$\mathbf{L}^{\mathbf{p}}$
$L p$ -Sobolev Estimates for a Class of Integral Operators with Folding Canonical Relations

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L^p-Sobolev Estimates for a Class of Integral Operators with Folding Canonical Relations

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Abstract

We prove a sharp L^p-Sobolev regularity result for a class of generalized Radon transforms for families of curves in a three-dimensional manifold, with folding canonical relations. The proof relies on decoupling inequalities by Wolff and by Bourgain–Demeter for plate decompositions of thin neighborhoods of cones.

1 Introduction

In this paper we continue the study [20] of L^p regularity properties of integral operators along families of curves in $\mathbb{R}^3$ satisfying suitable curvature and torsion conditions. The previous article dealt with the translation invariant case, i.e., the integrals

$$\mathcal{A}f(x) = \int f(x - \gamma(s))\chi(s)ds \qquad (1.1)$$

In memory of Eli Stein.

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where γ is a curve in $\mathbb{R}^3$ with nonvanishing curvature and torsion and χ is smooth and compactly supported. The authors showed an optimal result with a gain of $1/p$ derivatives for sufficiently large p, namely that for large p the operator $\mathcal{A}$ maps $L^p(\mathbb{R}^3)$ into the L^p-Sobolev space $L^p_{1/p}$. The usual combination of damping of oscillatory integrals arguments and improved L^∞ bounds, as employed in [24], does not apply to averaging operators for curves in three or higher dimensions. Instead the authors had to apply a deep result of Wolff [26] on decompositions of cone multipliers in $\mathbb{R}^3$ which is now known as an ℓ^p-decoupling inequality. The result in [20] can be combined with a recent result by Bourgain and Demeter [5] which extends the decoupling result for the cone in $\mathbb{R}^3$ to the optimal L^p range $p > 6$; this combination immediately yields $\mathcal{A} : L^p(\mathbb{R}^3) \to L^p_{1/p}(\mathbb{R}^3)$ for $p > 4$. A result by Oberlin and Smith [15] shows that this range is optimal, up to possibly the endpoint $p = 4$.
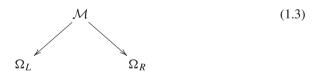
In the current work we shall treat extensions of these results for operators which are not of convolution type. Let Ω_L, Ω_R be three-dimensional smooth manifolds and consider families of curves $\mathcal{M}_x \subset \Omega_R$ parametrized by and smoothly depending on $x \in \Omega_L$. Let $d\sigma_x$ be arclength measure on $\mathcal{M}_x$ and $\chi_\circ \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. We define the generalized Radon transform operator $\mathcal{R} : C_c^\infty(\Omega_R) \to C^\infty(\Omega_L)$ by

$$\mathcal{R}f(x) = \int_{\mathcal{M}_x} f(y)\chi_\circ(x, y)d\sigma_x(y). \tag{1.2}$$

In order to formulate our results we use the double fibration formalism of Gelfand and Helgason (see, e.g., [11], p. 340 ff.). Assume

$$\mathcal{M}_x = \{y \in \Omega_R : (x, y) \in \mathcal{M}\}$$

where $\mathcal{M}$ is a submanifold of $\Omega_L \times \Omega_R$ of codimension 2 such that the projections

$$\begin{array}{ccc} & \mathcal{M} & \\ & \swarrow \quad \searrow & \\ \Omega_L & & \Omega_R \end{array} \tag{1.3}$$

have surjective differentials. The surjectivity assumption on the differential of $\mathcal{M} \to \Omega_L$ implies that the $\mathcal{M}_x$ are smooth immersed curves in Ω_R (depending smoothly on x). Similarly the corresponding assumption on the differential of $\mathcal{M} \to \Omega_R$ implies that $\mathcal{M}^y = \{x \in \Omega_L : (x, y) \in \mathcal{M}\}$ are smooth immersed curves in Ω_L (depending smoothly on y).

The operator $\mathcal{R}$ can be realized as a Fourier integral operator of order $-1/2$ belonging to the Hörmander class $I^{-\frac{1}{2}}(\Omega_L, \Omega_R; (N^*\mathcal{M})')$ where

$$(N^*\mathcal{M})' = \{(x, \xi, y, \eta) : (x, y, \xi, -\eta) \in N^*\mathcal{M}\}$$

with $N^*\mathcal{M}$ the conormal bundle given by

$$N^*\mathcal{M} := \left\{(x, y, \eta, \xi)) \in T^*(\Omega_L \times \Omega_R) \setminus \{0\} : (\xi, \eta) \perp T_{(x,y)}\mathcal{M}\right\}.$$

(*cf.* Sect. 2).

The assumptions on the projections (1.3) imply that

$$\mathcal{C} := (N^*\mathcal{M})' \subset (T^*\Omega_L \setminus 0_L) \times (T^*\Omega_R \setminus 0_R)$$

with 0_L and 0_R referring to the zero sections of the cotangent spaces $T^*\Omega_L$ and $T^*\Omega_R$, respectively. $\mathcal{C}$ is a homogenous canonical relation, i.e., if σ_L and σ_R are the canonical two-forms on $T^*\Omega_L$ and $T^*\Omega_R$, respectively, then $\mathcal{C}$ is Lagrangian with respect to $\sigma_L - \sigma_R$. As is well known from the theory of Fourier integral operators (see [12], [17]), the L^2 Sobolev regularity properties of $\mathcal{R}$ are governed by the geometry of the projections

$$\begin{array}{ccc} & \mathcal{C} & \\ {}^{\pi_L}\swarrow & & \searrow^{\pi_R} \\ T^*\Omega_L & & T^*\Omega_R \end{array} \tag{1.4}$$

Since $\mathcal{C}$ is Lagrangian the differential $(D\pi_L)_P$ is invertible if and only if $(D\pi_R)_P$ is invertible [12]. For the canonical relations for averaging operators over curves in dimensions ≥ 3 the maps π_L (and π_R) fail to be diffeomorphisms, namely for every point $(x, y) \in \mathcal{M}$ there is $P = (x, \xi, y, \eta) \in (N^*\mathcal{M})'$ so that $(D\pi_L)_P$ and $(D\pi_R)_P$ are not invertible.

Statement of the Main Result

We shall assume that the only singularities π_L and π_R are *Whitney folds* and say that $\mathcal{C}$ projects with two-sided fold singularities. Recall the definition from [13, Appendix C4]. Given a C^∞ map $g : X \to Y$ between C^∞ manifolds and $P \in X$ the Hessian $g''(P)$ is invariantly defined as a map from $\ker(g')_P$ to $\text{Coker}\,(g')_{g(P)}$. Then g has a Whitney fold at P if $\dim(\ker(g')_P) = 1$, $\dim(\text{Coker}\,(g')_{g(P)} = 1$ and the Hessian at P is not equal to 0. Equivalently, g is such that for every point $P \in X$, Dg_P is either invertible or g has a Whitney fold; then $\mathcal{L} = \{P : \det(Dg)_P \neq 0\}$ is an immersed hypersurface of X and for any vector field V with $V_P \in \ker(Dg)_P$ for all $P \in \mathcal{L}$ (a "kernel field") we have $V(\det Dg) \neq 0$ at P.

Theorem 1.1 *Let $\mathcal{M} \subset \Omega_L \times \Omega_R$ be a four-dimensional manifold such that the projections (1.3) are submersions. Assume that the only singularities of $\pi_L : (\mathcal{N}^*\mathcal{M})' \to T^*\Omega_L$ and $\pi_R : (\mathcal{N}^*\mathcal{M})' \to T^*\Omega_R$ are Whitney folds. Let $\mathcal{L} \subset (N^*\mathcal{M})'$ be the conic hypersurface manifold where $D\pi_L$ and $D\pi_R$ drop rank by one, and let ϖ be the projection of $(\mathcal{N}^*\mathcal{M})'$ to the base $\mathcal{M}$. Suppose that its restriction to $\mathcal{L}$,*

$$\varpi : \mathcal{L} \mapsto \mathcal{M} \tag{1.5}$$

is a submersion. Then $\mathcal{R}$ extends to a continuous operator

$$\mathcal{R} : L^p_{comp}(\Omega_R) \mapsto L^p_{1/p,loc}(\Omega_L), \quad 4 < p < \infty.$$

The conclusion means that for any C^∞-function υ compactly supported in a coordinate chart of Ω_L and for any compact $K \subset \Omega_R$ we have for all L^p functions f supported in K

$$\|\upsilon \mathcal{R} f\|_{L^p_{1/p}} \leq C_p(\upsilon, K)\|f\|_p.$$

Here L^p_s is the standard Sobolev space consisting of tempered distributions g on $\mathbb{R}^3$ with $(I - \Delta)^{s/2}g \in L^p(\mathbb{R}^3)$. It is easy to see that the regularity index $s = 1/p$ cannot be improved. As mentioned above the result fails for $p < 4$, by a result in [15]. Regarding the hypotheses in Theorem 1.1, one may conjecture that the two-sided fold assumption can be weakened to a one-sided fold assumption, i.e., that the assumption of π_R being a Whitney fold can be dropped. See Sect. 4.2 for further discussion of relevant examples, and Sect. 10 for related results.

Using a theorem in [22] the regularity result can be further improved by using Triebel–Lizorkin spaces $F^s_{p,q}$, namely we have

$$\|\upsilon \mathcal{R} f\|_{F^{1/p}_{p,q}} \leq C_{p,q}(\upsilon, K)\|f\|_{F^0_{p,p}}, \quad 4 < p < \infty, \ q > 0, \tag{1.6}$$

for $f \in F^0_{p,p}$ supported in K, this is further discussed in Sect. 9. Recall that $F^s_{p,2} = L^p_s$ and $F^s_{p,q} \subset F^s_{p,2} \subset F^s_{p,p} = B^0_{p,p}$ for $q \leq 2 \leq p$, and any $s \in \mathbb{R}$.

Notation We shall use the notation $A \lesssim B$ for $A \leq CB$ with an unspecified constant C.

2 Generalized Radon Transforms and Fourier Integral Representations

We recall some basic facts on generalized Radon transforms and Fourier integrals. By localization we may assume that the Schwartz kernel of our operator is supported in a small neighborhood of a base point $P^\circ = (x^\circ, y^\circ) \in \mathcal{M}$. On the neighborhood the manifold $\mathcal{M}$ is given by a defining function Φ, i.e., $\mathcal{M} = \{(x, y) : \Phi(x, y) = 0\}$, where $\Phi = (\Phi^1, \Phi^2)^\mathsf{T}$ is a two-dimensional vector function defined on $\Omega_L \times \Omega_R$ and such that $\Phi(P_0) = 0$. The Schwartz kernel of our operator is given by the measure $\chi \, \delta \circ \Phi$ where δ is the Dirac measure in $\mathbb{R}^2$ and χ is C^∞ and compactly supported near the base point which can be chosen to be the origin in $\mathbb{R}^3 \times \mathbb{R}^3$. By the Fourier inversion formula the Schwartz kernel is an oscillatory integral distribution, formally written as [11,12,24]

$$\chi(x, y)\delta \circ \Phi(x, y) = (2\pi)^{-2} \iint e^{i(\tau_1 \Phi_1(x,y) + \tau_2 \Phi_2(x,y))} \chi(x, y) d\tau. \tag{2.1}$$

Since the projection $\mathcal{M} \to \Omega_L$ is a submersion, the 2×3 matrix Φ_y has rank 2, so by a linear change of variables in y, near y_0 we can assume that $\det[\nabla_{y'}\Phi^1, \nabla_{y'}\Phi^2] \neq 0$ where $y' = (y_1, y_2)^\mathsf{T}$. Then (x, y_3) can be chosen as the local coordinates on $\mathcal{M}$, so

that the equation $\Phi(x, y) = 0$ is equivalent to

$$y_i = S^i(x_1, x_2, x_3, y_3), \quad i = 1, 2. \tag{2.2}$$

Since $\Phi(x, S^1, S^2, y_3) = 0$, we can write

$$\Phi(x, y) = \sum_{i=1}^{2} (S^i(x, y_3) - y_i) B_i(x, y) \tag{2.3}$$

where

$$B_i(x, y) = -\int_0^1 \Phi_{y_i}\left(x, S(x, y_3) + s(y' - S(x, y_3)), y_3\right) ds.$$

Since Φ_{y_1} and Φ_{y_2} are linearly independent on $\mathcal{M}$, by choosing the cutoff χ to be supported sufficiently close to $\mathcal{M}$, we can ensure that B_1 and B_2 are linearly independent as well. The equation (2.3) can therefore be rewritten as

$$\begin{pmatrix} \Phi^1(x, y) \\ \Phi^2(x, y) \end{pmatrix} = B(x, y) \begin{pmatrix} S^1(x, y_3) - y_1 \\ S^2(x, y_3) - y_2 \end{pmatrix}$$

where $B(x, y)$ is the 2×2 invertible matrix whose column vectors are B_1 and B_2. Since the projection $\mathcal{M} \to \Omega_R$ is a submersion the x-gradients $S_x^1(x, y_3)$, $S_x^2(x, y_3)$ are linearly independent. Now (2.1) can be rewritten as

$$\begin{aligned}
\chi(x, y)\, \delta \circ \Phi(x, y) &= \chi(x, y) \int_{\tau \in \mathbb{R}^2} e^{i\langle \tau, \Phi(x, y) \rangle} d\tau \\
&= \frac{\chi(x, y)}{|\det B(x, y)|} \iint e^{i\langle \tau, S(x, y_3) - y' \rangle} d\tau.
\end{aligned} \tag{2.4}$$

Then in a neighborhood of the reference point P the canonical relation, that is the twisted conormal bundle $(N^*\mathcal{M})'$, is given by

$$\begin{aligned}
\{(x, \xi, y, \eta) : y_i = S^i(x, y_3),\ i = 1, 2, \quad \xi = \tau_1 S_x^1(x, y_3) + \tau_2 S_x^2(x, y_3), \\
\eta = (\tau_1, \tau_2, -\tau_1 S_{y_3}^1(x, y_3) - \tau_2 S_{y_3}^2(x, y_3))\}.
\end{aligned}$$

Thus using $(x_1, x_2, x_3, \tau_1, \tau_2, y_3)$ as coordinates on $(N^*\mathcal{M})'$ the projection $\pi_L : (N^*\mathcal{M})' \to T^*\Omega_L$ is identified with

$$\tilde{\pi}_L : (x_1, x_2, x_3, \tau_1, \tau_2, y_3) \mapsto (x, \tau_1 S_x^1(x, y_3) + \tau_2 S_x^2(x, y_3)). \tag{2.5}$$

Then

$$\det D\tilde{\pi}_L = \det(S_x^1, S_x^2, \tau_1 S_{xy_3}^1 + \tau_2 S_{xy_3}^2) = \tau_1 \Delta_1 + \tau_2 \Delta_2$$

with

$$\Delta_i(x, y_3) \equiv \Delta_i^S(x, y_3) := \det(S_x^1, S_x^2, S_{xy_3}^i)\big|_{(x, y_3)}, \quad i = 1, 2. \tag{2.6}$$

Hence $\mathcal{L}$ is the submanifold of $(\mathcal{N}^*\mathcal{M})'$ consisting of (x, ξ, y, η) such that

$$\xi = \tau_1 S_x^1(x, y_3) + \tau_2 S_x^2(x, y_3), \quad \eta = (\tau_1, \tau_2, -\tau_1 S_{y_3}^1(x, y_3) - \tau_2 S_{y_3}^2(x, y_3)),$$
$$y_i = S^i(x, y_3), \quad i = 1, 2, \qquad \tau_1 \Delta_1(x, y_3) + \tau_2 \Delta_2(x, y_3) = 0.$$

3 Curvature

We shall show that the assumptions in Theorem 1.1 imply a curvature condition on the fibers of $\mathcal{L}$, as formulated by Greenleaf and the second author in [8].

