RADON TRANSFORMS AND FINITE TYPE CONDITIONS

Andreas Seeger

1. Introduction.

The purpose of this paper is to study averaging operators of Radon transform type. We shall formulate suitable finite type conditions and prove L^p-Sobolev and $L^p \to L^q$ estimates. The results will be essentially sharp for operators associated with families of curves in the plane.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be smooth manifolds, $\dim \mathfrak{X} = n_L$, $\dim \mathfrak{Y} = n_R$ and let $\mathcal{M}$ be a submanifold in $\mathfrak{X} \times \mathfrak{Y}$ with conormal bundle $N^*\mathcal{M}$; we denote by ℓ the codimension of $\mathcal{M}$. We shall always assume that the projections $\pi_{\mathfrak{X}} : \mathcal{M} \to \mathfrak{X}$ and $\pi_{\mathfrak{Y}} : \mathcal{M} \to \mathfrak{Y}$ are surjective with rank $D\pi_{\mathfrak{X}} = n_L$, rank $D\pi_{\mathfrak{Y}} = n_R$. This in particular implies that $N^*\mathcal{M} \subset T^*\mathfrak{X} \setminus 0 \times T^*\mathfrak{Y} \setminus 0$ where 0 refers to the zero sections in $T^*\mathfrak{X}$ and $T^*\mathfrak{Y}$, respectively. This is the usual assumption for the canonical relation associated with Lagrangian distributions arising as kernels of Fourier integral operators.

The assumptions on $D\pi_{\mathfrak{X}}$, $D\pi_{\mathfrak{Y}}$ imply that for fixed $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$ the sets

$$
(1.1) \qquad
\begin{aligned}
\mathcal{M}_x &= \{ y \in \mathfrak{Y}; (x, y) \in \mathcal{M} \} \\
\mathcal{M}^y &= \{ x \in \mathfrak{X}; (x, y) \in \mathcal{M} \}
\end{aligned}
$$

are smooth immersed submanifolds of codimension ℓ in $\mathfrak{Y}$ and $\mathfrak{X}$, respectively.

Let $\chi \in C^\infty(\mathfrak{X} \times \mathfrak{Y})$ be compactly supported. We shall study the regularity of the averaging operator (or generalized Radon transform) given by

$$
(1.2) \qquad Rf(x) = \int_{\mathcal{M}_x} \chi(x, y) f(y) \, d\sigma_x(y);
$$

here $d\sigma_x$ is a smooth density on $\mathcal{M}_x$ depending smoothly on x.

The averaging operator R is a Fourier integral operator and its distribution kernel belongs to the class $I^m(\mathfrak{X}, \mathfrak{Y}, N^*\mathcal{M})$ with $m = \ell/2 - (n_L + n_R)/4$ (cf. [10]). Sharp estimates are well known in the case where the projections $\pi_L : N^*\mathcal{M} \to T^*\mathfrak{X}$, $\pi_R : N^*\mathcal{M} \to T^*\mathfrak{Y}$ are locally diffeomorphisms ([10]); then necessarily $n_L = n_R$. There has been considerable interest in obtaining sharp estimates in the degenerate case where this assumption is relaxed (see *e.g.* [4], [5], [6], [7], [8], [12], [13], [14], [15], [16], [17], [19], [20], [21]).

Finite type conditions may be formulated in terms of vector fields tangent to both $\mathcal{M}$ and $\mathfrak{X}$, or to both $\mathcal{M}$ and $\mathfrak{Y}$, and the commutatators of such vector fields. We use the notation

$$
\begin{aligned}
T_P^{1,0}\mathcal{M} &= T_P\mathcal{M} \cap T_P(\mathfrak{X} \times \{0\}) \\
T_P^{0,1}\mathcal{M} &= T_P\mathcal{M} \cap T_P(\{0\} \times \mathfrak{Y});
\end{aligned}
$$

so if $P = (x, y) \in \mathcal{M}$ then $T_P^{1,0}\mathcal{M}$ is tangent to $\mathcal{M}^y \times \{0\}$ and $T_P^{0,1}\mathcal{M}$ is tangent to $\{0\} \times \mathcal{M}_x$. Note that $\dim T_P^{1,0}\mathcal{M} = n_L - \ell$, $\dim T_P^{0,1}\mathcal{M} = n_R - \ell$ and $\dim T_P\mathcal{M} = n_L + n_R - \ell$. The finite type conditions will measure to which extent the subbundle $T^{1,0}\mathcal{M} \oplus T^{0,1}\mathcal{M}$ fails to be involutive.

Given the calculus of wavefront sets ([10]) it is also natural to formulate finite type conditions in terms of the twisted normal bundle or canonical relation

$$\mathcal{C} = (N^*\mathcal{M})' = \{(x, \xi, y, -\eta) : (x, \xi, y, \eta) \in N^*\mathcal{M}\}.$$

In fact $(N^*\mathcal{M})'$ can be identified with a subbundle $T^{*,\perp}\mathcal{M}$ of $T^*\mathcal{M}$ whose fiber at $P = (x, y) \in \mathcal{M}$ is

$$T_P^{*,\perp}\mathcal{M} = (T_P^{1,0}\mathcal{M} \oplus T_P^{0,1}\mathcal{M})^\perp,$$

the ℓ-dimensional space of all linear functionals in $T_P^*\mathcal{M}$ which annihilate vectors in $T_P^{1,0}\mathcal{M}$ and vectors in $T_P^{0,1}\mathcal{M}$. The identification is via restricting linear forms in $T_P^*(\mathfrak{X} \times \mathfrak{Y})$ to tangent vectors in $T_P\mathcal{M}$. More precisely if $\imath : \mathcal{M} \to \mathfrak{X} \times \mathfrak{Y}$ denotes the inclusion map and $\imath^*$ the pullback of $\imath$ acting on forms in $T^*(\mathfrak{X} \times \mathfrak{Y})$, then

$$T^{*,\perp}\mathcal{M} = \{(P, \imath_P^*\lambda) : (P, \lambda) \in \mathcal{C}\}$$

where $P = (x, y) \in \mathcal{M}$ and $\lambda = (\xi, -\eta) \in (N_P^*\mathcal{M})'$; see the discussion in §2.

Given vector fields V, W we denote the Lie-derivative of W with respect to V (or Lie product of V and W) by $[V, W]$; in coordinates it is identified with the commutator of the vector fields V and W. As customary in Lie-theory we shall also use the notation $\mathrm{ad}V(W) := [V, W]$. We first introduce the notion of a *vector field of type* $\preceq (\mu, \nu)$. Here $\prec$ refers to the partial order on pairs of integers defined by $(j, k) \prec (j', k')$ if and only if either $j \le j'$ and $k < k'$ or $j < j'$ and $k \le k'$. We say that $(j, k) \preceq (j', k')$ if $(j, k) \prec (j', k')$ or $(j, k) = (j', k')$.

Definition. *Let $\mathcal{U}$ be an open set in $\mathcal{M}$.*

(i) A vector field V is of type $(1, 0)$ in $\mathcal{U}$ if $V_P \in T_P^{1,0}\mathcal{M}$ for all $P \in \mathcal{U}$.

(ii) A vector field V is of type $(0, 1)$ in $\mathcal{U}$ if $V_P \in T_P^{0,1}\mathcal{M}$ for all $P \in \mathcal{U}$.

(iii) We set $\mathcal{V}^{(0,0)}(\mathcal{U}) = \{0\}$ and let $\mathcal{V}^{1,0}$, $\mathcal{V}^{0,1}$ be the $C^\infty(\mathcal{U})$-modules of all vector fields of type $(1, 0)$, $(0, 1)$, respectively. For a pair (μ, ν) of positive integers let $\mathcal{V}^{\mu,\nu}(\mathcal{U})$ be the $C^\infty(\mathcal{U})$-module generated by all vector fields of type $(1, 0)$ and $(0, 1)$ and all vector fields of the form $g\,\mathrm{ad}V_1 \cdots \mathrm{ad}V_{n-1}(V_n)$, where $g \in C^\infty(\mathcal{U})$, each V_i is of type $(1, 0)$ or of type $(0, 1)$, at most μ of the V_i are of type $(1, 0)$ and at most ν of the V_i are of type $(0, 1)$.

(iv) If $V \in \mathcal{V}^{\mu,\nu}(\mathcal{U})$ then we say that V is of type $\preceq (\mu, \nu)$ in $\mathcal{U}$. If $V \in \mathcal{V}^{\mu,\nu}(\mathcal{U})$ but $V \notin \mathcal{V}^{\mu',\nu'}(\mathcal{U})$ for any $(\mu', \nu') \prec (\mu, \nu)$ then we say that V is of type (μ, ν) in $\mathcal{U}$.

(v) Let $P \in \mathcal{M}$. Two vector fields V and $\widetilde{V}$ are said to be equivalent at P if there is a neighborhood $\mathcal{U}$ so that V and $\widetilde{V}$ coincide in $\mathcal{U}$. The set of equivalence classes of vector fields (or germs) at P is denoted by $\mathcal{V}(P)$. The submodule $\mathcal{V}^{\mu,\nu}(P)$ consists of the equivalence classes for which any representative is in $\mathcal{V}^{\mu,\nu}(\mathcal{U})$, for some suitable $\mathcal{U}$. By abuse of notation we shall say that a vector field V defined on $\mathcal{M}$ near P belongs to $\mathcal{V}^{\mu,\nu}(P)$ if $V \in \mathcal{V}^{\mu,\nu}(\mathcal{U})$ for some neighborhood $\mathcal{U}$ of P.

We remark that the notion of type (μ, ν) is invariant under special changes of variables of the form $\Psi(x, y) = (\Psi_L(x), \Psi_R(y))$ where Ψ_L is a diffeomorphism in $\mathfrak{X}$ and Ψ_R is a diffeomorphism in $\mathfrak{Y}$. Of course those are just the changes of variables which leave the L^p-Sobolev mapping properties of an operator invariant.

To see the invariance let Ψ^*V be the pullback of V under Ψ defined by $(\Psi^*V)_P = (D\Psi)_P^{-1} V_{\Psi(P)}$. Then the pullback of a vector field of type $(1,0)$ is of type $(1,0)$ (with respect to the manifold $\Psi^{-1}(\mathcal{M})$), in fact $(\Psi^*X)_P = (D\Psi_L \times \mathrm{Id})_P^{-1} X_{\Psi(P)}$ for $P \in \Psi^{-1}\mathcal{M}$. Similarly, the pullback of a vector field of type $(0,1)$ is of type $(0,1)$. Now $\Psi^*[X,Y] = [\Psi^*X, \Psi^*Y]$ and the latter equals $[\Psi_L^*X, \Psi_R^*Y]$, which is a vector field of type $(1,1)$. Inductively we see that the notion of a vector field of type (μ,ν) is invariant under change of variables in $\mathfrak{X}$ and $\mathfrak{Y}$.

Definition. *Let $P \in \mathcal{M}$ and $\lambda \in T_P^*\mathcal{M}$. Then $T^*\mathcal{M}$ is said to be of type (μ,ν) at (P,λ) if there is $V \in \mathcal{V}^{\mu,\nu}(P)$ such that $\langle \lambda, V_P \rangle \neq 0$ and if $\langle \lambda, \widetilde{V}_P \rangle = 0$ for all vector fields $\widetilde{V}$ of type $\prec (\mu,\nu)$. $\mathcal{M}$ is said to be of finite type at P if for every $\lambda \in T_P^*\mathcal{M}$ there are nonnegative integers μ, ν so that $T^*\mathcal{M}$ is of type (μ,ν) at (P,λ).*

By the above comments on the behavior of vector fields of type (μ,ν) under separate changes of variables in $\mathfrak{X}$ and $\mathfrak{Y}$ it is clear that the notion of type (μ,ν) is well defined. Note that $T^*\mathcal{M}$ is always of type $(1,0)$ at (P,λ) if λ does not annihilate vectors in $T_P^{1,0}\mathcal{M}$ and of type $(0,1)$ if λ does not annihilate vectors in $T_P^{0,1}\mathcal{M}$. Thus the finite type condition is only interesting if restricted to $T^{*,\perp}\mathcal{M}$. We note that the type is semicontinuous with respect to the partial order $\prec$; in the sense that if $T^*\mathcal{M}$ is of type (μ,ν) at (P,λ) then there is a neighborhood of (P,λ) in $T^*\mathcal{M}$, conic in the fiber, such that $T^*\mathcal{M}$ is of type $\preceq (\mu,\nu)$ in this neighborhood. The number

$$n(P) = \max_{\lambda \in T_P^*\mathcal{M}} \min\{\mu + \nu : T^*\mathcal{M} \text{ is of finite type } (\mu,\nu) \text{ at } (P,\lambda)\}$$

is called the Hörmander type. This terminology is suggested by conditions in [9]: If $n = n(P) < \infty$, then there are numbers μ_i, ν_i, with $\mu_i + \nu_i \leq n$ and vector fields V_i of type (μ_i, ν_i), $i = 1, \ldots, n_L + n_R - \ell$, so that the $(V_i)_P$ span the tangent space $T_P\mathcal{M}$.

In this paper we emphasize the case $\operatorname{codim} \mathcal{M} = 1$, so that $\mathcal{M}$ is a hypersurface in $\mathfrak{X} \times \mathfrak{Y}$ and the fibers of $T^{*,\perp}\mathcal{M}$ are one-dimensional. Then $T^*\mathcal{M}$ is of type (μ,ν) at (P,λ) for one $\lambda \in (T_P^{1,0}\mathcal{M} \oplus T_P^{0,1}\mathcal{M})^\perp$ if and only if it is of type (μ,ν) for all nonzero $\lambda \in (T_P^{1,0}\mathcal{M} \oplus T_P^{0,1}\mathcal{M})^\perp$. This justifies the following terminology.

Definition. *Suppose that $\mathcal{M}$ is a hypersurface in $\mathfrak{X} \times \mathfrak{Y}$, so that the projections to $\mathfrak{X}$ and $\mathfrak{Y}$ are submersions. Let $P \in \mathcal{M}$, $\mu \geq 1$, $\nu \geq 1$. Then $\mathcal{M}$ is said to be of type (μ,ν) at P if there is a vector field V of type (μ,ν) defined near P and a linear functional $\lambda \in T_P^*\mathcal{M}$ annihilating vectors in $T_P^{1,0}\mathcal{M}$ and vectors in $T_P^{0,1}\mathcal{M}$ so that*

$$\langle \lambda, V_P \rangle \neq 0.$$

Again $\mathcal{M}$ is said to be of finite type at P if $\mathcal{M}$ is of type (μ,ν) at P, for a pair (μ,ν) of positive integers.

We shall now formulate sharp theorems for curves in two dimensional manifolds and generalizations for hypersurfaces $\mathcal{M}$. For every $P \in \mathcal{M}$ let

(1.3) $$\tau(P) = \{(\mu,\nu) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \mathcal{M} \text{ is of type } (\mu,\nu) \text{ at } P.\}$$

Let $\overline{\mathcal{A}}(P)$ be the closed convex hull of the points in $\{(u,\beta) : 0 \leq u \leq 1, \beta \leq 0\}$ and the points

$$\left(\frac{\mu}{\mu+\nu}, \frac{1}{\mu+\nu} \right); \quad (\mu,\nu) \in \tau(P).$$

3

Let $\underline{\mathcal{A}}(P)$ be the closed convex hull of the points in $\{(u,\beta) : 0 \le u \le 1, \beta \le 0\}$ and the points

$$\left(\frac{\frac{\mu}{n_L - 1}}{\frac{\mu}{n_L - 1} + \frac{\nu}{n_R - 1}}, \frac{1}{\frac{\mu}{n_L - 1} + \frac{\nu}{n_R - 1}} \right); \quad (\mu, \nu) \in \tau(P).$$

Notice that

$$\overline{\mathcal{A}}(P) = \underline{\mathcal{A}}(P) \qquad \text{if } n_L = n_R = 2.$$

Theorem 1.1. *Suppose that* codim $\mathcal{M} = 1$, $P \in \mathcal{M}$ *and* $1 \le p \le \infty$.

(i) There is a neighborhood $\mathcal{U}$ of P such that if supp $\chi \subset \mathcal{U}$ *and if $(1/p, \alpha)$ belongs to the interior of $\overline{\mathcal{A}}(P)$ then R is bounded from $L^p_{s,\mathrm{comp}}(\mathfrak{Y})$ into $L^p_{s+\alpha,\mathrm{loc}}(\mathfrak{X})$, for all $s \in \mathbb{R}$.*

(ii) Suppose that R is bounded from $L^p_{s,\mathrm{comp}}(\mathfrak{Y})$ into $L^p_{s+\alpha,\mathrm{loc}}(\mathfrak{X})$, for some $s \in \mathbb{R}$ and that $\chi(P) \neq 0$. Then $(1/p, \alpha)$ belongs to $\underline{\mathcal{A}}(P)$.

