ENDPOINT INEQUALITIES FOR BOCHNER-RIESZ MULTIPLIERS IN THE PLANE

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Abstract. A weak-type inequality is proved for Bochner-Riesz means at the critical index, for functions in $L^p(\mathbb{R}^2)$, $1 \le p < 4/3$.

1. Introduction

For a Schwartz-function $f \in \mathcal{S}(\mathbb{R}^2)$ let $\widehat{f}(\xi) = \int f(y) e^{-i\langle y,\xi\rangle} dy$ denote the Fourier transform and define the Bochner-Riesz means by

$$S_R^\lambda f(x) \;=\; \frac{1}{(2\pi)^2} \int_{|\xi|\le R} \Big(1 - \frac{|\xi|^2}{R^2}\Big)^\lambda \widehat{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi;$$

we set $S^\lambda = S_1^\lambda$. It is a classical theorem of Bochner that S^λ extends to a bounded operator on $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$ if $\lambda > 1/2$. The theorem of Carleson and Sjölin [2] states that S^λ is bounded in $L^p(\mathbb{R}^2)$ if $0 < \lambda \le \frac{1}{2}$ and $\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda}$. It is well known that the L^p boundedness fails if $p \le \frac{4}{3+2\lambda}$ and C. Fefferman [11] showed that S^0 is not bounded in $L^p(\mathbb{R}^2)$ if $p \ne 2$.

In this paper we are concerned with endpoint estimates for the critical exponent $p_0(\lambda) = \frac{4}{3+2\lambda}$. In [4], [5] M. Christ proved that S^λ is of weak type $(p_0(\lambda), p_0(\lambda))$ if $1/6 < \lambda \le 1/2$ (for related results see also [6], [15]). A combination of L^2-variants of Calderón-Zygmund theory (as used first by Fefferman [10]) and the $L^p \to L^2$ restriction theorem for the Fourier transform (valid for $p \le 6/5 = p_0(1/6)$) is essential in Christ's analysis; this accounts for the restriction $\lambda > 1/6$. It had been an open problem whether the weak type inequality for the critical index $\lambda(p) = 2(1/p - 1/2) - 1/2$ is true for $6/5 \le p < 4/3$ (although for radial functions this was proved by Chanillo and Muckenhoupt [3]).

Theorem 1.1. *Suppose that $0 < \lambda \le 1/2$. Then for all $\alpha > 0$ there is the weak-type inequality*

$$\big|\{x \in \mathbb{R}^2 : |S^\lambda f(x)| > \alpha\}\big| \;\le\; C \frac{\|f\|_{p_0}^{p_0}}{\alpha^{p_0}}, \qquad p_0 = \frac{4}{3+2\lambda},$$

where C does not depend on f or α.

By scaling the same estimate holds for S_R^λ, uniformly in R, and a standard argument gives that $\lim_{R\to\infty} S_R^\lambda f = f$ in the topology of the weak type space $L^{p_0\infty}$ provided that $f \in L^{p_0}(\mathbb{R}^2)$.

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We shall also prove an L^p endpoint version of the Carleson-Sjölin theorem. Define

$$m_{\lambda,\gamma}(\xi) = \frac{(1-|\xi|^2)_+^\lambda}{(1-\log(1-|\xi|^2))^\gamma}. \tag{1.1}$$

Theorem 1.2. *Suppose that $1 \le p < 4/3$ and $\lambda(p) = 2(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then $m_{\lambda(p),\gamma}$ is a Fourier multiplier of $L^p(\mathbb{R}^2)$ if and only if $\gamma > \frac{1}{p}$.*

The necessity of the condition $\gamma > 1/p$ was proved in [14], the sufficiency for $p \le 6/5$ in [15].

In what follows c and C will always be positive numbers which may assume different values in different formulas.