Let Δ_i be as in (2.6) and $P^\circ = (a^\circ, S^1(a^\circ, b^\circ), S^2(a^\circ, b^\circ))$ be our reference point. The following preparatory observation is based on the assumption that ϖ in (1.5) is a submersion.

Lemma 3.1 *We have*

$$|\Delta_1(x, y_3)| + |\Delta_2(x, y_3)| \neq 0$$

for (x, y_3) near (a, b).

Proof By continuity we have to check $|\Delta_1| + |\Delta_2| \neq 0$ at P°.

Let $\tau^\circ \in \mathbb{R}^2 \setminus \{0\}$ and let $\xi^\circ = \tau_1^\circ S_x^1(a^\circ, b^\circ) + \tau_2^\circ S_x^2(a^\circ, b^\circ)$. Clearly if $(a^\circ, \xi^\circ) \notin \pi_L(\mathcal{L})$ then $\tau_1^\circ \Delta_1(a^\circ, b^\circ) + \tau_2^\circ \Delta_2(a^\circ, b^\circ) \neq 0$ and therefore we may assume that $(a^\circ, \xi^\circ) \in \pi_L(\mathcal{L})$, i.e.,

$$\tau_1^\circ \Delta_1(a^\circ, b^\circ) + \tau_2^\circ \Delta_2(a^\circ, b^\circ) = 0.$$

Let V_L be a kernel field which we may write as

$$V_L = \sum_{i=1}^{2} \alpha_i(x, \tau) \frac{\partial}{\partial \tau_i} + \alpha_3(x, \tau) \frac{\partial}{\partial y_3} + \sum_{i=1}^{3} \beta_i(x, y_3, \tau) \frac{\partial}{\partial x_i}$$

where $\beta_i = 0$, by (2.5). We have

$$V_L(\tau_1 \Delta_1 + \tau_2 \Delta_2)\big|_{(a^\circ, b^\circ, \tau^\circ)}$$
$$= \sum_{i=1}^{2} \alpha_i(a^\circ, \tau^\circ) \Delta_i(a^\circ, b^\circ) + \alpha_3(a^\circ, \tau^\circ) \sum_{i=1}^{2} \tau_i^\circ \frac{\partial \Delta_i}{\partial y_3}(a^\circ, b^\circ).$$

We argue by contradiction and assume that

$$\Delta_i(a^\circ, b^\circ) = 0, \quad i = 1, 2. \tag{3.1}$$

By assumption $V_L(\det \pi_L) \neq 0$ on $\mathcal{L}$. Using (3.1) we get

$$\tau_1^\circ \frac{\partial \Delta_1}{\partial y_3}(a^\circ, b^\circ) + \tau_2^\circ \frac{\partial \Delta_2}{\partial y_3}(a^\circ, b^\circ) \neq 0. \tag{3.2}$$

Hence we can, for $(\frac{\tau}{|\tau|}, x, y_3)$ near $(\frac{\tau^\circ}{|\tau^\circ|}, a^\circ, b^\circ)$, solve $\tau_1 \Delta_1 + \tau_2 \Delta_2 = 0$ in y_3 and obtain a function $\mathfrak{y}_3(\tau_1, \tau_2)$, homogeneous of degree 0, so that

$$\tau_1 \Delta_1(x, y_3) + \tau_2 \Delta_2(x, y_3) = 0 \iff y_3 = \mathfrak{y}_3(x, \tau).$$

Implicit differentiation gives

$$\frac{\partial \mathfrak{y}_3}{\partial \tau_i} = -\frac{\Delta_i(x, \tau, \mathfrak{y}_3)}{\tau_1 \partial_{y_3} \Delta_1 + \tau_2 \partial_{y_3} \Delta_2}, \quad i = 1, 2. \tag{3.3}$$

Now since we assume that $\varpi : \mathcal{L} \to \mathcal{M}$ is a submersion the differential of the map $(x, \tau) \mapsto (x, S^1(x, \mathfrak{y}_3(x, \tau)), S^2(x, \mathfrak{y}_3(x, \tau)), \mathfrak{y}_3(x, \tau))$ is surjective. This implies that

$$\text{rank} \begin{pmatrix} \partial_{y_3} S^1(x, \mathfrak{y}_3) \partial_{\tau_1} \mathfrak{y}_3 & \partial_{y_3} S^1(x, \mathfrak{y}_3) \partial_{\tau_2} \mathfrak{y}_3 \\ \partial_{y_3} S^2(x, \mathfrak{y}_3) \partial_{\tau_1} \mathfrak{y}_3 & \partial_{y_3} S^2(x, \mathfrak{y}_3) \partial_{\tau_2} \mathfrak{y}_3 \\ \partial_{\tau_1} \mathfrak{y}_3 & \partial_{\tau_2} \mathfrak{y}_3 \end{pmatrix} = 1$$

and thus $|\partial_{\tau_1} \mathfrak{y}_3| + |\partial_{\tau_2} \mathfrak{y}_3| > 0$. But by (3.3) this implies that at least one of the $\Delta_i(a^\circ, b^\circ)$ is nonzero, yielding a contradiction to (3.1). $\qquad\square$

It will be useful to explicitly construct a kernel field V_L in a conic neighborhood of $\mathcal{L}$. Notice that $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$ where $\mathcal{L}^\pm =$

$$\{(x, \pm\rho(-\Delta_2 S_x^1 + \Delta_1 S_x^2), S^1(x, y_3), S_{x,y_3}^2, y_3, \tau, \pm\rho(\Delta_2 S_{y_3}^1 - \Delta_1 S_{y_3}^2)) : \rho > 0\}.$$

We identify π_L with $\tilde{\pi}_L$ as in (2.5).

Lemma 3.2 *Define* $\Gamma_i(x, y_3)$, $i = 1, 2$, *by*

$$\Gamma_1 = \det \left(S_x^1 \ S_{x,y_3}^2 \ S_{xy_3}^1 \right), \tag{3.4a}$$
$$\Gamma_2 = \det \left(S_{xy_3}^1 \ S_x^2 \ S_{xy_3}^2 \right). \tag{3.4b}$$

Let

$$V_L^\pm = \frac{\pm|\tau|}{\sqrt{\Delta_1^2 + \Delta_2^2}} \left(\Gamma_2(x, y_3) \frac{\partial}{\partial \tau_1} - \Gamma_1(x, y_3) \frac{\partial}{\partial \tau_2} \right) + \frac{\partial}{\partial y_3}. \tag{3.5}$$

Then V_L^+, V_L^- *are kernel fields for* $\tilde{\pi}_L$ *near* $\mathcal{L}^+$, $\mathcal{L}^-$, *respectively.*

Proof We take $\tau = \pm\rho(-\Delta_2, \Delta_1)$ and then the assertion reduces to showing that

$$\Gamma_2 S_x^1 - \Gamma_1 S_x^2 - \Delta_2 S_{xy_3}^1 + \Delta_1 S_{xy_3}^2 \Big|_{(x, y_3)} = 0. \tag{3.6}$$

Denote the left-hand side by W. We first observe that

$$\det \left(S_x^1 \ S_x^2 \ \Delta_1 S_x^1 + \Delta_2 S_x^2 \right) = \Delta_1^2 + \Delta_2^2$$

which is nonzero, by Lemma 3.1. We use that three vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ form a basis of $\mathbb{R}^3$ if and only if the vector products $v_1 \wedge v_2$, $v_1 \wedge v_3$, $v_2 \wedge v_3$ form a basis, and apply this fact to $\{S_x^1, S_x^2, \Delta_1 S_{xy_3}^1 + \Delta_2 S_{xy_3}^2\}$. Now $W = 0$ follows by checking $\langle W, S_x^i \wedge (\Delta_1 S_{xy_3}^1 + \Delta_2 S_{xy_3}^2) \rangle = 0$, for $i = 1, 2$, and $\langle W, S_x^1 \wedge S_x^2 \rangle = 0$. These are straightforward to verify. □

We now consider the fibers in $T^* \Omega_L$ of $\pi_L(\mathcal{L})$, namely

$$\Sigma_x := \Big\{ (x, \tau_1 S_x^1(x, y_3) + \tau_2 S_x^2(x, y_3)) :$$
$$\tau_1 \Delta_1(x, y) + \tau_2 \Delta_2(x, y_3) = 0, \ |\tau| \neq 0 \Big\}. \tag{3.7a}$$

Σ_x is a cone which splits as $\cup_\pm \Sigma_x^\pm$ where

$$\Sigma_x^\pm = \{\pm \rho \, \Xi(x, y_3) : \rho > 0\} \tag{3.7b}$$

with

$$\Xi(x, y_3) = -\Delta_2(x, y_3) S_x^1(x, y_3) + \Delta_1(x, y_3) S_x^2(x, y_3). \tag{3.7c}$$

Next, for $\rho > 0$

$$V_L^\pm(\tau_1 \Delta_1(x, y_3) + \tau_2 \Delta_2(x, y_3)) \Big|_{\tau = \pm \rho(-\Delta_2, \Delta_1)} = \pm \rho \kappa(x, y_3) \tag{3.8a}$$

where

$$\kappa(x, y_3) = \Gamma_2 \Delta_1 - \Gamma_1 \Delta_2 + \Delta_1 \Delta_{2, y_3} - \Delta_2 \Delta_{1, y_3} \Big|_{(x, y_3)}. \tag{3.8b}$$

This quantity is nonzero, by the assumption that π_L projects with a fold singularity.

The following lemma will be crucial to establish the curvature properties of the cones Σ_x.

Lemma 3.3 *Let κ be as in (3.8b). Then*

$$\det \left(\Xi \ \Xi_{y_3} \ \Xi_{y_3 y_3} \right) \Big|_{(x, y_3)} = -[\kappa(x, y_3)]^2.$$

Proof We have

$$\Xi = -\Delta_2 S_x^1 + \Delta_1 S_x^2,$$
$$\Xi_{y_3} = -\Delta_{2, y_3} S_x^1 + \Delta_1 S_x^2 - \Delta_2 S_{xy_3}^1 + \Delta_1 S_{xy_3}^2,$$

and

$$\Xi_{y_3 y_3} = - \Delta_{2, y_3 y_3} S_x^1 + \Delta_{1, y_3 y_3} S_x^2$$
$$- 2\Delta_{2, y_3} S_{xy_3}^1 + 2\Delta_{1, y_3} S_{xy_3}^2 - \Delta_2 S_{xy_3 y_3}^1 + \Delta_1 S_{xy_3 y_3}^2$$

where all expressions are evaluated at (x, y_3). Also

$$\Xi \wedge \Xi_{y_3} = (\Delta_1 \Delta_{2, y_3} - \Delta_2 \Delta_{1, y_3})(S_x^1 \wedge S_x^2)$$
$$+ (\Delta_1 S_x^2 - \Delta_2 S_x^1) \wedge (\Delta_1 S_{xy_3}^2 - \Delta_2 S_{xy_3}^1).$$

Define

$$E = \Delta_1 \Delta_{2, y_3} - \Delta_2 \Delta_{1, y_3}.$$

Diligent computation yields

$$\langle \Xi \wedge \Xi_{y_3}, \Xi_{y_3 y_3} \rangle = \sum_{i=1}^{5} A_i \tag{3.9}$$

where

$$A_1 = -2E^2,$$
$$A_2 = E \big(\Delta_1 \det \big(S_x^1 \ S_x^2 \ S_{xy_3 y_3}^2 \big) - \Delta_2 \det \big(S_x^1 \ S_x^2 \ S_{xy_3 y_3}^1 \big) \big),$$
$$A_3 = (\Delta_2 \Delta_{1, y_3 y_3} - \Delta_1 \Delta_{2, y_3 y_3})(\Delta_1 \Delta_2 - \Delta_2 \Delta_1) = 0,$$
$$A_4 = 2E(\Delta_2 \Gamma_1 - \Delta_1 \Gamma_2)$$

and

$$A_5 = - \Delta_2 \Delta_1 \Big(S_x^1 \wedge S_{xy_3}^2 - \Delta_2 S_{xy_3 y_3}^1 + \Delta_1 S_{xy_3 y_3}^2 \Big)$$
$$+ \Delta_2^2 \Big(S_x^1 \wedge S_{xy_3}^1 - \Delta_2 S_{xy_3 y_3}^1 + \Delta_1 S_{xy_3 y_3}^2 \Big)$$
$$+ \Delta_1^2 \Big(S_x^2 \wedge S_{xy_3}^2 - \Delta_2 S_{xy_3 y_3}^1 + \Delta_1 S_{xy_3 y_3}^2 \Big)$$
$$- \Delta_1 \Delta_2 \Big(S_x^2 \wedge S_{xy_3}^1 - \Delta_2 S_{xy_3 y_3}^1 + \Delta_1 S_{xy_3 y_3}^2 \Big).$$

We rewrite the expression $A_5 = A_{5,1} + A_{5,2}$ where

$$A_{5,1} = \Delta_2^2 \det \big(S_x^1 \ \Delta_1 S_{xy_3}^2 \ S_{xy_3 y_3}^1 \big) - \Delta_2^2 \det \big(S_x^1 \ \Delta_2 S_{xy_3}^1 \ S_{xy_3 y_3}^1 \big)$$
$$- \Delta_1 \Delta_2 \det \big(S_x^2 \ \Delta_1 S_{xy_3}^2 \ S_{xy_3 y_3}^1 \big) + \Delta_1 \Delta_2 \det \big(S_x^2 \ \Delta_2 S_{xy_3}^1 \ S_{xy_3 y_3}^1 \big)$$

and

$$A_{5,2} = \Delta_1^2 \det \left(S_x^2 \ \Delta_1 S_{xy_3}^2 \ S_{xy_3y_3}^2 \right) - \Delta_1^2 \det \left(S_x^2 \ \Delta_2 S_{xy_3}^1 \ S_{xy_3y_3}^2 \right)$$
$$- \Delta_2\Delta_1 \det \left(S_x^1 \ \Delta_1 S_{xy_3}^2 \ S_{xy_3y_3}^2 \right) + \Delta_2\Delta_1 \det \left(S_x^2 \ \Delta_2 S_{xy_3}^2 \ S_{xy_3y_3}^2 \right).$$

Now by (3.6) we have

$$\Delta_1 S_{xy_3}^2 - \Delta_2 S_{xy_3}^1 = -\Gamma_2 S_x^1 + \Gamma_1 S_x^2.$$

We use this to simplify $A_{5,1}$ and $A_{5,2}$ to

$$A_{5,1} = \Delta_2(\Gamma_1\Delta_2 - \Gamma_2\Delta_1) \det \left(S_x^1 \ S_x^2 \ S_{xy_3y_3}^1 \right),$$
$$A_{5,2} = -\Delta_1(\Gamma_1\Delta_2 - \Gamma_2\Delta_1) \det \left(S_x^1 \ S_x^2 \ S_{xy_3y_3}^2 \right).$$

We combine these formulae with the previous ones for $A_1, \ldots, A_4$ and use that $A_3 = 0$. We get

$$\sum_{j=1}^{5} A_j = -2E^2 + 2E(\Delta_2\Gamma_1 - \Delta_1\Gamma_2)$$

$$+ (E + \Delta_1\Gamma_2 - \Delta_2\Gamma_1)\left(\Delta_1 \det \left(S_x^1 \ S_x^2 \ S_{xy_3y_3}^2 \right) - \Delta_2 \det \left(S_x^1 \ S_x^2 \ S_{xy_3y_3}^1 \right)\right).$$

Now using

$$\det \left(S_x^1 \ S_x^2 \ S_{xy_3y_3}^i \right) = \Delta_{i,y_3} - \Gamma_i, \quad i = 1, 2,$$

we obtain

$$\sum_{j=1}^{5} A_j = -2E^2 + 2E(\Delta_2\Gamma_1 - \Delta_1\Gamma_2)$$

$$+ (E + \Delta_1\Gamma_2 - \Delta_2\Gamma_1)(\Delta_1\Delta_{2,y_3} - \Delta_1\Gamma_2 - \Delta_2\Delta_{1,y_3} + \Delta_2\Gamma_1)$$
$$= -E^2 - 2E(\Delta_1\Gamma_2 - \Delta_2\Gamma_1) - (\Delta_1\Gamma_2 - \Delta_2\Gamma_1)^2$$
$$= -(E + \Delta_1\Gamma_2 - \Delta_2\Gamma_1)^2$$

which gives the assertion. □

We now examine the curvature properties of the cone $\Sigma_x = \{\rho \Xi(y_3)\}$. Lemma 3.3 implies that $\Xi \wedge \Xi_{y_3} \neq 0$. The second fundamental form at $\rho \Xi(x, y_3)$ with respect to the unit normal $N = \frac{\Xi \wedge \Xi_{y_3}}{|\Xi \wedge \Xi_{y_3}|}$ is given by

$$\begin{pmatrix} \rho\langle \Xi_{y_3y_3}, N \rangle & \langle \Xi_{y_3}, N \rangle \\ \langle \Xi_{y_3}, N \rangle & 0 \end{pmatrix} = \begin{pmatrix} \rho\langle \Xi_{y_3y_3}, N \rangle & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, by Lemma 3.3,

$$\rho \langle \Xi_{y_3 y_3}, N \rangle = \frac{\rho}{|\Xi \wedge \Xi_{y_3}|} \det \left(\Xi \;\; \Xi_{y_3} \;\; \Xi_{y_3 y_3} \right) = \frac{-\rho \kappa (x, y_3)^2}{|\Xi \wedge \Xi_{y_3}|}, \tag{3.10}$$

and the fold condition says that κ does not vanish. Hence Σ_x is a two-dimensional cone such that everywhere there is exactly one nonvanishing principal curvature, and it is given by (3.10).

