Estimates for generalized Radon transforms in three and four dimensions

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ABSTRACT. Sobolev and $L^p - L^q$ estimates for degenerate Fourier integral operators with fold and cusp singularities are discussed. The results for folds yield sharp estimates for restricted X-ray transforms and averages over non-degenerate curves in $\mathbb{R}^3$ and those for cusps give sharp L^2 estimates for restricted X-ray transforms in $\mathbb{R}^4$. In $\mathbb{R}^4$, sharp Lebesgue space estimates are proven for a class of model operators associated to rigid line complexes.

1. Introduction.

Some of the most basic questions in integral geometry concern the mapping properties of generalized Radon transforms. We will describe recent work, focusing on Lebesgue and Sobolev estimates for restricted X-ray transforms and averages over curves in $\mathbb{R}^n$; however, some the results extend to more general transforms.

Let X, Y be manifolds of dimensions n and $Z \subset X \times Y$ a submanifold of dimension $n + k$ such that

$$
\begin{array}{ccc}
& Z & \\
\pi_X \swarrow & & \searrow \pi_Y \\
X & & Y
\end{array}
$$

is a double fibration in the sense of Helgason and Gelfand. Thus, for each $x \in X$, the set $Y_x = \pi_Y \pi_X^{-1}(\{x\}) \subset Y$ is a submanifold of dimension k, and any smoothly varying family of submanifolds of Y can be described in this way. There is also the dual family $\{X^y\}$ of submanifolds of X, here $X^y = \pi_X \pi_Y^{-1}(\{y\})$ for $y \in Y$. Choosing smooth densities, we have generalized Radon transforms

$$
\mathcal{R} : C_0^\infty(Y) \to C^\infty(X), \quad \mathcal{R}f(x) = \int_{Y_x} f
$$

and

$$
\mathcal{R}^t : C_0^\infty(X) \to C^\infty(Y), \quad \mathcal{R}^t g(y) = \int_{X^y} g.
$$

One is interested in finding the optimal estimates for $\mathcal{R}$ of the following three types, listed in essentially increasing order of difficulty. (All of the estimates

considered here are local, mapping compactly supported functions to ones locally of the indicated regularity.)

• L^2-Sobolev estimates: How many derivatives does $\mathcal{R}$ add on L^2? Comparing with the canonical graph case discussed below, for which the answer is $k/2$, we rephrase this as: For which loss of smoothness r do we have

$$\mathcal{R} : L^2_s(Y) \to L^2_{s+\frac{k}{2}-r}(X), \quad \forall s \in \mathbb{R}?$$

Here L^2_s denotes the usual L^2-based Sobolev spaces.

• Lebesgue space estimates: For which p, q does $\mathcal{R} : L^p(Y) \to L^q(X)$? The *type set* $\mathcal{T}$ of $\mathcal{R}$ is the set of $(\frac{1}{p}, \frac{1}{q}) \in [0, 1]^2$ for which this holds. By Riesz-Thorin interpolation, $\mathcal{T}$ is a convex subset of the unit square. The operators we are dealing with are bounded from $L^p \to L^p, 1 \le p \le \infty$, so $\mathcal{T}$ contains the diagonal; since we are working locally, all points above the diagonal will then also be in $\mathcal{T}$, and we will ignore them in statements of results.

• L^p-Sobolev estimates: For $p \ne 2$, what is the sharp amount of smoothing

$$\mathcal{R} : L^p_s(Y) \to L^p_{s+\delta(\mathcal{R},p)}(X), \quad \forall s \in \mathbb{R}?$$

We are particularly interested in how the answers to these questions depend on the geometry underlying $\mathcal{R}$. This is best expressed in terms of the microlocalized diagram

$$
\begin{array}{ccc}
 & C & \\
\pi_L \swarrow & & \searrow \pi_R \\
T^*X \backslash 0 & & T^*Y \backslash 0
\end{array}
$$

where $C = N^*Z' = \{(x, \xi; y, \eta) \in T^*(X \times Y) \backslash 0 : (\xi, -\eta) \perp T_{(x,y)}Z\}$, the twisted conormal bundle of Z, is a canonical relation in $T^*X \times T^*Y$. Since its Schwartz kernel is a smooth density supported on Z, $\mathcal{R}$ is a Fourier integral operator of order $-\frac{k}{2}$ associated with C [16].

If the canonical relation C is a *local canonical graph*, i.e., if the projections $\pi_R : C \to T^*Y$ and $\pi_L : C \to T^*X$ are local diffeomorphisms, then the estimates for $\mathcal{R}$ are well understood. On L^2, there is no loss of derivatives: $\mathcal{R} : L^2_s \to L^2_{s+\frac{k}{2}}(X)$ locally [17]. Since $\mathcal{R}$ and $\mathcal{R}^*$ are bounded on L^∞, interpolation arguments and Littlewood-Paley arguments show that locally $\mathcal{R} : L^p_s(Y) \to L^p_{s+\frac{k}{2}-k|\frac{1}{p}-\frac{1}{2}|}(X), 1 < p < \infty$ (for related L^p results on more general Fourier integral operators see [36] and also [32]). Moreover $\mathcal{R}$ and $\mathcal{R}^*$ satisfy the same $L^p \to L^q$ estimates, so that the type set $\mathcal{T}$ is symmetric about the line of duality, $\{\frac{1}{p} + \frac{1}{q} = 1\}$, specifically the closed triangle, $\mathcal{T} = hull\{(0,0), (1,1), (\frac{n}{2n-k}, \frac{n-k}{2n-k})\}$ (see e.g. Strichartz[39], Littman [18], Oberlin and Stein [27] and Brenner [2]).