2. Strong type estimates

For an interval I on the real line denote by I^* the interval with same midpoint and double length. Suppose $\mathfrak{I} = \{I_j\}_{j\ge 0}$ is a collection of intervals such that $I_j \subset (1/4, 4)$ and $2^{-j-3} \le |I_j| \le 2^{-j}$ and such that

$$I_j^* \cap I_{j'}^* = \emptyset \quad \text{if } j \ne j'.$$

For each $j \ge 0$ let ψ_j be a C^2-function supported in I_j with bounds

$$\|\psi_j^{(\ell)}\|_\infty \le 2^{j\ell}, \quad \ell = 0, 1, 2.$$

Let $\eta \in C_0^\infty(\mathbb{R}^2)$ such supp $(\eta) \subset \{\xi \in \mathbb{R}^2 : |\xi_1/\xi_2| \le 10^{-1},\ \xi_2 > 0\}$.

Define the operator T_j by

$$\widehat{T_j f}(\xi) = \eta(\xi)\psi_j(|\xi|)\widehat{f}(\xi). \tag{2.1}$$

T_j is a bounded operator on L^1 with operator norm $O(2^{j/2})$, and Córdoba [8] showed that the $L^{4/3}$ operator norm of T_j is $O(j^{1/4})$. We note that in order to prove results such as Theorem 1.2 for $p > 1$ it is not sufficient to derive sharp L^p bounds for the individual operators T_j. Our main result is

Theorem 2.1. *Suppose that $1 \le p < 4/3$ and $\lambda(p) = 2(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ and $\mathfrak{I}$, T_j are as above. Then there is the inequality*

$$\Big\|\sum_j T_j f_j\Big\|_p \le C\Big(\sum_j \big[2^{j\lambda(p)}\|f_j\|_p\big]^p\Big)^{\frac{1}{p}}. \tag{2.2}$$

In particular if

$$m = \sum_j 2^{-j\lambda(p)} a_j \eta(\xi)\psi_j(|\xi|) \tag{2.3}$$

then m is a Fourier multiplier of L^p if $\{a_j\} \in \ell^p$ (simply apply Theorem 2.1 with $f_j = a_j 2^{-j\lambda(p)} f$). It is easy to see that the multiplier $m_{\lambda,\gamma}$ in (1.1) is a finite sum of a smooth compactly supported function and rotates of multipliers of the form (2.3), with $a_j = cj^{-\gamma}$. Therefore Theorem 2.1 implies Theorem 1.2.

Proof of Theorem 2.1. By duality the inequality (2.2) is equivalent to

$$\Big(\sum_j \big[2^{-j\lambda(q')}\|T_j f\|_q\big]^q\Big)^{\frac{1}{q}} \le C\|f\|_q, \qquad q > 4. \tag{2.4}$$

As in [8] one decomposes each $\psi_j(|\cdot|)$ into pieces which are essentially supported in rectangles of dimensions $(c2^{-j/2}, c2^{-j})$. To this end let $\beta \in C_0^\infty(\mathbb{R})$ be supported in $(-1,1)$ such that $\sum_{\nu=-\infty}^{\infty}\beta(s-\nu) = 1$ for all $s \in \mathbb{R}$. Then define T_j^ν by

$$\widehat{T_j^\nu f}(\xi) = \beta(2^{j/2}\xi_1 - \nu)\widehat{T_j f}(\xi).$$

For $n \le j/2$ let

$$\mathfrak{Z}_j^n = \{(\nu,\nu') \in \mathbb{Z}^2 : 2^{j/2-n-1} < |\nu-\nu'| \le 2^{j/2-n}\}.$$

Notice that $T_j^\nu f\, T_j^{\nu'} f = 0$ if $(\nu,\nu') \in \mathfrak{Z}_j^n$ and $n < 0$. Therefore

$$\begin{aligned}
&\Big(\sum_j \big[2^{-j\lambda(q')}\|T_j f\|_q\big]^q\Big)^{\frac{1}{q}}\\
&= \Big(\sum_j \Big[2^{-2j\lambda(q')}\Big\|\sum_\nu\sum_{\nu'} T_j^\nu f\, T_j^{\nu'} f\Big\|_{\frac{q}{2}}\Big]^{\frac{q}{2}}\Big)^{\frac{1}{q}}\\
&\le \sum_{n=0}^{\infty}\Big(\sum_{j\ge 2n}\Big[2^{-2j\lambda(q')}\Big\|\sum_{(\nu,\nu')\in\mathfrak{Z}_j^n} T_j^\nu f\, T_j^{\nu'} f\Big\|_{\frac{q}{2}}\Big]^{\frac{q}{2}}\Big)^{\frac{1}{q}}. \qquad (2.5)
\end{aligned}$$