We note that since supp χ is compact we are working with functions of compact support, and the distinction between the spaces L^p_α, $L^p_{\alpha,\mathrm{comp}}$, $L^p_{\alpha,\mathrm{loc}}$ becomes irrelevant.

We now turn to $L^p \to L^q$ estimates. Let $\overline{\mathcal{B}}(P)$ be the closed convex hull of the points in $\{(u,v)) : 0 \le u \le 1, v \ge u\}$ and the points

$$\left(\frac{\mu + 1}{\mu + \nu + 1}, \frac{\mu}{\mu + \nu + 1} \right), \quad (\mu, \nu) \in \tau(P)).$$

Let $\underline{\mathcal{B}}(P)$ be the closed convex hull of the points in $\{(u,v)) : 0 \le u \le 1, v \ge u\}$ and the points

$$\left(\frac{\frac{\mu}{n_L - 1} + 1}{\frac{\mu}{n_L - 1} + \frac{\nu}{n_R - 1} + 1}, \frac{\frac{\mu}{n_L - 1}}{\frac{\mu}{n_L - 1} + \frac{\nu}{n_R - 1} + 1} \right), \quad (\mu, \nu) \in \tau(P)).$$

Again

$$\overline{\mathcal{B}}(P) = \underline{\mathcal{B}}(P) \qquad \text{if } n_L = n_R = 2.$$

Theorem 1.2. *Suppose that* codim $\mathcal{M} = 1$, $P \in \mathcal{M}$ *and* $1 \le p \le \infty$.

(i) There is a neighborhood $\mathcal{U}$ of P such that if supp $\chi \subset \mathcal{U}$ *and if $(1/p, 1, q)$ belongs to the interior of $\overline{\mathcal{B}}(P)$ then R is bounded from $L^p_{\mathrm{comp}}(\mathfrak{Y})$ into $L^q_{\mathrm{loc}}(\mathfrak{X})$.*

(ii) Suppose dim $\mathfrak{X}$ = dim $\mathfrak{Y}$ = 2 *and that R is bounded from $L^p_{\mathrm{comp}}(\mathfrak{Y})$ into $L^q_{\mathrm{loc}}(\mathfrak{X})$. If $\chi(P) \neq 0$ then $(1/p, 1/q)$ belongs to $\underline{\mathcal{B}}(P)$.*

Consider now the two dimensional situation. If the principal symbol is multiplied by a suitable damping factor then one can obtain the same L^2 Sobolev estimates as in the nondegenerate case, without assumong finite type conditions, see [21]. An essentially sharp version of this is

Theorem 1.3. *Suppose* dim $\mathfrak{X}$ = dim $\mathfrak{Y}$ = 2 *and* codim $\mathcal{M}$ = 1. *Let $\mathcal{U}$ be an open set containing* supp χ *and let $X \in \mathcal{V}^{1,0}(\mathcal{U})$, $Y \in \mathcal{V}^{0,1}(\mathcal{U})$, nonvanishing in $\mathcal{U}$. Let ω be a section in $T^{*,\perp}\mathcal{M}$, nonvanishing over $\mathcal{U}$, and let $J_\gamma(x,y) = |\langle \omega, [X,Y]\rangle_{(x,y)}|^\gamma$. Define*

$$\mathcal{R}_\gamma f(x) = \int_{\mathcal{M}_x} \chi(x,y) J_\gamma(x,y) f(y)\, d\sigma_x(y).$$

If $\gamma > 1/2$ then $\mathcal{R}_\gamma$ maps L^2_{comp} boundedly to $L^2_{1/2,\mathrm{loc}}$.

Remarks. We state some immediate consequences and further comments, here codim $\mathcal{M}$ = 1 and in fact dim $\mathfrak{X}$ = dim $\mathfrak{Y}$ = 2, unless otherwise stated.

(1) If $n = n(P) = \min\{|\mu| + |\nu| : (\mu, \nu) \in \tau(P)\}$ is the Hörmander type of $\mathcal{M}$ at P then $\overline{\mathcal{A}}(P)$ and $\overline{\mathcal{B}}(P)$ are polygonal regions with at most $n + 1$ extreme points, see Lemma 5.3 below. The operator R is bounded from L^p_{comp} to $L^p_{\alpha,\mathrm{loc}}$ if $(\alpha, 1/p) \in \bigcap_{P \in \mathrm{supp}\ \chi} \mathrm{Int}(\overline{\mathcal{A}}(P))$, and this result is essentially sharp, up to endpoints.

(2) Suppose that $\mu \geq 1$ and $\nu \geq 1$ and suppose that $\mathcal{M}$ is of type $\preceq (\mu, \nu)$ at every $P \in \mathrm{supp}\ \chi$. Let $p = 1 + \nu\mu^{-1}$ and $\alpha < (\mu + \nu)^{-1}$. Then R is bounded from L^p_s into $L^p_{s+\alpha}$, moreover R is bounded from $L^{\frac{\mu+\nu+1}{\mu+1}}$ to $L^{\frac{\mu+\nu+1}{\mu} - \varepsilon}$. The first statement implies the second and then also the positive results of Theorems 1.1 and 1.2, by multiple interpolation.

(3) Suppose that R maps L^p to L^p_α and that $\chi(P) \neq 0$. Then necessarily $(1 - \alpha)^{-1} \leq p \leq \alpha^{-1}$; moreover there is a pair (μ, ν) of positive integers with $\mu + \nu \leq \alpha^{-1}$, so that $\mathcal{M}$ is of type (μ, ν) at P. Thus the reciprocal of the Hörmander type is the maximal gain of smoothness which can occur in some L^p space.

(4) Suppose that $1 \leq p < q \leq \infty$ and R maps L^p to L^q. Suppose also that $\chi(P) \neq 0$. Then $\mathcal{M}$ is of type (μ, ν) at (x_0, y_0) for some pair (μ, ν) of positive integers with $|\mu| + |\nu| + 1 \leq (1/p - 1/q)^{-1}$.

(5) By (3) and (4) the regions $\overline{\mathcal{A}}(P)$ and $\overline{\mathcal{B}}(P)$ have at least three extreme points if and only if $\mathcal{M}$ is of finite type at P. They are triangles in the model case in $\mathbb{R}^2$ where $\mathcal{M}$ is given by $y_2 = x_2 + x_1^\mu y_1^\nu$ and $P = (0, 0)$. Here $\mathcal{M}$ is of type (μ, ν) at $(0, 0)$. Also note that $\mathcal{M}$ is of type $(1, \nu)$ at all points with $x_1 \neq 0$, $y_1 = 0$, of type $(\mu, 1)$ for all points with $x_1 = 0$, $y_1 \neq 0$, and of type $(1, 1)$ for all points with $x_1 \neq 0$, $y_1 \neq 0$.

(6) Let $n = 2k$. The region $\overline{\mathcal{A}}((0, 0))$ has $n + 1$ extreme points if $\mathcal{M} \subset \mathbb{R}^2 \times \mathbb{R}^2$ is given by $y_2 = x_2 + x_1^k y_1^k + \sum_{j=1}^{k-1}(x_1^j y_1^{2^{k-j}k} + x_1^{2^{k-j}k} y_1^j)$. If $n = 2k + 1$, then an appropriate model example is given by $y_2 = x_2 + \sum_{j=1}^{k-1}(x_1^j y_1^{2^{k-j-1}k} + x_1^{2^{k-j-1}k} y_1^j)$.

(7) Theorem 1.1 is an extension of results in [19] since $\mathcal{M}$ satisfies a left finite type condition of order m, in the sense of [19], if and only if $\mathcal{M}$ is of type $\preceq (1, m - 1)$; and $\mathcal{M}$ satisfies a right finite type condition of order m if $\mathcal{M}$ is of type $\preceq (m - 1, 1)$. *Cf.* Corollary 3.7 below.

(8) Of course vector fields have been used before to describe various curvature conditions arising in analysis (see [9], [11], [18], [3], [15] etc.). In this paper we emphasize the distinction between vector fields tangent to X and vector fields tangent to Y; this distinction is important for the derivation of precise estimates for Radon transforms.

(9) Our theorems rely on a crucial L^2 estimate for Radon transforms proved in [19], see (5.3) below. In fact Theorem 1.3 is an almost immediate consequence of this estimate. We note that an erroneous argument in [19] is corrected in the appendix §7.

(10) An approach different from the one in this paper was previously used by Greenleaf and the author ([4], [5]) to obtain endpoint L^2 estimates for Fourier integral operators in specific situations (one sided folds and simple cusps). We plan to extend the methods of [19] to obtain L^2 estimates for Radon transforms associated with submanifolds of higher codimension, assuming finite type conditions.

(11) At this time it seems open in exactly which cases one can obtain sharp endpoint results, even in two dimensions. The case $\mu + \nu = 3$ is completely understood (see [12],[15], [13], [19], [20], [4]). Some endpoint estimate for operators of type $(1, \nu)$ were obtained by the author in [19]; these operators map L^p_s to $L^p_{s+1/p}$ if $p > \nu + 1$ and L^p to L^{2p} if $p > (\nu + 2)/2$; moreover by duality operators of type $(\mu, 1)$ map L^p_s to $L^p_{s+1/p'}$ if $p < (\mu + 1)/\mu$ and L^p to $L^{p/(2-p)}$ if $p \leq (\mu + 2)/(\mu + 1)$. Phong and Stein [16] established sharp L^p endpoint results for a number of interesting examples, namely when $\mathcal{M}$ is given by $y_2 = x_2 + h(x_1, y_1)$ where is a homogeneous polynomial of degree n,

$h(x_1, y_1) = \sum_{j=0}^{n} a_j x_1^j y_1^{n-j}$. If $a_k \neq 0$ and $1 \leq k \leq n-1$ then $\mathcal{M}$ is of type $(k, n-k)$ at $P = (0, 0)$. Recently they obtained sharp L^2 results for more general semi-translation invariant cases, namely when $h(x_1, y_1)$ is assumed to be real analytic, see [17]. At least one endpoint estimate is known to fail even in the translation invariant case; for $h(x_1, y_1) = (x_1 - y_1)^n$, M. Christ [2] showed that the corresponding averaging operator R does not map L_s^n to $L_{s+1/n}^n$. Observe that in this case $\mathcal{M}$ is of type (μ, ν), whenever $\mu + \nu = n$ and it is also known that R maps L_s^p to $L_{s+1/n}^p$ if $n/(n-1) < p < n$, cf. [19], [16] and [1].

(12) In [3] Christ, Nagel, Stein and Wainger prove L^p estimates for singular and maximal Radon transforms, under certain curvature assumptions. E. Stein and S. Wainger have informed me that [3] also contains the equivalence of those curvature hypotheses with $L^2 \to L_\varepsilon^2$ regularity for averages $\int_{-\delta}^{\delta} f(\gamma(x,t))\chi(t)dt$, where $\gamma(x,0) = x$. Moreover one of their conditions is closely related to the setup in this paper, although no specific type is defined (personal communication, Varenna 1997).

(13) Consider the special case where $\mathcal{M}$ is given by $y_2 = x_2 + h(x_1, y_1)$. Using suitable cutoff functions χ one can use Theorem 1.3 and Plancherel's theorem to derive an estimate for the damped oscillatory integral operator acting on functions on the real line, namely

$$T_{\lambda, \gamma} g(v) = \int e^{i\lambda h(v,w)} g(w) |h_{vw}''(v,w)|^\gamma \eta(v,w)dw;$$

here $\eta \in C_0^\infty(\mathbb{R} \times \mathbb{R})$. The result is that for $\lambda \geq 1$ the L^2 norm of $T_{\lambda, \gamma}$ is $O(|\lambda|^{-1/2})$ if $\gamma > 1/2$. Phong and Stein have recently shown that this bound even holds for $\gamma = 1/2$, provided that h is real analytic. This was announced by Stein at the harmonic analysis conference in Varenna (April 1997); for some model cases see [16]. It would be interesting to extend the endpoint result to classes of oscillatory integral operators and operators of Radon transform type satisfying finite type or suitable convexity assumptions, without the hypothesis of real analyticity.

In §2 we review the condition of type $(1, 1)$ and in §3 we give some convenient formulation of the finite type condition for hypersurfaces. §4 contains an elementary estimate for integral operators which relies on the finite type condition. In §5 we combine this with L^2 estimates from [19] to derive the L^p Sobolev estimates of Theorem 1.1. Necessary conditions are discussed in §6.

2. Preliminary remarks. In this section we consider coordinate patches on $\mathfrak{X}$ and $\mathfrak{Y}$; we may assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are itself open subsets of $\mathbb{R}^{n_L}$ and $\mathbb{R}^{n_R}$, respectively.

We assume that $\operatorname{codim} \mathcal{M} = \ell$ and that $\mathcal{M}$ is described by an $\mathbb{R}^\ell$-valued defining function,

$$\mathcal{M} = \{(x, y) : \Phi(x, y) = 0\}.$$

Here $\Phi = (\Phi^1, \ldots, \Phi^\ell)$ and

$$\operatorname{rank} \Phi_x' = \operatorname{rank} \Phi_y' = \ell.$$

Recall that the L^2 regularity properties are determined by the projections of $\mathcal{C} = (N^*\mathcal{M})' \setminus 0$ to $T^*\mathfrak{X}$ and $T^*\mathfrak{Y}$. Writing R as Fourier integral operator with frequency variable $\theta \in \mathbb{R}^\ell$ we see that $\mathcal{C}$ is parametrized in the usual way as

(2.1) $$\{(x, d_x\Psi, y, -d_y\Psi) : d_\theta\Psi = 0\},$$

with the phase function $\Psi(x, y, \theta) = \langle \theta, \Phi(x, y) \rangle = \sum_{i=1}^{\ell} \theta_i \Phi^i(x, y)$.

The situation of maximal nondegeneracy occurs if differentials of the projections π_L and π_R have maximal rank; that is if the rank of the $(n_L + \ell) \times (n_R + \ell)$ matrix

$$(2.2) \qquad \mathcal{J}_\Phi(x, y, \theta) = \begin{pmatrix} \Psi''_{xy} & \Psi''_{x\theta} \\ \Psi''_{\theta y} & \Psi''_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \langle\theta, \Phi\rangle''_{xy} & {\Phi'_x}^t \\ \Phi'_y & 0 \end{pmatrix}$$

is maximal, for all $\theta \neq 0$. In particular the twisted normal bundle $\mathcal{C} = (N^*\mathcal{M})'$ is a local canonical graph near P if $n_L = n_R$ and $\det(\mathcal{J}_\Phi) \neq 0$ for all $\theta \neq 0$.

It will be useful to reformulate the finite type condition in terms of the canonical relation $\mathcal{C}$ and the matrix $\mathcal{J}_\Phi$. We shall need the behavior of $\mathcal{J}_\Phi$ under changes of variables in $\mathfrak{X}$ and $\mathfrak{Y}$. Let ψ_L and ψ_R denote diffeomorphisms in $\mathfrak{X}$ and $\mathfrak{Y}$, respectively and define $\psi(w, z) = (\psi_L(w), \psi_R(z))$. Then

$$(2.3) \qquad \mathcal{J}_{\Phi\circ\psi}(w, z, \theta) = \begin{pmatrix} {D\psi_L}^t & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \langle\theta, \Phi\rangle''_{xy} & {\Phi'_x}^t \\ \Phi'_y & 0 \end{pmatrix} \begin{pmatrix} D\psi_R & 0 \\ 0 & I \end{pmatrix}$$

where the derivatives of Φ are evaluated at $(x, y) = (\psi_L(w), \psi_R(z))$.