An interesting aspect of generalized Radon transforms is that the associated canonical relations are often *not* local canonical graphs; indeed, they provide some of the most important examples of degenerate Fourier integral operators. For example, for a generalized Radon transform which averages over an n-dimensional family of curves in an n-dimensional manifold, $n \ge 3$, there must always be points on the canonical relation C where $d\pi_R$ (and thus $d\pi_L$) drops rank.

To see this, parametrize the curves Y_x as $Y_x = \{\gamma(x,t) : t \in \mathbb{R}\}$; then $Z = \{(x, \gamma(x,t)) : x \in X, t \in \mathbb{R})\}$. Thus, if we set $\Gamma(x,t) = (x, \gamma(x,t))$, then $(\xi, -\eta) \perp TZ \iff D_{x,t}\Gamma^*(\xi, -\eta) = (0,0)$. Hence,

$$C = \{(x, D_x\gamma^*(\eta); \gamma(x,t), \eta) : x \in X, t \in \mathbb{R}, \eta \cdot D_t\gamma = 0\},$$

whence we see that the set of critical values of π_R in T_y^*Y is the envelope of the one-dimensional family of hyperplanes $\{(D_t\gamma(x,t))^\perp : y = \gamma(x,t)\}$, which is necessarily nonempty if $n \geq 3$.

Ideally, one would understand for general canonical relations C how the singularity classes that π_L and π_R belong to control the mapping properties of FIOs associated with C. The operators that we consider have the maximal nondegeneracy possible given the dimensional restrictions, and can be thought of as prototypes for Fourier integral operators associated to canonical relations exhibiting the same types of singularities. Some of the estimates we describe are proved in the generality of Fourier integral operators whose canonical relations exhibit a fold or simple cusp singularity, with the image of the fold or cusp points satisfying a curvature or finite type condition. For $n = 3$, such general results supply the optimal L^2 Sobolev and Lebesgue space estimates for both averages over curves and restricted X-ray transforms. For the Lebesgue space estimates, it turns out that the sharp endpoint results are $L^{\frac{3}{2}} \to L^2$ and/or $L^2 \to L^3$; in higher dimensions the critical (conjectured) estimates do not involve L^2 and we are forced to use a different line of attack to obtain almost sharp results in $n = 4$ for model restricted X-ray transforms and averages over curves.

Situations where conditions are imposed on only one projection from the canonical relation, say π_L, while no assumption is imposed on the other projection, we refer to as *one-sided*, while those for which both projections belong to specified singularity classes are referred to as *two-sided*. Operators associated with two-sided canonical relations of a given singularity type should satisfy better estimates than those just satisfying a one-sided condition. This will be borne out by some of the results below. We begin by discussing two models which exhibit these behaviors.

2. Model examples.

In $\mathbb{R}^n$, let $M_{1,n}$ be the $(2n-2)$-dimensional grassmannian of all affine lines, and $\mathcal{R}_{1,n} : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(M_{1,n})$ the full X-ray transform. For an n-dimensional submanifold (or *line complex*) $\mathcal{C} \subset M_{1,n}$, consider the restricted X-ray transform $\mathcal{R}_\mathcal{C}f = \mathcal{R}f|_\mathcal{C}$; this is a generalized Radon transform associated with the *point-line relation* $Z = \{(x,y) \in \mathcal{C} \times \mathbb{R}^n : y \in x\}$ and thus an FIO of order $-1/2$ associated with $C = N^*Z' \subset T^*\mathcal{C} \times T^*\mathbb{R}^n$.

Now, writing $y = (y', y_n) \in \mathbb{R}^n$ and fixing $\psi \in C_0^\infty(\mathbb{R})$, set

$$\mathcal{R}_n f(x', x_n) = \int_\mathbb{R} f\left(x' + t(x_n, x_n^2, ..., x_n^{n-1}), t\right)\psi(t)dt.$$

Up to smooth factors, $\mathcal{R}_n$ is the restricted X-ray transform associated with a line complex $\mathcal{C}_0$ which is invariant under translations on $\mathbb{R}^n$. We refer to a line complex invariant under translations as *rigid*. Let $Z_n \subset \mathbb{R}^n \times \mathbb{R}^n$ be the corresponding point-line relation. One can show that, where it fails to be a local diffeomorphism, the projection $\pi_R : N^*Z_n' \to T^*\mathbb{R}^n$ has only singularities that belong to the singularity classes $S_{1_k,0}$, $1 \leq k \leq n-2$. For $k = 1$, these are Whitney folds; for $k = 2$, simple (or Whitney) cusps, etc. (See [8] for a discussion of the relevant singularity

theory.) Recall the fundamental result of Whitney [**40**] that mappings between two-manifolds are generically local diffeomorphisms, folds or simple cusps. In general, the $S_{1_k,0}$ singularities play an important role because they are the only stable classes of mappings between manifolds of the same dimension such that the rank of the differential drops by 1. We refer to a canonical relation for which the singularities of one projection are at most folds (without any assumption on the structure of the other projection) as a *one-sided fold*, those for which the singularities of one projection are folds or simple cusps as *one-sided simple cusps*, etc.