We shall show that for $q \ge 4$ the n^{th} term in (2.5) is bounded by $C2^{-n(1/2-2/q)}\|f\|_q$ from which (2.4) immediately follows. This is contained in

Proposition 2.2. *For $f, g \in \mathcal{S}(\mathbb{R}^2)$ let*

$$\mathcal{B}_j^n(f,g) = \sum_{(\nu,\nu')\in\mathfrak{Z}_j^n} T_j^\nu f\, T_j^{\nu'} g.$$

Then for $q \ge 4$ there is the inequality

$$\Big(\sum_{j\ge 2n}\big[2^{-2j\lambda(q')}\|\mathcal{B}_j^n(f,g)\|_{\frac{q}{2}}\big]^{\frac{q}{2}}\Big)^{\frac{2}{q}} \le C2^{-n(1-\frac{4}{q})}\|f\|_q\|g\|_q. \tag{2.6}$$

Proof. The inequality follows by complex interpolation for bilinear mappings from the cases $q = 4$ and $q = \infty$. The correct interpretation of (2.6) for $q = \infty$ is of course

$$\sup_j 2^{-j}\Big\|\sum_{(\nu,\nu')\in\mathfrak{Z}_j^n} T_j^\nu f\, T_j^{\nu'} g\Big\|_\infty \le C2^{-n}\|f\|_\infty\|g\|_\infty.$$

But this is immediate since each operator T_j^ν is bounded on L^∞ with norm independent of j and ν and since the cardinality of $\mathfrak{Z}_n^j$ is bounded by $C2^{j/2} \times 2^{j/2-n} = C2^{j-n}$.

We shall now prove the required estimate for $q = 4$ which is

$$\Big(\sum_{j\ge 2n} \|\mathcal{B}_j^n(f,g)\|_2^2\Big)^{1/2} \le C\|f\|_4\|g\|_4 \tag{2.7}$$

uniformly in n.

We first use Plancherel's theorem and C. Fefferman's basic observation ([12], [8]) that for fixed j the sets supp $(\widehat{T_j^\nu f})$ + supp $(\widehat{T_j^{\nu'} f})$ are essentially disjoint; that is each $\xi \in \mathbb{R}^2$ is contained in at most M of these sets where M is independent of j. This yields the inequality

$$\sum_{j\ge 2n} \|\mathcal{B}_j^n(f,g)\|_2^2 \le C \sum_{j\ge 2n} \sum_{(\nu,\nu')\in\mathfrak{Z}_j^n} \|T_j^\nu f\, T_j^{\nu'} g\|_2^2 \tag{2.8}$$

It is crucial for this proof that a finer decomposition can be made depending on how far apart the supports of $\widehat{T_j^\nu f}$ and $\widehat{T_j^{\nu'} g}$ are, that is, depending on n. We define operators $T_j^{\nu\mu}$ by

$$\widehat{T_j^{\nu\mu} f}(\xi) = \beta(2^{j-n}\xi_1 - \mu)\widehat{T_j^\nu f}(\xi)$$

so that $\widehat{T_j^{\nu\mu} f}$ is supported in a rectangle of dimensions $(C2^{-j+n}, C2^{-j})$. Again one can check that for fixed j and fixed $(\nu,\nu') \in \mathfrak{Z}_j^n$ each $\xi \in \mathbb{R}^2$ is contained in at most M of the sets $E_{jn\nu\nu'}^{\mu\mu'} =$ supp $(\widehat{T_j^{\nu\mu} f})$ + supp $(\widehat{T_j^{\nu'\mu'} g})$ where M is independent of j, ν, ν'. Each $E_{jn\nu\nu'}^{\mu\mu'}$ is contained in a rectangle of dimensions $(C'2^{-j+n}, C'2^{-j})$. For fixed j, ν, ν' there are no more than $C''2^{(j-2n)}$ of these rectangles and they form an essentially disjoint cover of supp $(\widehat{T_j^\nu f})$ + supp $(\widehat{T_j^{\nu'} g})$, the latter set being contained in a rectangle of dimensions $(C2^{-j/2}, C2^{-j/2-n})$. The disjointness property and Plancherel's theorem imply that

$$\sum_{j\ge 2n} \|\mathcal{B}_j^n(f,g)\|_2^2 \le C \sum_{j\ge 2n}\sum_{\mu,\mu'} \sum_{(\nu,\nu')\in\mathfrak{Z}_j^n} \|T_j^{\nu\mu} f\, T_j^{\nu'\mu'} g\|_2^2. \tag{2.9}$$

