the right hand side of (3.19.2) is bounded by $2^{2\ell}(1+|\tau|)^{-2}$ we obtain

$$\int_{\mathcal{A}_k} \sup_{1/2 \le t \le 2} |\mathcal{F}^{-1}[h_l \circ \frac{\rho}{t}](x)| dx \lesssim 2^{2\ell} k 2^{-k}; \tag{3.20}$$

this is certainly a favorable estimate if $k \ge 10\ell$. The Fourier transform of $b(\xi)\beta_2(|I|^{-1}(\xi_1 - c_I))$ is pointwise bounded by a constant times

$$H_I(x) = \frac{|I|}{(1+|I||x_1|)^2} \frac{1}{1+x_2^2}$$

and since $|I| \ge c2^{-\ell}$ it is straightforward to verify (3.15).

We now give a different estimate for the integral over the dyadic annulus $\mathcal{A}_k$ which is used to derive an improved bound for $|x| \le 2^{10\ell}$. Let $\widetilde{\beta}$ be a C_0^∞ function so that $\widetilde{\beta}(t) = 1$ for $|t| \le 9/16$ and $\widetilde{\beta}(t) = 0$ for $|t| \ge 5/8$. Let $\beta_I(t) = \widetilde{\beta}(|I|^{-1}(t - c_I))$ so that

$$\text{supp}(\beta_I) \subset I^*.$$

Note that $|\xi_1| \le 2^{-8M}$ on $\text{supp}(b)$ (since $|\xi| \le 2^M$ and $|\xi_1| \le 2^{-10M}|\xi_2|$). If also $|\xi_1 - c_I| \le |I|/2$ and $|1-\rho(\xi)| \le 2^{-\ell+1}$ then

$$\Big|\frac{\xi_1 - c_I\rho(\xi)}{\rho(\xi)}\Big| \le \Big|\frac{\xi_1}{\rho(\xi)}\Big| |1-\rho(\xi)| + |\xi_1 - c_I| \le 2^{-8M-\ell+2} + |I|/2 \le |I|(\tfrac{1}{2} + \tfrac{1}{2^7});$$

here we used that $|I| \ge 2^{-\ell-5M}$ and $M \ge 3$. Consequently

$$\beta_I(\xi_1/\rho(\xi)) = 1 \text{ if } \beta_2(|I|^{-1}(\xi_1 - c_I)) \ne 0 \text{ and } |1-\rho(\xi)| \le 2^{-\ell+1}.$$

Therefore we may write

$$m(\xi) = b(\xi)\beta_2(|I|^{-1}(\xi_1 - c_I))\widetilde{m}(\xi)$$

with $\widetilde{m}(\xi) = \beta_1(2^{\ell-1}(1-\rho(\xi))\beta_I(\xi_1/\rho(\xi))$ and estimate the Fourier inverse of $\widetilde{m}$.

Let η be as in the proof of Lemma 3.1, namely smooth and supported in $(-\varepsilon, \varepsilon)$ where ε is small (as in (3.4)). Let ϕ_0 be smooth and supported in $[-1,1]$, so that $\phi_0(s) = 1$ for $|s| \le 1/2$. Define

$$\Phi_0(x,\alpha) = \phi_0(|I|(x_1 + x_2\gamma'(\alpha)))\eta(\frac{x_1 + x_2\gamma'(\alpha)}{|x|})$$
$$\Phi_n(x,\alpha) = \big(\phi_0(2^{-n-1}|I|(x_1 + x_2\gamma'(\alpha))) - \phi_0(2^{-n}|I|(x_1 + x_2\gamma'(\alpha)))\big)\eta(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}).$$

We write the Fourier integral representing $\mathcal{F}^{-1}[\widetilde{m}(t^{-1}\cdot)](x)$ using coordinates $\xi = s(\alpha, \gamma(\alpha))$. Then we split the kernel

$$\mathcal{F}^{-1}[\widetilde{m}(t^{-1}\cdot)](x) = \frac{1}{(2\pi)^2}\big[\widetilde{K}_t(x) + \sum_{n\ge 0} K_{n,t}(x)\big] \tag{3.21}$$

where

$$K_{n,t}(x) = \int sh_\ell(s/t)\int \Phi_n(x,\alpha)\beta_I(\alpha)(\gamma(\alpha) - \alpha\gamma'(\alpha))e^{is(x_1\alpha + x_2\gamma(\alpha))}d\alpha\, ds \tag{3.22.1}$$

and

$$\widetilde{K}_t(x) = \int sh_\ell(s/t)\int (1 - \eta(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}))\beta_I(\alpha)(\gamma(\alpha) - \alpha\gamma'(\alpha))e^{is(x_1\alpha + x_2\gamma(\alpha))}d\alpha\, ds \tag{3.22.2}$$

Note that the sum in (3.21) has only $O(\log(1+|I||x|))$ terms since $K_{n,t}(x) = 0$ if $\varepsilon|x| \le 4|I|^{-1}2^n$. In particular if $x \in \mathcal{A}_k \cap \operatorname{supp}(K_{n,t})$ then $2^n \ll 2^k|I|$. The kernel $K_{0,t}$ is given by

$$K_{0,t}(x) = i^{-1}t^2\int \Lambda_\ell'(t(x_1\alpha + x_2\gamma(\alpha)))\Phi_0(x,\alpha)\beta_I(\alpha)(\gamma(\alpha) - \alpha\gamma'(\alpha))d\alpha.$$

Since $\det\begin{pmatrix}\alpha & 1\\ \gamma(\alpha) & \gamma'(\alpha)\end{pmatrix} \approx 1$ we may estimate

(3.23)
$$\int \sup_{1/2\le t\le 2}|K_{0,t}(x)|dx \lesssim 2^{-\ell}\int_I \iint_{\substack{|x_1+x_2\gamma'(\alpha)|\\ \le |I|^{-1}}} (1 + 2^{-\ell}|x_1\alpha + x_2\gamma(\alpha)|)^{-4}dx_1dx_2\, d\alpha \lesssim 1.$$

For $n > 0$ we integrate by parts in α to get

$$\begin{aligned} K_{n,t}(x) &= i\int h_\ell(s/t)\int \partial_\alpha g_n(x,\alpha)e^{is(x_1\alpha + x_2\gamma(\alpha))}d\alpha ds\\ &= i\int t\Lambda_\ell(t(x_1\alpha + x_2\gamma(\alpha)))\partial_\alpha g_n(x,\alpha)d\alpha \end{aligned}$$

where
$$g_n(x,\alpha) = \frac{\Phi_n(x,\alpha)\beta_I(\alpha)(\gamma(\alpha) - \alpha\gamma'(\alpha))}{x_1 + x_2\gamma'(\alpha)}.$$