Now define

$$\mathcal{A}_n f(x) = \int_{\mathbb{R}} f\Big(x - (t, t^2, ..., t^n)\Big)\psi(t)dt,$$

which is convolution on $\mathbb{R}^n$ with the singular measure $\mu_n = \psi(t)dt$ supported on the model nondegenerate curve $\{(t, t^2, ..., t^n) : t \in \mathbb{R}\}$. Then, $\mathcal{A}_n \in I^{-\frac{1}{2}}(C_n; \mathbb{R}^n, \mathbb{R}^n)$, with *both* projections $\pi_R, \pi_L : C_n \to T^*\mathbb{R}^n$ having singularities at most of type $S_{1_k,0}$, $1 \le k \le n-2$; we will refer to such canonical relations as *two-sided* folds, simple cusps, etc. To see that C_n is in fact has this structure, write

$$Z = \{(x, y) : f_2(x, y) = ... = f_n(x, y) = 0\},$$

where $f_j(x, y) = x_j - y_j - (x_1 - y_1)^j$, $2 \le j \le n$. The twisted conormal bundle is spanned by the $\nabla' f_j = (\nabla_x f_j(x, y), -\nabla_y f_j(x, y))$; letting $t = x_1 - y_1$, we find that

$$C = \Big\{\Big(x_1, ..., x_n, -\sum_{j=2}^{n} jt^{j-1}\theta_j, \theta_2, ..., \theta_n;$$

$$x_1 - t, ..., x_n - t^n, -\sum_{j=2}^{n} jt^{j-1}\theta_j, \theta_2, ..., \theta_n\Big)$$

$$: x \in \mathbb{R}^n, t \in \mathbb{R}, \theta \in \mathbb{R}^{n-1}\backslash 0\Big\}.$$

With $\phi(x, t, \theta) = -\sum_{j=2}^{n} jt^{j-1}\theta_j$, we have that $\nabla(\frac{\partial \phi}{\partial t}), ..., \nabla(\frac{\partial^{n-1}\phi}{\partial t^{n-1}})$ are linearly independent, and hence π_L has only singularities of type $S_{1_k,0}$, $1 \le k \le n-2$. Reparametrizing C with coordinates y, t, θ, one easily sees the same for π_R.

Since $S_{1_k,0}$ singularities are stable (with respect to perturbations in the C^{k+1} topology), restricted X-ray transforms and generalized Radon transforms given by perturbations of $\mathcal{R}_n$ and $\mathcal{A}_n$ will be one- and two-sided cusps, resp.

More generally, we may consider a line complex $\mathcal{C} \subset M_{1,n}$ with point-line relation $Z \subset \mathcal{C} \times \mathbb{R}^n$; then, the restricted X-ray transform $\mathcal{R}_{\mathcal{C}} = \mathcal{R}_{1,n}|_{\mathcal{C}}$ is a Fourier integral operator of order $-1/2$ with canonical relation $C = N^*Z'$. If we work on an open set in $\mathbb{R}^n$ over which $\pi_{\mathbb{R}^n} : Z \to \mathbb{R}^n$ is a submersion, then each $\mathcal{C}^y = \pi_{\mathcal{C}} \pi_{\mathbb{R}^n}^{-1}(y) \subset G_{1,n}^y \simeq \mathbb{R}P^{n-1}$ is a smooth curve. We define $\mathcal{C}$ to be *well-curved* if each $\mathcal{C}^y$ is a nondegenerate curve in $\mathbb{R}P^{n-1}$, i.e., for any parametrization $\sigma(t)$, the vectors $\sigma', \sigma'', ..., \sigma^{(n-1)}$ are linearly independent at each point. Thus, for $n = 3$, each $\mathcal{C}^y$ has nonzero curvature in $\mathbb{R}P^2$, while, for $n = 4$, each $\mathcal{C}^y$ has nonzero curvature and torsion in $\mathbb{R}P^3$. One can show [**10**] that $\mathcal{C}$ well-curved $\implies \pi_R : N^*Z' \to T^*\mathbb{R}^n\backslash 0$ has singularities of type $S_{1_k,0}$ for $1 \le k \le n-2$, and thus $C = N^*Z'$ is a one-sided cusp. Actually, C satisfies a somewhat stronger condition: For each y, $\pi_R|_{\pi_R^{-1}(T_y^*\mathbb{R}^n)}$ is an $S_{1_{n-2},0}$; we refer to C as a *strong* one-sided cusp. (The projections $\pi_R, \pi_L : C_n \to T^*\mathbb{R}^n$ are also strong cusps.) Folds

are automatically strong, but simple and higher cusps are not, and the fact that π_R is a strong cusp allows us to prove better $L^p \to L^2$ estimates than can be proven if π_R is merely a cusp.