For any integer κ with $|\kappa| \le 2^n$ let

$$\mathfrak{W}_{jn}^\kappa = \{\mu \in \mathbb{Z} : |2^{n-j}\mu - 2^{-n}\kappa| \le 2^{-n}\}.$$

Then observe that

$$T_j^{\nu\mu} f\, T_j^{\nu'\mu'} g = 0 \qquad \text{if } (\nu,\nu') \in \mathfrak{Z}_j^n,\ \mu \in \mathfrak{W}_{jn}^\kappa,\ \mu' \in \mathfrak{W}_{jn}^{\kappa'},\ |\kappa-\kappa'| \ge 8. \tag{2.10}$$

Indeed, if $\mu \in \mathfrak{W}_{jn}^\kappa$, $\mu' \in \mathfrak{W}_{jn}^{\kappa'}$, $T_j^{\nu\mu} f\, T_j^{\nu'\mu'} g \ne 0$ then $|2^{n-j}\mu - 2^{-j/2}\nu| \le 2^{-j/2+1}$ and $|2^{n-j}\mu' - 2^{-j/2}\nu'| \le 2^{-j/2+1}$. If $(\nu,\nu') \in \mathfrak{Z}_j^n$ this implies that $|2^{n-j}(\mu-\mu')| \le 2^{-j/2+2} + 2^{-n} \le 5\cdot 2^{-n}$ and therefore $|\kappa-\kappa'| \le 7$, hence (2.10). Moreover we note that for $\mu \in \mathfrak{W}_{jn}^\kappa$ the support of $\widehat{T_j^{\nu\mu} f}$ is essentially a rectangle with eccentricity 2^{-n} such that the directions of its sides depend on κ but not on μ.

By (2.9) and (2.10) we obtain that

$$\begin{aligned}
&\sum_{j\geq 2n}\|\mathcal{B}_j^n(f,g)\|_2^2\\
&\leq C\sum_{j\geq 2n}\sum_{\kappa}\sum_{\substack{\kappa'\\|\kappa'-\kappa|<8}}\Big\|\Big(\sum_{\mu\in\mathfrak{W}_{jn}^{\kappa}}\sum_{\nu}|T_j^{\nu\mu}f|^2\Big)^{\frac12}\Big(\sum_{\mu'\in\mathfrak{W}_{jn}^{\kappa'}}\sum_{\nu'}|T_j^{\nu'\mu'}g|^2\Big)^{\frac12}\Big\|_2^2\\
&\leq C'\sum_{j\geq 2n}\sum_{\kappa}\sum_{\substack{\kappa'\\|\kappa'-\kappa|<8}}\Big\|\Big(\sum_{\mu\in\mathfrak{W}_{jn}^{\kappa}}\sum_{\nu}|T_j^{\nu\mu}f|^2\Big)^{\frac12}\Big\|_4^2\Big\|\Big(\sum_{\mu'\in\mathfrak{W}_{jn}^{\kappa'}}\sum_{\nu'}|T_j^{\nu'\mu'}g|^2\Big)^{\frac12}\Big\|_4^2\\
&\leq C''\Big(\sum_{j\geq 2n}\sum_{\kappa}\Big\|\Big(\sum_{\mu\in\mathfrak{W}_{jn}^{\kappa}}\sum_{\nu}|T_j^{\nu\mu}f|^2\Big)^{\frac12}\Big\|_4^4\Big)^{\frac12}\Big(\sum_{j\geq 2n}\sum_{\kappa}\Big\|\Big(\sum_{\mu\in\mathfrak{W}_{jn}^{\kappa}}\sum_{\nu}|T_j^{\nu\mu}g|^2\Big)^{\frac12}\Big\|_4^4\Big)^{\frac12}.
\end{aligned}$$

Therefore the desired estimate (2.7) follows from the case $q=4$ of the following lemma.