Note that if $\Phi_n(x,\alpha) \ne 0$ then

$$|\partial_\alpha g_n(x,\alpha)| \lesssim \frac{(1 + |x_2|2^{-n}|I|)|\gamma''(\alpha)| + |I|^{-1}}{|x_1 + x_2\gamma'(\alpha)|}.$$

Therefore

$$\sup_{t\in[1,2]} |K_{n,t}(x)| \lesssim \int_{\substack{\alpha\in I^*: \\ |x_1+x_2\gamma'(\alpha)| \\ \approx 2^n|I|^{-1}}} \frac{(1+|x_2|2^{-n}|I|)|\gamma''(\alpha)|+|I|^{-1}}{|x_1+x_2\gamma'(\alpha)|} \frac{2^{-\ell}}{(1+2^{-\ell}|x_1\alpha+x_2\gamma(\alpha)|)^4} d\alpha \tag{3.24}$$

and

$$\begin{aligned}
&\int_{\mathcal{A}_k} \sup_{t\in[1,2]} |K_{n,t}(x)| dx \\
&\lesssim \int_{I^*} ((1+2^{k-n}|I|)|\gamma''(\alpha)|+|I|^{-1}) \int_{\substack{|u_2|\approx 2^n|I|^{-1} \\ |u|\approx 2^k}} |u_2|^{-1} \frac{2^{-\ell}}{(1+2^{-\ell}|u_1|)^4} du\, d\alpha \\
&\lesssim \min\{2^{k-\ell}, 2^{3(\ell-k)}\} \int_{I^*} (|\gamma''(\alpha)| + 2^{k-n}|I||\gamma''(\alpha)| + |I|^{-1}) d\alpha. \qquad (3.25)
\end{aligned}$$

In the evaluation of the integral we used that $|u|\approx 2^{-k}$ and $|u_2|\approx 2^n|I|^{-1}$ implies that $|u_1|\approx 2^k$ due to our assumption on $2^n|I|^{-1} \ll 2^k$. By assumption (3.12) we have $\int_{I^*} |I||\gamma''(\alpha)|d\alpha \le 2^{-\ell}$ and therefore

$$\int_{\mathcal{A}_k} \sup_{t\in[1,2]} |K_{n,t}(x)| dx \lesssim \min\{2^{k-\ell}, 2^{3(\ell-k)}\}(2^{k-\ell-n}+1);$$

again since $K_{n,t}$ vanishes on $\mathcal{A}_k$ if $n\ge k$ this yields

$$\sum_n \int_{\mathcal{A}_k} \sup_{t\in[1/2,2]} |K_{n,t}(x)| dx \lesssim k2^{-|k-\ell|}. \tag{3.26}$$

The estimate for $\widetilde{K}_t$ is similar to the estimate for the term $K_{t,2}$ in the proof of Lemma 3.1. Since in the support of $1-\eta(\cdots)$ we have $|x_1+x_2\gamma'(\alpha)|\approx|x|$ we obtain that

$$\begin{aligned}
&\int_{\mathcal{A}_k} \sup_{1/2\le t\le 2} |\widetilde{K}_t(x)| dx \\
&\lesssim \int_{I^*} (|I|^{-1}+|\gamma''(\alpha)|) \int |x|^{-1} \frac{2^{-\ell}}{(1+2^{-\ell}|x_1\alpha+x_2\gamma(\alpha)|)^4} dx\, d\alpha \le C. \qquad (3.27)
\end{aligned}$$

The estimate (3.16) for $\mathcal{F}^{-1}[m]$ follows immediately from (3.20), (3.26), (3.27) and the L^1 boundedness of H_I. Since m is supported in Ω and $t\in[1/2,2]$ we can write $\partial_{\xi_j}\mathcal{F}^{-1}[m(t^{-1}\cdot)] = \psi_j * \mathcal{F}^{-1}[m(t^{-1}\cdot)]$ for a suitable Schwartz function ψ_j depending only on Ω and j. The estimate (3.16) for $\nabla\mathcal{F}^{-1}[m]$ is then immediate.

We now turn to the (standard) estimation of the Calderón-Zygmund integral (3.17). Let $K_t = \mathcal{F}^{-1}[m(t^{-1}\cdot)]$. The integral in (3.17) can be decomposed as $\sum_{k\in\mathbb{Z}} \mathcal{E}_k$ where

$$\begin{aligned}
\mathcal{E}_k &= \int_{|x|>2|y|} \sup_{1\le t<2} |K_{2^k t}(x-y) - K_{2^k t}(x)|\, dx \\
&= \int_{|x|>2^{k+1}|y|} \sup_{1\le t<2} |K_t(x-2^k y) - K_t(x)|\, dx.
\end{aligned}$$

If $2^k|y| \ge 1$ then this integral is estimated by

$$\int_{|x|>2^{k+1}|y|} \sup_{1\le t<2} |K_t(x)|\, dx \lesssim \min\{(1+\ell),\, 2^{2\ell}(2^k|y|)^{-1}\log(2+2^k|y|)\};$$

this follows from (3.16) and (3.20). Therefore $\sum_{2^k|y|\ge 1} \mathcal{E}_k \lesssim (1+\ell)^2$. For $2^k|y| \le 1$ we use the estimate on the gradient in (3.16) and obtain

$$\mathcal{E}_k \le \int \sup_{1\le t<2} \Big| \int_0^1 \langle 2^k y, \nabla\rangle K_t(x - s2^k y) ds \Big|\, dx \lesssim 2^k|y|(1+\ell)$$

so that $\sum_{2^k|y|\le 1} \mathcal{E}_k \lesssim 1+\ell$. Thus (3.17) is proved.

Finally, an examination of the above arguments leading to (3.16) also yields the assertion for the maximal operator in (3.18). First let $|x| > 10\ell$; then we use (3.20), an estimate which does not depend on I. Then, from the shape of the kernels H_I, we obtain that

$$\sup_{\beta_1,\beta_2\in\mathfrak{B}} \sup_{I\in\mathfrak{I}_\ell} \big|(\chi_{\{|\cdot|\ge 2^{10\ell}\}}\mathcal{F}^{-1}[m_{\beta_1,\beta_2,I}]) * f(x)\big| \lesssim M_1(H_\ell * f)(x) \tag{3.28}$$

where H_ℓ is an kernel with L^1 norm $O(2^{-5\ell})$ and M_1 is the Hardy-Littlewood maximal function in the x_1 variable.