Similarly, if we look at generalized Radon transforms associated to families of curves in three and four dimensions, it is possible to give criteria for when π_R and π_L are folds or cusps. One way of describing families of curves is via the exponential map. Let $\{\gamma_x(\cdot) : x \in \mathbb{R}^n\}$ be a smooth family of curves in $\mathbb{R}^n$ such that $\gamma_x(0) = x, \quad \forall x$. There exist ([4]) unique vector fields $X_1, X_2, \ldots$ such that

$$\gamma_x(t) \simeq exp_x(tX_1 + t^2 X_2 + t^3 X_3 + \ldots)$$

to infinite order in t. Motivated by this, consider a generalized Radon transform in $\mathbb{R}^3$ associated to a family of curves $\gamma_x(t) = exp_x(tX + t^2 Y + t^3 Z)$, where we assume that X and Y are linearly independent vector fields. One has

PROPOSITION 2.1. *(Phong and Stein [29]) Let* $\Gamma = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : y = exp_x(tX + t^2 Y + t^3 Z), \quad some \quad t \in \mathbb{R}\}$. *Then* $\pi_R : N^*\Gamma' \to T^*\mathbb{R}^3$ *(resp.,* $\pi_L : N^*\Gamma' \to T^*\mathbb{R}^3$*), has at most Whitney fold singularities near the diagonal* $\{x = y\}$ *iff* $X, Y,$ *and* $Z - \frac{1}{6}[X,Y]$ *(resp.,* X, Y *and* $Z + \frac{1}{6}[X,Y]$*) are linearly independent.*

In $\mathbb{R}^4$, there is a similar result for simple cusps. We take as our family of curves $\gamma_x(t) = exp_x(tX + t^2 Y + t^3 Z + t^4 W)$ and form $\Gamma \subset \mathbb{R}^4 \times \mathbb{R}^4$ analogously.

PROPOSITION 2.2. *([10]) If* $X, Y, Z - \frac{1}{6}[X,Y]$ *and* $W - \frac{1}{4}[X,Z] + \frac{1}{24}[X,[X,Y]]$ *are linearly independent, then the singularities of* $\pi_R : N^*\Gamma' \to T^*\mathbb{R}^4$ *near the diagonal are Whitney folds or strong simple cusps. If* $X, Y, Z + \frac{1}{6}[X,Y]$ *and* $W + \frac{1}{4}[X,Z] + \frac{1}{24}[X,[X,Y]]$ *are linearly independent, then the same is true for* π_L.

3. L^2 estimates.

We now summarize some of the known estimates for generalized Radon transforms, or more general Fourier integral operators, with fold and cusp singularities.

- *Folds*

If X and Y are manifolds of the same dimension, and $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ is a two-sided fold (referred to as a *folding canonical relation* in [20]), then the fundamental results of Melrose and Taylor imply that there is a loss of 1/6 derivative on L^2:

$$A \in I^m(C; X, Y) \implies A : L_s^2(Y) \to L_{s-m-\frac{1}{6}}^2(X)$$

locally.

If C is merely a one-sided fold, then the loss of derivatives is higher: by the results of [9],

$$A \in I^m(C; X, Y) \implies A : L_s^2(Y) \to L_{s-m-\frac{1}{4}}^2(X).$$

Comech [5] has shown that if one projection is a Whitney fold and the other is an $S_{1_k,0}$, then there is a loss of $\frac{1}{2}\frac{k}{2k+1}$ derivatives; for $k = 1$ this is the result of [20], while as $k \to \infty$ the second projection becames increasingly degenerate and the loss tends toward the loss of [9] for one-sided folds. The latter is sharp if the other projection is a blowdown (*i.e.*, maximally degenerate); this situation had been already dealt with in Greenleaf and Uhlmann [14].

- *Cusps*

If C is a one-sided simple cusp, then it was shown in [10] that

$$A \in I^m(C; X, Y) \implies A : L^2_s(Y) \to L^2_{s-m-\frac{1}{3}}(X)$$

and an analysis of $\mathcal{R}_4$ shows that one cannot do better in general. Estimates for general two-sided simple cusps have not yet been established; the conjectured loss of derivatives is $1/4$.

Since restricted X-ray transforms are of order $-1/2$ the following result is a special case of the above estimates for Fourier integral operators with one sided fold or cusp singularities.

THEOREM 3.1. ([9,10]) *Let $\mathcal{R}_\mathcal{C}$ denote the restriction of the X-ray transform on $\mathbb{R}^n$ to a well-curved line complex $\mathcal{C} \subset M_{1,n}$.*
　(i) *If $n = 3$ then $\mathcal{R}_\mathcal{C}$ maps $L^2_s(\mathbb{R}^3) \to L^2_{s+\frac{1}{4}}(\mathcal{C})$ locally.*
　(ii) *If $n = 4$ then $\mathcal{R}_\mathcal{C}$ maps $L^2_s(\mathbb{R}^3) \to L^2_{s+\frac{1}{6}}(\mathcal{C})$ locally.*

Similarly, it follows from the work of Melrose and Taylor [20] that an averaging operator associated to a family of curves in $\mathbb{R}^3$ whose canonical relation is a two-sided fold (e.g., satisfying both conditions in Proposition 1) maps $L^2_s(\mathbb{R}^3) \to L^2_{s+\frac{1}{3}}(\mathbb{R}^3)$.