In order to reformulate the finite type condition in terms of $\mathcal{J}_\Phi$ we consider vector fields $X = \sum_j a_j(x, y)\partial/\partial x_j$ of type $(1, 0)$ and vector fields $Y = \sum_k b_k(x, y)\partial/\partial y_k$ of type $(0, 1)$ and extend them to smooth vector fields in a neighborhood of $\mathcal{M}$. For these vector fields we define the bilinear form

$$(2.4) \qquad B_\theta(X, Y)_P = a^t \langle\theta, \Phi\rangle''_{xy} b \Big|_P$$
$$= (\, a^t \quad 0\,) \begin{pmatrix} \langle\theta, \Phi\rangle''_{xy} & {\Phi'_x}^t \\ \Phi'_y & 0 \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} \Big|_P$$

which depends linearly on θ. (2.3) shows the invariance of B_θ, namely

$$(2.5) \qquad (D\psi_L a)^t \langle\theta, \Phi\circ\psi\rangle''_{wz}(D\psi_R b) = a^t \langle\theta, \Phi\rangle''_{xy} b \Big|_{(x,y)=(\psi_L(w), \psi_R(z))}.$$

Note also the behavior of this bilinear form under changes in the defining function. If $(x, y) \mapsto A(x, y)$ is smooth with values in the general linear group $GL(\mathbb{R}^\ell)$, and if $\widetilde{\Phi}(x, y) = A(x, y)\Phi(x, y)$, $\widetilde{\theta} = (A^t(x, y))^{-1}\theta$ then

$$(2.6) \qquad a^t \langle\widetilde{\theta}, \widetilde{\Phi}\rangle''_{xy} b = a^t \langle\theta, \Phi\rangle''_{xy} b$$

On $\mathfrak{X} \times \mathfrak{Y}$ we have exterior derivatives d_L and d_R defined as natural extensions to $\mathfrak{X} \times \mathfrak{Y}$ of d_x on $\mathfrak{X}$ and d_y on $\mathfrak{Y}$, by $d_L\omega_{(x,y)} = d_x(\omega_{(\cdot,y)})_x$ and $d_R\omega_{(x,y)} = d_y(\omega_{(x,\cdot)})_y$. Now consider the forms

$$(2.7) \qquad d_L\Phi^i - d_R\Phi^i, \quad i = 1, \dots, \ell,$$

on $T^*(\mathfrak{X} \times \mathfrak{Y})$, which span the fiber of the canonical relation $(N^*\mathcal{M})'$ at P. There are the corresponding forms $d_L\Phi^i + d_R\Phi^i$ that span the fiber of $N^*\mathcal{M}$ at P and in view of the linear independence of $\{d_L\Phi^i_P, d_R\Phi^i_P; i = 1, \dots \ell\}$ the set $\{d_L\Phi^i - d_R\Phi^i, d_L\Phi^i + d_R\Phi^i; i = 1, \dots, \ell\}$ is linearly independent at P. We restrict the forms in (2.7) to linear forms acting on tangent vectors in $T_P\mathcal{M}$;

$$(2.8) \qquad \omega^i = \imath^*(d_L\Phi^i - d_R\Phi^i)$$

where $\imath^*$ is the pullback map of the inclusion $\imath : \mathcal{M} \to \mathfrak{X} \times \mathfrak{Y}$. Then the ω^i_P are linearly independent in $T^*_P\mathcal{M}$, and since they annihilate vectors of type $(1, 0)$ and $(0, 1)$, they form a basis of $T^{*,\perp}_P\mathcal{M}$.

Lemma 2.1. *Suppose that* $\omega_\theta = \sum_{i=1}^\ell \theta_i \omega^i$, *with* ω^i *as in (2.8-9). Let* $X \in \mathcal{V}^{1,0}(P)$ *and* $Y \in \mathcal{V}^{0,1}(P)$. *Then*

$$(2.9) \qquad\qquad\qquad \langle \omega_\theta, [X, Y] \rangle = 2B_\theta(X, Y).$$

Proof. If $X = \sum_j a_j \partial/\partial x_j$, $Y = \sum_k b_k \partial/\partial y_k$ then $[X, Y] = \sum_j v_j \partial/\partial x_j + \sum_k w_k \partial/\partial y_k$ with $v = -(D_y a)b$ and $w = (D_x b)a$. Since $\Phi'_x a = 0$ and $\Phi'_y b = 0$ we obtain by differentiation

$$(2.10) \qquad\qquad \langle \theta, \Phi'_x \rangle (D_y a)b = -a^t \langle \theta, \Phi \rangle''_{xy} b = \langle \theta, \Phi'_y \rangle (D_x b)a.$$

By (2.8)

$$\begin{aligned}
\langle \omega_\theta, [X, Y] \rangle &= \sum_{i=1}^\ell \theta_i \langle d_L \Phi^i - d_R \Phi^i, [X, Y] \rangle \\
&= -\langle \theta, \Phi'_x \rangle (D_y a)b - \langle \theta, \Phi'_y \rangle (D_x b)a \\
&= 2a^t \langle \theta, \Phi \rangle''_{xy} b
\end{aligned}$$

where all the expressions are evaluated at $P \in \mathcal{M}$. $\square$

Remarks. (i) Let $(P, \lambda) \in T^{*,\perp}\mathcal{M}$, $\lambda = \sum_{i=1}^\ell \omega_P^i$. Then $T^*\mathcal{M}$ is of type $(1, 1)$ at (P, λ) if and only if $B_\theta(X, Y)_P \neq 0$ for some $X \in \mathcal{V}^{1,0}(P)$, $Y \in \mathcal{V}^{0,1}(P)$.

(ii) An alternative description of B_θ can be obtained using differential forms. Note that $d_L d_R \Phi^i = \sum_j \sum_k \frac{\partial^2 \Phi^i}{\partial x_j \partial y_k} dx_j \wedge dy_k$ so that for $X \in \mathcal{V}^{1,0}(P)$, $Y \in \mathcal{V}^{0,1}(P)$

$$(2.11) \qquad\qquad\qquad \langle d_L d_R \, \theta \cdot \Phi | X \wedge Y \rangle = B_\theta(X, Y);$$

here we use the standard inner product $\langle \cdot | \cdot \rangle$ for p-forms, with the normalization that for p-vectors $x = x_1 \wedge \cdots \wedge x_p$ and p forms $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_p$ we have $\langle \lambda | x \rangle = \det(\langle \lambda_i, x_j \rangle)$. Following Kohn [11] the result of Lemma 2.1 can be derived using a standard formula for the exterior derivative of a 1-form ω, namely $\langle d\omega | X \wedge Y \rangle = X(\langle \omega | Y \rangle) - Y(\langle \omega | X \rangle) - \langle \omega | [X, Y] \rangle$, see [23, p. 103].

Let $X \in \mathcal{V}^{1,0}(P)$, $Y \in \mathcal{V}^{0,1}(P)$ then $\langle d_L \Phi^i | X \rangle = 0$, $\langle d_R \Phi^i | X \rangle = 0$, $\langle d_R \Phi^i | Y \rangle = 0$, $\langle d_L \Phi^i | Y \rangle = 0$. Now $dd = d_L d_L = d_R d_R = 0$ and therefore $d_R d_L = -d_L d_R$. We compute $\langle d_L d_R \Phi^i | X \wedge Y \rangle = \langle dd_R \Phi^i | X \wedge Y \rangle = -\langle d_R \Phi^i | [X, Y] \rangle$ and $\langle d_L d_R \Phi^i | X \wedge Y \rangle = -\langle dd_L \Phi^i | X \wedge Y \rangle = \langle d_L \Phi^i | [X, Y] \rangle$. If $e_1, \ldots, e_\ell$ denote the standard unit vectors in $\mathbb{R}^\ell$ we see that on $\mathcal{M}$

$$2B_{e_i}(X, Y) = 2\langle d_L d_R \Phi^i | X \wedge Y \rangle = \langle d_L \Phi^i - d_R \Phi^i | [X, Y] \rangle = \langle \omega^i | [X, Y] \rangle$$

which is equivalent to the statement of Lemma 2.1.

(iii) There is a formal analogy with the Levi form in several complex variables, which is apparent from rewriting (2.9) as

$$(2.12) \qquad\qquad\qquad 2B_\theta(X, Y) = \frac{1}{2i} \langle \omega_\theta, [X - iY, X + iY] \rangle,$$

see also the discussion in [14, p. 114].

Lemma 2.2. *Let $T_1, \ldots, T_\ell$ be vector fields on $\mathcal{M}$ satisfying*

$$(2.13) \qquad \langle \omega^i, T_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $X_1, \ldots X_{n_L - \ell} \in \mathcal{V}^{1,0}(P)$, $Y_1, \ldots, Y_{n_R} \in \mathcal{V}^{0,1}(P)$ so that $\{X_i, Y_j, T_k\}_P$ form a basis of $T_P^* \mathcal{M}$. Let $\{e_1, \ldots, e_\ell\}$ be the standard basis of $\mathbb{R}^\ell$. Then there are C^∞ functions α_i, β_j so that

$$(2.14) \qquad [X, Y] = \sum_{j=1}^{n_L - \ell} \alpha_j X_j + \sum_{k=1}^{n_R - \ell} \beta_k Y_k + \sum_{s=1}^{\ell} B_{e_s}(X, Y) T_s.$$

Proof. This is an immediate consequence of Lemma 2.1. $\square$

It will also be convenient to note the following formula in the particular case where the manifolds $\mathcal{M}_x$, $\mathcal{M}^y$ are curves.

Lemma 2.3. *Suppose $\dim \mathfrak{X} = \dim \mathfrak{Y} = d$, $\operatorname{codim} \mathcal{M} = d - 1$, $P = (x, y) \in \mathcal{M}$. Let $\omega_\theta = \sum_{i=1}^{l} \theta_i \omega^i$, where ω^i is as in (2.8). Let $\Gamma_L^j(x, y)$ be the $(d-1) \times (d-1)$ matrix obtained from the $(d-1) \times d$ matrix Φ_x' by omitting the jth column, and let $\Gamma_R^j(x, y)$ be the $(d-1) \times (d-1)$ matrix obtained from $(B_{ij}) = \Phi_y'$ by omitting the jth column. Let*

$$(2.15) \qquad X = \sum_{j=1}^{d} a_j(x, y) \frac{\partial}{\partial x_j}, \quad Y = \sum_{k=1}^{d} b_k(x, y) \frac{\partial}{\partial y_k}$$

where

$$(2.16) \qquad a_j(x, y) = (-1)^{j-1} \det \Gamma_L^j(x, y), \quad b_k(x, y) = (-1)^{k-1} \det \Gamma_R^k(x, y).$$

Then X and Y are vector fields of type $(1, 0)$ and $(0, 1)$, respectively, and

$$(2.17) \qquad (-1)^{d-1} \langle \omega_\theta, [X, Y] \rangle = 2 \frac{|a|^2 |b|^2}{\det \Phi_x' \Phi_x'^{\,t} \det \Phi_y' \Phi_y'^{\,t}} \det \mathcal{J}_\Phi(\cdot, \cdot, \theta).$$

In particular, if $d = 2$ then $\langle \omega_\theta, [X, Y]_P \rangle = -2 \det \mathcal{J}_\Phi(x, y, \theta)$.

Proof. Let $A = \Phi_x'$, $B = \Phi_y'$, $M = \langle \theta, \Phi \rangle_{xy}''$. Then $Aa = Bb = 0$. Setting $\mathcal{A}^t = (a, A^t)$ and $\mathcal{B}^t = (b, B^t)$ we have the identity

$$\begin{pmatrix} \mathcal{A} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M & A^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathcal{B}^t & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} a^t M b & a^t M B^t & 0 \\ A M b & A M B^t & A A^t \\ 0 & B B^t & 0 \end{pmatrix}.$$

A cofactor expansion yields that $\det \mathcal{A} = |a|^2$, $\det \mathcal{B} = |b|^2$. Therefore taking determinants we see that

$$|a|^2 |b|^2 \det \mathcal{J}_\Phi = (-1)^{d-1} a^t \langle \theta, \Phi \rangle_{xy}'' b \det \Phi_x' \Phi_x'^{\,t} \det \Phi_y' \Phi_y'^{\,t}.$$

The asserted formula follows immediately from Lemma 2.1. $\square$

3. Finite type conditions for hypersurfaces. In this section we assume that $\mathcal{M}$ is a hypersurface in $\mathfrak{X} \times \mathfrak{Y}$, given by the equation $\Phi(x,y) = 0$. We shall need the notion of a differential operator of type $\preceq (\mu, \nu)$. The operator of multiplication by a function is said to be of type $(0,0)$. We say that X is of type $(1,0)$ if $X = \sum_j a_j \partial/\partial x_j$ is a vector field, whose restriction of X to $\mathcal{M}$ is tangential to both $\mathcal{M}$ and $\mathfrak{X}$. We say that Y is of type $(0,1)$ if $Y = \sum_j b_j \partial/\partial y_j$ is a vector field whose restriction to Y to $\mathcal{M}$ is tangential to $\mathcal{M}$ and $\mathfrak{Y}$. Let $\mu \geq 1$, $\nu \geq 1$; then the operator L is defined to be of type $\preceq (\mu, \nu)$ if L is of order $\leq \mu + \nu$ and if L is a linear combination over C^∞ of operators of order $(0,0)$ and differential operators of the form $gV_1 \ldots V_n$ where each V_i is of type $(1,0)$ or $(0,1)$, but at most μ of the V_i are of type $(1,0)$ and at most ν of the V_i are of type $(0,1)$. The C^∞ module of operators of type $\preceq (\mu, \nu)$ in an open set $\mathcal{U}$ is denoted by $\mathfrak{L}^{\mu,\nu}(\mathcal{U})$, similarly we may define germs of differential operators and use the notation $\mathfrak{L}^{\mu,\nu}(P)$.

We set

$$B(X,Y) = B_\theta(X,Y)\Big|_{\theta=1}, \qquad \omega = \omega_\theta\Big|_{\theta=1}.$$

We shall need to work with tangential vector fields T satisfying $\langle \omega, T \rangle_P = 1$; a natural choice is

$$(3.1) \qquad T = \frac{1}{2}\Big(\frac{1}{|\Phi'_x|^2} \sum_{j=1}^{n_L} \frac{\partial \Phi}{\partial x_j} \frac{\partial}{\partial x_j} - \frac{1}{|\Phi'_y|^2} \sum_{k=1}^{n_R} \frac{\partial \Phi}{\partial y_k} \frac{\partial}{\partial y_k} \Big).$$

Theorem 3.1. *Suppose $P = (x,y) \in \mathcal{M}$. The following conditions are equivalent.*

(i) $\mathcal{M}$ is of type (μ, ν) at P.

(ii) There are μ vector fields $V_1, \ldots, V_\mu \in \mathcal{V}^{1,0}(P)$ and ν vector fields $V_{\mu+1}, \ldots, V_{\mu+\nu} \in \mathcal{V}^{0,1}(P)$ so that

$$\langle \omega, \mathrm{ad}V_1 \cdots \mathrm{ad}V_{\mu+\nu-1}(V_{\mu+\nu}) \rangle_P \neq 0$$

but $\langle \omega, V \rangle_P = 0$ for any vector field V of type $\prec (\mu, \nu)$.

(iii) There are μ vector fields $V_1, \ldots, V_\mu \in \mathcal{V}^{1,0}(P)$ and ν vector fields $V_{\mu+1}, \ldots, V_{\mu+\nu} \in \mathcal{V}^{0,1}(P)$ so that the following holds. For any choice of C^∞ functions h_i with $h_i(P) \neq 0$, for any permutation π of $\{1, \ldots, \mu+\nu\}$ and for $W_i := h_i V_{\pi(i)}$

$$(3.2) \qquad \langle \omega, \mathrm{ad}W_1 \ldots \mathrm{ad}W_{\mu+\nu-1}(W_{\mu+\nu}) \rangle_P \neq 0$$

but $\langle \omega, V \rangle_P = 0$ for any vector field V of type $\prec (\mu, \nu)$.

(iv) There are vector fields $X \in \mathcal{V}^{1,0}(P)$ and $Y \in \mathcal{V}^{0,1}(P)$, and a differential operator L of type $\preceq (\mu-1, \nu-1)$ so that $L[B(X,Y)] \neq 0$, but $\widetilde{L}[B(\widetilde{X}, \widetilde{Y})] = 0$ for all differential operators $\widetilde{L}$ of type $\prec (\mu-1, \nu-1)$ and all $\widetilde{X} \in \mathcal{V}^{1,0}(P)$, $\widetilde{Y} \in \mathcal{V}^{0,1}(P)$.

A stronger variant of this theorem is obtained in Theorem 3.6 below.

We shall deduce Theorem 3.1 by induction over $n = \mu + \nu$ from various elementary lemmata.

Lemma 3.2. *Let $V_1, \ldots, V_n$ be vector fields in $\mathcal{V}(P)$ and let $\widetilde{V}_i = f_i V_i$ where $f_i \in C^\infty$, Then*

$$(3.3) \qquad \widetilde{V}_1 \widetilde{V}_2 \cdots \widetilde{V}_n = f_1 \cdots f_n V_1 \cdots V_n + \sum_{\substack{\emptyset \neq A \\ \subset \{1,\ldots,n\}}} \sum_{\mathfrak{B}} L_{\mathfrak{B}}(f_1, \ldots, f_n) V_A$$

here $V_A = V_{i_1} V_{i_2} \cdots V_{i_n}$ if $(i_1, \ldots, i_n) \in A$ and $i_1 < \cdots < i_n$. A^c denotes the complement of A in $\{1, \ldots, n\}$ and $L_{\mathfrak{B}}(f_1, \ldots, f_n)$ is a finite sum of expressions of the form

$$c \prod_{i \in B_0} f_i \prod_{k=1}^{N(\mathfrak{B})} V_{j_1^k} \cdots V_{j_{s(k)}^k} f_k$$

10

where $\mathfrak{B} = \{B_0, \ldots, B_{N(\mathfrak{B})}\}$ is a partition of the set A^c as a disjoint union, $A^c = B_0 \cup \cdots \cup B_{N(\mathfrak{B})}$, with $B_k = \{j_1^k, \ldots, j_{s(k)}^k\}$ and $j_\nu^k < k$.