4 Some Model Operators

The examples motivating the present paper originate from problems in harmonic analysis and integral geometry. We list a few of them below. The notation used in each of these examples is self-contained.

4.1 Averages Along Curves with Nonvanishing Curvature and Torsion

Let $\gamma : I \to \mathbb{R}^3$ be a compact space curve with nonvanishing curvature and torsion. Then, the integral operator $\mathcal{A}$ in (1.1) is an example of a Fourier integral operator of order $-1/2$ with two-sided fold singularities. Clearly the projection ϖ in (1.5) is a submersion. Thus we recover the result that $\mathcal{A} : L^p \to L^p_{1/p}$ for $p > 4$ which is known by a combination of the results in [5,20]. The theorem in this paper shows that the $L^p_{\text{comp}} \to L^p_{1/p}$ estimate holds true for small variable perturbations of the translation invariant case.

4.2 Restricted *X*-Ray Transforms in $\mathbb{R}^3$

A restricted X-ray transform in $\mathbb{R}^3$ is the restriction of the X-ray transform to a *line complex*, that is, a three-dimensional manifold of lines. Under a suitable *well-curvedness assumption* it was shown in [9] (see also [8]) that (local versions) of this operator are Fourier integral operators of order $-1/2$ for which the projection π_R has Whitney folds. For a class of generic line complexes we have two-sided fold singularities but this is not the case for the important class satisfying Gelfand's admissibility condition (see [9]) which is relevant for invertibility of the restricted X-ray transform. The optimal $L^2 \to L^2_{1/4}$-Sobolev regularity for the latter was obtained in [10], and can also be seen as a part of a result on more general Fourier integral operators with one-sided fold singularities in [8].

We discuss a model case. Let I be a compact interval and $\gamma : I \to \mathbb{R}^2$ a smooth regular curve $\gamma(t) = (t, g(t)), t \in I$. We assume that γ has *nonvanishing curvature*, i.e.,

$$g''(t) \neq 0, \quad t \in I. \tag{4.1}$$

Let $\beta \in (-1, 1)$ and let $e_2 = (0, 1) \in \mathbb{R}^2$. For $f \in C_0^\infty(\mathbb{R}^3)$ we define

$$\mathcal{R}_\beta f(x_1, x_2, t) = \chi_1(t) \int f(x_1 + st, x_2 + s(\beta x_2 + g(t)), s)\chi_2(s)ds \qquad (4.2)$$

where $x' = (x_1, x_2)$ and χ_1, χ_2 are smooth real-valued functions, with χ_1 supported in the interior of I and χ_2 with compact support contained in $\mathbb{R} \setminus \{-\beta^{-1}\}$.

We examine the adjoint operator which is given by

$$R_\beta^* h(x) = \int h(S^1(x, y_3), S^2(x, y_3), y_3)\chi_2(x_2)\chi_1(y_3)dy_3$$

with

$$S^1(x, y_3) = x_1 - x_3 y_3, \qquad S^2(x, y_3) = \frac{x_2 - x_3 g(y_3)}{1 + x_3\beta}.$$

Then $(S_x^1 \; S_x^2 \; \tau_1 S_{xy_3}^1 + \tau_2 S_{xy_3}^2)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 + x_3\beta)^{-1} & 0 \\ -y_3 & \frac{-g(y_3)-\beta x_2}{(1+x_3\beta)^2} & -\tau_1 - \tau_2\frac{g'(y_3)}{(1+x_3\beta)^2} \end{pmatrix}$$

and hence

$$\det \pi_L = \tau_1\Delta_1 + \tau_2\Delta_2 = -(1 + x_3\beta)^{-1}\left(\tau_1 + \tau_2\frac{g'(y_3)}{(1 + x_3\beta)^2}\right).$$

Now $\partial/\partial y_3$ is a kernel field and the fold condition holds iff and only if (4.1) holds on I.

The cones Σ_x are given by

$$\Sigma_x = \left\{\xi \in \mathbb{R}^3 : \xi = \lambda(-g'(t), 1 + \beta x_3, tg'(t) - g(t) - \beta x_2, \; \lambda \in \mathbb{R}, \; t \in I\right\}.$$

To check that the cone Σ_x has one nonvanishing curvature everywhere one verifies that the plane curve $\Gamma(t) = (-g'(t), tg'(t) - g(t))$ has nonvanishing curvature. This holds since $\Gamma_1'\Gamma_2'' - \Gamma_1''\Gamma_2'(t) = -(g''(t))^2$.

In order to apply our main result one also needs to check that the projection π_R (for the adjoint R_β^*) has only fold singularities; this turns out to be the case when $\beta \neq 0$. π_R is given by

$$(x_1, x_2, x_3, \tau_1, \tau_2, y_3) \mapsto (S^1(x, y_3), S^2(x, y_3), y_3, \tau_1, \tau_2, \tau_1 S_{y_3}^1 + \tau_2 S_{y_3}^2)$$

and a calculation shows that $V_R = y_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$ is a kernel field. Then

$$V_R(\tau_1 \Delta_1 + \tau_2 \Delta_2) = -\tau_2 g'(y_3) \frac{2\beta}{(1 + \beta x_3)^3} .$$

Thus, if $\beta \neq 0$, π_R has only fold singularities. Now Theorem 1.1 implies that for $\beta \neq 0$ the operator R_β^* maps L^p to $L_{1/p}^p$ for $p > 4$, and more generally L_α^p to $L_{\alpha+1/p}^p$. Hence

$$R_\beta : L^p \rightarrow L_{1/p'}^p, \ 1 < p < 4/3, \tag{4.3}$$

when $\beta \neq 0$. Our theorem does not apply to the case $\beta = 0$, when π_R has maximal degeneracy (a blowdown singularity). However by a rather straightforward argument it was shown in [19] that (4.3) remains valid if $\beta = 0$ (provided one uses the result by Bourgain and Demeter in conjunction with [19]). This leads one to conjecture that the assumption on $D\pi_R$ in Theorem 1.1 can be dropped.

4.3 Averages Along Curves in $\mathbb{H}^1$

Convolution operators on noncommutative groups can often be analyzed as generalized Radon transforms. Let us consider the Heisenberg group $\mathbb{H}^1$ which is $\mathbb{R}^3$ with the group multiplication defined by

$$x \cdot y = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\right).$$

4.3.1 Measures on Curves in the Plane

Let I be a bounded open interval and $g \in C^\infty$. We consider convolution on $\mathbb{H}^1$ with a measure on $\mathbb{R}^2 \times \{0\}$ supported on $\{(t, g(t), 0) : t \in I\}$ where $g''(t) \neq 0$ for $t \in I$. For $\chi \in C_0^\infty(I)$ define μ by

$$\langle \mu, f \rangle := \int_\gamma f(t, g(t), 0) \chi(t) dt$$

and the convolution

$$\mathcal{A}_1 f(x) := f * \mu(x) = \int f\left(x' - y', x_3 - \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\right) d\mu_0(y').$$

Then $\mathcal{A}_1 f(x)$ can be written as $\int f(y_1, S^2(x, y_1), S^3(x, y_1)) \chi(x_1 - y_1) dy_1$ with

$$S^2(x, y_1) = x_2 - g(x_1 - y_1),$$
$$S^3(x, y_1) = x_3 - \frac{x_1}{2} g(x_1 - y_1) + \frac{x_2}{2}(x_1 - y_1).$$

As observed in [14], $\mathcal{A}_1$ is a Fourier integral operator with folding canonical relation (i.e., π_L and π_R) project with folds. Moreover

$$\pi_L(N^*\mathcal{M}) = \left\{(x, \tau_2 S_x^2(x, y_1) + \tau_3 S_x^3(x, y_1))\right\}$$

and $\det \pi_L = g''(x_1 - y_1)(\tau_2 + \tau_3 x_1/2)$ and thus Σ_x is given by the parametrization

$$\Xi(\tau_3, t) = \frac{\tau_3}{2}(x_2 - g(t), -x_1 + t, 1).$$

This example and higher dimensional versions were considered in [14] for the L^2-Sobolev category, together with some refinements, that yield sharp maximal function estimates on $\mathbb{H}^n$, $n \geq 2$. The measure in the horizontal plane can also be replaced by other measures in other planes transversal to the center, in which case the estimates in [14] yield less satisfactory results for maximal function bounds. However in this case sharp L^p-Sobolev estimates and maximal function bounds for $n \geq 2$ have been recently established in [1], using methods which are closely related to the current paper. For a more recent result on circular maximal functions on the Heisenberg group see [2] where the case of Heisenberg radial functions is considered.

4.3.2 Averages Along Space Curves in $\mathbb{H}^1$

A closely related example was considered by Phong and Stein [18] and Secco [23]. Let $\gamma_\alpha : I \to \mathbb{H}^1$ be the curve given by $\gamma_\alpha(s) = (s, s^2, \alpha s^3)$, where α is a real-valued parameter, and I a bounded interval. Given a cutoff function $\chi \in C_0^\infty(I)$, let us consider the singular measure μ_α on $\mathbb{H}^1$ supported on γ_α given by

$$\langle \mu_\alpha, f \rangle = \int_I f(\gamma_\alpha(s))\chi(s)\,ds,$$

and the right convolution operator by μ_α:

$$\mathcal{A}_{2,\alpha}f(x) = f * \mu_\alpha(x) := \int f(x \cdot \gamma_\alpha(s)^{-1})\,ds, \qquad x \in \mathbb{H}^1. \qquad (4.4)$$

As shown in [18] $\mathcal{A}_{2,\alpha}$ is a Fourier integral operator, with two-sided folds for $\alpha \neq \pm\frac{1}{6}$ and with one-sided folds for $\alpha = \pm\frac{1}{6}$. A special role of the parameters $\pm\frac{1}{6}$ has also been observed by Secco [23] in the context of $L^p \to L^q$ estimates. It is straightforward to verify that the projection π_L in this problem is a fold if and only if $\alpha = \frac{1}{6}$, and the cone $\Sigma_x \subseteq \mathbb{R}_\xi^3$ is generated by the parabola

$$\xi_1 = \frac{x_2}{2} + 2(6\alpha - 1)t^2, \quad \xi_2 = -\frac{x_1}{2} - (6\alpha - 1)t, \quad \xi_3 = 1.$$

Our result yields the sharp L^p regularity properties for all $\alpha \in \mathbb{R} \setminus \{\pm 1/6\}$ but it does not cover the cases $\alpha = \pm 1/6$. Bentsen [4] obtained a sharp L^p regularity results for

a class of averaging operators over curves in the Heisenberg group for which one of π_L, π_R is a fold and the other is a blowdown. It turns out that in the case $\alpha = -1/6$ of (4.4) the local regularity results follow by changes of variables directly from the regularity results for the restricted X-ray transform in (4.2) when $\beta = 0$ (i.e., the case considered in [19]).

5 Basic Decompositions

We decompose dyadically in τ (for large τ). Then for $|\tau| \approx 2^k$ we decompose further according to the size of $2^{-k} \det \pi_L$ which is approximately the size of $2^{-k}(\tau_1 \Delta_1 + \tau_2 \Delta_2)$, which is also approximately the distance to the fold surface. This decomposition is standard and goes back to [18] (with earlier precursors).

Let $\eta_0 \in C_c^\infty(\mathbb{R})$ be an even function so that $\eta_0(s) = 1$ for $|s| \leq \frac{1}{2}$ and $\mathrm{supp}(\eta_0) \subset (-1, 1)$, and set $\eta_1(s) = \eta_0(\frac{s}{2}) - \eta_0(s)$. Then $\eta_0(s) + \sum_{k \geq 1} \eta_1(2^{1-k}s) \equiv 1$ for $s \geq 0$. Define

$$\chi_k(x, y, \tau) := \chi(x, y)\eta_1(2^{1-k}|\tau|) \quad \text{for } k \geq 1, \tag{5.1a}$$

$$\chi_0(x, y, \tau) := \chi(x, y)\eta_0(|\tau|), \tag{5.1b}$$

and, after changing variables in τ

$$\mathcal{R}_k f(x) := 2^{2k} \iint e^{i2^k \langle \tau, S(x, y_3) - y' \rangle} \chi_k(x, y, 2^k \tau) d\tau \, f(y) dy, \tag{5.1c}$$

with $\langle \tau, S(x, y_3) - y' \rangle = \sum_{i=1}^2 \tau_i (S^i(x, y_3) - y_i)$ and now $|\tau| \approx 1$. We then have

$$\mathcal{R} f = \sum_{k \geq 0} \mathcal{R}_k f$$

for all Schwartz functions f. For $0 \leq \ell \leq \lfloor k/3 \rfloor$, let

$$\chi_{k,\ell}(x, y, \tau) := \begin{cases} \chi_k(x, y, 2^k \tau)\eta_1\big(2^\ell(\tau_1 \Delta_1 + \tau_2 \Delta_2)\big), & \text{if } \ell < \lfloor k/3 \rfloor, \\ \chi_k(x, y, 2^k \tau)\eta_0\big(2^{\lfloor \frac{k}{3} \rfloor}(\tau_1 \Delta_1 + \tau_2 \Delta_2)\big), & \text{if } \ell = \lfloor k/3 \rfloor, \end{cases}$$

and

$$R_{k,\ell}(x, y) := 2^{2k} \int e^{i2^k \langle \tau, y' - S(x, y_3) \rangle} \chi_{k,\ell}(x, y, 2^k \tau) \, d\tau, \tag{5.2a}$$

$$\mathcal{R}_{k,\ell} f(x) := \int R_{k,\ell}(x, y) f(y) \, dy. \tag{5.2b}$$

so that $\mathcal{R}_k = \sum_{\ell \leq \frac{k}{3}} \mathcal{R}_{k,\ell}$. For $k > 0$ the τ-integration is extended over a subset of the annulus $\{1/2 < |\tau| < 2\}$ (indeed the intersection of this annulus with a

$C2^{-\ell}$-neighborhood of a line $l(x, y_3)$). The quantity $\tau_1 \Delta_1 + \tau_2 \Delta_2$, when $|\tau| \approx 1$ is comparable to the distance to the fold surface $\mathcal{L}$.