That these estimates are sharp can be seen for the model restricted X-ray transforms $\mathcal{R}_n$ by considering test functions whose Fourier transforms are supported in a tubular neighborhood of a ray in the fold [13] or cusp [10] directions. For the averaging operator $\mathcal{A}_n$, simply note that $\widehat{\mathcal{A}_n f}(\xi) = \widehat{\mu_n}(\xi)\widehat{f}(\xi)$ and $|\widehat{\mu_n}(\xi)| \le c(1+|\xi|)^{-\frac{1}{n}}$ by van der Corput's lemma (and no better.)

4. $L^p \to L^q$ estimates.

If $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ is a one-sided fold, with singular set $\Sigma^1 = \Sigma^1(\pi_R)$, where π_R drops rank by 1, then Σ^1 is a smooth hypersurface. Furthermore, if $\pi_Y : C \to Y$ is a submersion, each $\Sigma_y = \pi_R(\Sigma^1) \cap T^*_y Y$ is a smoothly immersed conic hypersurface in $T^*_y Y$. If $C \subset T^*\mathcal{C} \times T^*\mathbb{R}^3$ is the canonical relation for a well-curved line complex $\mathcal{C} \subset M_{1,3}$, then in fact each Σ_y has one nonzero principal curvature at each point ([9]). (This is maximal, since Σ_y is flat in the radial direction.) The sharp Lebesgue space estimates for $\mathcal{R}_\mathcal{C}$ in three dimensions then follow from

THEOREM 4.1. ([9]) (a) *Let $n = 3$ and assume that $\pi_R : C \to T^*Y\backslash 0$ is a Whitney fold. If $\mathcal{F} \in I^{-\frac{1}{2}}(C; X, Y)$ then, locally, $\mathcal{F} : L^{\frac{8}{5}}(Y) \to L^2(X)$.*
(b) *Furthermore, if each $\Sigma_y = \pi_R(\Sigma^1) \cap T^*_y Y$ has one nonzero principal curvature at each point, then $\mathcal{F} : L^{\frac{3}{2}}(Y) \to L^2(X)$.*

This is actually a special case of a more general result in [9], valid for general n and assuming that at least k principal curvatures of Σ_y are $\ne 0$, $0 \le k \le n-2$.

That Theorem 4.1(b) gives the best possible $L^p \to L^q$ estimates for restricted X-ray transforms on $\mathbb{R}^3$ can be seen by considering the model operators $\mathcal{R}_n$. For each $0 < \delta < 1$, let $f_\delta(y) = \chi_{\{|y|<\delta\}}$. One sees easily that $\mathcal{R}_n f(x) \ge c\delta$ on the rectangle $\{|x'| < \delta, |x_n| < 1\}$; thus,

$$\|f_\delta\|_{L^p} \simeq c\delta^{n/p} \quad and \quad \|\mathcal{R}_n f_\delta\|_{L^q} \ge c\delta^{1+\frac{n-1}{q}}.$$

In order for $\|\mathcal{R}_n f\|_{L^q} \le c\|f\|_{L^p}$ to hold for all $f \in L^p$, we must have $c'\delta^{\frac{n}{p}} \ge c''\delta^{1+\frac{n-1}{q}}$; letting $\delta \to 0$, we find the necessary condition $\frac{n}{p} \le 1 + \frac{n-1}{q}$. On the

other hand, if we use the family of test functions

$$g_\epsilon(y) = \chi_{\{|y_1| \le \epsilon, |y_2| \le \epsilon^2, \ldots, |y_{n-1}| \le \epsilon^{n-1}, |y_n| \le 1\}},$$

we see that $\mathcal{R}_n g_\epsilon(x) \ge c$ on

$$\{|x_1| \le \epsilon, |x_2| \le \epsilon^2, \ldots, |x_{n-1}| \le \epsilon^{n-1}, |x_n| \le \epsilon\},$$

so that

$$||g_\epsilon||_{L^p} \simeq \epsilon^{\frac{n(n-1)}{2p}} \quad and \quad ||\mathcal{R}_n g_\epsilon||_{L^q} \ge c\epsilon^{\frac{n^2-n+2}{2q}},$$

and taking $\epsilon \to 0$ we find that we must have $\frac{1}{q} \ge \frac{n(n-1)}{n^2-n+2}\frac{1}{p}$. Combining these two restrictions, we see that $\mathcal{R}_n : L^p_{\text{comp}}(\mathbb{R}^n) \to L^q_{\text{loc}}(\mathcal{C}), p \le q \implies (\frac{1}{p}, \frac{1}{q}) \in \mathcal{T}_n$, where $\mathcal{T}_n$ is the closed triangle

$$\mathcal{T}_n = hull\{(0,0), (1,1), (\frac{n^2-n+2}{n(n+1)}, \frac{n-1}{n+1})\}.$$

The conjectured sharp estimate in n dimensions, from which the optimal results would follow by interpolation, is thus $\mathcal{R}_\mathcal{C} : L^{\frac{n(n+1)}{n^2-n+2}} \to L^{\frac{n+1}{n-1}}$. In three dimensions, the desired $L^{\frac{3}{2}} \to L^2$ estimate is given by Theorem 4.1(b) yielding $L^p \to L^q$ boundedness in all of $\mathcal{T}_3$.