Proof. This follows by a straightforward computation from repeated applications of the product rule. $\square$

For $j \geq 0$, $k \geq 0$ let $\mathcal{I}^{j,k}$ be the ideal in $\mathcal{V}(P)$ generated by vector fields of the form $L(B(X,Y))W$, where $W \in \mathcal{V}(P)$ and $L \in \mathcal{L}^{j,k}(P)$, $X \in \mathcal{V}^{1,0}(P)$, $Y \in \mathcal{V}^{0,1}(P)$. We set $\mathcal{I}^{-1,0} = \mathcal{I}^{0,-1} = \{0\}$.

Lemma 3.3. Suppose $n = \mu + \nu$ and suppose $V_1, \ldots, V_n$ are vector fields, μ of them in $\mathcal{V}^{1,0}(P)$, ν of them in $\mathcal{V}^{0,1}(P)$. Let $T \in \mathcal{V}(P)$ so that $\langle \omega, T \rangle = 1$. Then

(3.4) $$\mathrm{ad}V_1 \cdots \mathrm{ad}V_{n-1}(V_n) = 2 V_1 \cdots V_{n-2} B(V_{n-1}, V_n) T + W_1 + W_2 + Z_1 + Z_2$$

where $W_1 \in \mathcal{V}^{\mu-1,\nu}(P)$, $W_2 \in \mathcal{V}^{\mu,\nu-1}(P)$, $Z_1 \in \mathcal{I}^{\mu-2,\nu-1}(P)$, $Z_2 \in \mathcal{I}^{\mu-1,\nu-2}(P)$.

Proof. This easily follows by induction using the formula $\langle \omega, [X,Y] \rangle = 2B(X,Y)$. $\square$

The leading terms in (3.4) are not commutative in $V_1, \ldots, V_n$. The following Lemma shows that commutativity holds modulo suitable "negligeable" terms.

Let $\mathfrak{I}_{j,k}^n(P)$ consist of all linear combinations of the expressions $g L_1 [B_k(X,Y)] L_2$ where $X \in \mathcal{V}^{1,0}(P)$, $Y \in \mathcal{V}^{0,1}(P)$, L_1 is of type (j',k') where $(j',k') \preceq (j,k)$ and L_2 is a differential operator so that $j' + k' + \mathrm{order}(L_2) \leq n$.

Lemma 3.4. Let $V_1, \ldots, V_n$ be vector fields, μ of them in $\mathcal{V}^{1,0}(P)$, ν of them in $\mathcal{V}^{0,1}(P)$. Let π, π' be permutations of the set $\{1, 2, \ldots, n\}$.

Then
$$V_{\pi(1)} \cdots V_{\pi(n)} = V_{\pi'(1)} \cdots V_{\pi'(n)} + L_1 + L_2 + E;$$

here $L_1 \in \mathcal{L}^{\mu-1,\nu}$, $L_2 \in \mathcal{L}^{\mu,\nu-1}$; moreover $E \in \mathfrak{I}_{\mu-1,\nu-1}^{\mu+\nu-1}$.

Proof. It suffices to prove this theorem for π' being the identity. We note by Lemma 2.2 that

$$V_1 \cdots V_{k-2} V_{k-1} V_k V_{k+1} \cdots V_n = V_1 \cdots V_{k-2} V_k V_{k-1} V_{k+1} \cdots V_n$$
$$+ V_1 \cdots V_{k-2} X V_{k+1} \cdots V_n$$
$$+ V_1 \cdots V_{k-1} Y V_{k+1} \cdots V_n$$

(3.5) $$+ 2 \sum_{k=1}^{\ell} V_1 \cdots V_{k-2} \mathcal{B}_k(V_{k-1}, V_k) T V_{k+1} \cdots V_n$$

where $X \in \mathcal{V}^{1,0}(P)$, $Y \in \mathcal{V}^{0,1}(P)$, $\mathcal{B}_k(V_{k-1}, V_k) = B_{e_k}(V_{k-1}, V_k)$ if $V_{k-1} \in \mathcal{V}^{1,0}(P)$, $V_k \in \mathcal{V}^{0,1}(P)$ and $\mathcal{B}_k(V_{k-1}, V_k) = -B_{e_k}(V_k, V_{k-1})$ if $V_k \in \mathcal{V}^{1,0}(P)$, $V_{k-1} \in \mathcal{V}^{0,1}(P)$, moreover $\mathcal{B}_k(V_{k-1}, V_k) = 0$ if V_{k-1} and V_k are both in $\mathcal{V}^{1,0}(P)$ or both in $\mathcal{V}^{0,1}(P)$. (3.5) and an application of Lemma 3.2 imply the assertion for transpositions π. A repetition of this argument shows the assertion for the permutation π with $\pi(1) = k$, $\pi(j) = j - 1$ for $2 \leq j \leq k$ and $\pi(j) = j$ for $j > k$. From this the general case follows easily by induction. In the induction step one uses that for $X \in \mathcal{V}^{1,0}(P)$, $Y \in \mathcal{V}^{0,1}(P)$, $L \in \mathcal{L}^{\mu,\nu}(P)$, $E \in \mathfrak{I}_{\mu-1,\nu-1}^{\mu+\nu-1}$ we have that $XL \in \mathcal{L}^{\mu+1,\nu}(P)$, $YL \in \mathcal{L}^{\mu,\nu+1}(P)$, $XE \in \mathfrak{I}_{\mu,\nu-1}^{\mu+\nu}$, and $YE \in \mathfrak{I}_{\mu-1,\nu}^{\mu+\nu}$. $\square$

Proof of Theorem 3.1. (i) is equivalent with (ii) by the definition of type (μ, ν). (ii) is equivalent with (iii) by Lemma 3.2 and Lemma 3.4. (iii) is equivalent with (iv) by Lemma 3.3. $\quad\square$

We now wish to show that in statements (ii), (iii) of Theorem 3.1 it is possible to work with a single vector field of type $(1,0)$ and a single vector field of type $(0,1)$. Our calculations are facilitated by a particular choice of coordinates in $\mathfrak{X}$ and $\mathfrak{Y}$ vanishing at the point P.

We shall use the notation $u = x_{n_L}$, $v = y_{n_R}$ and $y' = (y_1, \ldots, y_{n_R-1})$, $x' = (x_1, \ldots, x_{n_L-1})$, so that $x = (x', u)$, $y = (y', v)$. We may assume that near $P = (0,0)$ the manifold is given by

$$(3.6) \qquad v = S(x, y') \iff u = \mathfrak{S}(y, x').$$

By suitable choices of the coordinates in $\mathfrak{X}$ and in $\mathfrak{Y}$ we can assume that

$$(3.7) \qquad \begin{aligned} S(x, 0) &= x_{n_L} \equiv u, \\ \mathfrak{S}(y, 0) &= y_{n_R} \equiv v. \end{aligned}$$

In particular it follows by implicit differentiation that $\partial_y^\alpha S(0,0) = 0$ for all multiindices $\alpha \neq (0, \ldots, 0, 1)$.

Set $\Phi(x, y) = y_{n_L} - S(x, y')$. We shall work with the vector fields

$$(3.8) \qquad X_j = -\Phi'_{x_{n_L}} \frac{\partial}{\partial x_j} + \Phi'_{x_j} \frac{\partial}{\partial x_{n_L}} = S'_u \frac{\partial}{\partial x_j} - S'_{x_j} \frac{\partial}{\partial u}$$

and

$$(3.9) \qquad Y_k = \Phi'_{y_{n_R}} \frac{\partial}{\partial y_k} - \Phi'_{y_k} \frac{\partial}{\partial y_{n_R}} = \frac{\partial}{\partial y_k} + S'_{y_k} \frac{\partial}{\partial v};$$

then a short computation using (2.4) shows that for $P = (x, y', S(x, y'))$

$$(3.10) \qquad B(X_j, Y_k)_P = S'_u S''_{x_j y_k} - S'_{x_j} S''_{u y_k} \Big|_{(x, y')}.$$

If

$$(3.11) \qquad X = \sum_{j=1}^{n_L-1} a_j(x, y) X_j; \qquad Y = \sum_{k=1}^{n_R-1} b_k(x, y) Y_k$$

with smooth functions a_j, b_k and $a = (a_1, \ldots, a_{n_L-1})$, $b = (b_1, \ldots, b_{n_R-1})$ then

$$B(X, Y) = S'_u \langle \alpha, \nabla_{x'} \rangle \langle \beta, \nabla_{y'} \rangle S - \langle \alpha, \nabla_{x'} S \rangle \langle \beta, \nabla_{y'} S_u \rangle \Big|_{\substack{\alpha = a(x,y) \\ \beta = b(x,y)}}.$$

$$(3.12) \qquad = \sum_{j=1}^{n_L-1} \sum_{k=1}^{n_R-1} a_j(x, y) b_k(x, y) [S'_u S''_{x_j y_k} - S'_{x_j} S''_{u y_k}]_{(x,y)}.$$

Lemma 3.5. *Suppose that $\mu \geq 1$, $\nu \geq 1$ and suppose that in a neighborhhod $\mathcal{U}$ of P coordinates vanishing at P are chosen so that $\mathcal{M}$ is given by (3.6), (3.7). Let $a^l : \mathcal{U} \to \mathbb{R}^{n_L-1}$, $b^m : \mathcal{U} \to \mathbb{R}^{n_R-1}$ be smooth functions, $l = 1, \ldots, \mu$, $m = 1, \ldots, \nu$. For each l, m define $X^l = \sum_{j=1}^{n_L-1} a_j^l(x,y) X_j$, $Y^m = \sum_{k=1}^{n_R-1} b_k^m(x,y) Y_k$, where X_j and Y_k are as in (3.8), (3.9). Let $V_1 = X^1$, $V_2 = Y^1$ and, if $\mu + \nu > 1$, let $V_3, \ldots, V_{\mu+\nu}$ a permutation of the vector fields $X^2, \ldots, X^\mu, Y^2, \ldots, Y^\nu$. Let*

$$(3.13) \qquad \mathcal{L} = V_3 \cdots V_{\mu+\nu}$$

if $(1,1) \prec (\mu,\nu)$ and let $\mathcal{L}$ be the identity if $(1,1) \preceq (\mu,\nu)$. For fixed (x,y) let L_{xy} be the constant coefficient differential operator

$$(3.14) \qquad L_{xy} = \langle \alpha^1, \nabla_{x'} \rangle \cdots \langle \alpha^l, \nabla_{x'} \rangle \langle \beta^1, \nabla_{y'} \rangle \cdots \langle \beta^m, \nabla_{y'} \rangle \Big|_{\substack{\alpha^l = a^l(x,y) \\ \beta^m = b^m(x,y)}}.$$

Let $\mathcal{I}_{1,1}$ be the smallest ideal in $C^\infty(\mathcal{U})$ containing the functions $(\partial/\partial x')^\gamma S$ with $1 \leq |\gamma| \leq \mu$ and the functions $(\partial/\partial y')^\delta S$ with $1 \leq |\delta| \leq \nu$. For $(1,1) \prec (\mu,\nu)$ let $\mathcal{I}_{\mu,\nu}$ be the smallest ideal in $C^\infty(\mathcal{U})$ containing $\mathcal{I}_{1,1}$ and the functions $\bar{L} B(\bar{X}, \bar{Y})_{(x,y)}$ where $\bar{L}$ is a differential operator of type $\prec (\mu-1, \nu-1)$, $\bar{X}$ is of type $(1,0)$ and $\bar{Y}$ is of type $(0,1)$. Then

    (i) $[S_u']^\mu L_{xy} S - \mathcal{L}[B(X^1, Y^1)] \in \mathcal{I}_{\mu,\nu}$.

    (ii) Let $\widetilde{X} = \sum_{j=1}^{n_L-1} \widetilde{a}_j(x,y) X_j$ and $\widetilde{Y} = \sum_{k=1}^{n_R-1} \widetilde{b}_k(x,y) Y_k$. If $g \in \mathcal{I}_{\mu,\nu}$ then $\widetilde{X} g \in \mathcal{I}_{\mu+1,\nu}$ and $\widetilde{Y} g \in \mathcal{I}_{\mu,\nu+1}$.

    (iii) If $\mathcal{M}$ is not of type $\prec (\mu,\nu)$ at $P = (0,0)$ then $L_{xy} S(0,0) = \mathcal{L}[B(X^1, Y^1)](0,0)$ and this expression does not vanish if $\mathcal{M}$ is of type (μ,ν).

Proof. We prove (i)= (i)$_{\mu,\nu}$ and (ii)= (ii)$_{\mu,\nu}$ simultaneously by induction over $n = \mu + \nu$. For $n = 2$, *i.e.* $\mu = \nu = 1$ the assertion (i)$_{1,1}$ follows from (3.12).

Clearly $\widetilde{Y} S_{y_k} \in \mathcal{I}_{1,2}$, $\widetilde{X} S_{x_j} \in \mathcal{I}_{2,1}$. Next by (i)$_{1,1}$ we have that $S_u' S_{x_j y_k}'' - B(X_j, Y_k) \in \mathcal{I}_{1,1}$ and since $B(X_j, Y_k) \in \mathcal{I}_{1,2} \cap \mathcal{I}_{2,1}$ and $-S_{x_j} S_{u y_k}'' \in \mathcal{I}_{1,1}$ we see that $X_j S_{y_k}' \in \mathcal{I}_{2,1}$ and $Y_k S_{x_j}' \in \mathcal{I}_{1,2}$. This implies also that $\widetilde{X} S_{y_k}' \in \mathcal{I}_{2,1}$ and $\widetilde{Y} S_{x_j}' \in \mathcal{I}_{1,2}$ and therefore (ii) for $\mu = \nu = 1$.

For the induction step let $\mu' + \nu' = n + 1$. We consider the case $\mu' > 1$ and set $\mu = \mu' - 1$, $\nu' = \nu$. Let $\mathcal{L}$ and L_{xy} be as in (3.13), (3.14). Consider $\widetilde{\mathcal{L}} = \widetilde{X} \mathcal{L}$, and the constant coefficient operator $\widetilde{L}_{xy} = \langle \widetilde{\alpha}, \nabla_{x'} \rangle \langle \alpha^1, \nabla_{x'} \rangle \cdots \langle \beta^m, \nabla_{y'} \rangle$ with $\widetilde{\alpha} = \widetilde{a}(x,y)$, $\alpha^l = a^l(x,y)$ and $\beta^m = b^m(x,y)$. Then $S_u' \langle \widetilde{a}, \nabla_{x'} \rangle - \sum_j \widetilde{a}_j X_j = \sum_j \widetilde{a}_j S_{x_j}' \frac{\partial}{\partial u}$ and therefore

$$
\begin{aligned}
(S_u')^{\mu+1} \langle \widetilde{a}, \nabla_{x'} \rangle L_{xy} S - \widetilde{X} \mathcal{L}[B(X^1, Y^1)] &= (S_u')^\mu \sum_{j=1}^{n_L-1} \widetilde{a}_j S_{x_j}' \frac{\partial}{\partial u}(L_{xy} S) \\
&\quad + (S_u')^\mu \widetilde{X}\Big(\frac{1}{(S_u')^\mu}\Big)\big(((S_u')^\mu L_{xy} S - \mathcal{L}(B(X^1, Y^1)))\big) \\
&\quad + \widetilde{X}\big((S_u')^\mu L_{xy} S - \mathcal{L}(B(X^1, Y^1))\big) \\
&\quad + (S_u')^\mu \widetilde{X}\Big(\frac{1}{(S_u')^\mu}\Big)\mathcal{L}(B(X^1, Y^1))
\end{aligned}
$$

The first term on the right hand side belongs to the ideal generated by the S_{x_j}', $1 \leq j \leq n_L - 1$, hence to $\mathcal{I}_{1,1} \subset \mathcal{I}_{\mu+1,\nu} = \mathcal{I}_{\mu',\nu'}$. The third belongs to $\mathcal{I}_{\mu,\nu}$ by the induction hypothesis for (i),

13

hence also to $\mathcal{I}_{\mu',\nu'}$. The fourth term involves $\mathcal{L}(B(X_1,Y_1))$ and therefore also belongs to $\mathcal{I}_{\mu',\nu'}$. The second term belongs to $\widetilde{X}\mathcal{I}_{\mu,\nu} \subset \mathcal{I}_{\mu+1,\nu} = \mathcal{I}_{\mu',\nu'}$ by the induction hypothesis for (ii).

There is another case with $\mu' + \nu' = n + 1$, namely $\nu' > 1$ and $\mu' = \mu$, $\nu = \nu' - 1$, and now $\widetilde{\mathcal{L}}$ is replaced by $\widetilde{Y}\mathcal{L}$ and the relevant constant coefficient operator is $\langle \widetilde{\beta}, \nabla_{y'} \rangle \langle \alpha^1, \nabla_{x'} \rangle \cdots \langle \beta^m, \nabla_{y'} \rangle$ with $\widetilde{\beta} = \widetilde{b}(x,y)$, $\alpha^l = a^l(x,y)$, $\beta^m = b^m(x,y)$. One now shows that the difference of $\widetilde{Y}\mathcal{L}(B(X^1,Y^1))$ and the constant coefficient operator belongs to $\mathcal{I}_{\mu,\nu+1} = \mathcal{I}_{\mu',\nu'}$. The argument is the same as before.