Lemma 2.3. *For $q\geq 2$ there is the inequality*

$$\Big(\sum_{j\geq 2n}\sum_{\kappa}\Big\|\Big(\sum_{\mu\in\mathfrak{W}_{jn}^{\kappa}}\sum_{\nu}|T_j^{\nu\mu}f|^2\Big)^{\frac12}\Big\|_q^q\Big)^{\frac1q}\leq C\|f\|_q \tag{2.11}$$

where C does not depend on n.

Proof. It suffices to prove (2.11) for $q=2$ and $q=\infty$. Let $h_j^{\nu\mu}$ be the Fourier multiplier defining $T_j^{\nu\mu}$.

For fixed μ and j there are at most three ν such that $T_j^{\nu\mu}\neq 0$ and since the supports of the functions ψ_j are disjoint it follows that each $\xi\in\mathbb{R}^2$ is contained in at most 6 of the sets supp $h_j^{\mu\nu}$. Moreover for fixed μ and j there are at most two κ such that $\mu\in\mathfrak{W}_{jn}^{\kappa}$. Now (2.11) for $q=2$ is an immediate consequence of Plancherel's theorem.

In order to check the required estimate for $q=\infty$ we consider for a fixed $\mathfrak{a}=\{a_{\nu\mu}\}\in\ell^2(\mathbb{Z}^2)$ the multiplier

$$m_{\mathfrak{a}}^{j\kappa}(\xi)=\sum_{\mu\in\mathfrak{W}_{jn}^{\kappa}}\sum_{\nu}a_{\nu\mu}h_j^{\nu\mu}(\xi)$$

and denote by $K_{\mathfrak{a}}^{j\kappa}$ its inverse Fourier transform.

Let $e_1^{\kappa}=(2^{-n}\kappa,\sqrt{1-2^{-2n}\kappa^2})$ and $e_2^{\kappa}=(-\sqrt{1-2^{-2n}\kappa^2},2^{-n}\kappa)$ and let L_{jn}^{κ} be the symmetric linear transformation in $\mathbb{R}^2$ with $L_{jn}^{\kappa}e_1^{\kappa}=2^je_1^{\kappa}$, $L_{jn}^{\kappa}e_2^{\kappa}=2^{j-n}e_2^{\kappa}$. Then $h_j^{\nu\mu}(L_{jn}^{\kappa}\cdot)$ is supported in a cube $Q_j^{\nu\mu}$ of sidelength 10 and for fixed j the cubes $Q_j^{\nu\mu}$ have finite overlap, uniformly in j. Moreover it is easy to see that for $\mu\in\mathfrak{W}_{jn}^{\kappa}$

$$\Big\|\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\big[h_j^{\nu\mu}(L_{jn}^{\kappa}\cdot)\big]\Big\|_{\infty}\leq C,\qquad |\alpha|\leq 2.$$

Since the Sobolev-space L^2_2 is a subspace of $\widehat{L^1}$ we obtain that

$$\begin{aligned}\|K^{j\kappa}_{\mathfrak{a}}\|_1 &= \|2^{-2j+n}K^{j\kappa}_{\mathfrak{a}}((L^{\kappa}_{jn})^{-1}\cdot)\|_1\\ &\le C\sum_{|\alpha|\le 2}\Big\|\sum_{\mu,\nu}a_{\nu\mu}\frac{\partial^\alpha}{\partial\xi^\alpha}\big[h^{\nu\mu}_j(L^{\kappa}_{jn}\cdot)\big]\Big\|_2\\ &\le C'\Big(\sum_{\mu,\nu}|a_{\nu\mu}|^2\Big)^{\frac12}\end{aligned}$$