Now, if C is a two-sided fold with the images of the fold surface under both projections well-curved, i.e., with all $\Sigma_y = \pi_R(\Sigma^1) \cap T_y^* Y$ and $\Sigma^x = \pi_L(\Sigma^1) \cap T_x^* X$ having one principal curvature nonzero, then Thm. 1(b) can be applied to both $\mathcal{F} \in I^{-\frac{1}{2}}(C; X, Y)$ and $\mathcal{F}^* \in I^{-\frac{1}{2}}(C^t; Y, X)$; the $L^{3/2} \to L^2$ boundedness of F^* then implies the $L^2 \to L^3$ boundedness of $\mathcal{F}$ and thus we have that the type set of $\mathcal{F}$ contains the trapezoid $hull\{(0,0), (1,1), (\frac{1}{2}, \frac{2}{3}), (\frac{1}{3}, \frac{1}{2})\}$. This was first proved by Oberlin [22] for the model operator $\mathcal{A}_3$ and similar translation-invariant operators. Modifying the test functions f_δ, g_ϵ above, one can show that a necessary condition for $(\frac{1}{p}, \frac{1}{q})$ to be in the type set of $\mathcal{A}_n$ is

$$(\frac{1}{p}, \frac{1}{q}) \in \tilde{\mathcal{T}}_n = hull\{(0,0), (1,1), (\frac{n^2-n+2}{n(n+1)}, \frac{n-1}{n+1}), (\frac{2}{n+1}, \frac{n-2}{n^2-n+2})\},$$

which is a closed trapezoid symmetric about the line of duality. (This restriction was apparently first observed by Carbery and Christ.) Thus, the result of Oberlin (which was extended to curves in $\mathbb{R}^3$ with nonzero curvature and torsion in [28]) is sharp, and Theorem 4.1 provides the same estimates for averages over not necessarily translation invariant families of curves in $\mathcal{R}^3$ having the same underlying geometry.

An instructive example is obtained as follows. Equip $\mathbb{R}^3$ with the Heisenberg group structure $x \cdot y = \left(x' + y', x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right)$ and define the right translations $\tau_y^R(x) = x \cdot y$. For a curve γ, form the convolution operator

$$A_\gamma f(x) = \int f(x \cdot (\gamma(s))^{-1})\chi(s)ds.$$

Define the auxiliary curve $s \mapsto \gamma_R'(s)$ by $\gamma_R'(s) = (d\tau_{\gamma(s)}^R)^{-1}\gamma'(s)$, so that γ_R' takes values in the Lie algebra and hence can be differentiated further. Secco [33] showed that the optimal $L^{3/2} \to L^2$ estimate holds if one assumes the linear independence of $\gamma_R', \gamma_R'', \gamma_R'''$. (For the model family $\gamma_\alpha(s) = (s, s^2, \alpha s^3)$, $\alpha \in \mathbb{R}$, this condition holds if and only if $\alpha \ne -1/6$.) The operator A_γ is a Fourier integral

operator of order $-1/2$ and it can be shown that the linearly independence of $\gamma_R', \gamma_R'', \gamma_R'''$ implies the validity of the assumptions in Theorem 4.1(b).

In higher dimensions, one can obtain the sharp $L^p \to L^2$ estimates for the model restricted X-ray transform $\mathcal{R}_n$ ([10]), namely $\mathcal{R}_n : L^{\frac{2n(n-1)}{n^2-n+2}} \to L^2$. Moreover, in [10] the sharp $L^{12/7} \to L^2$ estimate is established for general X-ray transforms associated to well-curved complexes. Furthermore, Oberlin[24] has shown that the model operator $\mathcal{R}_n$ is of restricted weak type, from $L^{\frac{n}{n-1},1} \to L^{\frac{n-1}{n-2},\infty}$, where $L^{p,q}$ denote the standard Lorentz spaces ([38]). Interpolating these with the $L^1 \to L^1$ and $L^\infty \to L^\infty$ estimates, one obtains $L^p \to L^q$ boundedness in an asymmetric trapezoid strictly contained in $\tilde{\mathcal{T}}_n$ (but containing some of the boundary points). However, the (almost) sharp $L^p \to L^q$ estimats are currently known only for rigid line complexes in four dimensions [11]; although such $\mathcal{R}_\mathcal{C}$'s are models for Fourier integral operators with one-sided simple cusps, the following result is not known in that generality.