We dominate $(K_{n,t}\chi_{\mathcal{A}_k}) * f$ by a Besicovich type maximal function; here $\chi_{\mathcal{A}_k}$ is of course the characteristic function of the annulus $\mathcal{A}_k$. A straightforward analysis of (3.24) yields that for $k \le 10\ell$ (and hence $2^n|I|^{-1} \le 2^{10\ell}$)

$$\big|(K_{t,n}\chi_{\mathcal{A}_k}) * f(x)\big| \lesssim (1+\ell)\Big(\frac{1}{|I^*|}\int_{I^*} \big[(1+2^{k-n}|I|)|\gamma''(\alpha)| + |I|^{-1}\big] \min\{1, 2^{2(\ell-k)}\}\mathcal{M}_{\ell,\alpha} f(x) d\alpha\Big) \tag{3.29}$$

where $\mathcal{M}_{\ell,\alpha}$ is the maximal operator associated to all rectangles centered at the origin which have eccentricity less then $2^{10\ell}$ and one side parallel to $(\alpha, \gamma(\alpha))$. To verify this one notes that if $|x_1\alpha + x_2\gamma(\alpha)| \ll |x_1 + x_2\gamma'(\alpha)|$ in (3.24) then $|x_1 + x_2\gamma'(\alpha)| \approx |x|$.

Let $\mathfrak{M}_\ell$ denote the maximal function associated to all rectangles centered at the origin, with eccentricity $\le 2^{10\ell}$. Then it follows from (3.29) that

$$\big|(K_{t,n}\chi_{\mathcal{A}_k}) * f(x)\big| \lesssim (1+\ell)\mathcal{M}_\ell f(x).$$

A similar (easier) pointwise estimate holds for $\widetilde{K}_t$. By Córdoba's result [4] the L^2 bound for $\mathcal{M}_\ell$ is $O(1+\ell)$. Summing in $n \le 10\ell$ and $k \le 10\ell$ we can dominate $\chi_{\{|\cdot|\le 2^{10\ell}\}}|\mathcal{F}^{-1}[m]|(x)$ by $C(1+\ell)^3$ times $M_1(\mathcal{M}_\ell f)(x)$ and this together with (3.28) implies (3.18). □

L^1 estimates and decompositions of the multiplier. We shall now prove the statements of Theorem 1.1 and 1.2 for $p = 1$. We first recall the standard dyadic decomposition of the Riesz multiplier in terms of $1-\rho$.

Let $\phi_0 \in C_0^\infty(\mathbb{R})$ so that $\phi_0(t) = 1$ for $|t| \le 1/2$ and $\phi_0(t) = 0$ for $|t| > 3/4$. Define $\phi_\ell(t) = \phi_0(2^\ell t) - \phi_0(2^{\ell-1}t)$ for $\ell \ge 1$ and

$$m_{\lambda,\ell}(\xi) = \phi_\ell(1-\rho(\xi))(1-\rho(\xi))_+^\lambda$$

for $\ell \ge 0$. By Lemma 3.1 $\|\mathcal{F}^{-1}[m_{\lambda,\ell}]\|_1 = O(2^\ell)$ for $\ell \ge 0$ and also

$$\begin{aligned} &\int_{|x|\approx 2^k} \sup_{1/2\le t\le 2} |\mathcal{F}^{-1}[m_{\lambda,\ell}](tx)| \lesssim 2^{\ell-\frac{k}{2}} \\ &\int \sup_{1/2\le t\le 2} |\nabla\mathcal{F}^{-1}[m_{\lambda,\ell}](tx)| dx \lesssim 2^\ell \end{aligned} \tag{3.30}$$

This estimate is used for small ℓ; the statement about the gradient follows since $m_{\ell,\lambda}$ has compact support.

To improve the estimate for large ℓ, say $\ell \ge 10M$, we need to introduce a further decomposition, refining the one in §2. It suffices to consider $\chi m_{\ell,\lambda}$ where χ is supported in the half strip $\mathfrak{S}_u$ as defined in (2.2); without loss of generality $u = (0,1)$.

We fix $\delta = 2^{-\ell}$ and refine the partition $\mathfrak{A}_u(2^{-\ell}) = \{a_0, a_1, \dots, a_Q\}$ (for notational simplicity we do not indicate the dependence of this decomposition on ℓ). Define

$$a_{j,\nu} = \begin{cases} a_{j+1} - 2^{-\nu-1}(a_{j+1} - a_j) & \text{if } \nu = 1, \dots, 2M + \ell - 1 \\ \frac{1}{2}(a_{j+1} + a_j) & \text{if } \nu = 0 \\ a_j + 2^{-|\nu|-1}(a_{j+1} - a_j) & \text{if } \nu = -2M - \ell + 1, \dots, -1; \end{cases}$$

also set $a_{j,2M+\ell} = a_{j+1}$, $a_{j,-2M-\ell} = a_j$. Let $I^\ell_{j,\nu} = [a_{j,\nu}, a_{j,\nu+1}]$, $\nu = -2M - \ell, \dots, 2M + \ell - 1$. Note that two consecutive intervals $I^\ell_{j,\nu}$ have comparable length. Moreover if I is the union of such two consecutive intervals, then I satisfies the hypothesis (3.12) of Proposition 3.2. In order to see this simply note that $(s-t)(\gamma_R'(s) - \gamma_L'(t)) \le 2^{-\ell}$ for $t < s$; $t, s \in (a_j, a_{j+1})$; moreover if I denotes the union of two subsequent closed subintervals $I^\ell_{j,\nu}$, both of them contained in (a_j, a_{j+1}) then the associated interval I^* (blown up by a factor of $4/3$) is contained in (a_j, a_{j+1}). In the remaining case, if one of the two subsequent intervals contains a_j or a_{j+1}; then the length of I^* is $\le 2^{-2M-\ell+2}|a_{j+1} - a_j|$ and therefore in this case the quantity $(s-t)(\gamma_R'(s) - \gamma_L'(t))$ can be estimated by $2\|\gamma'\|_\infty 2^{-2M+2-\ell} < 2^{-\ell}$.