where C' does not depend on j, κ and $\mathfrak{a}$. This implies

$$\begin{aligned}&\sup_{j\ge 2n}\sup_{\kappa}\Big\|\Big(\sum_{\mu\in\mathfrak{W}^{\kappa}_{jn}}\sum_{\nu}|T^{\nu\mu}_j f|^2\Big)^{\frac12}\Big\|_\infty\\ &=\sup_{j\ge 2n}\sup_{\kappa}\sup_{x\in\mathbb{R}^2}\sup_{\|\mathfrak{a}\|_{\ell^2(\mathbb{Z}^2)}\le 1}|K^{j\kappa}_{\mathfrak{a}}*f(x)|\\ &\le\sup_{j\ge 2n}\sup_{\kappa}\sup_{\|\mathfrak{a}\|_{\ell^2(\mathbb{Z}^2)}\le 1}\|K^{j\kappa}_{\mathfrak{a}}\|_1\|f\|_\infty\le C\|f\|_\infty\end{aligned}$$

which is the desired estimate for $q=\infty$. □

Remarks.

(a) For $q=\infty$ the inequality (2.11) is closely related to an estimate on square-functions with respect to an equally spaced decomposition, see *e.g.* [9], [13]; in fact it can be obtained from these estimates.

(b) A variant of the above proof can be used to obtain the known sharp L^4 bound $\|T_j\|_{L^4\to L^4}=O(j^{1/4})$ without making use of the sharp L^2 bounds for Kakeya-maximal functions.

(c) The observation concerning the overlapping properties of supp $T^{\nu\mu}_j$+supp $T^{\nu'\mu'}_j$ can be used to improve on some bounds for sectorial square-functions in Córdoba [9]. This has been observed by A. Carbery and the author.

(d) The decomposition in terms of the bilinear operators $\mathcal{B}^n_j$ is related to a decomposition used by Carbery [1] in his work on weighted inequalities for the maximal Bochner-Riesz operator S^λ_*. The techniques above can be used to prove new weighted inequalities for S^λ_*.

3. Weak type estimates

Let $\mathfrak{I}$ be a family of disjoint intervals as introduced in §2 and let T_j be as in (2.1). Define

$$T^\lambda f=\sum_{j\ge 0}2^{-j\lambda}T_j f.$$

We shall prove the estimate

$$\big|\{x\in\mathbb{R}^2:\,|T^{\lambda(p)}f(x)|>\alpha\}\big|\le C\frac{\|f\|_p^p}{\alpha^p},\qquad p<\frac43 \tag{3.1}$$

where $\lambda(p)=2(1/p-1/2)-1/2$ and C does not depend on f or α. Of course Theorem 1.1 is a consequence of (3.1).

As in [5] the proof is based on an interpolation. The argument uses Theorem 2.1 and known estimates previously obtained in the proof of weak-type (1,1) inequalities (see [4], [7], [15]).

Let $f \in L^p(\mathbb{R}^2)$ where $1 \le p < \frac{4}{3}$ and let $\alpha > 0$. In order to estimate the quantity on the left hand side of (3.1) we apply the Calderón-Zygmund decomposition to $|f|^p$ at height α^p. We obtain a decomposition $f = g + b$ where $\|g\|_\infty \le C\alpha$, $\|g\|_p \le C\|f\|_p$, $b = \sum_Q b_Q$, supp $b_Q \subset Q$, the squares Q are pairwise disjoint, $\|b_Q\|_p^p \le C\alpha^p|Q|$, $\sum_Q |Q| \le C\alpha^{-p}\|f\|_p^p$; and as a consequence $\alpha^{p-2}\|g\|_2^2 + \|b\|_p^p \le C\|f\|_p^p$.

Let $l(Q)$ be the sidelength of Q and $B_j = \sum_{Q:l(Q)=2^j} b_Q$ if $j > 0$ and $B_0 = \sum_{Q:l(Q)\le 0} b_Q$. Then

$$\{x \in \mathbb{R}^2 : |T^{\lambda(p)}f(x)| > \alpha\} \subset \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$$