To complete the induction step we have to show (ii)$_{\mu',\nu'}$. We argue as in the first step. Assume that $\mu' > 1$. Clearly $\widetilde{X}(\partial/\partial x')^\gamma S \in \mathcal{I}_{\mu',1} \subset \mathcal{I}^{\mu',\nu'}$ if $|\gamma| = \mu$. If δ is a multiindex with $|\delta| = \nu'$ then by (i)$_{(1,\nu')}$ we see as in the first step that $\widetilde{X}(\partial/\partial y')^\delta S \in \mathcal{I}_{2,\nu'} \subset \mathcal{I}_{\mu',\nu'}$. Also if $\bar{L}$ is a differential operator of type $\prec (\mu - 1, \nu - 1)$ then $\widetilde{X}\bar{L}$ is a differential operator of type $\prec (\mu, \nu - 1)$. This shows $\widetilde{X}\mathcal{I}_{\mu'-1,\nu'} \subset \mathcal{I}_{\mu',\nu'}$. Similarly one checks $\widetilde{Y}\mathcal{I}_{\mu',\nu'-1} \subset \mathcal{I}_{\mu',\nu'}$, and the proof of the induction step is complete.

Finally, (iii) is an immediate consequence of (i) since if $\mathcal{M}$ is not of type $\prec (\mu, \nu)$ at P then all functions in $\mathcal{I}_{\mu,\nu}$ vanish at the point $(0,0)$. $\square$

Given Lemma 3.5 we can refine Theorem 3.1 to

Theorem 3.6. *Suppose $P = (x,y) \in \mathcal{M}$.*
(1) The following conditions are equivalent.

(i) $\mathcal{M}$ is of type (μ,ν) at P.

(ii) There are vector fields $X \in \mathcal{V}^{1,0}(P)$ and $Y \in \mathcal{V}^{0,1}(P)$ with the following property: If $L = V_1 \cdots V_{\mu+\nu-2}$ where $\mu - 1$ of the V_i are equal to X and $\nu - 1$ of the V_i are equal to Y, then $L[B(X,Y)]_P \neq 0$; but $\widetilde{L}[B(\widetilde{X},\widetilde{Y})] = 0$ for all differential operators $\widetilde{L}$ of type $\prec (\mu - 1, \nu - 1)$ and all $\widetilde{X} \in \mathcal{V}^{1,0}(P)$, $\widetilde{Y} \in \mathcal{V}^{0,1}(P)$.

(iii) There are vector fields $X \in \mathcal{V}^{1,0}(P)$ and $Y \in \mathcal{V}^{0,1}(P)$ with the following property: If $V = \mathrm{ad}V_1 \cdots \mathrm{ad}V_{\mu+\nu-2}[X,Y]$ where $\mu - 1$ of the V_i are equal to X and $\nu - 1$ of the V_i are equal to Y, then $\langle \omega, V \rangle_P \neq 0$; but $\langle \omega, \widetilde{V} \rangle_P = 0$ for all vector fields $\widetilde{V}$ of type $\prec (\mu, \nu)$.
(2) Suppose $\mathcal{M}$ is not of type $\prec (\mu, \nu)$ at P. Then coordinates x in $\mathfrak{X}$, y in $\mathfrak{Y}$ vanishing at P can be chosen near P, such that $\mathcal{M}$ is given by (3.6), so that (3.7) holds and

$$(3.15) \qquad \frac{\partial^{|\gamma|+|\delta|} S}{(\partial x')^\gamma (\partial x')^\delta}(0,0) = 0$$

for multiindices γ, δ with $(|\gamma|, |\delta|) \prec (\mu, \nu)$.

Moreover if $\mathcal{M}$ is of type (μ,ν) at P then the coordinates can be chosen so that in addition

$$(3.16) \qquad \frac{\partial^{\mu+\nu} S}{(\partial x_1)^\mu (\partial y_1)^\nu}(0,0) \neq 0$$

Proof. The implications (ii) $\implies$ (i), (iii) $\implies$ (i) have been already proved in Theorem 3.1. Consider the vector space $\mathbb{V}$ of all linear combinations of monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n_L-1}^{\alpha_{n_L}-1} y_1^{\beta_1} y_2^{\beta_2} \cdots y_{n_R-1}^{\beta_{n_R}-1}$$

14

with the property that $\sum_j \alpha_j = \mu$, $\sum_k \beta_k = \nu$. Then $\mathbb{V}$ is spanned by the polynomials of the form $\langle \xi', x' \rangle^\mu \langle \eta', y' \rangle^\nu$, $\xi' \in \mathbb{R}^{n_L - 1}$, $\eta' \in \mathbb{R}^{n_R - 1}$. This is seen by a straightforward adaptation of a corresponding argument in Stein's book [22, p.343] for polynomials of a single set of variables.

Now suppose that $\mathcal{M}$ is of type (μ, ν) at P. Then we can choose coordinates such that $\mathcal{M}$ is given by (3.6), (3.7). The observation about $\mathbb{V}$ implies that

$$\langle \xi', \nabla_{x'} \rangle^{\mu-1} \langle \eta', \nabla_{y'} \rangle^{\nu-1} S(0,0) \neq 0$$

for suitable $\xi' \in \mathbb{R}^{n_L - 1}$, $\eta' \in \mathbb{R}^{n_R - 1}$. Then by Lemma 3.5 for $X_{\xi'} := \sum_{j=1}^{n_L - 1} \xi_j X_j$, $Y_{\eta'} := \sum_{k=1}^{n_R - 1} \eta_k Y_k$

$$X_{\xi'}^{\mu-1} Y_{\eta'}^{\nu-1} B(X_{\xi'}, Y_{\eta'}) \neq 0;$$

moreover $\widetilde{L}[B(\widetilde{X}, \widetilde{Y})] = 0$ for all $\widetilde{X} \in \mathcal{V}^{1,0}(P)$, $\widetilde{Y} \in \mathcal{V}^{0,1}(P)$ and all differential operators $\widetilde{L}$ of type $\prec (\mu - 1, \nu - 1)$, by Theorem 3.1. We have proved the implication (i) $\implies$ (ii). The equivalence of the conditions (ii) and (iii) follows from Lemma 3.3. (3.15) follows from Lemma 3.5 and (3.16) can be achieved by separate rotations in the x and the y coordinates. $\square$

Corollary 3.7. *Suppose* $\dim \mathfrak{X} = \dim \mathfrak{Y} = 2$, $\dim \mathcal{M} = 3$ *and* $P \in \mathcal{M}$. *Let* $X \in \mathcal{V}^{1,0}(P)$ *and* $Y \in \mathcal{V}^{0,1}(P)$, *non vanishing at* P. *Then*

(i) $\mathcal{M}$ *is of type* (μ, ν) *at* P *if and only if* $X^{\mu-1} Y^{\nu-1} (\det \mathcal{J}_\Phi)_P \neq 0$ *but* $X^{\mu'-1} Y^{\nu'-1} (\det \mathcal{J}_\Phi)_P \neq 0$ *for all* $(\mu', \nu') \prec (\mu, \nu)$.

(ii) $\mathcal{M}$ *is of type* $(1, m-1)$ *at* $P = (x, y)$ *if and only if* $\det J_\Phi\big|_{\mathcal{M}_x}$ *vanishes of order* $m - 2$ *at* $y \in \mathcal{M}_x$.

(iii) $\mathcal{M}$ *is of type* $(m-1, 1)$ *at* $P = (x, y)$ *if and only if* $\det J_\Phi\big|_{\mathcal{M}^y}$ *vanishes of order* $m - 2$ *at* $x \in \mathcal{M}^y$.

The proof is immediate from Theorem 3.1, Lemma 2.3 and Lemma 3.4.

4. L^p-estimates for integral operators in the plane. We assume that $\mathcal{M}$ is imbedded in $\mathbb{R}^2 \times \mathbb{R}^2$ and that in a neighborhood of supp $\chi \cap \mathcal{M}$ the manifold $\mathcal{M}$ is given by $\Phi(x, y) = S(x_1, x_2, y_1) - y_2 = 0$ so that $S'_{x_2} \neq 0$ and that

$$(4.1) \qquad y_2 = S(x, y_1) \quad \Longleftrightarrow \quad x_2 = \mathfrak{S}(y, x_1),$$

with $S'_{x_2} \neq 0$ and $\mathfrak{S}'_{y_2} \neq 0$. Moreover we assume that $\mathcal{M}$ is of type $\preceq (\mu, \nu)$ at $P^0 \in \text{supp } \chi \cap \mathcal{M}$. Our operator is then defined by

$$(4.2) \qquad Rf(x_1, x_2) = \int f(y_1, S(x, y_1)) \chi(x, y) dy_1$$

The Monge-Ampère determinant is now given by

$$(4.3) \qquad \det \mathcal{J}_\Phi(x, y) = S''_{x_1 y_1}(x, y_1) S'_{x_2}(x, y_1) - S''_{x_2 y_1}(x, y_1) S'_{x_1}(x, y_1);$$

in particular it is independent of y_2.

Define vector fields X, Y

$$(4.4) \qquad \begin{aligned} X &= \frac{\partial}{\partial x_1} + \mathfrak{S}'_{x_1} \frac{\partial}{\partial x_2}, \\ Y &= \frac{\partial}{\partial y_1} + S'_{y_1} \frac{\partial}{\partial y_2}. \end{aligned}$$

15

Then X is of type $(1, 0)$ and Y is of type $(0, 1)$ on $\mathcal{M}$. We note that the vector field

$$(4.5) \qquad T = \frac{\partial}{\partial y_2} + S'_{x_2} \frac{\partial}{\partial x_2}.$$

is tangent to $\mathcal{M}$ and X, Y and T are linearly independent at any particular point $P \in \mathcal{M}$. An analogue to Lemma 2.2 is the formula

$$(4.6) \qquad [aX, bY] = aX(b)Y - bY(a)X + ab\Delta T + W$$

where

$$\Delta = \frac{\det \mathcal{J}_\Phi}{S_{x'_2}}$$

and $W = 0$ on $\mathcal{M}$. It follows from Lemma 2.3 and Theorem 3.6 that the finite type condition can be restated in terms of Δ. The assumption of type $\preceq (\mu, \nu)$ implies that there is $(j, k) \preceq (\mu - 1, \nu - 1)$ such that

$$(4.7) \qquad X^j Y^k \Delta \neq 0$$

for $(x, y_1, S(x, y_1))$ in supp $\chi \cap \mathcal{M}$.

Note that Δ does not depend on y_2. Therefore if

$$\Gamma_k(y_1, y_2, x_1) = \frac{\partial^k \Delta}{(\partial y_1^k)}(x_1, \mathfrak{S}(y_1, y_2, x_1), y_1)$$

then (4.7) just means that

$$\frac{\partial^j \Gamma_k}{(\partial x_1)^j}(y_1, y_2, x_1) \neq 0$$

for $(x_1, \mathfrak{S}(y, x_1), y_1) = (x, y_1, S(x, y_1)) \in$ supp $\chi \cap \mathcal{M}$.

We introduce a localization into regions where $|\det \mathcal{J}_\Phi| \approx |\Delta| \approx 2^{-l}$. Let

$$\eta_l(x, y_1) = \eta(2^l |\Delta(x, y_1)|) \chi(x, y_1, S(x, y_1))$$

where $\eta \in C_0^\infty((1/2, 2))$. We shall need L^p estimates for the integral operator

$$(4.8) \qquad R_l f(x) = \int \eta_l(x, y_1) f(y_1, S(x, y_1)) \, dy_1$$

Proposition 4.1. *Suppose that $\mathcal{M}$ is of type $\preceq (\mu, \nu)$ in supp $\chi \cap \mathcal{M}$. Let $p = \frac{\mu + \nu - 2}{\mu - 1}$ and suppose that $\gamma < (\mu + \nu - 2)^{-1}$. Then the inequality*

$$\|R_l f\|_p \leq C_\gamma 2^{-l\gamma} \|f\|_p$$

holds for all $f \in L^p$.

We shall use the following elementary

Lemma 4.2. *Let k and N be positive integers, such that $k \le N$. For an interval J and $f \in C^N(J)$ let $M_N(f) = \max_{x \in J} |f^{(N)}(x)|$.*

Then there is a constant $A_{k,N}$ such that for all L, for all intervals J of length $L > 0$, for all functions $f \in C^N(J)$ and for all β, γ satisfying $\gamma \ge \frac{1}{N!}(L/2)^N M_N(f)$, $\beta \ge \frac{2}{(N-k)!}(L/2)^{N-k} M_N(f)$

$$\mathrm{meas}\{x \in J : |f(x)| \le \gamma,\ |f^{(k)}(x)| \ge \beta\} \le A_{k,N}\left(\frac{\gamma}{\beta}\right)^{1/k}.$$

This estimate is an easy consequence of a lemma by M. Christ [1] which is closely related to van der Corputs Lemma on oscillatory integrals. It states that *for any $k \in \mathbb{Z}_+$ there is a constant A_k such that for any interval $I \subset \mathbb{R}$, any $f \in C^k(I)$ and any $\gamma > 0$*

$$(4.9) \qquad \mathrm{meas}\{x \in I : |f(x)| \le \gamma\} \le A_k \gamma^{1/k} \inf_{x \in I} |D^k f(x)|^{-1/k}.$$

Proof of Lemma 4.2. Let a be the midpoint of J and let P the Taylor polynomial of degree $N-1$ expanded about $x = a$. With the specifications on β and γ it follows that $|P(x) - f(x)| \le \gamma$, $|P^{(k)}(x) - f^{(k)}(x)| \le \beta/2$ for all $x \in J$. If $E_{\beta,\gamma}$ denotes the set of all $x \in J$ with $|P(x)| \le 2\gamma$ and $|f^{(k)}(x)| \ge \beta/2$ then

$$\{x \in J : |f(x)| \le \gamma,\ |f^{(k)}(x)| \ge \beta\} \subset E_{\beta,\gamma}.$$

Since P is a polynomial of degree $N-1$ the set $E_{\beta,\gamma}$ is the union of $O(N)$ disjoint intervals. We may apply Christ's estimate (4.9) to each of those intervals and as a result obtain

$$\mathrm{meas}(E_{\beta,\gamma}) \le CNA_k(4\gamma/\beta)^{1/k}.$$

This implies the asserted inequality.

Proof of Proposition 4.1. Fix N so that $2N^{-1} < (\mu + \nu - 2)^{-1} - \gamma$. Let m_N be the maximum of the $C^{N+\nu}$ norm of Δ and the $C^{N+\mu}$ norm of Γ_k in supp χ. Recall that $\partial^j \Gamma_k / (\partial x_1)^j \ne 0$ for $(x, y_1, S(x, y_1)) \in$ supp χ where $(j, k) \preceq (\mu - 1, \nu - 1)$.

Let $A^l(x) = \{y_1 : |\Delta(x, y_1)| \le 2^{-l+1}\}$. If $\mu = 1$ then $j = 0$ so that $\partial_{y_1}^k \Delta \ne 0$ and hence by (4.9) $|A^l(x)| \le C2^{-l/k}$. The asserted L^∞ estimate is immediate.

In what follows we assume that $\mu > 1$ and $\nu > 1$ (the case $\nu = 1$ is easier). Let

$$A_{m,l}(x) = \left\{y_1 : |\Delta(x, y_1)| \le 2^{-l+1},\ 2^{-m} \le \left|\frac{\partial^k \Delta}{(\partial y_1^k)}(x, y_1)\right| \le 2^{-m+1},\ (x, y_1, S(x, y_1)) \in \text{supp } \chi\right\}$$

$$A_{l,l}(x) = \left\{y_1 : \left|\frac{\partial^k \Delta}{(\partial y_1^k)}(x, y_1)\right| \le 2^{-l+1},\ (x, y_1, S(x, y_1)) \in \text{supp } \chi\right\}$$

and let

$$B_m(y) = \{x_1 : |\Gamma_k(y, x_1)| \le 2^{-m+1},\ (x_1, \mathfrak{S}(y, x_1), y) \in \text{supp } \chi\}.$$

For any interval J of length $2^{-l/N} m_N^{-1}$ the set $A_{m,l}(x) \cap J$ has measure $\le C2^{(m-l)/k}$, by Lemma 4.2. Adding these estimates we obtain

$$(4.10) \qquad \mathrm{meas}(A_{m,l}(x)) \le C_N 2^{l/N} 2^{(m-l)/k}$$

uniformly in x. Similarly by (4.9)

$$\text{(4.11)} \qquad\qquad \text{meas}(B_m(y)) \leq C_j 2^{-m/j}$$

uniformly in y.