THEOREM 4.2. ([11]) Let $\Gamma \subset \mathbb{R}P^3$ be a curve with nonzero curvature and torsion, and let $\mathcal{C} \subset M_{1,4}$ be the rigid line complex consisting of all lines parallel to lines in Γ. Then $\mathcal{R}_\mathcal{C} : L^p(\mathbb{R}^4) \to L^q(\mathcal{C})$ locally for $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{T}_4 \backslash \{(\frac{7}{10}, \frac{3}{5})\}$, and $\mathcal{R}_\mathcal{C}$ is restricted weak type, $\mathcal{R}_\mathcal{C} : L^{\frac{10}{7},1}(\mathbb{R}^4) \to L^{\frac{5}{3},\infty}(\mathcal{C})$.

The proof of Theorem 4.2 which will be sketched below, is based on an argument of Oberlin [23,II] dealing with the two-sided operator $\mathcal{A}_4$. In [23,II], it was shown that $\mathcal{A}_4 : L^p(\mathbb{R}^4) \to L^q(\mathbb{R}^4)$ for all $(\frac{1}{p}, \frac{1}{q})$ in the interior of $\tilde{\mathcal{T}}_4$ and some points on the boundary; modifying the proof, replacing an analytic family of operators with a dyadic decomposition and using an interpolation argument of Bourgain [1], it is possible to show ([11]) that $\mathcal{A}_4$ is bounded for all $(\frac{1}{p}, \frac{1}{q}) \in \tilde{\mathcal{T}}_4 \backslash \{(\frac{7}{10}, \frac{3}{5}), (\frac{2}{5}, \frac{3}{10})\}$, with a substitute restricted weak type result at the vertices: $\mathcal{A}_4 : L^{\frac{10}{7},1} \to L^{\frac{5}{3},\infty}$ and $L^{\frac{5}{2},1} \to L^{\frac{10}{3},\infty}$. M. Christ has now established the analogous result in all dimensions:

THEOREM 4.3. (Christ [3]) Let $P_n = (p_n, q_n) = (\frac{n^2-n+2}{n(n+1)}, \frac{n-1}{n+1})$ and $P_n^* = (q_n', p_n')$. Then $\mathcal{A}_n : L^P(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ locally for all $(\frac{1}{p}, \frac{1}{q}) \in \tilde{\mathcal{T}}_n \backslash \{P_n, P_n^*\}$, and at P_n, P_n^* $\mathcal{A}_n$ is restricted weak type: $\mathcal{A}_n : L^{p_n,1} \to L^{q_n,\infty}$ and $L^{q_n',1} \to L^{p_n',\infty}$.

The sharp $L^p \to L^{p'}$ boundedness of $\mathcal{A}_n$ had previously been obtained by McMichael [19]. The $L^p \to L^q$ boundedness of $\mathcal{A}_n$ at P_n, P_n^*, or failure thereof, is currently unknown for $n \geq 4$.

5. L^p Sobolev estimates.

Very little about the $L^p \to L^p_\delta$ mapping properties of degenerate Fourier integral operators is understood, except in two dimensions ([29,30,31][34,35]). Estimates for two-sided folds, for more general Fourier integral operators, have been established in [37]. These results have been extended in [6] to allow one of the projections to satisfy a type k condition. These results are optimal when the singular support is a hypersurface, but presumably not in the case of higher codimension singular support which is relevant for restricted X-ray transforms and averages over curves.

Recently, Oberlin and Smith [25] have shown that some of the L^p Sobolev estimates that one might have expected for a variant of $\mathcal{A}_3$, namely, convolution with the measure $\mu = dt$ on the helix $\gamma(t) = (\cos(t), \sin(t), t)$, in fact fail: For $0 < |\frac{1}{p} - \frac{1}{2}| < \frac{1}{4}$, the loss must be worse than simply $\max(\frac{1}{6}, |\frac{1}{p} - \frac{1}{2}|)$ derivative. Obtaining the correct estimates is related to some central problems in harmonic analysis ([26]) and seems to be very difficult.

6. Sketch of the proof of Theorem 4.2.

We now indicate the outlines of the argument of [23,II], as modified to prove Theorem 4.2 ([11]). We use coordinates $y = (y', y_4) \in \mathbb{R}^4$, $x = (x', x_4) \in \mathcal{C}$. For simplicity, we may assume that Γ lies in $\{|y_4| > c|y'|\}$, and thus rewrite

$$\mathcal{R}_\mathcal{C} f(x', x_4) = \int_\mathbb{R} f(x' - t\gamma(x_4), t)\psi(t)dt,$$

where $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ is a smooth curve with nonzero curvature and torsion, and $\psi(t) \in C_0^\infty$ is a fixed cut-off function. To show that $\mathcal{R}_\mathcal{C} : L^{\frac{10}{7},1}(\mathbb{R}^4) \to L^{\frac{5}{3},\infty}(\mathcal{C})$, it suffices to show that $\mathcal{R}_\mathcal{C}^*$ is restricted weak type between the dual spaces: $\mathcal{R}_\mathcal{C}^* : L^{\frac{5}{2},1}(\mathcal{C}) \to L^{\frac{10}{3},\infty}(\mathbb{R}^4)$. In other words, we need to show that, for any measurable set $E \subset \mathcal{C}$ and any $\lambda > 0$,

$$(1) \qquad |\{y \in \mathbb{R}^4 : |\mathcal{R}_\mathcal{C}^*(\chi_E)(y)| > \lambda\}| \le c\left(\frac{\|\chi_E\|_{L^{\frac{5}{2}}}}{\lambda}\right)^{\frac{10}{3}} = c\frac{|E|^{\frac{4}{3}}}{\lambda^{\frac{10}{3}}}.$$