The by now standard L^2 estimate for the operators $\mathcal{R}_{k,\ell}$ is

$$\|\mathcal{R}_{k,\ell}\|_{L^2 \to L^2} \lesssim 2^{\frac{\ell-k}{2}} \tag{5.3}$$

for $\ell = 0, 1, \ldots \lfloor k/3 \rfloor$, see [7]. The following estimates will be the main ingredient for the proof of Theorem 1.1.

Theorem 5.1 *Let* $0 < \epsilon < 1/6$. *For* $\ell \leq \lfloor k/3 \rfloor$ *we have*

$$\|\mathcal{R}_{k,\ell}\|_{L^p \to L^p} \leq C_{\epsilon,p} \cdot \begin{cases} 2^{\ell(\epsilon + \frac{2}{p} - \frac{1}{2})} 2^{-\frac{k}{p}}, & 4 < p \leq 6, \\ 2^{-\ell(\frac{1-6\epsilon}{p})} 2^{-\frac{k}{p}}, & 6 \leq p \leq \infty. \end{cases} \tag{5.4}$$

The endpoint Sobolev bound will follow from this theorem with some additional arguments, see §9.

The main important tool in the proof is the following decoupling inequality.

Theorem 5.2 *Let* $\ell \leq k/3$ *and let* $\epsilon > 0$. *Let, for* $\nu \in \mathbb{Z}$

$$f_\nu(y) = f(y) \mathbb{1}_{[2^{-\ell}\nu, 2^{-\ell}(\nu+1)]}(y_3). \tag{5.5}$$

Then for $2 \leq p \leq 6$,

$$\left\| \sum_\nu \mathcal{R}_{k,\ell} f_\nu \right\|_p \leq C_\epsilon 2^{\ell\left(\epsilon + \frac{1}{2} - \frac{1}{p}\right)} \left(\sum_\nu \|\mathcal{R}_{k,\ell} f_\nu\|_p^p \right)^{1/p} + C_\epsilon 2^{-k} \|f\|_p. \tag{5.6}$$

Theorem 5.2 will be proved by induction, see §8. In each induction step we will combine a standard application of the Wolff–Bourgain–Demeter decoupling theorem in combinations with suitable changes of variables.

Proof that Theorem 5.2 implies Theorem 5.1 We first note that for $g_\nu \in L^\infty$ and with $\mathbb{1}_{\nu,\ell}(y_3) := \mathbb{1}_{[2^{-\ell}\nu, 2^{-\ell}(\nu+1)]}(y_3)$

$$\sup_\nu \|\mathcal{R}_{k,\ell}[\mathbb{1}_{\nu,\ell} g_\nu]\|_\infty \lesssim 2^{-\ell} \sup_\nu \|g_\nu\|_\infty. \tag{5.7}$$

To see this one derives an estimate for the Schwartz kernel $R_{k,\ell}(x, y)$ by integrating by parts, distinguishing the directions (Δ_1, Δ_2) and $(-\Delta_2, \Delta_1)$. This shows that $|R_{k,\ell}(x, y)| \leq C_N \prod_{i=1}^2 U_{k,\ell,i}(x, y)$ where

$$U_{k,\ell,1}(x, y) = \frac{2^{k-\ell}}{(1 + 2^{k-\ell} |\Delta_1(y_1 - S^1) + \Delta_2(y_2 - S^2)|)^N}$$

$$U_{k,\ell,2}(x, y) = \frac{2^k}{(1 + 2^k | -\Delta_2(y_1 - S^1) + \Delta_1(y_2 - S^2)|)^N}$$

where S^1, S^2, Δ_1, Δ_2 are evaluated at (x, y_3). We integrate in (y_1, y_2) first and then use that the y_3 integration is extended over an interval of length $2^{-\ell}$. This yields (5.7).

From (5.3) and averaging with Rademacher functions we also get

$$\left(\sum_\nu \left\|\mathcal{R}_{k,\ell}[\mathbb{1}_{\nu,\ell}g_\nu]\right\|_2^2\right)^{1/2} \lesssim 2^{\frac{\ell-k}{2}}\left(\sum_\nu \|g_\nu\|_2^2\right)^{1/2},$$

and by interpolation,

$$\left(\sum_\nu \left\|\mathcal{R}_{k,\ell}[\mathbb{1}_{\ell,\nu}g_\nu]\right\|^p\right)^{\frac{1}{p}} \lesssim 2^{\ell(\frac{3}{p}-1)}2^{-\frac{k}{p}}\left(\sum_\nu \|g_\nu\|_p^p\right)^{\frac{1}{p}}, \quad 2 \le p \le \infty. \quad (5.8)$$

Combining this with (5.6) we obtain

$$\left\|\mathcal{R}_{k,\ell}f\right\|_p \lesssim C_\varepsilon 2^{\ell\left(\epsilon+\frac{2}{p}-\frac{1}{2}\right)}2^{-\frac{k}{p}}\|f\|_p, \quad 2 \le p \le 6.$$

Finally from (5.7) we also have the bound $\|\mathcal{R}_{k,\ell}\|_{L^\infty \to L^\infty} = O(1)$ and a further interpolation gives the inequality asserted in (5.4) for $6 \le p \le \infty$. $\qquad\square$

An Estimate in Besov Spaces

Theorem 5.1 implies an estimate in Besov spaces. To see that we let L_k be the operator defined by $\widehat{L_k f} = \beta(2^{-k}\xi)\widehat{f}$ where $\beta \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. Integration by parts arguments show that there exists a constant C such that

$$\|L_{k'}\mathcal{R}_{k,\ell}L_{k''}\|_{L^p \to L^p} \le C_N 2^{-N\max\{k,k',k''\}} \text{ if } \max\{|k-k'|, |k-k''|\} > C \quad (5.9)$$

whenever $\min\{k, k', k''\} \ge 3\ell$. This, together with the main estimate

$$\|\mathcal{R}_k\|_{L^p \to L^p} \le C(p)2^{-k/p} \text{ for } p > 4. \quad (5.10)$$

implies the boundedness result

$$\mathcal{R} : (B_{p,q}^s)_{\text{comp}} \to (B_{p,q}^{s+1/p})_{\text{loc}}, \text{ for } p > 4.$$

For the more sophisticated Sobolev bounds, and improvements, see §9.

6 The Decoupling Step in a Model Case

In this section we consider a model version of the operator $\mathcal{R}_{k,\ell}$ defined in (5.2), where the functions S^i are replaced by $\mathfrak{S}^i$ satisfying additional assumptions at the origin, see (6.9). These normalizing assumptions will enable us to carry out a decoupling step

as suggested by the Bourgain–Demeter decoupling theorem which we review in §6.1. The reduction of the general case to the model case will be carried out later in Sect. 8, using suitable changes of variables discussed in Sect. 7.

6.1 The Bourgain–Demeter Decoupling Theorem

Let $\kappa_0 \neq 0$ be a constant. We use the decoupling result in [5], for the part of the cone

$$\Sigma = \left\{ \xi : \kappa_0 \xi_2 \xi_3 + \tfrac{1}{2}\xi_1^2 = 0 \right\}$$

where $|\xi_2| \approx 1$, $|\xi_1| \ll 1$. A parametrization is given by

$$\xi(b, \lambda) = \lambda(-\kappa_0 b e_1 + e_2 - \tfrac{1}{2}\kappa_0 b^2 e_3)$$

where $|\lambda| \approx 1$, $|b| \ll |b| M_0 \ll 1$. Let

$$T_1(b) = \frac{\partial}{\partial \lambda}\xi(b, \lambda) = -\kappa_0 b e_1 + e_2 - \tfrac{1}{2}\kappa_0 b^2 e_3, \tag{6.1}$$

be the tangent vector pointing towards the origin and let

$$\widetilde{T}_2(b) = -\kappa_0^{-1}\lambda^{-1}\frac{\partial}{\partial b}\xi(b, \lambda) = e_1 + b e_3. \tag{6.2}$$

Then $T_1(b)$ and $\widetilde{T}_2(b)$ form a basis of the tangent space of Σ at $\lambda\xi(b)$. A normal vector is given by

$$N(b) = T_1(b) \wedge \widetilde{T}_2(b) = b e_1 + \tfrac{1}{2}\kappa_0 b^2 e_2 - e_3. \tag{6.3}$$

For the definition of our plate we need to replace $\widetilde{T}_2(b)$ by a vector in the span of $T_1(b)$ and $\widetilde{T}_2(b)$ that is perpendicular to $T_1(b)$. Such a vector is given by

$$T_2(b) = \left(1 - \frac{1}{4}\kappa_0^2 b^4\right) e_1 + \kappa_0 b \left(1 + \tfrac{1}{2}b^2\right) e_2 + \left(b + \tfrac{1}{2}\kappa_0^2 b^3\right) e_3$$
$$= e_1 + \kappa_0 b e_2 + b e_3 + O(b^3). \tag{6.4}$$

Let $A > 1$. For $\delta \ll 1$ let

$$\Pi_{A,b}(\delta)$$
$$= \left\{ \xi \in \mathbb{R}^3 : A^{-1} \le \left| \left\langle \tfrac{T_1(b)}{|T_1(b)|}\xi \right\rangle \right| \le A, \quad \left| \left\langle \tfrac{T_2(b)}{|T_2(b)|}\xi \right\rangle \right| \le A\delta, \quad \left| \left\langle \tfrac{N(b)}{|N(b)|}\xi \right\rangle \right| \le A\delta^2 \right\}. \tag{6.5}$$

One refers to the sets $\Pi_{A,b}(\delta)$ as plates; they are unions of $A(1, \delta, \delta^2)$-boxes with the long, middle, short side parallel to $T_1(b)$, $T_2(b)$, $N(b)$, respectively.

Theorem [5] *Let $\epsilon > 0$, $A > 1$. There exists a constant $C(\epsilon, A)$ such that the following holds for $0 < \delta_1 < \delta_0 < 1$.*

Let $B = \{b_\nu\}_{\nu=1}^{M}$ be a set of points in $[-1, 1]$ such that $|b_\nu - b_{\nu'}| \geq \delta_1$ for $b_\nu, b_{\nu'} \in B$, $\nu \neq \nu'$, and B is contained in an interval of length δ_0. Let $2 \leq p \leq 6$. Let $f_\nu \in L^p(\mathbb{R}^3)$ such that the Fourier transform of f_ν is supported in $\Pi_{A, b_\nu}(\delta_1)$. Then

$$\left\| \sum_\nu f_\nu \right\|_p \leq C(\epsilon, A)(\delta_0/\delta_1)^\epsilon \left(\sum_\nu \|f_\nu\|_p^2 \right)^{1/2}. \tag{6.6}$$

One also has

$$\left\| \sum_\nu f_\nu \right\|_p \leq C(\epsilon, A) \begin{cases} (\delta_0/\delta)^{\epsilon+1/2-1/p} \left(\sum_\nu \|f_\nu\|_p^p \right)^{1/p}, & p \leq 6, \\ (\delta_0/\delta)^{\epsilon+1-4/p} \left(\sum_\nu \|f_\nu\|_p^p \right)^{1/p}, & 6 \leq p \leq \infty. \end{cases} \tag{6.7}$$

This is the ℓ^p-decoupling result that was first proved for large p by Wolff [26]. (6.7) follows from (6.6) by Hölder's inequality and interpolation arguments. Our proof of Theorem 1.1 will be based on (6.6) but could also be based on the case $p > 6$ of (6.7), as it was in [20], in the case of convolution operators. A variant of this argument was also given in the manuscript [21] on the variable case, an unpublished precursor to the current paper, with only a preliminary result.

6.2 The Model Case

For $i = 1, 2$ consider C^∞ functions $(w, z_3) \mapsto \mathfrak{S}^i(w, z_3)$ defined on a neighborhood U of $[-r, r]^4$, for some $r \in (0, 1)$. Assume that M_0 satisfies

$$M_0 \geq 2 + \|\mathfrak{S}^1\|_{C^5([-r,r]^4)} + \|\mathfrak{S}^2\|_{C^5([-r,r]^4)} \tag{6.8}$$

where the C^5 norm is the maximum of the supremum of all derivatives of order $0, \ldots, 5$. We assume that for $w \in [-r, r]^3$

$$(\mathfrak{S}^1, \mathfrak{S}^2, \mathfrak{S}^1_{z_3})\big|_{(w,0)} = (w_1, w_2, w_3); \tag{6.9a}$$

moreover

$$\mathfrak{S}^2_{wz_3}(0, 0) = 0, \tag{6.9b}$$

and

$$\mathfrak{S}^2_{w_3 z_3 z_3}(0, 0) = \kappa_0. \tag{6.9c}$$

Let in (5.1a) the function χ_0 be supported in a neighborhood V of $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}^3$ which is of diameter $\leq 10^{-10} M_0 \ll r$ and let $(w, z) \mapsto \alpha(w, z)$ be a C^∞ function satisfying

$$M_0^{-1} \leq |\alpha(w, z)| \leq M_0 \tag{6.10}$$

and with the higher derivatives of α depending on M_0 and the order of differentiation.

Let $(w, z_3, \mu) \mapsto \zeta(w, z_3, \mu)$ belong to a bounded family of C^∞ functions supported where $-r \le w_i, z_3 \le r$ and $1/4 \le |\mu| \le 4$. Let η be C^∞ and supported in $(-2, 2)$ and let $\mathcal{T}_{k,\ell}$ be the operator with Schwartz kernel

$$\mathcal{T}_{k,\ell}(w, z) := 2^{2k} \int_{\mathbb{R}^2} e^{i2^k \langle \mu, \mathfrak{S}(w, z_3) - z' \rangle}$$
$$\times \eta \big(2^\ell \alpha(w, z)(\mu_1 \Delta_1^{\mathfrak{S}}(w, z_3) + \mu_2 \Delta_2^{\mathfrak{S}}(w, z_3))\big) \zeta(w, z, \mu) \, d\mu.$$
(6.11)

Here $\Delta_i^{\mathfrak{S}}(w, z_3) = \det(\mathfrak{S}_w^1, \mathfrak{S}_w^2, \mathfrak{S}_{wz_3}^i)$. We shall omit the superscript and assume throughout this subsection §6.2 that $\Delta_i \equiv \Delta_i^{\mathfrak{S}}$. The operator $\mathcal{T}^{k,\ell}$ is a version of $\mathcal{R}^{k,\ell}$ defined before under the additional assumptions in (6.9). We need to include the function α in the localization to provide added flexibility in the later stages of the proof of Theorem 5.2 when we apply repeated changes of variables (*cf.* formula (7.15)).

The basic decoupling step is summarized in

Proposition 6.1 *Let $0 < \epsilon < 1/2$. There is a constant C_ϵ so that the following holds. Let $\ell \le \lfloor k/3 \rfloor$ and let*

$$\delta_0, \delta_1 \in \big(M_0^2 2^{20 - \ell(1 - \epsilon^2)}, 2^{-\ell \epsilon^2 - 20} M_0^{-2} \big)$$
(6.12a)

such that

$$2^{100} M_0 \max \big\{ (2^{-\ell} \delta_0)^{1/2}, \delta_0^{3/2} \big\} < \delta_1 < \delta_0.$$
(6.12b)

Let $\mathcal{I}_J$ be a collection of intervals of length δ_1 which have disjoint interior and which are contained in $[0, \delta_0]$. Let $a \in \mathbb{R}^3$, $\varsigma \in C_c^\infty$ supported in $(-1, 1)^3$ and $\varsigma_{\ell,0}(w) = \varsigma(2^\ell w)$. Then for $2 \le p \le 6$, for $g \in L^p(\mathbb{R}^3)$ and $g_I(y) := g(y) \mathbb{1}_I(z_3)$ we have

$$\left\| \varsigma_{\ell,0} \sum_{I \in \mathcal{I}_J} \mathcal{T}_{k,\ell} g_I \right\|_p \le C_\epsilon (\delta_0/\delta_1)^\epsilon \left(\sum_{I \in \mathcal{I}_J} \left\| \varsigma_{\ell,0} \mathcal{T}_{k,\ell} g_I \right\|_p^2 \right)^{1/2} + C_\epsilon 2^{-10k} \|g\|_p.$$
(6.13)

In order to apply (6.6) in this situation we need to consider the Fourier transforms of $\varsigma_{\ell,0} \sum_{I \in \mathcal{I}_J} \mathcal{T}_{k,\ell} g_I$ and show that they are concentrated on the plates $2^k \Pi_{A,b_I}(\delta_1)$ for $b_I \in I$ and suitable $A > 1$. We establish this plate localization in Sect. 6.4 and conclude the proof of Proposition 6.1 in §6.5.