It is now straightforward to construct C^∞ functions $\beta^\ell_{j,\nu}$ so that each $\beta^\ell_{j,\nu}$ is supported in the union of two consecutive intervals containing $a_{j,\nu}$ so that

$$\sum_{j,\nu} [\beta^\ell_{j,\nu}(t)]^2 = 1, \qquad |t| \le 1$$

and

$$\Big|\Big(\frac{d}{dt}\Big)^n \beta^\ell_{j,\nu}(t)\Big| \le C_0 |I^\ell_{j,\nu}|^{-n}, \quad n = 1, 2, 3, 4.$$

Define $S^\ell_{j,\nu}$ by $\widehat{S^\ell_{j,\nu} f}(\xi) = \beta^\ell_{j,\nu}(\xi)\widehat{f}(\xi)$ and $K^\ell_{j,\nu}$ by

$$\widehat{K^\ell_{j,\nu}}(\xi) = \beta^\ell_{j,\nu}(\xi) b(\xi) m_{\ell,\lambda}(\xi);$$

then $\mathcal{F}^{-1}[b m_{\ell,\lambda}] * f = \sum_{j,\nu} K^\ell_{j,\nu} * S^\ell_{j,\nu} f$. The L^p operator norm of $S^\ell_{j,\nu}$ is uniformly bounded in ℓ, j, ν. After renormalization we may apply Proposition 3.2 to get that

$$\|K^\ell_{j,\nu}\|_1 \lesssim (1+\ell) 2^{-\ell\lambda}. \tag{3.31}$$

For fixed ℓ the sum in ν contains less than $2^{4M+\ell}$ terms; hence

$$\begin{aligned} \big\|F^{-1}[\chi m_{\ell,\lambda}]\big\|_1 &\lesssim (1+\ell)^2 Q_u(2^{-\ell}) 2^{-\ell\lambda} \\ &\lesssim (1+\ell)^3 N(2^{-\ell}, \Omega) 2^{-\ell\lambda} \end{aligned} \tag{3.32}$$

by Lemma 2.3. Now the asserted L^1 bound for $\lambda > \kappa_\Omega$ follows from

$$N(2^{-\ell}, \Omega) \le C_\varepsilon 2^{\ell(\kappa_\Omega + \varepsilon)} \tag{3.33}$$

by definition of κ_Ω.

We now define the maximal operator M_ℓ by

$$M_\ell f(x) = \sup_{t>0} |\mathcal{F}^{-1}[m_{\lambda,\ell}(t^{-1}\cdot)] * f(x)|.$$

Standard estimates (see [21, ch. VII], [5]) show that M_ℓ is bounded on L^2 with norm $O(2^{-\ell\lambda}(1+\ell))$. Moreover standard arguments and (3.30) yield

$$\int_{|x|>2|y|} \sup_{t>0} \big|\mathcal{F}^{-1}[m_{\lambda,\ell}(t^{-1}\cdot)](x-y) - \mathcal{F}^{-1}[m_{\lambda,\ell}(t^{-1}\cdot)](x)\big| dx \lesssim 2^\ell;$$

furthermore for $\ell \geq 10M$ we deduce

$$\int_{|x|>2|y|} \sup_{t>0} |\mathcal{F}^{-1}[m_{\lambda,\ell}(t^{-1}\cdot)](x-y) - \mathcal{F}^{-1}[m_{\lambda,\ell}(t^{-1}\cdot)](x)| dx \lesssim (1+\ell)^4 N(2^{-\ell},\Omega) 2^{-\ell\lambda}.$$

from (3.17) and Lemma 2.3. This means that M_ℓ is of weak type (1,1) and more precisely for $\alpha > 0$

$$\big|\{x : |M_\ell f(x)| \geq \alpha\}\big| \lesssim (1+\ell)^4 N(2^{-\ell},\Omega) 2^{-\ell\lambda} \frac{\|f\|_1}{\alpha};$$

cf. [25], [20]. Using the familiar result by Stein and N. Weiss [22] on summing functions in weak L^1 we obtain the weak type $(1,1)$ inequality for maximal function $\sup_{t>0} |\mathcal{F}^{-1}[(1-\rho/t)_+^\lambda] * f|$ for $\lambda > \kappa_\Omega$. Since pointwise convergence holds for Schwartz functions the assertion of Theorem 1.2 for general L^1 functions follows. $\square$

***L^p* estimates.** The L^p estimates

$$\|\mathcal{F}^{-1}[\chi m_{\ell,\lambda} \widehat{f}]\|_p \lesssim 2^{-\ell\lambda}(1+\ell)^{c_p}[N(\Omega, 2^{-\ell})]^{\frac{4}{p}-3}\|f\|_p$$

for $1 < p \leq 4/3$ are obtained by interpolation from the cases $p = 1$ (see (3.32) above) and $p = 4/3$. The $L^{4/3}$ estimate follows by duality from the L^4 estimate

$$\|\mathcal{F}^{-1}[\chi m_{\ell,\lambda}\widehat{f}]\|_4 \lesssim 2^{-\ell\lambda}(1+\ell)^c\|f\|_4 \tag{3.34}$$

for suitable c. We have made no attempt to optimize the power c here; $c = 6$ certainly works but is far from being optimal.

In order to obtain (3.34) it suffices to consider

$$\widetilde{m}(\xi) := \chi(\xi) m_{\ell,\lambda}(\xi) \sum_{j: j\in\mathfrak{a}} [\beta^\ell_{j,\nu(j)}(\xi_1)]^2 \tag{3.35}$$

where $\mathfrak{a}$ is a subset of $\mathfrak{A}_u(2^{-\ell}, r)$, $1 \leq 2^r \leq 2^{M+\ell}$, $u = (0,1)$ so that the property (2.13) is satisfied with $B = 2^M$, and the function $j \mapsto \nu(j)$ takes integer values in $[-2M-\ell+1, 2M+\ell-1]$.

We then have to show that $\widetilde{m}$ is a Fourier multiplier on L^4 with norm $\lesssim 2^{-\ell\lambda}(1+\ell)^3$. Since m is a sum of no more than $O((1+\ell)^3)$ such multipliers the assertion follows.