By Hölder's and Minkowski's inequality

$$\left(\int |R_l f(x)|^p dx \right)^{1/p} \leq \sum_{m \leq l} \left(\int \left[\int_{A_{m,l}(x)} |f(y_1, S(x,y_1))| dy_1 \right]^p dx \right)^{1/p}$$

$$\text{(4.12)} \qquad\qquad \leq C_N 2^{l/N} \sum_{m \leq l} 2^{(m-l)(p-1)/(kp)} \left(\int \int_{A_{m,l}(x)} |f(y_1, S(x,y_1))|^p dy_1 dx \right)^{1/p}.$$

Now note that if $y_1 \in A_{m,l}(x)$ and $y_2 = S(x, y_1)$ (or equivalently $x_2 = \mathfrak{S}(y, x_1)$) then also $x_1 \in B_m(y)$. We interchange the order of integration in (4.12) and perform the change of variable $y_2 = \mathfrak{S}(x_1, x_2, y_1)$, for fixed x_1, y_1:

$$\iint \int_{A_{m,l}(x)} |f(y_1, S(x,y_1))|^p dy_1\, dx_1 dx_2$$

$$= \iint \int_{\{x_2 : y_1 \in A_{m,l}(x_1, x_2)\}} |f(y_1, S(x,y_1))|^p dx_2\, dx_1 dy_1$$

$$= \iint \int_{\{y_2 : y_1 \in A_{m,l}(x_1, \mathfrak{S}(y_1, y_2, x_1))\}} |f(y_1, y_2))|^p \left| \frac{\partial \mathfrak{S}}{\partial y_2} \right| dy_2\, dx_1 dy_1$$

$$\leq \iint \int_{B_m(y)} dx_1 |f(y_1, y_2))|^p \left| \frac{\partial \mathfrak{S}}{\partial y_2} \right| dy_2 dy_1$$

$$\text{(4.13)} \qquad\qquad \leq C_j 2^{-m/j} \iint |f(y_1, y_2))|^p dy_2 dy_1$$

by (4.10). From (4.12) and (4.13) it follows that

$$\|R_l f\|_p \leq C_N 2^{l/N} \sum_{-c \leq m \leq l} 2^{-\frac{m}{jp}} 2^{-\frac{l-m}{k}\left(1 - \frac{1}{p}\right)} \|f\|_p$$

$$\leq C_N 2^{l/N} \sum_{-c \leq m \leq l} 2^{-\frac{m}{(\mu-1)p}} 2^{-\frac{l-m}{\nu-1}\left(1 - \frac{1}{p}\right)} \|f\|_p.$$

If $p = p_0 = (\mu + \nu - 2)/(\mu - 1)$ then this yields

$$\|R_l f\|_{p_0} \leq C(1 + l) 2^{l/N} 2^{-l/(\mu+\nu-2)} \|f\|_{p_0} \leq C_\gamma 2^{-l\gamma} \|f\|_{p_0}.$$

Finally note that after notational changes the above argument also proves the easier L^1 estimate for the case $\nu = 1$. $\square$

5. Regularity of Radon transforms. We shall first consider the case of Radon transforms associated with a family of curves in $\mathbb{R}^2$, where $\mathcal{M}$ is given by the equation $y_2 = S(x, y_1)$, see (4.2). We assume that S is defined in a neighborhood $\mathcal{U}$ of supp χ.

Introducing a dyadic decomposition in the frequency variable we consider the operator

$$\text{(5.1)} \qquad R_l^k f(x) = \iint e^{i\tau(y_2 - S(x,y_1))} \eta(2^l |\Delta(x, y_1)|) a_k(x, y, \tau) f(y)\, d\tau\, dy.$$

where the a_k are symbols of order 0 (with uniform bounds in k), so that $\tau \mapsto a_k(x, y, \tau)$ is supported in the union of dyadic intervals $\pm[2^k, 2^{k+1}]$. The C^∞ function η is assumed to have support in $(1/2, 2)$. The operator R_l^k is bounded on $L^p(\mathbb{R}^2)$, uniformly in l and k, since

$$(5.2) \qquad |R_l^k f(x)| \leq C_N \int 2^k (1 + 2^k |y_2 - S(x, y_1)|)^{-N} |\eta_l(\Delta(x, y_1))| f(y) \, dy.$$

The main estimate of [19] is that

$$(5.3) \qquad \|R_l^k f\|_2 \leq C_\varepsilon 2^{l(\varepsilon + 1/2)} 2^{-k/2} \|f\|_2$$

This estimate is uniform in S, if S varies over a compact subset of $C^\infty(\mathcal{U})$.

Let R_l be defined as in (4.8).

Proposition 5.1. *For* $l \in \mathbb{Z}$

$$\|R_l f\|_{L_{1/2}^2} \leq C_\epsilon 2^{l(\epsilon + \frac{1}{2})} \|f\|_2$$

Proof. Let $\rho \in C_0^\infty(\mathbb{R}^2)$ be supported where $|\xi| \leq 1$ so that $\rho(\xi) = 1$ if $|\xi| \leq 1/2$. Let $\beta_0 = \rho$ and $\beta_k(\xi) = \rho(2^{-k-1}|\xi|) - \rho(2^{-k}|\xi|)$ for $k \geq 1$.

Suppose $l \leq k$. Then for a suitable choice of a_k and a fixed integer M (independent of l and k)

$$\beta_k(D)R_l = \sum_{\kappa=k-M}^{k+M} \sum_{m=k-M}^{k+M} R_l^\kappa \beta_m(D) + E_l^k$$

where

$$(5.4) \qquad \|E_l^k\|_{L^2 \to L^2} \leq C_N 2^{-(k-l)N},$$

(for the proof see estimate (5.7) in [19]). The inequalities (5.3) and (5.4) are used for $k \geq l(1 + \varepsilon)$ and for $k \leq l(1 + \varepsilon)$ we simply use the trivial bounds $\|\beta_k(D)R_l\|_{L^2 \to L^2} = O(1)$. We obtain

$$\|R_l f\|_{L_{1/2}^2} \leq C \Big(\sum_k 2^k \|\beta_k(D)R_l f\|_2^2 \Big)^{1/2}$$

$$\leq C_1 2^{l(1+\varepsilon)/2} \|f\|_2 \; + C_2 \sum_{n_1, n_2 = -M}^{M} \Big(\sum_{k \geq l(1+\varepsilon)} 2^k \|R_l^{k+n_1} \beta_{k+n_2}(D) f\|_2^2 \Big)^{1/2}$$

$$+ C_3 \Big(\sum_{k \geq l(1+\varepsilon)} 2^k \|E_l^k f\|_2^2 \Big)^{1/2}$$

By (5.3) and an orthogonality argument the second expression is dominated by

$$C 2^{l(\varepsilon + 1/2)} \sum_{n_2 = -M}^{M} \Big(\sum_k \|\beta_{k+n_2}(D) f\|_2^2 \Big)^{1/2} \leq C' 2^{l(\varepsilon + 1/2)} \|f\|_2$$

while the third expression is bounded by

$$C \sum_{k \geq l(1+\varepsilon)} 2^{(k-2kN)/2} 2^{lN} \|f\|_2 \leq C' 2^{l/2} \|f\|_2.$$

The assertion follows. $\square$

Proof of Theorem 1.3. By Lemma 2.3 and (2.6) we have $\langle \omega, [X, Y] \rangle_{(x,y)} = a(x, y) \Delta(x, y)$ where a is a C^∞ function, nonvanishing in $\mathcal{U}$. Let $\chi_\gamma = |a|^\gamma \chi$, then $\tilde{\chi}_\gamma$ is smooth and the asserted estimate follows if we can show that for $\gamma > 1/2$ the operator

$$(5.5) \qquad R_\gamma f(x) = \int |\Delta(x, y_1)|^\gamma \chi_\gamma(x, y_1, S(x, y_1)) f(y_1, S(x, y_1)) \, dy_1$$

maps L^2 boundedly to $L_{1/2}^2$.

Let $\eta \in C_0^\infty(1/2, 2)$ so that $\sum_{l=-\infty}^\infty \eta(2^l s) = 1$ for $s > 0$. Define $\tilde{\eta}_\gamma(s) = \eta(s)|s|^\gamma$ (so that η_γ is an admissible cutoff function in (5.1)). Define

$$R_{\gamma,l} f(x) = \int \tilde{\eta}_\gamma(2^l |\Delta(x, y_1)|) \chi_\gamma(x, y_1, S(x, y_1)) f(y_1, S(x, y_1)) \, dy_1.$$

The operator $R_{\gamma,l}$ maps L^2 to $L_{1/2}^2$ with norm bounded by $C_\varepsilon 2^{l(\varepsilon + 1/2)}$, by Proposition 5.1. However $R_\gamma = \sum_l 2^{-l\gamma} R_{\gamma,l}$ and $R_{\gamma,l} = 0$ for large negative l, so the asserted bound for R_γ follows.

Estimates for Radon transforms associated with hypersurfaces. Assume now that $\mathcal{M}$ is a hypersurface in $\mathfrak{X} \times \mathfrak{Y}$. Suppose that $(\mu, \nu) \in \tau(P)$ (that is $\mathcal{M}$ is of type (μ, ν) at P). Then there is a neighborhood $\mathcal{U}$ so that $\mathcal{M}$ is of type $\preceq (\mu, \nu)$ in $\mathcal{U}$ and we assume that the cutoff function χ is supported in $\mathcal{U}$.

The L^p Sobolev estimates of Theorem 1.1 follow from the following result by multiple interpolation.

Proposition 5.2. Let $p = (\mu + \nu)/\mu$, $\alpha < (\mu + \nu)^{-1}$. Then R maps $L_{\mathrm{comp}}^p(\mathfrak{Y})$ boundedly to $L_{\alpha, \mathrm{loc}}^p(\mathfrak{X})$.

Proof. A duality argument shows that it is sufficient to consider the case $\mu \leq \nu$, and since the nondegenerate case is well understood we may assume that $\nu \geq 2$. We may choose coordinates x in $\mathfrak{X}$ and y in $\mathfrak{Y}$ so that $x(P) = 0$, $y(P) = 0$ and $\mathcal{M}$ is given by (3.6), (3.7) near P. Let

$$R^k f(x', v) = \iint \int e^{i\tau(v - S(x', u, y'))} a_k(x, y, \tau) d\tau \, f(y', v) dy' dv$$

where $a_k(x, y, \tau) = \beta_k(\tau)\chi(x, y)$ is a symbol of order 0 supported where $|\tau| \approx 2^k$. Since we do not attempt to prove endpoint results it suffices to show that the $L^p \to L^p$ bound of R^k is $O(2^{-k\alpha})$ for $\alpha < (\mu + \nu)^{-1}$ and $p = (\mu + \nu)/\mu$. Let X_1, Y_1 be as in (3.8), (3.9), and by Theorem 3.6 we may assume that

$$X_1^{\mu-1} Y_1^{\nu-1} B(X_1, Y_1) \neq 0$$

where

$$B(X_1, Y_1) = S_u' S_{x_1 y_1}'' - S_{x_1}' S_{u y_1}''.$$

Split $x' = (x_1, x'')$, $y' = (y_1, y'')$ and define for functions $g(y_1, v)$

$$R_{x'' y''}^k g(x_1, u) = \iint \int e^{i\tau(v - S(x_1, x'', u, y_1, y''))} a_k(x_1, x'', u, y_1, y'', v, \tau) d\tau \, g(y_1, v) dy_1 dv.$$

20

Moreover put $\Delta = B(X_1, Y_1)/S'_u$, then also $X_1^{\mu-1} Y_1^{\nu-1} \Delta \neq 0$. Define

$$\zeta_l(x_1, x'', u, y_1, y'', v) = \eta(2^l |\Delta(x_1, x'', u, y_1, y'')|) \chi(x_1, x'', u, y_1, y'', v),$$

where η satisfies $\sum_l \eta(2^l s) = 1$ for $s > 0$, and

$$R^k_{x''y'',l} g(x_1, u) = \iint \int e^{i\tau(v - S(x_1, x'', u, y_1, y''))} \zeta_l \beta_k(\tau) d\tau \, g(y_1, v) dy_1 dv,$$

$$R_{x''y'',l} g(x_1, u) = \int \zeta_l \chi(x_1, x'', u, y_1, y'', S(x_1, x'', u, y_1, y'')) g(y_1, v) dy_1.$$

By (5.3)

(5.7) $$\|R^k_{x''y'',l} g\|_{L^2(\mathbb{R}^2)} \leq C 2^{-k/2} 2^{l(\varepsilon+1/2)} \|g\|_{L^2(\mathbb{R}^2)}$$

uniformly in x'', y''. By (5.1) we have

$$|R^k_{x''y''} g| \leq C_N \int \frac{2^k}{1 + 2^k |s|} |R_{x''y'',l}[g(\cdot, \cdot + s)]| ds$$

and from Proposition 4.1 it follows that for $p_0 = (\mu + \nu - 2)/(\mu - 1) > 2$

(5.8) $$\|R^k_{x''y'',l} g\|_{L^{p_0}(\mathbb{R}^2)} \leq C_\gamma 2^{-l\gamma} \|g\|_{L^{p_0}(\mathbb{R}^2)}, \quad \gamma < (\mu + \nu - 2)^{-1}.$$

Since $R^k_{x''y'',l}$ is $L^1 \to L^1$ bounded with uniform bound in l, k, x'', y'' we also obtain that

(5.9) $$\|R^k_{x''y'',l} g\|_{L^r(\mathbb{R}^2)} \leq C_\beta 2^{-l\beta} \|g\|_{L^r(\mathbb{R}^2)}, \quad \beta < \frac{1 - r^{-1}}{1 - p_0^{-1}} \frac{1}{\mu + \nu - 2}, \quad 1 < r < p_0.$$

Interpolation of (5.7) and (5.8) yields that for $p = (\mu + \nu)/\mu \in (2, p_0)$

(5.10) $$\|R^k_{x''y'',l} g\|_{L^p(\mathbb{R}^2)} \leq C_\epsilon 2^{-k/(\mu+\nu)} 2^{l\varepsilon} \|g\|_{L^p(\mathbb{R}^2)}.$$

We take a suitable geometric mean of (5.9) and (5.10) and obtain that

(5.11) $$\|R^k_{x''y'',l} g\|_{L^p(\mathbb{R}^2)} \leq C_\alpha 2^{-l\epsilon(\alpha)} 2^{-k\alpha} \|g\|_{L^p(\mathbb{R}^2)}$$

where $\alpha < (\mu + \nu)^{-1}$ and $\varepsilon(\alpha) > 0$ in this range of α. Observe that

$$R^k f(x) = \sum_l \int R^k_{x''y'',l}[f(\cdot, y'', \cdot)] dy''.$$

Since supp χ is compact we see from (5.11) and applications of Minkowski's and Hölder's inequalities that

$$\|R^k\|_{L^p \to L^p} = O(2^{-k\alpha})$$

for $\alpha < (\mu + \nu)^{-1}$. $\quad\square$

We now use the Sobolev estimates to deduce $L^p \to L^q$ estimates. The argument follows known patterns.

Lemma 5.3. *Suppose that R is bounded from L^{p_i} to $L^{p_i}_{\alpha_i}$, with $\alpha_i > 0$, $i = 1, \ldots, N$. Suppose $(1/p, 1/q)$ belongs to the interior of the convex hull of the points $((\alpha_i + p_i^{-1})/(1 + \alpha_i), p_i^{-1}/(1 + \alpha_i))$. Then R is bounded from L^p_{comp} to L^q_{loc}.*

Proof. One uses the obvious fact that for $k > 0$ the $L^1 \to L^\infty$ bound of R^k is $O(2^k)$ (this does not rely on any finite type assumption). The assumed boundedness in Sobolev spaces implies that the $L^{p_i} \to L^{p_i}$ operator norm is $O(2^{-k\alpha_i})$. An interpolation shows that the $L^p \to L^q$ operator norm of R^k is bounded by $C2^{-k\varepsilon(p,q)}$ with $\varepsilon(p,q) > 0$ if $(1/p, 1/q)$ belongs to the half open line segment connecting the points $(1/p_i, 1/p_i)$ and $((\alpha_i + p_i^{-1})/(1 + \alpha_i), p_i^{-1}/(1 + \alpha_i))$ excluding the latter. Interpolating with the trivial $L^1 \to L^1$ and $L^\infty \to L^\infty$ estimates yields $L^p \to L^q$ boundedness for $(1/p, 1/q)$ in the interior of the triangle with corners $(0, 0), (1, 1), ((\alpha_i + p_i^{-1})/(1 + \alpha_i), p_i^{-1}/(1 + \alpha_i))$ and obvious further interpolations yield the assertion. $\square$

The Lemma applied with $p_i = (\mu_i + \nu_i)/\mu_i$, $\alpha_i = (\mu_i + \nu_i)^{-1}$ and then $(\alpha_i + p_i^{-1})/(1 + \alpha_i) = (\mu + 1)/(\mu + \nu + 1)$, $p_i^{-1}/(1 + \alpha_i) = \mu/(\mu + \nu + 1)$ implies the $L^p \to L^q$ inequalities asserted in Theorem 1.2.