6.3 Derivatives of $\mathfrak{S}$ and Δ

We use this section to record some facts needed later in §6.4, about various derivatives of $\mathfrak{S}^i(w, z_3)$ and $\Delta_i(w, z_3)$, under the assumption that

$$|w|_\infty \le 2^{-\ell} \le \delta_0, \quad |z_3| \le \delta_0,$$
(6.14)

under the specifications in (6.12).

6.3.1 Taylor Expansion of $\mathfrak{S}_w^1$ and $\mathfrak{S}_w^2$

Lemma 6.2 *Let w, z_3, $2^{-\ell}$, δ_0 be as in (6.14). Then*

$$
\begin{aligned}
\mathfrak{S}_w^1(w, z_3) &= e_1 + z_3 e_3 + E^1(w, z_3) \\
\mathfrak{S}_w^2(w, z_3) &= e_2 + \tfrac{1}{2}\kappa_0 z_3^2 e_3 + E^2(w, z_3)
\end{aligned}
\tag{6.15}
$$

where

$$
|\langle e_i, E^1(w, z_3)\rangle| \le 8M_0\delta_0^2, \quad i = 1, 2, 3,
\tag{6.16}
$$

and

$$
|\langle e_i, E^2(w, z_3)\rangle| \le 8M_0\delta_0^2, \quad i = 1, 2,
\tag{6.17a}
$$

$$
|\langle e_3, E^2(w, z_3)\rangle| \le M_0(8\delta_0 2^{-\ell} + 2\delta_0^3),
\tag{6.17b}
$$

Proof We expand using conditions (6.9a) and obtain

$$
\begin{aligned}
\mathfrak{S}_w^1(w, z_3) &= e_1 + z_3 e_3 + \widetilde{E}^1(w, z_3) \\
\mathfrak{S}_w^2(w, z_3) &= e_2 + \widetilde{E}^2(w, z_3)
\end{aligned}
$$

where for $\nu = 1, 2$ we have $\langle e_i, \widetilde{E}^\nu\rangle = I_{i,\nu} + II_{i,\nu} + III_{i,\nu}$ with

$$
I_{i,\nu}(w, z_3) = \int_0^1 (1-s)\sum_{j=1}^3\sum_{k=1}^3 \mathfrak{S}_{w_i w_j w_k}^\nu(sw, sz)w_j w_k \, ds
$$

$$
II_{i,\nu}(w, z_3) = \int_0^1 (1-s)2\sum_{j=1}^3 \mathfrak{S}_{w_i w_j z_3}^\nu(sw, sz)w_j z_3 \, ds
$$

$$
III_{i,\nu}(w, z_3) = \int_0^1 (1-s)\mathfrak{S}_{w_i z_3 z_3}^\nu(sw, sz)z_3^2 \, ds
$$

and obtain the bounds

$$
\begin{aligned}
|I_{i,\nu}(w, z_3)| &\le \tfrac{9}{2}M_0|w|_\infty^2 \le \tfrac{9}{2}M_0 2^{-2\ell} \\
|II_{i,\nu}(w, z_3)| &\le \tfrac{6}{2}M_0|w|_\infty|z_3| \le 3M_0 2^{-\ell}\delta_0 \\
|III_{i,\nu}(w, z_3)| &\le \tfrac{1}{2}M_0|z_3|^2 \le \tfrac{1}{2}M_0\delta_0^2.
\end{aligned}
$$

Recall $\kappa_0 = \mathfrak{S}_{w_3 z_3 z_3}^2(0, 0)$. For $i = 3$, $\nu = 2$ we expand further

$$
III_{3,2}(w, z_3) = \tfrac{1}{2}\kappa_0 z_3^2 + E_{3,2}(w, z_3)
$$

where

$$
|E_{3,2}(w, z_3)| \le M_0\big(\tfrac{3}{2}|w|_\infty|z_3|^2 + \tfrac{1}{2}|z_3|^3\big) \le 2M_0\delta_0^3
$$

(where we used $2^{-\ell} \leq \delta_0$). Combining terms we obtain the stated error estimates. $\square$

6.3.2 Computations Involving Δ_1 and Δ_2

By the assumption $\delta_0 \leq 2^{-10} M_0^{-1}$ we have from above $|\mathfrak{S}^1_{w_i}(w, z_3)| \leq 2$, $|\mathfrak{S}^2_{w_i}(w, z_3)| \leq 2$ and $|\mathfrak{S}^1_{w_i z_3}(w, z_3)| \leq 2$. Moreover $|\mathfrak{S}^2_{wz_3}(w, z_3)| \leq 4M_0\delta_0 \ll 1$. Using upper bounds for $\mathfrak{S}^i_w$, $\mathfrak{S}^i_{wz_3}$ and higher derivatives, the permutation formula for determinants, trilinearity of the determinants and differentiation of products, we see that any first-order partial derivative of $\pm\Delta_i$ is a sum of $3 \cdot 6$ terms, each bounded by $4M_0$. Hence any first-order partial derivative of $\pm\Delta_i^{\mathfrak{S}}$ is bounded by $72M_0$, and similarly, by the structure of the Δ_i, any second-order partial derivative of $\pm\Delta_i^{\mathfrak{S}}$ is bounded by $216M_0$. Moreover, any third-order partial derivative of $\pm\Delta_i^{\mathfrak{S}}$ is bounded by $54 \cdot 2M_0^2$. These observations also yield

$$|\Delta_1(w, z_3) - 1|, |\Delta_2(w, z_3)| \leq 72M_0\delta_0 \leq 2^{-10}.$$

In Sect. 6.4 we shall use a Taylor expansion and rely on the conditions (6.9). This yields

$$\Delta_1(w, 0) = 1, \quad \Delta_2(w, 0) = 0,$$

and straightforward computations give

$$\Delta_{1,z_3}(w, 0) = \mathfrak{S}^1_{w_3 z_3 z_3}(w, 0) + \mathfrak{S}^2_{w_2 z_3}(w, 0)$$
$$\Delta_{2,z_3}(w, 0) = \mathfrak{S}^2_{w_3 z_3 z_3}(w, 0) + \mathfrak{S}^2_{w_1 z_3}(w, 0)$$

and thus

$$\Delta_{1,z_3}(0, 0) = \mathfrak{S}^1_{w_3 z_3 z_3}(0, 0), \quad \Delta_{2,z_3}(0, 0) = \mathfrak{S}^2_{w_3 z_3 z_3}(0, 0) = \kappa_0.$$

Further calculations give

$$\Delta_{1,z_3 z_3}(0, 0) = 3\mathfrak{S}^1_{w_1 z_3 z_3}(0, 0) + \mathfrak{S}^2_{w_2 z_3 z_3}(0, 0) + \mathfrak{S}^1_{w_3 z_3 z_3 z_3}(0, 0)$$
$$\Delta_{2,z_3 z_3}(0, 0) = 2\mathfrak{S}^2_{w_1 z_3 z_3}(0, 0) + \mathfrak{S}^2_{w_3 z_3 z_3 z_3}(0, 0),$$
$$\Delta_{1,w_j z_3}(0, 0) = \mathfrak{S}^2_{w_2 w_j z_3}(0, 0) + \mathfrak{S}^1_{w_3 w_j z_3 z_3}(0, 0)$$
$$\Delta_{2,w_j z_3}(0, 0) = \mathfrak{S}^2_{w_3 w_j z_3 z_3}(0, 0),$$

and

$$\Delta_{1,w_j w_k}(0, 0) = 0$$
$$\Delta_{2,w_j w_k}(0, 0) = \mathfrak{S}^2_{w_3 z_3 w_j w_k}(0, 0).$$

6.4 Plate Localization in the Model Case

The following lemma contains the information that will allow us to apply the decoupling inequality (6.6).

Lemma 6.3 *Let δ_0, δ_1 be as in (6.12). Let $2^{-\ell} \ll r$, $M_0 2^{-\ell} \le 2^{-10}$, $w \in [-2^{-\ell}, 2^{-\ell}]$, $|z_3| \le \delta_0$. Suppose $1/4 < |\mu| \le 4$ and*

$$\left| \mu_1 \Delta_1(w, z_3) + \mu_2 \Delta_2(w, z_3) \right| \le M_0 2^{-\ell}. \tag{6.18}$$

Then

$$\mu_1 \mathfrak{S}_w^1(w, z_3) + \mu_2 \mathfrak{S}_w^2(w, z_3) \in \Pi_{A,b}(\delta_1), \quad A = 2(1 + |\kappa_0|). \tag{6.19}$$

Proof We examine the quantity $\mu_1 \mathfrak{S}_w^1 + \mu_2 \mathfrak{S}_w^2$, for $1/4 < |\mu| \le 4$ and under the condition (6.18), and rewrite it as

$$\frac{1}{\Delta_1} \left((\mu_1 \Delta_1 + \mu_2 \Delta_2) \mathfrak{S}_w^1 + \mu_2 (\Delta_1 \mathfrak{S}_w^2 - \Delta_2 \mathfrak{S}_w^1) \right). \tag{6.20}$$

The assumption (6.18) and $|\mu| \in (1/4, 4)$ implies that $|\mu_1| \le 2^{-8}$ and hence $|\mu_2| \in (1/5, 4)$.

The second expression in (6.20) is the main term for our analysis. We use a Taylor expansion:

$$\Delta_1 \mathfrak{S}_w^2 - \Delta_2 \mathfrak{S}_w^1 \Big|_{(w, z_3)} = e_2 + v_0 z_3 + \sum_{j=1}^3 v_j x_j$$

$$+ \frac{1}{2} \left(v_{0,0} z_3^2 + 2 \sum_{j=1}^3 v_{0,j} z_3 w_j + \sum_{j=1}^3 \sum_{k=1}^3 v_{j,k} w_j w_k \right)$$

$$+ \mathcal{E}(w, z_3) \tag{6.21}$$

where $\mathcal{E}(w, z_3)$ is the Taylor reminder which vanishes of third order. Since $\Delta_1(w, 0) = 1$, $\Delta_2(w, 0) = 0$ the leading term is e_2. For the $\mathbb{R}^3$-valued coefficients of the linear term we get (with all terms on the right-hand side evaluated at 0, and using input from Sect. 6.3)

$$v_0 = \Delta_{1,z_3} \mathfrak{S}_w^2 + \Delta_1 \mathfrak{S}_{w z_3}^2 - \Delta_{2,z_3} \mathfrak{S}_w^1 - \Delta_2 \mathfrak{S}_{w z_3}^1 \Big|_{0,0}$$

$$= \mathfrak{S}_{w_3 z_3 z_3}^1(0, 0) e_2 - \kappa_0 e_1$$

and, for $j = 1, 2, 3$,

$$v_j = \Delta_{1,w_j} \mathfrak{S}_w^2 + \Delta_1 \mathfrak{S}_{w w_j}^2 - \Delta_{2,w_j} \mathfrak{S}_w^1 - \Delta_2 \mathfrak{S}_{w w_j}^1 \Big|_{(0,0)} = 0.$$

For the coefficients of the quadratic terms we have

$$
\begin{aligned}
v_{0,0} &= \Delta_{1,z_3 z_3}\mathfrak{S}^2_w + 2\Delta_{1,z_3}\mathfrak{S}^2_{w z_3} + \Delta_1 \mathfrak{S}^2_{w z_3 z_3} \\
&\quad - \Delta_{2,z_3 z_3}\mathfrak{S}^1_w - 2\Delta_{2,z_3}\mathfrak{S}^1_{w z_3} - \Delta_2 \mathfrak{S}^1_{w z_3 z_3}\Big|_{(0,0)} \\
&= (\mathfrak{S}^2_{w_1 z_3 z_3} - \Delta_{2,z_3 z_3})e_1 + (\mathfrak{S}^2_{w_2 z_3 z_3} + \Delta_{1,z_3 z_3})e_2 + (\mathfrak{S}^2_{w_3 z_3 z_3} - 2\Delta_{2,z_3})e_3\Big|_{(0,0)};
\end{aligned}
$$

in particular

$$
\langle v_{0,0}, e_3\rangle = -\kappa_0. \tag{6.22}
$$

Moreover, for $j = 1, 2, 3$,

$$
\begin{aligned}
v_{0,j} &= \Delta_1 \mathfrak{S}^2_{w w_j z_3} + \Delta_{1,w_j}\mathfrak{S}^2_{w z_3} + \Delta_{1,z_3}\mathfrak{S}^2_{w w_j} + \Delta_{1,w_j z_3}\mathfrak{S}^2_w \\
&\quad - \Delta_2 \mathfrak{S}^1_{w w_j z_3} - \Delta_{2,w_j}\mathfrak{S}^1_{w z_3} - \Delta_{2,z_3}\mathfrak{S}^1_{w w_j} - \Delta_{2,w_j z_3}\mathfrak{S}^1_w\Big|_{(0,0)} \\
&= (\mathfrak{S}^2_{w_1 w_j z_3} - \Delta_{2,w_j z_3})e_1 + (\mathfrak{S}^2_{w_2 w_j z_3} + \Delta_{1,w_j z_3})e_2 + \mathfrak{S}^2_{w_3 w_j z_3}e_3\Big|_{(0,0)},
\end{aligned}
$$

and, for $j, k = 1, 2, 3$,

$$
\begin{aligned}
v_{j,k} &= \Delta_1 \mathfrak{S}^2_{w w_j w_k} + \Delta_{1,w_j}\mathfrak{S}^2_{w w_k} + \Delta_{1,w_k}\mathfrak{S}^2_{w w_j} + \Delta_{1,w_j w_k}\mathfrak{S}^2_w \\
&\quad - \Delta_2 \mathfrak{S}^1_{w w_j w_k} - \Delta_{2,w_j}\mathfrak{S}^1_{w w_k} - \Delta_{2,w_k}\mathfrak{S}^1_{w w_j} - \Delta_{2,w_j w_k}\mathfrak{S}^1_w\Big|_{(0,0)} \\
&= \Delta_{1,w_j w_k}(0,0)e_2 - \Delta_{2,w_j w_k}(0,0)e_1.
\end{aligned}
$$

Gathering terms in the above Taylor expansion leads to

$$
\begin{aligned}
\Delta_1\mathfrak{S}^2_w - \Delta_2\mathfrak{S}^1_w\Big|_{(w,z_3)} &= e_2 - \kappa_0 z_3 e_1 - \tfrac{1}{2}\kappa_0 z_3^2 e_3 \\
&\quad + \mathfrak{S}^1_{w_3 z_3 z_3}(0,0)z_3 e_2 + \sum_{j=1}^{3}\mathfrak{S}^2_{w_3 w_j z_3}(0,0)w_j z_3 e_3 + \sum_{i=1}^{2} r_i(w, z_3)e_i + \mathcal{E}_3(w, z_3)
\end{aligned}
\tag{6.23}
$$

where we get

$$
\left|\mathfrak{S}^1_{w_3 z_3 z_3}(0,0)z_3\right| \leq M_0\delta_0, \tag{6.24a}
$$

and by assumption (6.12b),

$$
\sum_{j=1}^{3}\left|\mathfrak{S}^2_{w_3 w_j z_3}(0,0)w_j z_3\right| \leq 3M_0 2^{-\ell}\delta_0 \ll \delta_1^2. \tag{6.24b}
$$