Let G_j be as in (2.14), with $B = 2^M$. If Σ_ρ is parametrized by $(t,\gamma(t))$ in $\mathfrak{S}_u$ then $1-\rho(\xi) \geq 2^{-M}|\xi_2 - \gamma(\xi_1)|$ and therefore the j^{th} term in the sum (3.35) is supported in G_j. Using Lemma 2.4 we may use the familiar argument from [6], [4] to obtain the estimate

$$\|\mathcal{F}^{-1}[\widetilde{m}\widehat{f}]\|_4 \lesssim \Big\|\Big(\sum_{a_j\in\mathfrak{a}} |K^\ell_{j,\nu(j)} * S^\ell_{j,\nu(j)} f|^2\Big)^{1/2}\Big\|_4.$$

We continue arguing as in Córdoba [4]. By (3.31) and the bound $O((1+\ell)^4)$ for the L^2 norm of the maximal operator $\mathfrak{M}_\ell$ in (3.18) we obtain for nonnegative $\omega \in L^2$

$$\begin{aligned}
&\int \sum_{a_j \in \mathfrak{a}} |K^\ell_{j,\nu(j)} * S^\ell_{j,\nu(j)} f(x)|^2 \omega(x) dx \\
&\lesssim 2^{-2\ell\lambda}(1+\ell)^2 \int \sum_{a_j \in \mathfrak{a}} |S^\ell_{j,\nu(j)} f(x)|^2 \mathfrak{M}_\ell \omega(x) dx \\
&\lesssim 2^{-2\ell\lambda}(1+\ell)^6 \Big\| \Big(\sum_{a_j \in \mathfrak{a}} |S^\ell_{j,\nu(j)} f|^2 \Big)^{1/2} \Big\|_4 \|\omega\|_2
\end{aligned}$$

By Rubio de Francia's theorem on square functions for an arbitrary collection of intervals [16] (or a more elementary version of it where all intervals have comparable length) we know that

$$\Big\| \Big(\sum_{a_j \in \mathfrak{a}} |S^\ell_{j,\nu(j)} f|^2 \Big)^{1/2} \Big\|_4 \lesssim \|f\|_4.$$

Putting these estimates together we deduce that

$$\big\| \mathcal{F}^{-1}[\widetilde{m} \widehat{f}] \big\|_4 \lesssim 2^{-\ell\lambda}(1+\ell)^3 \|f\|_4$$

which implies (3.34) and finishes the proof of the L^p boundedness of $\mathcal{R}_\lambda$. $\square$

Convergence in L^p. Given the uniform boundedness of the operators $\mathcal{R}_{\lambda,t}$ we sketch the routine proof of the convergence result as stated in Theorem 1.1. Denote by $\mathcal{S}_0$ the space of Schwartz-functions with compactly supported Fourier transform; $\mathcal{S}_0$ is dense in L^p if $1 \le p < \infty$ and dense in C^0. Suppose that $g \in \mathcal{S}_0$ so that $\widehat{g}$ is supported where $|\xi| \le R$. Let $\Phi \in C_0^\infty(\mathbb{R})$ so that $\Phi(s) = 1$ if $|s| \le 1/2$, $\Phi(s) = 0$ if $|s| > 3/4$. Define $S_{\lambda,t}$ by

$$\widehat{S_{\lambda,t} f}(\xi) = \Phi(\rho(\xi)/t)(1 - \rho(\xi)/t)_+^\lambda \widehat{f}(\xi);$$

then $S_{\lambda,t} g = \mathcal{R}_{\lambda,t} g$ for $t \ge 2R$. By Lemma 3.1 the convolution kernel of $S_{\lambda,t}$ is an L^1 kernel, for all $\lambda \in \mathbb{R}$, and the family $\{S_{\lambda,t}\}$ is a standard approximation of the identity. Therefore $S_{\lambda,t} g \to g$ uniformly, and in L^p, $1 \le p < \infty$. For general $f \in L^p$ (or C_0) the convergence result follows by approximating f by functions in $\mathcal{S}_0$ and the uniform boundedness of the operators $\mathcal{R}_{\lambda,t}$. $\square$

4. Examples

Given two parameters $\eta \in (0,\infty)$, $\alpha \in (0,1)$ we consider a convex domain $\Omega = \Omega(\eta,\alpha)$ with $C^{1,\alpha}$ boundary so that $\kappa_\Omega = \max\{\frac{\alpha}{\alpha+1}, \frac{\eta}{2(1+\eta)}\}$ for which Theorem 1.1 is sharp. We may think of Ω as a polygonal region with infinitely many vertices; however near the vertices the boundary is regularized using primitives of suitable Lebesgue functions of class C^α.

The set Ω is contained in $\{x : 4 \le |x| \le 8\}$ and symmetric with respect to the reflections $(x_1,x_2) \mapsto (x_1,-x_2)$ and $(x_1,x_2) \mapsto (-x_1,x_2)$. The portion of the boundary which lies in $\{x : |x_1| > 1, |x_2| > 1\}$ is given by segments of the lines $x_2 = \pm 8 \pm x_1$. It is then enough to parametrize the boundary in $\{x : |x_1| \le 4, x_2 < 0\}$ by an even convex function γ with $\gamma(0) = -15/2$, so that $\gamma(t) = -8 + t$ for $1 \le t \le 4$.

Fix $\eta > 0$ and define for $k \ge 0$

$$\mu_k = 1 + [e^{\log^2(1+k)} 2^{k\eta}] \tag{4.1}$$

where $[x]$ denotes the largest integer $\leq x$.

We define a doubly indexed sequence $x_{j,k}$ by

$$x_{j,k} = 2^{-k} - \frac{j}{2^{k+1}\mu_k};$$

here k is a nonnegative integer and $j = 0, \dots, \mu_k - 1$. We also define

$$x_{\mu_k,k} = x_{0,k+1} \; (= 2^{-k-1}) \; .$$

Note that $x_{j,k+1} \leq x_{j',k}$ if $j' \leq \mu_k$, $j \geq 0$, and $x_{j+1,k} < x_{j,k}$ if $0 \leq j < \mu_k$.

Let

$$\sigma_{j,k} = \frac{x_{j,k} + x_{j+1,k}}{2} = 2^{-k} - \frac{2j+1}{2} 2^{-k-1}\mu_k^{-1}, \quad j = 0, \dots, \mu_k - 1;$$
$$\sigma_{\mu_k,k} = \sigma_{0,k+1} = 2^{-k-1} - \frac{1}{2} 2^{-k-2}\mu_{k+1}^{-1},$$

Then $\sigma_{j,k}$ is the slope of the secant connecting the points $(x_{j+1,k}, \frac{x_{j+1,k}^2}{2})$ and $(x_{j,k}, \frac{x_{j,k}^2}{2})$, and it is of course also the midpoint of the interval $[x_{j+1,k}, x_{j,k}]$. On a substantial portion of this interval containing the midpoint we shall define γ so that its graph coincides with the secant, and near the endpoints we shall replace it by a more regular $C^{1,\alpha}$ function. We also set

$$\sigma_{-1,k+1} = \sigma_{\mu_k-1,k};$$

moreover

$$\tau_{j,k} = \sigma_{j,k} - \sigma_{j+1,k}, \quad j = 0, \dots, \mu_k - 1.$$
$$\tau_{-1,k+1} = \sigma_{\mu_k-1,k} - \sigma_{0,k+1}.$$

Note that for $0 \leq j \leq \mu_k - 1$ the expression $\tau_{j,k}$ is actually independent of j, namely equal to $2^{-k-1}\mu_k^{-1}$.