where Ω_1 is the union of the double squares Q^* and

$$\begin{aligned}
\Omega_2 &= \{x \in \mathbb{R}^2 : |T^{\lambda(p)}g(x)| > \frac{\alpha}{5}\}\\
\Omega_3 &= \Big\{x \in \mathbb{R}^2 : \Big|\sum_{s\ge 0}\sum_{j>s} 2^{-j\lambda(p)}T_jB_{j-s}(x)\Big| > \frac{\alpha}{5}\Big\}\\
\Omega_4 &= \Big\{x \in \mathbb{R}^2 : \Big|\sum_{j\ge 0} 2^{-j\lambda(p)}T_jB_0(x)\Big| > \frac{\alpha}{5}\Big\}\\
\Omega_5 &= \Big\{x \in \mathbb{R}^2 \setminus \Omega_1 : \Big|\sum_{\sigma>0}\sum_{j\ge 0} 2^{-j\lambda(p)}T_jB_{j+\sigma}(x)\Big| > \frac{\alpha}{5}\Big\}.
\end{aligned}$$

By the disjointness of the squares Q we have

$$|\Omega_1| \le \sum_Q |Q^*| \le C\frac{\|f\|_p^p}{\alpha^p}$$

and Chebyshev's inequality and the L^2-boundedness of T^λ imply

$$|\Omega_2| \le C\frac{\|T^\lambda g\|_2^2}{\alpha^2} \le C'\frac{\|g\|_2^2}{\alpha^2} \le C''\frac{\|f\|_p^p}{\alpha^p}.$$

Next we choose r such that $p < r < 4/3$. We shall show that the following estimates hold with $\epsilon = \frac{1}{2}(\frac{r}{p} - 1)$.

$$\Big\|\sum_{j>s} 2^{-j\lambda(p)}T_jB_{j-s}\Big\|_r^r \le C2^{-\epsilon s}\alpha^{r-p}\|b\|_p^p, \qquad s \ge 0, \tag{3.2}$$

$$\|2^{-j\lambda(p)}T_jB_0\|_r^r \le C2^{-\epsilon j}\alpha^{r-p}\|b\|_p^p, \qquad j \ge 0, \tag{3.3}$$

$$\Big\|\sum_{j\ge 0} 2^{-j\lambda(p)}T_jB_{j+\sigma}\Big\|_{L^p(\mathbb{R}^2\setminus\Omega_1)}^p \le C2^{-\varepsilon\sigma}\|b\|_p^p, \qquad \sigma \ge 0. \tag{3.4}$$

From (3.2-4) it follows by applications of Minkowski's and Chebyshev's inequalities that

$$|\Omega_3| + |\Omega_4| + |\Omega_5| \le C\frac{\|b\|_p^p}{\alpha^p} \le C'\frac{\|f\|_p^p}{\alpha^p}.$$

In order to prove (3.2-4) we use analytic interpolation (*i.e.* the Phragmen-Lindelöf principle) similarly as in [5]. For $\mathrm{Re}\,(z) \in [0,1]$ define

$$B_{j,z}(x) \;=\; |B_j(x)|^{p[(1-z)+z/r]}\mathrm{sign}(B_j(x))$$

and

$$\gamma(z) = 2(1 - z + \frac{z}{r} - \frac{1}{2}) - \frac{1}{2}.$$

Since $2^{-j\gamma(1+i\tau)}T_j$ is a bounded operator on L^1 with norm independent of j we obtain

$$\Big\|\sum_{j>s} 2^{-j\gamma(1+i\tau)}T_j B_{j-s,1+i\tau}\Big\|_1 \;\le C\sum_{j>s}\|B_{j-s,1+i\tau}\|_1 \;\le\; C'\|b\|_p^p \tag{3.5}$$

$$\|2^{-j\gamma(1+i\tau)}T_j B_{0,1+i\tau}\|_1 \;\le C\|B_0\|_p^p \;\le C'\,\|b\|_p^p. \tag{3.6}$$