We conclude with an observation concerning the boundedness regions $\mathcal{A}(P)$ and $\mathcal{B}(P)$, mentioned in the introduction.

Lemma 5.4. *The regions $\overline{\mathcal{A}}(P)$, $\underline{\mathcal{A}}(P)$ and $\overline{\mathcal{B}}(P)$, $\underline{\mathcal{B}}(P)$ are convex polygonal regions. They have at least three extreme points if and only if $\mathcal{M}$ is of finite type at P. In this case let $n = \min\{\mu + \nu : (\mu, \nu) \in \tau(P)\}$ be the Hörmander type of $\mathcal{M}$ at P. Then there are at most $n - 1$ different pairs (μ_s, ν_s) so that $\mathcal{M}$ is of type (μ_s, ν_s) at $\mathcal{M}$, and the regions $\overline{\mathcal{A}}(P)$, $\underline{\mathcal{A}}(P)$, $\overline{\mathcal{B}}(P)$, $\underline{\mathcal{B}}(P)$ have at most $n + 1$ extreme points .*

Proof. Fix a pair $(\mu, \nu) \in \tau(P)$ so that $\mu + \nu = n$. Let E_1 consist of those integers j with $1 \le j \le \mu$ such that $(j, k) \in \tau(P)$ for some $k \ge 1$. Let E_2 consist of those integers k with $1 \le k \le \nu - 1$ such that $(j, k) \in \tau(P)$ for some $j \ge 1$. For $j \in E_1$ let $k_j = \min\{k : (j, k) \in \tau(P)\}$ and for $k \in E_2$ let $j_k = \min\{j : (j, k) \in \tau(P)\}$. If $\mathcal{M}$ is of type (μ_s, ν_s) at P then (μ_s, ν_s) has to be one of the points (j, k_j), $j \in E_1$ or (j_k, k), $k \in E_2$, and there are at most $n - 1$ such points. Moreover it is easy to see that $\overline{\mathcal{A}}(P)$ is the convex hull of the points (u, β) with $\beta \le 0$ and $0 \le u \le 1$ and the points $(0, 0), (1, 0), (j/(j + k_j), 1/(j + k_j))$, $j \in E_1$ and $(j_k/(j_k + k), 1/(j_k + k))$, $k \in E_2$. Every extreme point of $\overline{\mathcal{A}}(P)$ has to be among these points. Similarly $\underline{\mathcal{B}}(P)$ is the convex hull of the points (u, v) with $v \ge u$ and $0 \le u \le 1$ and the points $(0, 0), (1, 0), ((j + 1)/(j + k_j + 1), j/(j + k_j + 1))$, $j \in E_1$, and $((j_k + 1)/(j_k + k + 1), j_k/(j_k + k + 1))$, $k \in E_2$, and every extreme point of $\overline{\mathcal{B}}(P)$ has to be among these points. A similar consideration applies to $\underline{\mathcal{A}}(P)$, $\underline{\mathcal{B}}(P)$. $\square$

6. Necessary conditions.

Fixing P, we may introduce coordinates vanishing at P, so that (3.6) and (3.7) hold. From Taylor's formula we obtain

$$(6.1) \qquad S(x', x_{n_L}, y')) = \sum_{|\alpha| \le N} \sum_{|\beta| \le N} \rho_{\alpha\beta}(x')^\alpha (y')^\beta + O((|y'| + |x'|)^{N+1} + |x_{n_L}|),$$

and if $\mathcal{M}$ is of not of type (j, k) at P, for $(j, k) \prec (\mu, \nu)$, then

$$(6.2) \qquad \qquad \rho_{\alpha\beta} = 0 \text{ if } (|\alpha|, |\beta|) \prec (\mu, \nu)$$

by Theorem 3.6.

In what follows we use the notation

$$c_{jk} = \Big(\frac{\frac{j}{n_L-1}+1}{\frac{j}{n_L-1}+\frac{k}{n_R-1}+1}, \frac{\frac{j}{n_L-1}}{\frac{j}{n_L-1}+\frac{k}{n_R-1}+1} \Big).$$

Note that except $(0,0)$ and $(1,1)$ all the extreme points of the region $\underline{\mathcal{A}}(P)$ are of the form $c_{\mu\nu}$ for suitable integers $\mu \geq 1$, $\nu \geq 1$.

It suffices to show the necessity of the condition $(1/p, 1/q) \in \underline{\mathcal{B}}(P)$ in Theorem 1.2 since it implies the necessity of the condition $(1/p, \alpha) \in \underline{\mathcal{A}}(P)$ in Theorem 1.1, by Lemma 5.3 and a simple computation.

Lemma 6.1. *Assume $P \in \mathcal{M}$ and $\chi(P) \neq 0$. Suppose that $1 \leq p \leq \infty$ and that R is bounded from $L^p \to L^q$. Then*

(i) $n_R/p \leq 1/q + n_R - 1$ so that $(1/p, 1/q)$ is on or above the line connecting $c_{\mu 1}$ and $(1,1)$, for all $\mu > 0$, and

(ii) $q \leq n_L p$, so that $(1/p, 1/q)$ is on or above the line connecting $c_{1\nu}$ and $(0,0)$.

(iii) Suppose that $\mathcal{M}$ is not of type (μ', ν') for all (μ', ν') with either $\mu' < \mu$ and $\nu' > 0$ or $\mu' = \mu$ and $\nu' < \nu$. Then $\frac{n_R-1}{\nu} + \frac{1}{q} \geq \frac{n_R-1+\nu}{\nu p}$, i.e. $(1/p, 1/q)$ is on or above the line connecting $c_{\mu\nu}$ and $(1,1)$.

(iv) Suppose that $\mathcal{M}$ is not of type (μ', ν') for all (μ', ν') with either $\mu' > 0$ and $\nu' < \nu$ or $\mu' < \mu$ and $\nu' = \nu$. Then $q \leq (n_L - 1 + \mu)p/\mu$; i.e. $(1/p, 1/q)$ is on or above the line connecting $c_{\mu\nu}$ and $(0,0)$.

(v) Suppose that $\mathcal{M}$ is not of finite type at P. Then $p \geq q$.

(vi) Suppose that $c_{\mu\nu}$ and $c_{\mu'\nu'}$ are two extreme points of $\mathcal{B}(P)$ (different from $(0,0)$, $(1,1)$), so that the line segment connecting $c_{\mu\nu}$ and $c_{\mu'\nu'}$ is an edge of $\mathcal{B}(P)$. Then $(1/p, 1/q)$ lies on or above the straight line through $c_{\mu\nu}$ and $c_{\mu'\nu'}$.

Proof. For assertion (i) one chooses $f = \chi_\delta$ to be the characteristic function of a ball of radius δ. Then $|Rf(x)| \geq c\delta^{n_R-1}$ on a set of measure $\geq c_1\delta$ and therefore $\|Rf_\delta\|_q \geq \delta^{n_R-1+1/q}$ while $\|f_\delta\|_p = O(\delta^{n_R/p})$. (ii) follows from (i) by applying the first to the adjoint operator. Similarly (iv) follows from (iii).

To see (iii) we assume that $P = (0,0)$ and assume that $\mathcal{M}$ near P is given by (3.6), (3.7) with $u = x_2$, $v = y_2$. We use (6.1) for large N. Let f be the characteristic function of the cylinder $\{|y'| \leq c_1\delta^{1/\nu}, |y_{n_R}| \leq \delta\}$. Then $|Rf(x)| \geq c\delta^{(n_R-1)/\nu}$ if $|x'| \leq c_2\delta^{1/(N+1)}$ (here we apply (6.1), (6.2)) and $|x_{n_L}| \leq c_3\delta$ for suitable c_1, c_2, c_3. Therefore $\|Rf\|_q \geq \delta^{(n_R-1)/\nu}\delta^{(N+n_R)/(N+1)q}$, but $\|f_\delta\|_p \leq C\delta^{(n_R-1+\nu)/\nu p}$. Therefore the assumed $L^p \to L^q$ boundedness yields the restriction $(n_R-1)/\nu + (N+n_R)/q(N+1) \geq (n_R-1+\nu)/(\nu p)$. (iii) follows from letting $N \to \infty$. Moreover (v) follows immediately from a combination of (i)-(iv).

Finally we turn to the proof of (vi). Let $\widetilde{\mu} = \frac{\mu}{n_L-1}$, $\widetilde{\nu} = \frac{\nu}{n_R-1}$. Suppose that $c_{\mu\nu}$ and $c_{\mu'\nu'}$ are two extreme points of $\underline{\mathcal{B}}(P)$ so that $c_{\mu\nu}$ lies to the left of $c_{\mu'\nu'}$, that is $(\widetilde{\mu}+1)/(\widetilde{\mu}+\widetilde{\nu}+1) \leq (\widetilde{\mu}'+1)/(\widetilde{\mu}'+\widetilde{\nu}'+1)$; this condition is equivalent with $(\widetilde{\mu}+1)\widetilde{\nu}' < (\widetilde{\mu}'+1)\widetilde{\nu}$. Since $c_{\mu\nu}$ is an extreme point it lies below the line through $(0,0)$ and $c_{\mu'\nu'}$ which implies that $\widetilde{\mu} < \widetilde{\mu}'$. Since $c_{\mu'\nu'}$ is an extreme point it lies below the line through $c_{\mu\nu}$ and $(1,1)$ which implies that $\widetilde{\nu}' < \widetilde{\nu}$. Both conditions also imply that $\widetilde{\mu}'\widetilde{\nu} - \widetilde{\mu}\widetilde{\nu}' > 0$. Let

$$a = \frac{1}{n_L-1}\frac{\widetilde{\nu}-\widetilde{\nu}'}{\widetilde{\mu}'\widetilde{\nu}-\widetilde{\mu}\widetilde{\nu}'}, \quad b = \frac{1}{n_R-1}\frac{\widetilde{\mu}'-\widetilde{\mu}}{\widetilde{\mu}'\widetilde{\nu}-\widetilde{\mu}\widetilde{\nu}'}.$$

23

A computation shows that the point (u, v) is below the line through $c_{\mu\nu}$ and $c_{\mu'\nu'}$ if and only if $(b(n_R - 1) + 1)u - (a(n_L - 1) + 1)v > b(n_R - 1)$. Moreover the point c_{jk} is below this line if and only if $aj + bk < 1$. Therefore, $\mathcal{M}$ is not of finite type (j, k) if $aj + bk < 1$, by the definition of $\underline{\mathcal{B}}(P)$.

Now let f_δ be the characteristic function of the cylinder $\{|y'| \leq \delta^b, |y_{n_R}| \leq \delta\}$. By (6.1) with $N \gg a, b$ we see that

$$|S(x, y')| \leq C \sum_{\substack{(j,k) \in \tau(P) \\ 1 \leq j \leq N \\ 1 \leq k \leq N}} |x'|^j \delta^{kb} + O(\delta^{b(N+1)} + |x'|^{N+1}\delta^b + |x_{n_L}|).$$

We choose N large and evaluate for $|x'| \leq c_1 \delta^a$, $|x_{n_L}| \leq c_2 \delta$ to see that $|Rf_\delta(x)| \geq \delta^{b(n_R-1)}$ on a set of measure $> c_2 \delta^{a(n_L-1)+1}$. Therefore $\|Rf_\delta\|_q \geq c\delta^{b(n_R-1)+(a(n_L-1)+1)/q}$ while $\|f_\delta\|_p = O(\delta^{(b(n_R-1)+1)/p})$. Assuming that R is bounded from L^p to L^q and letting $\delta \to 0$ we obtain the restriction $(b(n_R - 1) + 1)/p - (a(n_L - 1) + 1)/q \leq b(n_R - 1)$ and this is precisely the assertion. $\qquad\square$

7. Appendix. We have noticed that our argument used in [19, Lemma 3.6] is correct only in the semitranslation-invariant case. Since this lemma is a technical step leading to inequality (5.3) above we give the proof valid in the general case in this appendix.

Let ρ denote a C^∞ function of the variables $(w, \tau, y_1) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$, supported near $(w, y_1, \tau)_0 = (0, 0, 1)$, in particular $\rho(w, \tau, y_1) = 0$ for $\tau < 1/2$. Let $(w, y_1) \mapsto S(w, y_1)$ be a C^∞ function such that $1/4 \leq \partial_{w_2} S \leq 4$. We assume that if $w, z \in \text{supp } \rho(\cdot, \tau, y_1)$ for any τ, y_1 then the equation $S(z, y_1) - S(w, y_1) = 0$ can be solved in z_1 and in w_1; that is, there is a function v such that

$$S(z_1, v(z_1, w, y_1), y_1) - S(w, y_1) = S(z, y_1) - S(w_1, v(w_1, z, y_1)) = 0.$$

Note that then

$$S(z, y_1) - S(w, y_1) \approx z_2 - v(z_1, w, y_1) \approx v(w_1, z, y_1)) - w_2$$

Let n be a fixed positive integer and $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ such that $0 \leq r_i \leq l + n + C$ for suitable C. Let

$$\sigma_j(x, y_1) = \frac{1}{(j+1)!}\left(\frac{\partial}{\partial z_1}\right)^{j+1}\left[S'_{y_1}(z_1, v(z_1, x, y_1), y_1)\right]\Big|_{z_1 = y_1}.$$

Let $\chi_l^0(s) = \chi_0(2^{l+n+4}s)$ and $\chi_l^{r_i}(s) = \chi_0(2^{l-r_i+n+4}s) - \chi_0(2^{l-r_i+n+3}s)$ and let $\gamma_l^r(x, y_1) = \chi_l(\sigma_0(x, y_1))\prod_{i=1}^n \chi_l^{r_i}(\sigma_i(x, y_1))$. Define

$$(7.1) \qquad K_\lambda^{l,r}(w, z) = \iint e^{i\lambda\tau[S(z,y_1)-S(w,y_1)]}\rho_l^r(w, z, \tau, y_1)\, d\tau\, dy_1$$

with $\rho_l^r(w, z, \tau, y_1) = \rho(w, y_1, \tau)\gamma_l^r(w, y_1)\overline{\rho(z, y_1, \tau)\gamma_l^r(z, y_1)}$. The following is a reformulation of [19, Lemma 3.6].

Lemma 7.1. *Fix $\lambda > 1$ and l, k such that $0 < l < k$ and $2^{k-1} < \lambda \leq 2^k$, and let*

$$I_m^{lk}(z_1) = \{w_1; |w_1 - z_1| \in [2^{l-k}m, 2^{l-k}(m+1)]\}.$$

Let $m \geq 2^{l\varepsilon_1}$ for suitable small $\varepsilon_1 > 0$ and assume that

$$(7.2) \qquad |S_{y_1}(w_1, u(w_1, z, y_1), y_1) - S_{y_1}(z_1, z_2, y_1)| \geq c2^{-l}|w_1 - z_1|$$

for $w_1 \in I_m^{lk}(z_1)$ and for $z_1 \in I_m^{lk}(w_1)$. Then

$$(7.3) \qquad \iint_{w_1 \in I_m^{lk}(z_1)} |K_\lambda^{l,r}(w, z)|\, dw + \iint_{z_1 \in I_m^{lk}(w_1)} |K_\lambda^{l,r}(w, z)|\, dz \leq C_{\epsilon_1}(1 + m)^{-1}2^{l(1+\epsilon_1)}\lambda^{-2}.$$

We shall need

24

Lemma 7.2. *Let $N \geq 2$ and let a be a C^{N+1} function on the interval $[-R, R]$, $R \leq 1$. For $l \geq 1$ let $E_l = \{t \in [-R, R] : |a(t)| > 2^{-l}\}$ and $^c E_l = [-R, R] \setminus E_l$. Suppose that for some $B \geq 1$*

$$\sup_{t \in [-R,R]} |a^{(N)}(t)| \leq B.$$

Then

$$\int_{E_l} \frac{|a'(t)|}{|a(t)|} dt \leq C_{B,N} \, l 2^{l/N}.$$

and

$$\int_{^c E_l} |a'(t)| dt \leq C_{B,N} \, l 2^{-l(1-1/N)}.$$

The first inequality is proved in [19 , Lemma 3.8], the second can be deduced from the first, noticing that

$$\int_{^c E_l} |a'(t)| dt \leq C \sum_{m \geq 0} 2^{-l-m} \int_{E_{l+m}} \frac{|a'(t)|}{|a(t)|} dt \leq C' \sum_{m \geq 0} 2^{-l-m}(l+m) 2^{(l+m)/N} \leq C'' l 2^{-l(1-1/N)}.$$

Actually by a similar argument one could also obtain the first inequality from the second.