We further split the interval $[x_{j+1,k}, x_{j,k}]$ using points $x_{j+1,k} < d_{j,k} < \sigma_{j,k} < b_{j,k} < x_{j,k}$ where

$$b_{j,k} = x_{j,k} - 2^{-k-3}\mu_k^{-1} \qquad \text{if } 0 \leq j \leq \mu_k - 1$$
$$d_{j,k} = x_{j,k} - 3 \cdot 2^{-k-3}\mu_k^{-1} \quad \text{if } 0 \leq j < \mu_k - 1$$

and

$$d_{\mu_k-1,k} \equiv d_{-1,k+1} = x_{\mu_k-1,k} + 2^{-k-2}\mu_k^{-1} - 2^{-k-4}\mu_{k+1}^{-1} = \sigma_{\mu_k-1,k} - 2^{-k-4}\mu_{k+1}^{-1}.$$

One may then verify that for $0 \leq j \leq \mu_k - 1$

$$\frac{b_{j,k} + d_{j,k}}{2} = \sigma_{j,k} \quad \text{and} \quad \frac{b_{j,k} + d_{j-1,k}}{2} = \sigma_{j,k} + \frac{\tau_{j-1,k}}{2}. \tag{4.2}$$

Let g_α be the Lebesgue function on $[0,1]$ associated to the symmetrical perfect sets of Cantor type, with constant ratio of dissection $= 2^{-1/\alpha}$ (see Zygmund [26, ch.V, 3]; the dissections are of type $[2; 0, 1-2^{-1/\alpha}, 2^{-1/\alpha}]$ in the notation of [26]). Note that g_α is a monotone function on $[0,1]$ with

$$g_\alpha(0) = 0, \qquad g_\alpha(1) = 1;$$
$$\int_0^1 g_\alpha(t)dt = \frac{1}{2}; \tag{4.3}$$

the integral can be evaluated since $g_\alpha(1/2 + s) - 1/2 = 1/2 - g_\alpha(1/2 - s)$ for $0 \leq s \leq 1/2$.

Note that $b_{0,0} = 15/16$. On the interval $[0, 15/16]$ we define γ by

$$(4.4)\qquad \gamma(t) = \begin{cases} -\frac{15}{2} + \frac{d_{j,k}^2}{2} + \sigma_{j,k}(t - d_{j,k}) & \text{if } d_{j,k} \le t \le b_{j,k} \\ -\frac{15}{2} + \frac{b_{j,k}^2}{2} + \sigma_{j,k}(t - b_{j,k}) + \tau_{j-1,k} \int_{b_{j,k}}^{t} g_\alpha(\frac{s - b_{j,k}}{d_{j-1,k} - b_{j,k}}) ds & \text{if } b_{j,k} \le t \le d_{j-1,k} \end{cases}$$

On $[15/16, 4]$ we define

$$(4.5)\qquad \gamma(t) = \begin{cases} -\frac{15}{2} + \frac{225}{512} + \frac{7}{8}(t - \frac{15}{16}) + \frac{1}{256}\int_0^{32t-30} g_\alpha(s)ds & \text{if } \frac{15}{16} \le t \le \frac{31}{32} \\ -8 + t & \text{if } \frac{31}{32} \le t \le 4 \end{cases}$$

One verifies that for $t < 15/16$ the function γ is C^1 with $\gamma(b_{j,k}) = -15/2 + b_{j,k}^2/2$, $\gamma'(b_{j,k}) = \sigma_{j,k}$, $\gamma'(d_{j,k-1}) = \tau_{j-1,k} + \sigma_{j,k} = \sigma_{j-1,k}$ and in view of (4.2), (4.3) also $\gamma(d_{j-1,k}) = -15/2 + d_{j-1,k}^2/2$. Moreover, since $b_{0,0} = 15/16$ and $\sigma_{0,0} = 7/8$ it is easily checked that (4.4) and (4.5) together define a C^1 function on $[0, 4]$.

The function g_α belongs to $C^\alpha([0,1])$ (see [26, p.197]); from this it easily follows that actually $\gamma \in C^{1,\alpha}$ in $[0, 4]$; in fact

$$(4.6)\qquad |\gamma'(t_1) - \gamma'(t_2)| \lesssim \begin{cases} (2^{-k}\mu_k^{-1})^{1-\alpha}|t_1 - t_2|^\alpha & \text{if } |t_1 - t_2| \le 2^{-k}\mu_k^{-1} \\ |t_1 - t_2| & \text{if } |t_1 - t_2| \ge 2^{-k}\mu_k^{-1} \end{cases}$$

if $t_1, t_2 \approx 2^{-k}$.

We now estimate the covering numbers $N(\Omega, \delta)$, for small δ. Let m be so that

$$(4.7)\qquad 2^{-m(1+\eta)} e^{-\log^2(1+m)} \approx \delta^{1/2};$$

then we can cover the graph over $[0, 2^{-m}]$ with

$$\approx e^{\log^2(1+m)} 2^{m\eta} \le C_\varepsilon \delta^{-\frac{\eta}{2(1+\eta)} - \varepsilon}$$

adjacent rectangles with sidelengths $(\delta^{1/2}, \delta)$; here $\varepsilon > 0$. Moreover there are

$$(4.8)\qquad \approx \sum_{k \le m} e^{\log^2(1+k)} 2^{k\eta} \lesssim e^{\log^2(1+m)} 2^{m\eta}$$

points $x_{j,k}$ in $[2^{-m}, 1]$. Therefore if $\widetilde{\Omega}$ denotes the polygon with symmetry about the x_1 and x_2 axes interpolating the points $(x_{j,k}, x_{j,k}^2/2)$ then $N(\widetilde{\Omega}) \lesssim \delta^{-\frac{\eta}{2(1+\eta)} - \varepsilon}$. Similarly one can obtain the appropriate lower bound to see that $\kappa_{\widetilde{\Omega}} = \frac{\eta}{2(1+\eta)}$. Note that the drawback of working with $\widetilde{\Omega}$ is that the boundary is merely Lipschitz. To remedy this situation we interpolated using the Cantor-Lebesgue functions. Since the covering numbers over the interval $[2^{-m}, 1]$ may now increase we shall have to impose the restriction $\frac{\alpha}{\alpha+1} \le \frac{\eta}{2(1+\eta)}$.