From estimates in [7] (or [15]) it follows that

$$\Big\|\sum_{j>s} 2^{-j\gamma(1+i\tau)}T_j B_{j-s,1+i\tau}\Big\|_2^2 \;\le C 2^{-s/2}\alpha^p\|b\|_p^p \tag{3.7}$$

$$\|2^{-j\gamma(1+i\tau)}T_j B_{0,1+i\tau}\|_2^2 \;\le C 2^{-j/2}\|b\|_p^p \tag{3.8}$$

and also that
(3.9)

$$\Big\|\sum_{j\ge 0} 2^{-j\gamma(1+i\tau)}T_j B_{j+\sigma,1+i\tau}\Big\|_{L^1(\mathbb{R}^2\setminus\Omega_1)} \;\le\; C2^{-\sigma}\sum_{j\ge 0}\|B_{j+\sigma,1+i\tau}\|_1 \;\le\; C'2^{-\sigma}\|b\|_p^p.$$

Using the inequality $\|F\|_r \le C\|F\|_1^{\frac{2}{r}-1}\|F\|_2^{2-\frac{2}{r}}$ we get from (3.5), (3.7) and from (3.6), (3.8) that

$$\Big\|\sum_{j>s} 2^{-j\gamma(1+i\tau)}T_j B_{j-s,1+i\tau}\Big\|_r^r \;\le C 2^{-s\frac{r-1}{2}}\alpha^{p(r-1)}\|b\|_p^p \tag{3.10}$$

$$\|2^{-j\gamma(1+i\tau)}T_j B_{0,1+i\tau}\|_r^r \;\le C 2^{-j\frac{r-1}{2}}\alpha^{p(r-1)}\|b\|_p^p. \tag{3.11}$$

Now by Theorem 2.1 it follows that

$$\Big\|\sum_{j>s} 2^{-j\gamma(i\tau)}T_j B_{j-s,i\tau}\Big\|_r^r \;\le C\sum_{j>s}\|B_{j-s,i\tau}\|_r^r \;\le\; C'\|b\|_p^p \tag{3.12}$$

$$\|2^{-j\gamma(i\tau)}T_j B_{0,i\tau}\|_r^r \;\le C\|B_{0,i\tau}\|_r^r \;\le\; C'\|b\|_p^p \tag{3.13}$$

$$\Big\|\sum_{j\ge 0} 2^{-j\gamma(i\tau)}T_j B_{j+\sigma,i\tau}\Big\|_r^r \;\le C\sum_{j\ge 0}\|B_{j+\sigma,i\tau}\|_r^r \;\le\; C'\|b\|_p^p. \tag{3.14}$$

Now let h be arbitrary function in $L^{p'}$, $p' = p/(p-1)$, with $\|h\|_{p'} \le 1$ and define

$$h_z(x) \;=\; |h(x)|^{zp'/r'}\mathrm{sign}(h(x)).$$

Moreover let g be an arbitrary function in $L^{r'}$ with $\|g\|_{r'} \le 1$. We then apply the Phragmen-Lindelöf principle to the functions

$$z \mapsto W_{1,s}(z) = \int \sum_{j>s} 2^{-j\gamma(z)} T_j B_{j-s,z}(x) g(x) dx$$
$$z \mapsto W_{2,j}(z) = \int 2^{-j\gamma(z)} T_j B_{0,z}(x) g(x) dx$$
$$z \mapsto W_{3,\sigma}(z) = \int \sum_{j\ge 0} 2^{-j\gamma(z)} T_j B_{j+\sigma,z}(x) h_z(x) dx$$

and estimate these functions at $z = \theta$ chosen such that $1/p = (1-\theta) + \theta/r$. From (3.10), (3.12), from (3.11), (3.13) and from (3.9), (3.14) it follows that

$$|W_{1,s}(\theta)| \le C\alpha^{r-p} 2^{-\frac{s}{2}(\frac{r}{p}-1)} \|b\|_p^p$$
$$|W_{2,j}(\theta)| \le C\alpha^{r-p} 2^{-\frac{j}{2}(\frac{r}{p}-1)} \|b\|_p^p$$
$$|W_{3,\sigma}(\theta)| \le C 2^{-\sigma(\frac{r}{p}-1)} \|b\|_p^p$$

and an application of the converse of Hölder's inequality yields (3.2), (3.3) and (3.4).

Remark. Endpoint versions for more general classes of multiplier transformations have been formulated in [15]. By combining arguments in this and the present paper one can prove similar results for radial Fourier multipliers of $L^p(\mathbb{R}^2)$, for the full range $1 \le p < 4/3$.

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