For the quadratic error terms in the first two coordinates we have

$$
|r_i(w, z_3)| \leq 8M_0\delta_0^2, \quad i = 1, 2, \tag{6.24c}
$$

and finally for the cubic error terms we have the straightforward estimate

$$|\mathcal{E}_3(w, z_3)| \leq 2^{20} M_0^2 \delta_0^3. \tag{6.24d}$$

Now consider the situation where $|z_3 - b| \leq \delta_1$. Let $T_2(b)$ be as in (6.4), that is, $T_2(b) = e_1 + \kappa_0 b e_2 + b e_3 + O(b^3)$ with $1/2 \leq |T_2(b)| \leq 2$. We compute

$$\left\langle \frac{T_2(b)}{|T_2(b)|} \Delta_1 \mathfrak{S}_w^2 - \Delta_2 \mathfrak{S}_w^1 \right\rangle\Big|_{(w, z_3)} = \frac{1}{|T_2(b)|} \kappa_0 (b - z_3) + \mathcal{E}_{T_2}(w, z_3) \tag{6.25}$$

where (*cf.* (6.12b))

$$|\mathcal{E}_{T_2}(w, z_3)| \leq 2^{13} M_0 \delta_0^2 \leq 2^{13} M_0 \left(2^{-100} M_0^{-1} \delta_1\right)^{4/3} \ll \delta_1.$$

The computation for the normal component is more subtle. With $N(b) = b e_1 + \frac{1}{2} \kappa_0 b^2 e_2 - e_3$, we consider the contributions of the terms in the above Taylor expansion to $\langle N(b), \Delta_1 \mathfrak{S}_w^2 - \Delta_2 \mathfrak{S}_w^1 \rangle$. We get

$$\left\langle N(b) \Delta_1 \mathfrak{S}_w^2 - \Delta_2 \mathfrak{S}_w^1 \right\rangle = \frac{1}{2} \kappa_0 b^2 - \kappa_0 b z_3 + \frac{1}{2} \kappa_0 z_3^2$$

$$+ \frac{1}{2} \kappa_0 b^2 z_3 \mathfrak{S}_{w_3 z_3 z_3}^1(0, 0) - \sum_{j=1}^{3} \mathfrak{S}_{w_3 w_j z_3}^2(0, 0) w_j z_3$$

$$+ b r_1(w, z_3) + \frac{1}{2} \kappa_0 b^2 r_2(w, z_3) + \langle N(b), \mathcal{E}_3(w, z_3) \rangle.$$

By (6.24),

$$\left\langle \frac{N(b)}{|N(b)|} \Delta_1 \mathfrak{S}_w^2 - \Delta_2 \mathfrak{S}_w^1 \right\rangle = \frac{1}{|N(b)|} \left(\frac{1}{2} \kappa_0 (z_3 - b)^2 + \mathcal{E}_N(w, z_3) \right)$$

$$\text{with } |\mathcal{E}_N(w, z_3)| \leq 2^{21} M_0^2 \delta_0^3 + 2^{2-\ell} M_0 \delta_0 \ll \delta_1^2 \tag{6.26}$$

where for the error estimate we have used (6.12b). Clearly the main term on the right-hand side is $\leq |\kappa_0| \delta_1^2 / 2$.

This finishes the analysis of the second term in (6.20). Finally consider the first term in (6.20), again under the assumption (6.18). We get the estimates

$$|(\mu_1 \Delta_1 + \mu_2 \Delta_2) \langle \mathfrak{S}_w^1(w, z_3), T_i(b) \rangle| \leq 10 M_0 2^{-\ell} \ll \delta_1, \quad i = 1, 2$$

and

$$|(\mu_1 \Delta_1 + \mu_2 \Delta_2) \langle \mathfrak{S}_w^1(w, z_3), N(b) \rangle|$$
$$\leq 2^{-\ell} |\langle e_1 + z_3 e_3, b e_1 - e_3 \rangle| + 10^2 M_0^2 2^{-\ell} \delta_0^2 \leq 2^{2-\ell} \delta_1 \ll \delta_1^2.$$

The proof is completed by combining terms. $\qquad \square$

6.5 Proof of the Decoupling Step in the Model Case

Fix b and let $m_{k,\delta_1,b}$ be a multiplier that is equal to 1 on $\Pi_{2A,b}(\delta_1)$ and equal to 0 on $\mathbb{R}^3 \setminus \Pi_{3A,b}(\delta_1)$, and satisfies the natural differentiability properties

$$\left| \langle T_1(b), \nabla \rangle^{\alpha_1} \langle T_2(b), \nabla \rangle^{\alpha_2} \langle N(b), \nabla \rangle^{\alpha_3} m_{k,\delta_1,b}(\xi) \right|$$
$$\lesssim_\alpha 2^{-k\alpha_3}(2^k\delta_1)^{-\alpha_2}(2^k\delta_1^2)^{-\alpha_3}.$$

Let $P_{k,\delta_1,b}$ be defined by $\widehat{P_{k,\delta_1,b}f} = m_{k,\delta_1,b}\widehat{f}$. Let I be an interval of length δ_1 and let $f_I(y) = f(y)\mathbb{1}_I(y_3)$. The Schwartz kernel of

$$f \mapsto (I - P_{k,\delta,b})[\varsigma_{\ell,0}\mathcal{T}_{k,\ell}f]$$

is given as a sum of oscillatory integrals $\sum_{n=0}^\infty K_{n,k,\ell}$ where for $n > 0$

$$K_{n,k,\ell}(w,z) = 2^{2k}\iiint e^{i(\langle w-v,\xi\rangle+2^k\langle\tau,\mathfrak{S}(v,z_3)-z'\rangle)}\varsigma_{\ell,0}(v)$$
$$\times(1 - m_{k,\delta_1,b}(\xi))\eta_1(|\xi|2^{-n})\chi_{k,\ell}(v,z,2^k\tau)dv\,d\xi d\tau \,\mathbb{1}_I(z_3),$$

with

$$\chi_{k,\ell}(v,z,2^k\tau) := \eta\big(2^\ell\alpha(w,z)(\tau_1\Delta_1^\mathfrak{S}(w,z_3) + \tau_2\Delta_2^\mathfrak{S}(w,z_3))\big)\zeta(w,z,\tau,k)\,d\tau$$

and the family $\zeta(\cdot,\cdot,\cdot,k)$ is bounded uniformly in C_c^∞. If $|n - k| > 10$, then repeated integration by parts in the v-variables (followed by subsequent integration by parts in the ξ-variables) shows that

$$|K_{n,k,\ell}(w,z)| \lesssim \min\{2^{-10n}, 2^{-10k}(1 + |w - z|)^{-N}\}, \quad |k - n| \geq C.$$

For $|k - n| \leq C$ a similar argument applies to the assumption that on the support of $(1 - m_{k,\delta_1,b})$ we have $2^{-k}\xi \notin \Pi_{3A,b}(\delta_1)$. That means

$$\left| \nabla_v[-\langle v,\xi\rangle + \langle 2^k\tau, \mathfrak{S}(v,z_3)\rangle] \right| \geq c2^k\delta_1^2.$$

Differentiating the amplitude gives a factor of 2^ℓ with each differentiation. Thus for $|k - n| \leq C$ an N-fold integration by parts in the v variables followed by integration by parts in the ξ-variables shows that

$$|K_{n,k,\ell}(w,z)| \lesssim_N \big(2^k\delta_1^2 2^{-\ell}\big)^{-N}(1 + |w - z|)^{-N_1}, \quad |k - n| \leq C.$$

Notice that by $\ell \leq k/3$ and $\delta_1 \geq 2^{-(1-\varepsilon^2)\ell}$ we have

$$2^k\delta_1^2 2^{-\ell} \geq 2^{k\varepsilon^2/3}.$$

Thus a $\lfloor 40/\varepsilon^2 \rfloor$-fold integration by parts in v (again followed by multiple integration by parts in ξ) yields

$$|K_{n,k,\ell}(w,z)| \lesssim 2^{-11k}(1+|w-z|)^{-N_1}.$$

Let b_I be the left endpoint of the interval I. We decompose the left-hand side of (6.13) as

$$\left\| \sum_{I \in \mathcal{I}_J} P_{k,\delta_1,b_I}[\varsigma_{\ell,0}\mathcal{T}_{k,\ell}g_I] \right\|_p + \left\| \sum_{I \in \mathcal{I}_J} (I - P_{k,\delta_1,b_I})[\varsigma_{\ell,0}\mathcal{T}_{k,\ell}g_I] \right\|_p \tag{6.27}$$

By Lemma 6.3 we can apply the decoupling inequality (6.6) (with ϵ replaced by ϵ^2) to bound the first term in (6.27) by

$$C(\varepsilon^2, A)\delta^{-\varepsilon^2} \left(\sum_{I \in \mathcal{I}_J} \left\| P_{k,\delta_1,b_I}[\varsigma_{\ell,0}\mathcal{T}_{k,\ell}g_I] \right\|_p^p \right)^{1/p}$$

$$\lesssim C(\varepsilon^2, A)\delta^{-\varepsilon^2} \left(\sum_{I \in \mathcal{I}_J} \left\| \varsigma_{\ell,0}\mathcal{T}_{k,\ell}g_I \right\|_p^p \right)^{1/p}.$$

For the second term in (6.27) we use the above error estimates, apply Minkowski's inequality and get the bound (6.27) by

$$2^{-11k} \sum_{I \in \mathcal{I}_J} \left(\int \left| \int (1+|w-z|)^{-N}|g(z)\mathbb{1}_I(z_3)|dz \right|^p dw \right)^{1/p} \lesssim 2^{-10k}\|g\|_p.$$

This finishes the proof of Proposition 6.1. □

7 Families of Changes of Variables

Let $P^\circ = (a^\circ, y^\circ) \in \mathcal{M}$, with $y^\circ = S^1(a^\circ, b^\circ), S^2(a^\circ, b^\circ), b^\circ)$. For $r > 0$ let

$$Q(r) := \{(x, y_3) : |x - a^\circ|_\infty \leq r\} \text{ and } I(r) := \{y_3 : |y_3 - b^\circ| \leq r\}.$$

Let S^i be smooth functions in a neighborhood of $Q(2r_0) \times I(2r_0)$, for some $r_0 > 0$. After possibly permuting the variables y_1, y_2 we may assume, by Lemma 3.1 that $\Delta_1(x, y_3) = \det(S_x^1, S_x^2, S_{xy_3}^1) \neq 0$ on $Q(2r_0) \times I(2r_0))$. Choose M so that

$$M > 2 + \|S\|_{C^5(Q(2r_0) \times I(2r_0))} + \max_{(x,y_3) \in Q(2r_0)} |\Delta_1(x, y_3)|^{-1}.$$

We now consider (a, b) close to (a°, b°) and construct changes of variables so that in the new coordinates the constant coefficient decoupling theorem in Proposition 6.1

can be applied at suitable scales. The idea of applying a constant coefficient decoupling theorem in a variable coefficient situation also appears in [3].

For $a \in Q(2r_0)$, $b \in I(2r_0)$ let Γ_1, Γ_2 be as in (3.4), and let $\rho \equiv \rho(a, b) \in \mathbb{R}^3$ be defined by

$$(\rho_1, \rho_2, \rho_3) = \frac{1}{\Delta_1(a, b)}\big(-\Gamma_2(a, b), \Gamma_1(a, b), \Delta_2(a, b)\big). \tag{7.1}$$

For $(x, y_3), (a, y_3) \in Q(r_0)$ and $(a, y_3) \in I(2r_0)$ consider the function

$$(x, a, y_3) \mapsto \mathfrak{w}(x, a, y_3)$$
$$Q(r_0) \times Q(r_0) \times I(2r_0) \to \mathbb{R}^3$$

defined by

$$\begin{pmatrix} \mathfrak{w}_1 \\ \mathfrak{w}_2 \\ \mathfrak{w}_3 \end{pmatrix} = \begin{pmatrix} S^1(x, b) - S^1(a, b), \\ S^2(x, b) - \rho_3(a, b)S^1(x, b) - S^2(a, b) + \rho_3(a, b)S^1(a, b) \\ S^1_{y_3}(x, b) - S^1_{y_3}(a, b) \end{pmatrix}. \tag{7.2}$$

We have

$$\det(D\mathfrak{w}(x, a, b)/Dx) = \det\big(S^1_x, S^2_x - \rho_3 S^1_x, S^1_{x, y_3}\big)|_{(x,b)} = \Delta_1(x, b). \tag{7.3}$$

By the implicit function theorem there exists $r_1 > 0$ with $r_1 < r_0$ such that for $|w|_\infty < 2r_1$, $|a - a^\circ| < 2r_1$, $b - b_\circ| < 2r_1$ the equation $\mathfrak{w}(x, a, b) = w$ is solved by a unique C^∞ function

$$x = \mathfrak{x}(w, a, b). \tag{7.4}$$

Note the estimate

$$|\rho_i(a, b)| \leq 6M^4, \text{ for } a \in Q(2r_0), b \in I(2r_0). \tag{7.5}$$

By the definition of $\mathfrak{w}$ and the mean value theorem for the coordinate functions, this implies $|\mathfrak{w}(x, a, b)|_\infty \leq 3M(1 + 6M^4)|x - a|_\infty$ for $x, a \in Q(r_0), b \in I(2r_0)$. Hence if $r_2 < r_1$ and if $|x - a^\circ|_\infty < r_2$ and $|a - a^\circ|_\infty < r_2$, then $|\mathfrak{w}(x, a, b)|_\infty \leq 42M^5 r_2$ and if we define

$$r_2 = (50M^5)^{-1}r_1 \tag{7.6}$$

we get $|\mathfrak{w}(x, a, b)|_\infty < r_1$ for $x, a \in Q(r_2), b \in I(2r_1)$. By the uniqueness of the function $\mathfrak{x}$, we thus see that $\mathfrak{x}(\mathfrak{w}(x, a, b)) = x$ for $x, a \in Q(r_2)$ for $x, a \in Q(r_2)$, $b \in I(2r_1)$.