Fix an interval $[b_{j,k}, d_{j-1,k}]$ and $n > 0$. Then there are $\approx 2^n$ subintervals $I_{s,n}$ of length $(2^{-1/\alpha})^n 2^{-k-1}\mu_k^{-1}$ so that $|(\gamma'(t_1) - \gamma'(t_2))(t_1 - t_2)| \lesssim 2^{-n(1+\frac{1}{\alpha})}(2^{-k-1}\mu_k^{-1})^2$ for $t_1, t_2 \in I_{s,n}$ and so that γ' is constant on the complimentary intervals; here we used (4.6). Given small δ find n so that $2^{-n(1+\frac{1}{\alpha})}(2^{-k-1}\mu_k^{-1})^2 \approx \delta$; then $2^n \approx (2^{2k}\mu_k^2\delta)^{-\frac{\alpha}{\alpha+1}}$. Therefore the sum in (4.8) is now replaced by

$$\sum_{k \le m} e^{\log^2(1+k)} 2^{k\eta} (2^{2k(1+\eta)} e^{2\log^2(1+k)} \delta)^{-\frac{\alpha}{\alpha+1}} \le \begin{cases} C\delta^{-\frac{\alpha}{\alpha+1}} & \text{if } \frac{\alpha}{\alpha+1} > \frac{\eta}{2+2\eta} \\ C_\varepsilon \delta^{-\frac{\alpha}{\alpha+1} - \varepsilon} & \text{if } \frac{\alpha}{\alpha+1} = \frac{\eta}{2+2\eta} \\ C_\varepsilon \delta^{-\frac{\eta}{2\eta+2} - \varepsilon} & \text{if } \frac{\alpha}{\alpha+1} < \frac{\eta}{2+2\eta} \end{cases}.$$

This implies in particular that

$$(4.9)\qquad \kappa_\Omega \le \max\{\frac{\eta}{2 + 2\eta}, \frac{\alpha}{\alpha + 1}\};$$

in fact a more careful examination of the previous argument would show that (4.9) holds with equality.

Lower bounds. Let $\phi \in C^\infty(\mathbb{R})$ supported in $(1/2, 2)$ so that $\widehat{\phi}(\tau) \geq 1$ for $|\tau| \leq 2^{10}$. Let T_ℓ be defined by

$$\widehat{T_\ell f}(\xi) = \phi(2^\ell(1-\rho(\xi)))\widehat{f}(\xi) \tag{4.10}$$

where ρ is the Minkowski functional associated to the set $\Omega = \Omega(\eta, \alpha)$ defined above.

Lemma 4.1. *The following holds for large positive k. If $\ell = \ell(k)$ is chosen so that*

$$2^{-\ell(k)} < (2^k \mu_k)^{-2} < 2^{-\ell(k)+1} \tag{4.11}$$

then there is $c > 0$ (independent of k) so that

$$\|T_{\ell(k)}\|_{L^p \to L^p} \geq c\mu_k^{\frac{4}{p}-3} k^{1-\frac{2}{p}}$$

for $p \geq 1$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$ so that $\chi(s) = 0$ for $|s| > 2$ and $\chi(s) = 1$ for $|s| \leq 1$. Then $|\widehat{\chi}(\tau)| \leq C_0(1+|\tau|)^{-2}$ and $\mathrm{Re}(\widehat{\chi}(\tau)) > 1/2$ for $|\tau| \leq 2{-}R$ with suitable $R > 10$. Fix a small $\varepsilon > 0$ and a large integer L, and assume that $\ell(k) \gg 100L$.

We test $T_{\ell(k)}$ on the function f_k defined by

$$\widehat{f_k}(\xi) = \chi(2^{2k}(\xi_2 + \tfrac{15}{2})) \sum_{\nu=1}^{N_k} \chi(2^{\frac{\ell(k)}{2}+R}(\xi_1 - \sigma_{L\nu,k}))$$

where $N_k = [\mu_k/10L]$. $\widehat{f_k}$ is bounded and supported on a rectangle of sidelengths $c_0 2^{-k}$ and 2^{-2k}. One may think of $\widehat{f_k}$ as a modified bump function; however we localize to tiny strips containing the lines $\xi_1 = \sigma_{j,\ell(k)}$ where $j = 0 \mod L$.

Clearly $\|f_k\|_2 \lesssim 2^{-3k/2}$, by Plancherel's theorem. Moreover

$$|f_k(x)| \lesssim \frac{2^{-2k}}{(1+2^{-2k}|x_2|)^2} \frac{2^{-R-\ell(k)/2}}{(1+2^{-R-\ell(k)/2}|x_1|)^2} \Big| \sum_{\nu=1}^{N_k} e^{i\nu L 2^{-k-1}\mu_k^{-1} x_1} \Big|$$

and the geometric sum is dominated by $\min\{N_k, |e^{iL2^{-k-1}\mu_k^{-1}x_1} - 1|^{-1}\}$. Since $2^{-k-1}\mu_k^{-1} \approx 2^{-\ell(k)/2}$ and $\log N_k \approx k$ a straightforward computation shows that $\|f_k\|_1 \lesssim k$. By interpolation

$$\|f_k\|_p \lesssim k^{-1+2/p} 2^{-3k/p'}, \qquad 1 \leq p \leq 2. \tag{4.12}$$

Since $\sigma_{j,k}$ is the midpoint of $[x_{j+1,k}, x_{j,k}]$ only the definition of γ in $[d_{j,k}, b_{j,k}]$ will be relevant in computing $T_{\ell(k)} f_k$. We write out the Fourier integral for $T_{\ell(k)} f_k$ and introduce homogeneous coordinates $\xi = s(t, \gamma(t))$. Set $\chi_{\nu,k}(s) = \chi(2^{R+\frac{\ell(k)}{2}}(s - \sigma_{L\nu,k})))$. Notice $\gamma(t) = -\frac{15}{2} + \frac{d_{L\nu,k}^2}{2} + \sigma_{L\nu,k}(t - d_{L\nu,k})$ if $st \in \mathrm{supp}\chi_{\nu,k}$ and that $t\gamma'(t) - \gamma(t) = g_{\nu,k}$ where the constant $g_{\nu,k}$ satisfies $7 \leq g_{\nu,k} \leq 8$. Then