Note that, if $\mathcal{R}_\mathcal{C}^*$ were weak type $(L^{\frac{5}{2}}, L^{\frac{10}{3}})$, then this inequality would hold with the characteristic function $\chi_E(x)$ replaced with any $g \in L^{\frac{5}{2}}(\mathcal{C})$; restricted weak type means we only have this inequality for characteristic functions. Now,

$$\mathcal{R}_\mathcal{C}^* g(y', y_4) = \int_\mathbb{R} f(y' + y_4\gamma(t), t)\psi(t)dt;$$

for each $y \in \mathbb{R}^4$, this integrates g over a curve in $\mathcal{C}$, namely the dual curve $\mathcal{C}^y$. Following [23,II], we extend this curve to a two-dimensional surface in order to take advantage of the nondegeneracy of γ. For this problem, the appropriate way to do this is as follows

Let η be an even Schwartz function on $\mathbb{R}$ with the property that $\hat{\eta}(\tau) = 1$ for $|\tau| \le 1/2$ and $\hat{\eta}(\tau) = 0$ for $|\tau| \ge 1$. For $k \in \mathbb{Z}$ define

$$B_k g(y', y_4) = \iint g(x' + y_4\gamma(t) + u\gamma'(t), t)\psi(t)2^k\eta(2^k u)dt\,du.$$

The family $\{2^k\eta(2^k \cdot) : k \in \mathbb{Z}\}$ forms an approximate identity as $k \to +\infty$ and so, for $g \in C_0^\infty$, $\mathcal{R}_\mathcal{C}^* g = \lim_{k \to \infty} B_k g$. Theorem 4.2 is a consequence of the following two estimates, which hold for all $k \in \mathbb{Z}$:

$$(2) \qquad \|(\mathcal{R}_\mathcal{C}^* - B_k)g\|_{L^2(\mathbb{R}^4)} \lesssim 2^{-k/2}\|g\|_{L^2(\mathcal{C})}$$

$$(3) \qquad \|B_k g\|_{L^6(\mathbb{R}^4)} \lesssim 2^{k/3}\|g\|_{L^3(\mathcal{C})}.$$

Theorem 4.2 follows by applying Tshebyshev's inequality and (2), (3) with the choice $2^k \simeq \lambda^{4/3}/|E|^{1/3}$ (see [1] for a previous application of this argument).

To obtain (3) and (2) one may assume that the lines $\{l_{x,\alpha}\}$ in $\mathcal{C}$, for $x \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ are given by $y = (x + t\gamma(\alpha), t)$ where $\gamma', \gamma'', \gamma'''$ are linearly independent. Estimate (3) is obtained by modifying the proof in [23,II] for the operator $\mathcal{A}_4$

to the present context. Estimate (2) is proved by using a T^*T argument from [21]. Denote by $\widetilde{f}$ the partial Fourier transform of f with respect to the variables $x' \in \mathbb{R}^3$ and let $\widehat{\zeta}(\tau) = \eta(\tau) - 2\eta(2\tau)$. Then

$$\|B^*_{\ell+1}f - B^*_\ell f\|^2_2 = (2\pi)^{-3}2^{-\ell} \iiint \widetilde{f}(\xi,t)\overline{\widetilde{f}(\xi,t')}h_\ell(t'-t,\xi)\,dt'dt\,d\xi$$

where

$$h_\ell(\sigma,\xi) = 2^\ell \int \chi_0(\alpha)|\zeta(2^{-\ell}\langle\gamma'(\alpha),\xi\rangle)|^2 e^{i\sigma\langle\gamma(\alpha),\xi\rangle}\,d\alpha;$$

here, χ_0 is a suitable smooth cutoff with small support. To prove the desired estimate one has to show that the functions $\sigma \mapsto h_\ell(\sigma,\xi)$ are in $L^1(\mathbb{R})$, uniformly in $\ell \in \mathbb{Z}$ and $\xi \in \mathbb{R}^3$. The behavior of $h_\ell(\sigma,\xi)$ can be expressed by using the distance $d_\Sigma(\xi)$ to the binormal cone Σ consisting of all $\xi \in \mathbb{R}^3$ for which $\langle\gamma'(s),\xi\rangle = \langle\gamma''(s),\xi\rangle = 0$ for some $s \in \operatorname{supp}\chi_0$. Specifically one can prove the inequality [11]

$$|h_\ell(\sigma,\xi)| \lesssim \frac{a_\ell(\xi)}{(1+a_\ell(\xi)\sigma)^2} \quad \text{where } a_\ell(\xi) = \frac{2^{2\ell}}{\sqrt{|\xi|(2^\ell+d_\Sigma(\xi))}}$$

which of course implies the required L^1 estimate.

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