We will also need to change variables in the y-variables, in a more explicit form. Define

$$\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3) : \mathbb{R}^2 \times Q(2r_0) \times I(2r_0) \to \mathbb{R}^3 \tag{7.7}$$

by

$$\mathfrak{z}_1(y, a, b) = y_1 - S^1(a, y_3), \qquad \mathfrak{z}_3(y, a, b) = y_3 - b,$$

and

$$\mathfrak{z}_2(y, a, b) = y_2 - \rho_3(a, b)y_1 - S^2(a, y_3) + \rho_3(a, b)S^1(a, y_3)$$

$$- (y_3 - b) \sum_{i=1}^{2} \rho_i(y_i - S^i(a, y_3)).$$

We have

$$\det(D\mathfrak{z}/Dy) = (1 - \rho_2(y_3 - b)). \tag{7.8}$$

By (7.5) this quantity lies in $(1/2, 3/2)$ provided that $y_3, b \in I(2r_3)$ with

$$r_3 < \min\{r_1, (24M^4)^{-1}\}. \tag{7.9}$$

The inverse $z \mapsto \mathfrak{y}(z, a, b)$, defined for $|z_3| \leq r_3$, $|b - b^\circ| \leq r_3$, $|a - a^\circ| \leq 2r_0$, is given by

$$\mathfrak{y}_1(z, a, b) = z_1 + S^1(a, b + z_3),$$

$$\mathfrak{y}_2(z, a, b) = \frac{z_2 + z_1(\rho_3(a, b) + \rho_1(a, b)z_3) + (1 - z_3)S^2(a, b + z_3)}{1 - \rho_2(a, b)z_3}, \tag{7.10}$$

$$\mathfrak{y}_3(z, a, b) = b + z_3.$$

Lemma 7.1 *The functions $\mathfrak{x}$, $\mathfrak{y}$ defined above have the following properties.*

(i) $\mathfrak{x}(0, a, b) = a$, $\mathfrak{y}(0, a, b) = (S^1(a, b), S^2(a, b), b)$, $\mathfrak{y}_3(z, a, b) = b + z_3$.
(ii) $\det\left(\frac{D\mathfrak{x}(w,a,b)}{Dw}\right) = \frac{1}{\Delta_1(\mathfrak{x}(w,a,b),b)}$.
(iii) *Let $\rho \equiv \rho(a, b)$ be as in (7.1) and let*

$$B(z_3, a, b) = \begin{pmatrix} 1 & 0 \\ -\rho_3 - \rho_1 z_3 & 1 - \rho_2 z_3 \end{pmatrix}. \tag{7.11}$$

Then for $|z_3| \leq r_3$, $|a - a^\circ|_\infty \leq r_2$, $|w| \leq r_2$

$$B(z_3, a, b) \begin{pmatrix} S^1(\mathfrak{x}(w, a, b), b + z_3) - \mathfrak{y}_1(z, a, b) \\ S^2(\mathfrak{x}(w, a, b), b + z_3) - \mathfrak{y}_2(z, a, b) \end{pmatrix} = \begin{pmatrix} \mathfrak{S}^1(w, z_3, a, b) - z_1 \\ \mathfrak{S}^2(w, z_3, a, b) - z_2 \end{pmatrix} \tag{7.12}$$

where $\mathfrak{S}^i$ are C^∞ with

$$\mathfrak{S}^1(w, 0) = w_1, \quad \mathfrak{S}^2(w, 0) = w_2, \quad \mathfrak{S}^1_{z_3}(w, 0) = w_3; \tag{7.13}$$

moreover

$$\mathfrak{S}^2_{wz_3}(0, 0, a, b) = 0. \tag{7.14}$$

(iv) *Let*

$$\Delta_i^S(x, y_3) = \det(S_x^1, S_x^2, S_{xy_3}^1)|_{(x,y_3)},$$

$$\Delta_i^{\mathfrak{S}}(w, z_3) = \det(\mathfrak{S}_w^1, \mathfrak{S}_w^2, \mathfrak{S}_{wz_3}^1)|_{(w,z_3)}.$$

Then, for $(\tau_1, \tau_2) = (\mu_1, \mu_2) B(z_3, a, b)$,

$$\sum_{i=1}^{2} \tau_i \, \Delta_i^S(\mathfrak{x}(w, a, b), b + z_3) = \frac{\Delta_1^S(\mathfrak{x}(w, a, b), b)}{1 - \rho_2(a, b) z_3} \sum_{i=1}^{2} \mu_i \Delta_i^{\mathfrak{S}}(w, z_3). \quad (7.15)$$

(v) *Let* κ *be as in* (3.8b). *Then*

$$\mathfrak{S}_{w_3 z_3 z_3}^2(0, 0, a, b) = \frac{\kappa(a, b)}{\Delta_1(a, b)^2}. \quad (7.16)$$

Proof We write for $i = 1, 2$

$$
\begin{aligned}
S^i(x, y_3) - y_i = {} & S^i(a, b) + S^i(x, b) - S^i(a, b) \\
& + S^i(x, y_3) - S^i(a, y_3) - S^i(x, b) + S^i(a, b) \\
& + S^i(a, y_3) - S^i(a, b) - y_i
\end{aligned}
$$

and set

$$
\begin{aligned}
\tilde{x}_i &= S^i(x, b) - S^i(a, b), \quad i = 1, 2, \\
\tilde{x}_3 &= S_{y_3}^1(x, b) - S_{y_3}^1(a, b)
\end{aligned}
$$

so that

$$\det\left(\frac{D\tilde{x}}{Dx}\right) = \Delta_1^S(x, b).$$

Also let

$$
\begin{aligned}
\tilde{y}_i &= y_i - S^i(a, y_3), \quad i = 1, 2. \\
\tilde{y}_3 &= y_3 - b
\end{aligned}
$$

Note that $\tilde{x}^\mathsf{T} = (x - a)^\mathsf{T} A^\mathsf{T} + O(|x - a|^2)$ where A^T is the matrix with column vectors $(S_x^1, S_x^2, S_{xy_3}^1)|_{(a,b)}$. We then expand

$$S^1(x, y_3) - y_1 = \tilde{x}_1 - \tilde{y}_1 + \tilde{x}_3 \tilde{y}_3 + R_{1,1}(\tilde{x}, \tilde{y}_3, a, b)$$

$$S^2(x, y_3) - y_2 = \tilde{x}_2 - \tilde{y}_2 + \tilde{y}_3 \sum_{i=1}^{3} \underline{\rho}_i \tilde{x}_i + R_{2,1}(\tilde{x}, \tilde{y}_3, a, b)$$

where

$$\underline{\rho}_i = \langle A^{-1} e_i, S_{x, y_3}^2(a, b) \rangle, \quad (7.17)$$

and where $R_{1,1}, R_{2,1}$ vanish to third order with no pure $\tilde{x}$ or pure $\tilde{y}_3$ terms; moreover $\partial_{\tilde{y}_3} R_{1,1}$ has no pure $\tilde{x}$ terms. We label $R_{1,1}$ an error term of *type I* and $R_{2,1}$ an error

term of *type II*. Precisely, an error term of type *I* is of the form

$$\tilde{y}_3^2 \sum_{i=1}^{3} \tilde{x}_i \tilde{\beta}_i(\tilde{x}, \tilde{y}_3, a, b) \tag{7.18a}$$

with $\tilde{\beta}_i$ smooth, and a term of type *II* is of the form

$$y_3 \sum_{j=1}^{3} \sum_{k=1}^{3} \tilde{x}_j \tilde{x}_k \tilde{\beta}_{jk}(\tilde{x}, \tilde{y}_3, a, b) + \text{term of type } I, \tag{7.18b}$$

with $\tilde{\beta}_{jk}$ smooth. Note that $(\underline{\rho}_1, \underline{\rho}_2, \underline{\rho}_3)$ satisfies

$$\underline{\rho}_1 S_x^1(a, b) + \underline{\rho}_2 S_x^2(a, b) + \underline{\rho}_3 S_{xy_3}^1(a, b) = S_{xy_3}^2(a, b)$$

and hence, by Cramer's rule, we see that $\Delta_1(\underline{\rho}_1, \underline{\rho}_2, \underline{\rho}_3) = (-\Gamma_2, \Gamma_1, \Delta_2)$, i.e.,

$$\underline{\rho}_i = \rho_i$$

where ρ_i is as in (7.1).

Given $c_1, c_2, c_3 \in \mathbb{R}$ we compute

$$(c_3 + c_1 \tilde{y}_3)(S^1(x, y_3) - y_1) + (1 + c_2 \tilde{y}_3)(S^2(x, y_3) - y_2)$$
$$= (\tilde{x}_2 + c_3 \tilde{x}_1) - (\tilde{y}_2 + c_3 \tilde{y}_1)$$

$$- \tilde{y}_3 (\sum_{i=1}^{3} \tilde{x}_i(\rho_i + c_i)) - c_1 \tilde{y}_1 \tilde{y}_3 - c_2 \tilde{y}_2 \tilde{y}_3 + R_{2,2}(\tilde{x}, \tilde{y}_3, a, b)$$

where $R_{2,2}$ is an error term of type *II*. We choose $c_i = -\rho_i(a, b)$ so that the mixed quadratic terms drop out.

We now change variable in $\tilde{x}$ and in $\tilde{y}$ separately, setting

$$z_1 = \tilde{y}_1, \quad z_2 = \tilde{y}_2 - \rho_3 \tilde{y}_1 - \rho_1 \tilde{y}_1 \tilde{y}_3 - \rho_2 \tilde{y}_2 \tilde{y}_3, \quad z_3 = \tilde{y}_3$$

and

$$w_1 = \tilde{x}_1, \quad w_2 = \tilde{x}_2 - \rho_3 \tilde{x}_1, \quad w_3 = \tilde{x}_3.$$

Define

$$\mathfrak{S}^i(w, z_3, a, b) = S^i(\mathfrak{x}(w, a, b), b + z_3), \quad i = 1, 2.$$

Setting

$$B(z_3, b) = \begin{pmatrix} 1 & 0 \\ -\rho_3 - \rho_1 z_3 & 1 - \rho_2 z_3 \end{pmatrix}$$

we obtain

$$B(y_3 - b, b) \begin{pmatrix} S^1(x, y_3) - y_1 \\ S^2(x, y_3) - y_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{S}^1(w, z_3, a, b) - z_1 \\ \mathfrak{S}^2(w, z_3, a, b) - z_2 \end{pmatrix} \qquad (7.19)$$

if $w = \mathfrak{w}(x, a, b)$ and $y = \mathfrak{y}(z, a, b)$ and $\mathfrak{w}$ and $\mathfrak{y}$ are as in (7.2) and (7.10). Now (7.19) implies (7.12).

The functions $\mathfrak{S}^1$, $\mathfrak{S}^2$ satisfy

$$\mathfrak{S}^1(w, z_3, a, b) = w_1 + w_3 z_3 + R_{1,3}(w, z_3, a, b)$$
$$\mathfrak{S}^2(w, z_3, a, b) = w_2 + R_{2,3}(w, z_3, a, b)$$

where $R_{1,3}$ is an error term of type I (with $(\tilde{x}, \tilde{y}_3)$ replaced by (w, z_3), cf. (7.18a)) and $R_{2,3}$ is a term of type II (again in the (w, z_3)-variables, *cf.* (7.18b)). We see that (7.12) and (7.13), (7.14) hold.

In order to obtain (7.15) we calculate

$$\sum_{i=1}^{2} (B^{\mathsf{T}} \mu)_i \Delta_i^S(\mathfrak{x}(w, a, b), b + z_3)$$

$$= \det \left(\nabla_w(S^1(\mathfrak{x}, y_3)), \nabla_w(S^2(\mathfrak{x}, y_3)), \nabla_w(\langle B^{\mathsf{T}} \mu, S_{y_3}(\mathfrak{x}, y_3) \rangle) \right) \det \frac{D\mathfrak{w}}{Dx}(\mathfrak{x})$$

with $\mathfrak{x} \equiv \mathfrak{x}(w) \equiv \mathfrak{x}(w, a, b)$. We have

$$\nabla_w \mathfrak{S}^1(\mathfrak{x}(w), z_3) = \nabla_w(S^1(\mathfrak{x}(w), b + z_3)),$$

and, with $b_{22}(z_3) = 1 - \rho_2 z_3$,

$$\nabla_w \mathfrak{S}^2(\mathfrak{x}(w), z_3) = b_{22}(z_3) \nabla_w(S^2(\mathfrak{x}(w), b + z_3)) - b_{21}(z_3) \nabla_w(S^1(\mathfrak{x}(w), b + z_3));$$

moreover

$$\nabla_w \mathfrak{S}^1_{z_3}(\mathfrak{x}(w), b + z_3) = \nabla_w(S^1_{y_3}(\mathfrak{x}(w), b + z_3)),$$

and

$$\nabla_w \mathfrak{S}^2_{z_3}(\mathfrak{x}(w), b + z_3) = - \rho_1 \nabla_w(S^1(\mathfrak{x}(w), b + z_3)) - \rho_2 \nabla_w(S^2(\mathfrak{x}(w), b + z_3))$$
$$+ b_{21}(z_3) \nabla_w(S^1_{y_3}(\mathfrak{x}(w), b + z_3))$$
$$+ b_{22}(z_3) \nabla_w(S^1_{y_3}(\mathfrak{x}(w), b + z_3)).$$

A quick calculation with determinants and (7.3) yields the asserted identity (7.15).

For the curvature calculation we start with the equation (7.12) for the second component and differentiate with respect to w_3. This yields

$$\mathfrak{S}^2_{w_3}(w, z_3)$$
$$= (-\rho_3 - \rho_1 z_3)\left(\frac{D\mathfrak{x}}{Dw}e_3\right)^{\mathsf{T}} S^1_x(\mathfrak{x}, b + z_3) + (1 - \rho_2 z_3)\left(\frac{D\mathfrak{x}}{Dw}e_3\right)^{\mathsf{T}} S^2_x(\mathfrak{x}, b + z_3).$$

Here the Jacobian $D\mathfrak{x}/Dw$ is evaluated at (w, b). We differentiate twice with respect to z_3 to obtain

$$\mathfrak{S}^2_{w_3 z_3 z_3}(w, z_3) = -2\rho_1\left(\frac{D\mathfrak{x}}{Dw}e_3\right)^{\mathsf{T}} S^1_{xy_3}(\mathfrak{x}, b + z_3) - 2\rho_2\left(\frac{D\mathfrak{x}}{Dw}e_3\right)^{\mathsf{T}} S^2_{xy_3}(\mathfrak{x}, b + z_3)$$

$$- (\rho_3 + \rho_1 z_3)\left(\frac{D\mathfrak{x}}{Dw}e_3\right)^{\mathsf{T}} S^1_{xy_3 y_3}(\mathfrak{x}, b + z_3)$$

$$+ (1 - \rho_2 z_3)\left(\frac{D\mathfrak{x}}{Dw}e_3\right)^{\mathsf{T}} S^2_{xy_3 y_3}(\mathfrak{x}, b + z_3)$$

where $\mathfrak{x} \equiv \mathfrak{x}(w, a, b)$. Using Cramer's rule we find that

$$\frac{D\mathfrak{x}}{Dw}e_3\Big|_{(w,a,b)} = \frac{1}{\Delta_1(\mathfrak{x}(w, a, b), b)} S^1_x \wedge S^2_x\Big|_{(\mathfrak{x}(w,a,b),b)}.$$

We evaluate the previous identity at $z_3 = 0$, and $w = 0$ to obtain

$$\mathfrak{S}^2_{w_3 z_3 z_3}(0, 0) = \frac{1}{\Delta_1(a, b)}\Big(-2\rho_1\langle S^1_x \wedge S^2_x, S^1_{xy_3}\rangle - 2\rho_2\langle S^1_x \wedge S^2_x, S^2_{xy_3}\rangle$$

$$- \rho_3\langle S^1_x \wedge S^2_x, S^1_{xy_3 y_3}\rangle + \langle S^1_x \wedge S^2_x, S^2_{xy_3 y_3}\rangle\Big|_{(a,b)}\Big).$$

Using (7.1) we see that $\mathfrak{S}^2_{w_3 z_3 z_3}(0, 0)$ equals

$$\frac{1}{\Delta_1}\Big(-2\rho_1\Delta_1 - 2\rho_2\Delta_2 - \rho_3(\Delta_{1,y_3} - \Gamma_1) + (\Delta_{2,y_3} - \Gamma_2)\Big)\Big|_{(a,b)}$$

$$= \frac{1}{\Delta_1^2}\Big(2\Gamma_2\Delta_1 - 2\Gamma_1\Delta_2 - (\Delta_{1,y_3} - \Gamma_1)\Delta_2 + (\Delta_{2,y_3} - \Gamma_2)\Delta_1\Big)\Big|_{(a,b)}$$

which equals $\kappa(a, b)/\Delta_1(a, b)^2$ so that (7.16) is proved. $\qquad\square$

8 Decoupling in the General Case

We consider the operator $\mathcal{R}_{k,\ell}$ as in (5.2). With χ is as in (5.1a) we assume that χ is zero if $x \notin [-r_2/2, r_2/2]$ or if $y_3 \notin [r_3/2, r_3/2]$ (see the paragraph leading to (7.4) and (7.6), (7.9)).