$$\begin{aligned}(2\pi)^2 T_{\ell(k)} f_k(x) &= \iint \phi(2^{\ell(k)}(1-s)) \sum_{\nu=1}^{N_k} g_{\nu,k}\chi_{\nu,k}(st) e^{is(x_1 t + x_2\gamma(t))} ds dt \\ &= \sum_{\nu=1}^{N_k} g_{\nu,k} F_{\nu,k}(x) \qquad (4.13)\end{aligned}$$

where

$$F_{\nu,k}(x) = \int \frac{\phi(2^{\ell(k)}(1-s))}{s} e^{isx_2(\frac{d^2_{L\nu,k}}{2} - \sigma_{L\nu,k} d_{L\nu,k} - \frac{15}{2})} ds \int \chi(2^{R+\ell(k)/2}(u - \sigma_{L\nu,k})) e^{i(x_1 u + x_2 \sigma_{L\nu,k} u)} du.$$

In the first integral we expand $1/s = 1 + (1-s)/s$ and obtain that

$$F_{\nu,k}(x) = \Gamma_{k,\nu}(x)\, A_{\ell(k)}(x_1 + x_2\sigma_{L\nu,k}) \big(B_{\nu,k}((\tfrac{d^2_{L\nu,k}}{2} - \sigma_{L\nu,k} d_{L\nu,k} - \tfrac{15}{2})x_2) + E_k(x_2)\big)$$

where

$$\begin{aligned} A_\ell(\tau) &= 2^{-R-\frac{\ell}{2}} \widehat{\chi}(-2^{-R-\frac{\ell}{2}}\tau) \\ B_{\nu,k}(\tau) &= 2^{-\ell(k)} \widehat{\phi}(-2^{-\ell(k)}\tau) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &|\Gamma_{k,\nu}(x)| = 1 \\ &|E_k(x_2)| \le 2\|\phi\|_1 2^{-2\ell(k)}. \end{aligned} \tag{4.15}$$

We derive an estimate for x in

$$S_{\nu,k} = \{x : 2^{\ell(k)+2} \le |x_2| \le 2^{\ell(k)+3}, \quad |x_1 + \sigma_{L\nu,k} x_2| \le 2^{\ell(k)/2}\}.$$

From (4.14-15) we see that

$$|F_{\nu,k}(x)| \ge \frac{1}{2} 2^{-3\ell(k)/2}, \quad \text{if } x \in S_{\nu,k}. \tag{4.16}$$

If $x \in S_{\nu',k}$ for $\nu \neq \nu'$ then $|x_1 + \sigma_{L\nu,k}x_2| \ge |x_2(\sigma_{L\nu,k} - \sigma_{L\nu',k})| - O(2^{\ell(k)/2})$. Therefore, if L is sufficiently large,

$$|F_{\nu',k}(x)| \lesssim 2^{-3\ell(k)/2}(L|\nu - \nu'|)^{-2}, \quad \text{if } x \in S_{\nu,k}. \tag{4.17}$$

and we see from (4.16-17)

$$\|T_{\ell(k)} f_k\|_p \ge c 2^{-3\ell(k)/2p'} N_k^{1/p} \ge c'(2^k \mu_k)^{-3/p'} \mu_k^{1/p}. \tag{4.18}$$

Comparing (4.12) and (4.18) we obtain the asserted lower bound for the L^p operator norm of $T_{\ell(k)}$. □

Proof of Theorem 1.3. For given $\kappa \in (0, 1/2)$ choose $\eta = \frac{2\kappa}{1-2\kappa}$ and $\alpha = \frac{\kappa}{1-\kappa}$, and define $\Omega_\kappa = \Omega(\eta, \alpha)$ as above. Note that $\kappa = \frac{\eta}{2+2\eta} = \frac{\alpha}{\alpha+1}$ so that by formula (4.9) and Theorem 1.1 we know that $\mathcal{R}_\lambda$ is bounded on L^p if $1 < p < 4/3$ and $\lambda > \kappa(-3 + 4/p)$.

For the converse fix $\lambda > 0$ and *assume* that $\mathcal{R}_\lambda$ is bounded on L^p. Let ϕ be as in (4.10). For $m(s) = \phi(2^\ell(1-s))$ we use the familiar formula

$$m(\rho) = \frac{(-1)^{[\lambda]+1}}{\Gamma(\lambda+1)} \int_0^\infty s^\lambda m^{(\lambda+1)}(s) \big(1 - \frac{\rho}{s}\big)_+^\lambda ds \tag{4.19}$$

where the derivative is defined by $\widehat{m^{(\gamma)}}(\tau) = (-1)^{[\gamma]}(-i\tau)^\gamma \widehat{m}(\tau)$; see [23] for the proof of (4.19) for fractional λ.

A scaling argument shows that for $\lambda+1>0$

$$\int_0^\infty s^\lambda \big|[\phi(2^\ell(1-\cdot))]^{(\lambda+1)}(s)\big| ds \lesssim 2^{\ell\lambda}$$

and the assumed boundedness of $\mathcal{R}_\lambda$ and dilation invariance implies that

$$\|T_{\ell(k)}\|_{L^p\to L^p} \lesssim 2^{\lambda\ell(k)}.$$

By Lemma 4.1 it follows that $\mu_k^{-3+4/p} k^{1-2/p} \lesssim (2^k\mu_k)^{2\lambda}$ for large positive k. Taking into account the definition of μ_k and η, it follows that

$$k^{1-\frac{2}{p}} e^{(\frac{4}{p}-3-2\lambda)\log^2(1+k)} \lesssim 4^{\frac{k}{1-2\kappa}(\lambda+\kappa(3-\frac{4}{p}))}. \tag{4.20}$$

Note that $(4p^{-1}-3-2\lambda) = (4p^{-1}-3)(1-2\kappa)$ in the critical case $\lambda=\kappa(-3+4/p)$. Since we assume $p<4/3$ and $\kappa<1/2$ the necessity of the condition $\lambda>\kappa(-3+4/p)$ follows from (4.20). □

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Andreas Seeger, University of Wisconsin-Madison, WI 53706
E-mail address: seeger@math.wisc.edu

Sarah Ziesler, Dominican University, River Forest, IL 60305
E-mail address: ziessara@email.dom.edu