Proof of Lemma 7.1. By symmetry it suffices to consider the first term in (7.3), for fixed z_1. In (7.1) we perform first one integration by parts with respect to y_1 and then many integrations by parts with respect to τ. Let $\mathcal{L}$ be the differential operator defined by

$$\mathcal{L}g(y_1, \tau) = \frac{1 - i\lambda\tau(S'_{y_1}(z, y_1) - S'_{y_1}(w, y_1))g'_{y_1}}{1 + |\lambda^2 \tau^2[S'_{y_1}(z, y_1) - S'_{y_1}(w, y_1)]|^2}$$

and $^t\mathcal{L}$ its formal transpose. Then integration by parts yields

$$K_\lambda^{l,r}(w, z) = \int e^{i\lambda\tau[S(z,y_1) - S(w,y_1)]} \frac{\left(1 - \frac{\partial^2}{(\partial\tau)^2}\right)^N [^t\mathcal{L}\gamma_l^r](w, z, \tau, y_1)}{(1 + \lambda^2|S(w, y_1) - S(z, y_1)|^2)^N} d\tau \, dy_1.$$

Let

$$J_{lr}(w, z) = \{y_1; \gamma_l^r(w, y_1)\gamma_l^r(z, y_1) \neq 0\}$$

and for $0 \leq j \leq n$

$$J_{lr}^{j1}(w, z) = \{y_1 \in J_{lr}(w, z); \frac{\partial}{\partial y_1}[\chi_l^{r_j}(\sigma_j(w, y_1))] \neq 0\}.$$

$$J_{lr}^{j2}(w, z) = \{y_1 \in J_{lr}(w, z); \frac{\partial}{\partial y_1}[\chi_l^{r_j}(\sigma_j(z, y_1))] \neq 0\}.$$

Arguing as in [19] one obtains the estimate

$$|K_\lambda^{l,r}(w, z)| \leq C \Bigg[\int_{J_{lr}(w,z)} H_{\lambda,1}^{l,r}(w, z, y_1) \, dy_1 + \int_{J_{lr}(w,z)} H_{\lambda,2}^{l,r}(w, z, y_1) \, dy_1$$

$$+ \sum_{j=0}^{n} \int_{J_{lr}^{j1}(w,z)} H_{\lambda,3}^{l,r,j}(w, z, y_1) \, dy_1 + \sum_{j=0}^{n} \int_{J_{lr}^{j2}(w,z)} H_{\lambda,4}^{l,r,j}(w, z, y_1) \, dy_1 \Bigg]$$

25

where

$$H_{\lambda,1}^{l,r}(w,z,y_1) = (1+\lambda|S_{y_1}'(w,y_1) - S_{y_1}'(z,y_1)|)^{-1}(1+\lambda^2|S(w,y_1) - S(z,y_1)|^2)^{-N}$$

$$H_{\lambda,2}^{l,r}(w,z,y_1) = \frac{\lambda|S_{y_1 y_1}''(w,y_1) - S_{y_1 y_1}''(z,y_1)|}{1+\lambda^2|S_{y_1}'(w,y_1) - S_{y_1}'(z,y_1)|^2}\frac{1}{1+\lambda^2|S(w,y_1) - S(z,y_1)|^2)^N}$$

(7.4)

$$H_{\lambda,j,3}^{l,r}(w,z,y_1) = \frac{2^{l-r_j}|\partial_{y_1}\sigma_j(w,y_1)|}{(1+\lambda|S_{y_1}'(w,y_1) - S_{y_1}'(z,y_1)|)(1+\lambda^2|S(w,y_1) - S(z,y_1)|^2)^N}$$

$$H_{\lambda,j,4}^{l,r}(w,z,y_1) = \frac{2^{l-r_j}|\partial_{y_1}\sigma_j(z,y_1)|}{(1+\lambda|S_{y_1}'(w,y_1) - S_{y_1}'(z,y_1)|)(1+\lambda^2|S(w,y_1) - S(z,y_1)|^2)^N}.$$

We now estimate separately each term in (7.4). For fixed z set $u(w_1,y_1) := v(w_1,z,y_1)$. In the argument to follow the constants in the various estimates will be chosen independently of z. Integrations will always be performed over subsets of a fixed compact set, in particular we shall always assume that $w_1 \in I_m^{lk}(z_1)$, so that $\lambda|w_1 - z_1| \approx 2^l m$, moreover the length of $I_m^{lk}(z_1)$ is $\approx 2^l \lambda^{-1}$. Then

(7.5) $$|S_{y_1}'(w,y_1) - S_{y_1}'(z,y_1)| \geq c_0\big[2^{-l}|w_1 - z_1| - C_0|w_2 - u(w_1,y_1)|,\big]$$

hence

(7.6) $$|S_{y_1}'(w,y_1) - S_{y_1}'(z,y_1)| \geq c'\lambda^{-1}(m+1) \quad \text{if } |w_2 - u(w_1,y_1)| \leq C_0^{-1}2^{-l}|w_1 - z_1|.$$

Note also that
(7.7)
$$\int_{\substack{|w_2-u(w_1,y_1)| \\ \geq C_0^{-1}2^{-l}|w_1-z_1|}} (1+\lambda|w_2 - u(w_1,y_1)|)^{-2N}dw_2\,dy_1\,dw_1 \leq C\lambda^{-2N}(|w_1-z_1|2^{-l})^{1-2N} \leq C'm^{1-2N}\lambda^{-1}.$$

It follows that

$$\iiint_{\substack{w_1 \in I_m^{lk}(z_1) \\ y_1 \in J_{lr}(w,z)}} H_{\lambda,1}^{l,r}(w,z,y_1)\,dy_1\,dw \leq C\Big[\int \frac{2^l}{\lambda|w_1 - z_1|}\int\int (1+\lambda|w_2 - u(w_1,y_1)|)^{-2N}dw_2\,dy_1\,dw_1$$

$$+ \iiint_{\substack{|w_2-u(w_1,y_1)| \\ \geq C_0^{-1}2^{-l}|w_1-z_1|}} (1+\lambda|w_2 - u(w_1,y_1)|)^{-2N}dw_2\,dy_1\,dw_1\Big]$$

$$\leq C\big[(1+m)^{-1} + m^{1-2N}\big]2^l\lambda^{-2}.$$

In order to estimate the second term in (7.4) we wish to apply the Lemma 7.2 (with $N \gg 1/\epsilon_2$, some $\epsilon_2 \in (0,\varepsilon_1)$) to the function

$$y_1 \mapsto a(w_1,y_1) = \frac{S_{y_1}'(w_1,u(w_1,y_1),y_1) - S_{y_1}'(z,y_1)}{w_1 - z_1}.$$

We have $|a(w_1,y_1)| \geq c2^{-l}$ for $w_1 \in I_m^{lk}(z_1)$ and

(7.8) $$\big|a(w_1,y_1)(w_1 - z_1) - [S_{y_1}'(w,y_1) - S_{y_1}'(z,y_1)]\big| \leq C_1|w_2 - u(w_1,y_1)|.$$

26

Moreover

$$
\begin{aligned}
S''_{y_1 y_1}&(w, y_1) - S''_{y_1 y_1}(z, y_1) \\
&= \partial_{y_1} a(w_1, y_1)(w_1 - z_1) - S''_{y_1 x_2}(w_1, u(w_1, y_1), y_1) \partial_{y_1} u(w_1, y_1) \\
&\quad + S''_{y_1 y_1}(w, y_1) - S''_{y_1 y_1}(w_1, u(w_1, y_1), y_1) \\
&= \partial_{y_1} a(w_1, y_1)(w_1 - z_1) + O(|S'_{y_1}(w, y_1) - S'_{y_1}(w_1, u(w_1, y_1), y_1)|) + O(w_2 - u(w_1, y_1)).
\end{aligned}
$$

Here we have used that $\partial_{y_1} u = O(|S'_{y_1}(w, y_1) - S'_{y_1}(w_1, u(w_1, y_1), y_1)|)$; this follows by differentiation of the identity $S(w_1, u(w_1, y_1), y_1) - S(z, y_1) = 0$. Consequently if $|w_2 - u| \le c2^{-l}|w_1 - z_1|$ for small c then

$$
\frac{\lambda |S''_{y_1 y_1}(w, y_1) - S''_{y_1 y_1}(z, y_1)|}{1 + \lambda^2 |S'_{y_1}(w, y_1) - S'_{y_1}(z, y_1)|^2} \le C \frac{2^l}{\lambda |w_1 - z_1|} \left[\frac{|\partial_{y_1} a(w_1, y_1)|}{|a(w_1, y_1)|} + 1 + \lambda |w_2 - u(w_1, y_1)| \right]
$$

while in the case $|w_2 - u| \ge c2^{-l}|w_1 - z_1|$ we have

$$
\frac{\lambda |S''_{y_1 y_1}(w, y_1) - S''_{y_1 y_1}(z, y_1)|}{1 + \lambda^2 |S'_{y_1}(w, y_1) - S'_{y_1}(z, y_1)|^2} \le C \lambda [|w_1 - z_1| + C'|w_2 - u(w_1, y_1)|].
$$

Therefore using Lemma 7.2 we obtain

$$
\iiint_{\substack{w_1 \in I_m^{lk}(z_1) \\ y_1 \in J_{lr}(w,z)}} H_{\lambda,2}^{l,r}(w, z, y_1)\, dy_1\, dw
$$

$$
\begin{aligned}
&\le C \int \frac{2^l}{\lambda |w_1 - z_1|} \int \left[\frac{|\partial_{y_1} a(w_1, y_1)|}{|a(w_1, y_1)|} + 1 \right] \int \frac{1}{(1 + \lambda |w_2 - u(w_1, y_1)|)^{2N-1}} dw_2\, dy_1\, dw_1 \\
&\quad + C \int_{\substack{|w_1 - z_1| \\ \approx 2^l \lambda^{-1} m}} \int \int_{\substack{|w_2 - u| \ge \\ c2^{-l}|w_1 - z_1|}} \frac{2^l m}{(1 + \lambda |w_2 - u(w_1, y_1)|)^{2N-1}} dw_2\, dy_1\, dw_1
\end{aligned}
$$

(7.9)

$$
\le C(\epsilon_2) 2^{l(1+\epsilon_2)} \lambda^{-2} m^{-1} + C' 2^{2l} \lambda^{-2} m^{3-2N}
$$

where $\epsilon_2 > 0$ is arbitrary (we choose $\epsilon_2 < \epsilon_1$). Since we assume that $m \ge 2^{l\epsilon_1}$ we may choose N large enough to obtain the desired estimate for $H_{\lambda,2}^{l,r}(w, z, y_1)$.

The estimate for the term $H_{\lambda,3}^{l,r,j}(w, z, y_1)$ is similar. This time we shall apply Lemma 7.2 to the function

$$
y_1 \mapsto \alpha_j(w_1, y_1) = \sigma_j(w_1, u(w_1, y_1), y_1).
$$

Note that

$$
\partial_{y_1} \sigma_j(w, y_1) = \partial_{y_1} \alpha_j(w, y_1) - \partial_{w_2} \sigma_j(w_1, u(w_1, y_1), y_1) \partial_{y_1} u + \partial_{y_1} \sigma(w, y_1) - \partial_{y_1} \sigma(w_1, u(w_1, y_1), y_1)
$$

and therefore

$$
|\partial_{y_1} \sigma_j(w, y_1) - \partial_{y_1} \alpha_j(w_1, y_1)| \le C_0 \left[|S'_{y_1}(w, y_1) - S'_{y_1}(w_1, u(w_1, y_1), y_1)| + |w_2 - u(w_1, y_1)| \right]
$$

and

$$|\sigma_j(w, y_1) - \alpha_j(w_1, y_1)| \le C_0 |w_2 - u(w_1, y_1)|.$$

In the support of $\partial_{y_1} \chi_l^r(\sigma_j(w, y_1))$ the function $\sigma_j(w, y_1)$ is bounded above by $C' 2^{l-r_i}$. Therefore we have either $|\alpha_j(w_1, y_1)| \le C' 2^{l-r_i+1}$ or $|w_2 - u(w_1, y_1)| \ge c|\alpha_j(w_1, y_1)| \ge C' 2^{l-r_i+1}$. Consequently

$$\iiint_{\substack{w_1 \in I_m^{lk}(z_1) \\ y_1 \in J_{lr}(w,z)}} H_{\lambda,3}^{l,r,j}(w, z)\, dy_1\, dw$$

$$\le C \Big[\int \frac{2^l}{\lambda|w_1 - z_1|} \int_{|\alpha_j| \le C 2^{-l+r_i}} \int \frac{1 + 2^{l-r_i}|\partial_{y_1} \alpha_j(w_1, y_1)|}{(1 + \lambda|w_2 - u(w_1, y_1)|)^{2N-1}} dw_2\, dy_1\, dw_1$$

$$+ \int \frac{2^l}{\lambda|w_1 - z_1|} \int_{|\alpha_j| \ge C 2^{-l+r_i}} \int_{\substack{|w_2 - u| \ge \\ c|\alpha_j|}} \frac{1 + 2^{l-r_i}|\partial_{y_1} \alpha_j(w_1, y_1)|}{(1 + \lambda|w_2 - u(w_1, y_1)|)^{2N-1}} dw_2\, dy_1\, dw_1$$

$$+ C \int_{\substack{|w_1 - z_1| \\ \approx 2^l \lambda^{-1} m}} \int \int_{\substack{|w_2 - u| \ge \\ c2^{-l}|w_1 - z_1|}} \frac{2^l m}{(1 + \lambda|w_2 - u(w_1, y_1)|)^{2N-1}} dw_2 dy_1 dw_1.$$

We first perform the w_2 integration, then use the second Lemma and we obtain, by a straightforward computation, the bound (7.9) for the sum of the three previous expressions. The analogous estimate for $H_{\lambda,4}^{l,r,j}(w, z, y_1)$ is similar (actually simpler, since the terms $\sigma_j(z_1, y_1)$ do not vary in w). The proof is concluded by combining the desired estimates for the terms in (7.4). $\square$

REFERENCES

1. M. Christ, *Hilbert transforms along curves, I. Nilpotent groups*, Ann. Math. **122** (1985), 575–596.

2. _____, *Failure of an endpoint estimate for integrals along curves*, Fourier analysis and partial differential equations, ed. by J. Garcia-Cuerva, E. Hernandez, F. Soria and J. L. Torrea, CRC Press, 1995.

3. M. Christ, A. Nagel, E. M. Stein and S. Wainger, *Singular and maximal Radon transforms*, preprint 1997.

4. A. Greenleaf and A. Seeger, *Fourier integral operators with fold singularities*, J. reine ang. Math. **455** (1994), 35–56.

5. _____, *Fourier integral operators with simple cusps*, preprint (1997).

6. A. Greenleaf and G. Uhlmann, *Composition of some singular Fourier integral operators and estimates for the X-ray transform, I*, Ann. Inst. Fourier (Grenoble) **40** (1990), 443–466.

7. _____, *Composition of some singular Fourier integral operators and estimates for the X-ray transform, II*, Duke Math. J. **64** (1991), 413–419.

8. V. Guillemin, *Cosmology in $(2+1)$-dimensions, cyclic models and deformations of $M_{2,1}$*, Ann. of Math. Studies 121, Princeton Univ. Press, Princeton, 1989.

9. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **19** (1967), 147–171.

10. _____, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.

11. J. J. Kohn, *Boundary behavior of $\overline{\partial}$ on weakly pseudo-convex manifolds of dimension two*, J. Diff. Geom. **6** (1972), 523–542.

12. R. Melrose and M. Taylor, *Near peak scattering and the correct Kirchhoff approximation for a convex obstacle*, Adv. in Math. **55** (1985), 242–315.

13. D. H. Phong, *Singular integrals and Fourier integral operators*, Essays on Fourier analysis in honor of Elias M. Stein, edited by C. Fefferman, R. Fefferman and S. Wainger, Princeton University Press, 1995.

14. D. H. Phong and E.M. Stein, *Hilbert integrals, singular integrals and Radon transforms I*, Acta Math. **157** (1986), 99–157.

15. _____, *Radon transforms and torsion*, International Mathematics Research Notices (1991), 49–60.

16. _____, *Models of degenerate Fourier integral operators and Radon transforms*, Ann. Math. **140** (1994), 703–722.

17. _____, *The Newton polyhedron and oscillatory integral operators*, Acta Math. (to appear).

18. L. P. Rothschild and E. M. Stein, *Hypoelliptic operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

19. A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), 685–745.

20. H. Smith and C. D. Sogge, L^p *regularity for the wave equation with strictly convex obstacles*, Duke Math. J. **73** (1994), 97–153.

21. C.D. Sogge and E.M. Stein, *Averages of functions over hypersurfaces: smoothness of generalized Radon transforms*, J. Analyse Math. **54** (1990), 165–188.

22. E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, 1993.

23. S. Sternberg, *Differential Geometry*, second edition, Chelsea, 1983.

UNIVERSITY OF WISCONSIN, MADISON, WI 53706