Proposition 8.1 *Let $0 < \epsilon < 1/2$, $\ell \leq \lfloor k/3 \rfloor$. Let $\delta_0, \delta_1 \in (2^{-\ell(1-\epsilon^2)}, 2^{-\ell\epsilon^2})$ such that*

$$\max\left\{(2^{-\ell}\delta_0)^{1/2}, \delta_0^{3/2}\right\} < \delta_1 < \delta_0. \tag{8.1}$$

Let J be an interval of length δ_0, near b°, and let $\mathcal{I}_J$ be a collection of intervals of length δ_1 which have disjoint interior and which intersect J. For each I, let f_I be defined by $f_I(y) = f(y)\mathbb{1}_I(y_3)$. Let $a \in \mathbb{R}^3$, $\varepsilon_0 = (10M)^{-4}$, $\vartheta \in C_c^\infty$ supported in $(-r_2, r_2)^3$ and $\vartheta_{\ell,a}(x) = \vartheta(2^\ell \varepsilon_0^{-1}(x - a))$. Then for $2 \leq p \leq 6$,

$$\left\| \vartheta_{\ell,a} \sum_{I \in \mathcal{I}_J} \mathcal{R}_{k,\ell} f_I \right\|_p \leq C_\epsilon (\delta_0/\delta_1)^\epsilon \left(\sum_{I \in \mathcal{I}_J} \left\| \vartheta_{\ell,a} \mathcal{R}_{k,\ell} f_I \right\|_p^2 \right)^{1/2} + C_{N,\epsilon} 2^{-kN} \|f\|_p. \tag{8.2}$$

The constants do not depend on the choice of J and $\mathcal{I}_J$.

Proof. Fix a near a° and $b \in J$. We apply (7.12) and then the changes of variables $y = \mathfrak{y}(z, a, b)$ in (7.10) and $\tau = B^{\mathsf{T}-1}(z_3, a, b))\mu$. Note from (7.8), (7.11) that $\det(D\mathfrak{y}/Dz) \det B = 1$. Let $f(y) = \sum_{I \in \mathcal{J}_I} f \mathbb{1}_I(y_3)$ and $g(z, a, b) = f(\mathfrak{y}(z, a, b))$.

Let

$$M_1 \geq 1 + \sum_{i=1}^{2} \sup_{(a,b) \in [-r_0, r_0]^4} \|\mathfrak{S}^i(\cdot, a, b)\|_{C^5([-r_0, r_0]^4)} \tag{8.3}$$

which is just the uniform version of the condition (6.8). By applications of Hölder's inequality it suffices to prove (8.2) under a slightly more restrictive assumptions on δ_0, δ_1, namely

$$\delta_0, \delta_1 \in \left(M_1^2 2^{20-\ell(1-\epsilon^2)}, 2^{-\ell\epsilon^2 - 20} M_1^{-2} \right)$$
$$2^{100} M_1 \max\left\{ (2^{-\ell}\delta_0)^{1/2}, \delta_0^{3/2} \right\} < \delta_1 < \delta_0.$$

These are the uniform versions of (6.12) which will allow us to apply Proposition 6.1.

We have

$$\mathcal{R}_{k,\ell} f(x) = 2^{2k} \iint e^{i2^k \langle \mu, \mathfrak{S}(w(x,a,b), z_3) - z' \rangle} \tilde{\chi}_{k,\ell}(x, z, \mu, a, b) g(z, a, b) d\mu dz$$

with

$$\tilde{\chi}_{k,\ell}(x, z, \mu, a, b) = \chi(x, \mathfrak{y}(z, a, b)) \eta_1(|B^{\mathsf{T}-1}(z_3, a, b)\mu|)$$
$$\times \eta\left(2^\ell \frac{\Delta_1(x)}{1 - \rho_3(a,b)z_3} (\mu_1 \Delta_1^{\mathfrak{S}}(w, z_3, a, b) + \mu_2 \Delta_2^{\mathfrak{S}}(w, z_3, a, b)) \right).$$

Hence we get, with $\varsigma_{\ell,a}(w) := \vartheta_{\ell,a}(\mathfrak{x}(w, a, b))$,

$$\vartheta_{\ell,a}(\mathfrak{x}(w, a, b)) \sum_I \mathcal{R}_{k,\ell} f_I(\mathfrak{x}(w, a, b)) = \varsigma_{\ell,0}(w) \sum_{I \in \mathcal{J}_I} \mathcal{T}_{k,\ell,a,b} g_I(w)$$

where $g_I(z, a, b) = g(z, a, b)\mathbb{1}_{-b+I}(z_3)$ and $\mathcal{T}_{k,\ell} \equiv \mathcal{T}_{k,\ell,a,b}$ is as in (6.11).

We can now write the left-hand side of (5.6) as

$$\left(\int \left| \vartheta_{\ell,a}(\mathfrak{x}(w, a, b)) \sum_{I \in \mathcal{I}_J} \mathcal{R}_{k,\ell} f_I(\mathfrak{x}(w, a, b)) \right|^p \left| \det\left(\tfrac{D\mathfrak{x}}{Dw} \right| dw \right)^{1/p}$$

$$\lesssim \left\| \varsigma_{\ell,0} \sum_{I \in \mathcal{I}_J} \mathcal{T}_{k,\ell} g_I \right\|_p$$

where we used uniform upper bounds on $|\det(\tfrac{D\mathfrak{x}}{Dw})|$. By Proposition 6.1 we can bound

$$\left\| \varsigma_{\ell,0} \sum_{I \in \mathcal{I}_J} \mathcal{T}_{k,\ell} g_I \right\|_p \leq C_\epsilon (\delta_0/\delta_1)^\epsilon \left(\sum_{I \in \mathcal{I}_J} \left\| \varsigma_{\ell,0} \mathcal{T}_{k,\ell} g_I \right\|_p^2 \right)^{1/2} + C_\epsilon 2^{-10k} \|g\|_p.$$

Undoing the above change of variable (and using uniform lower bounds on $|\det(\tfrac{D\mathfrak{x}}{Dw})|$) we may bound this, using Proposition 6.1, by

$$C'_\epsilon (\delta_0/\delta_1)^\epsilon \left(\sum_{I \in \mathcal{I}_J} \left\| \vartheta_{\ell,a} \mathcal{R}_{k,\ell} f_I \right\|_p^2 \right)^{1/2} + C_\epsilon 2^{-10k} \|f\|_p. \quad \square$$

Proof of Theorem 5.2 We may assume $\varepsilon < 1/10$. Let $\vartheta \in C_c^\infty(\mathbb{R}^3)$ supported in $(-1, 1)^3$ such that $\vartheta \geq 0$ everywhere and $\sum_{n \in \mathbb{Z}^3} \vartheta(\cdot - n) = 1$.

Let, for $n \in \mathbb{Z}^3$, $\zeta^{\ell,n}(x) = \upsilon(x)\zeta(2^\ell \varepsilon_0^{-1} x - n)$. Thus

$$\|\upsilon \mathcal{R}_{k,\ell} f\|_p \lesssim \left(\sum_{n \in \mathbb{Z}^3} \left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f \right\|_p^p \right)^{1/p}. \tag{8.4}$$

Now let $\mathcal{I}(m)$ be the family of dyadic intervals with length 2^{-m}. Let I' be a dyadic interval of length $\geq 2^{-m}$; then we denote by $\mathcal{I}(m, I')$ the collection of dyadic intervals which are of length 2^{-m} and are contained in I'. For any dyadic interval define $f_I(y) = f(y)\mathbb{1}_I(y_3)$. Let $m_0 = \lfloor \ell\varepsilon^2 \rfloor$. By Hölder's inequality,

$$\left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f \right\|_p \leq 2^{m_0(1-\frac{1}{p})} \left(\sum_{J \in \mathcal{I}(m_0)} \left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f_J \right\|_p^p \right)^{1/p}. \tag{8.5}$$

It is not hard to see that we can pick a sequence of integers

$$m_1, \ldots, m_{N(\ell)}$$

such that $m_j \leq m_{j+1} \leq \ell$ for $j = 0, \ldots, N(\ell) - 1$, and such that

$$m_{j+1} \leq \min \left\{ \lfloor \tfrac{3m_j}{2} \rfloor, \lfloor \tfrac{m_j + \ell}{2} \rfloor \right\}; \tag{8.6a}$$

moreover

$$m_N \geq \lfloor \ell(1 - \varepsilon^2) \rfloor, \text{ and } N(\ell) \leq C_\varepsilon \log_2(\ell). \tag{8.6b}$$

We claim that for $j = 0, \ldots, N(\ell) - 1$

$$\left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f \right\|_p \leq C_{\varepsilon^2}^j 2^{m_0 \left(1 - \frac{1}{p}\right) + (m_j - m_0)\left(\frac{1}{2} - \frac{1}{p} + \varepsilon^2\right)} \left(\sum_{I \in \mathcal{I}(m_j)} \left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f_I \right\|_p^p \right)^{1/p}$$

$$+ 2^{-9k} \left(\sum_{v=0}^{j-1} C_{\varepsilon^2}^v \right) \| f \|_p. \tag{8.7}$$

We show this by induction. The case $j = 0$ is covered by (8.5). For the induction step assume (8.7) for some $j < N(\ell) - 1$. Observe that for $I \in \mathcal{I}(m_j)$ Proposition 8.1 and Hölder's inequality give

$$\left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f_I \right\|_p = \left\| \vartheta^{\ell,n} \sum_{I' \in \mathcal{I}(m_{j+1}, I)} \mathcal{R}_{k,\ell} f_{I'} \right\|_p$$

$$\leq C_{\varepsilon^2} 2^{(m_{j+1} - m_j)\varepsilon^2} \left(\sum_{I' \in \mathcal{I}(m_{j+1}, I)} \left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f_{I'} \right\|_p^2 \right)^{1/2} + C_{\varepsilon^2} 2^{-10k} \| f_I \|_p$$

$$\leq C_{\varepsilon^2} 2^{(m_{j+1} - m_j)\left(\frac{1}{2} - \frac{1}{p} + \varepsilon^2\right)} \left(\sum_{I' \in \mathcal{I}(m_{j+1}, I)} \left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f_{I'} \right\|_p^p \right)^{1/p} + C_{\varepsilon^2} 2^{-10k} \| f_I \|_p.$$

We use the induction hypothesis (8.7) and by the last inequality we bound $\left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f \right\|_p$ by

$$C_{\varepsilon^2}^{j+1} 2^{m_0 \left(1 - \frac{1}{p}\right) + (m_{j+1} - m_0)\left(\frac{1}{2} - \frac{1}{p} + \varepsilon^2\right)} \left(\sum_{I' \in \mathcal{I}(m_{j+1})} \left\| \vartheta^{\ell,n} \mathcal{R}_{k,\ell} f_{I'} \right\|_p^p \right)^{1/p}$$

$$+ C_{\varepsilon^2}^j 2^{m_0 \left(1 - \frac{1}{p}\right) + (m_{j+1} - m_0)\left(\frac{1}{2} - \frac{1}{p} + \varepsilon^2\right)} 2^{-10k} \left(\sum_{I \in \mathcal{I}(m_j)} \| f_I \|_p^p \right)^{1/p}$$

$$+ 2^{-9k} \left(\sum_{v=0}^{j-1} C_{\varepsilon^2}^v \right) \| f \|_p.$$

Since $m_0(1 - \frac{1}{p}) + (m_{j+1} - m_0)(\frac{1}{2} - \frac{1}{p} + \varepsilon^2) \leq k$ and $(\sum_{I \in \mathcal{I}(m_j)} \|f_I\|_p^p)^{1/p} \leq \|f\|_p$, we obtain the case for $j + 1$ of (8.7).

We consider the case $j = N(\ell)$ of (8.7). Observe that each interval $I \in \mathcal{I}(m_{N(\ell)})$ is the union of $2^{\ell - m_{N(\ell)}}$ dyadic intervals of length $2^{-\ell}$. We also sum in $n \in \mathbb{Z}^3$ and use the finite overlap of the supports of $\zeta^{\ell,n}$. Observe that the cardinality of the index set of n which give a nonzero contribution is $O(2^{3\ell}) = O(2^k)$. We get

$$\|\mathcal{R}_{k,\ell} f\|_p$$

$$\lesssim \left(\sum_{n \in \mathbb{Z}^3} \|\zeta^{\ell,n} \mathcal{R}_{k,\ell} f\|_p^p \right)^{1/p}$$

$$\leq C_{\varepsilon^2}^j 2^{m_0 \left(1 - \frac{1}{p}\right) + (m_j - m_0)\left(\frac{1}{2} - \frac{1}{p} + \varepsilon^2\right) + (\ell - m_{N(\ell)})\left(1 - \frac{1}{p}\right)} \left(\sum_{I \in \mathcal{I}(\ell)} \sum_{n \in \mathbb{Z}^3} \|\zeta^{\ell,n} \mathcal{R}_{k,\ell} f_I\|_p^p \right)^{1/p}$$

$$+ 2^{-8k} \left(\sum_{\nu=0}^{N(\ell)-1} C_{\varepsilon^2}^\nu \right) \|f\|_p.$$

Observe

$$m_0 \left(1 - \frac{1}{p}\right) + (m_j - m_0)\left(\frac{1}{2} - \frac{1}{p} + \varepsilon^2\right) + (\ell - m_{N(\ell)})\left(1 - \frac{1}{p}\right)$$

$$\leq \ell(\varepsilon^2)\left(1 - \frac{1}{p}\right) + (1 - 2\varepsilon^2)\left(\frac{1}{2} - \frac{1}{p} + \varepsilon^2\right) \leq \ell(2\varepsilon^2 + 1/2 - 1/p)$$

and (with $N(\ell)$ as in (8.6b))

$$\sum_{l=0}^{N(\ell)-1} C_{\varepsilon^2}^l \lesssim (1 + \ell)^{B(\varepsilon)}$$

for some large constant $B(\varepsilon)$. This yields the assertion of the theorem. $\qquad \square$

9 L^p-Sobolev Estimate

In order to prove our Sobolev estimate we have to combine the estimates for the operators $\mathcal{R}_k$. Here we use a special case of Theorem 1.1. in [22]. In what follows the operators P_k are defined by $\widehat{P_k f}(\xi) = \phi(2^{-k}\xi)\widehat{f}$, where ϕ is supported in $\{\xi : \frac{1}{2} < |\xi| < 2\}$

Proposition 9.1 [22] *Assume $\varepsilon > 0$, $p_0 < p < \infty$ and $\lambda > 1$. We are given operators T_k, $k > 0$, with smooth Schwartz kernels K_k (acting on functions in $\mathbb{R}^3$) satisfying*

$$\sup_{k>0} 2^{k/p} \|T_k\|_{L^p \to L^p} \le A \tag{9.1a}$$

$$\sup_{k>0} 2^{k/p_0} \|T_k\|_{L^{p_0} \to L^{p_0}} \le B_0. \tag{9.1b}$$

Assume that for each cube Q there is a measurable exceptional set E_Q such that

$$\text{meas}(E_Q) \le \lambda \max\{diam(Q)^2, |Q|\} \tag{9.1c}$$

and such that for every $k > 0$ and every cube Q with $2^k diam(Q) \ge 1$ we have

$$\int_{\mathbb{R}^3 \setminus E_Q} |K_k(x,y)| dy \le B_1 \max\{(2^k diam(Q)^{-\varepsilon}, 2^{-k\varepsilon}\} \text{ for a.e. } x \in Q. \tag{9.1d}$$

Then for $q > 0$,

$$\left\| \left(\sum_{k>0} 2^{kq/p} |P_k T_k f_k|^q \right)^{1/q} \right\|_p$$

$$\lesssim A \left[\log \left(3 + \frac{B_0^{\frac{p_0}{p}} (A\lambda^{1/p} + B_1)^{1 - \frac{p_0}{p}}}{A} \right) \right]^{\frac{1}{q} - \frac{1}{p}} \left(\sum_k \|f_k\|_p^p \right)^{\frac{1}{p}}. \tag{9.2}$$

We claim that for $\ell > 0$

$$\left\| \left(\sum_{k: \lfloor k/3 \rfloor \ge \ell} 2^{kq/p} |P_k \mathcal{R}_{k,\ell} f_k|^q \right)^{1/q} \right\|_p \le C_p 2^{-\ell \varepsilon(p)} \left(\sum_k \|f_k\|_p^p \right)^{\frac{1}{p}}, \quad p > 4 \tag{9.3}$$

which can be used, together with (5.9) to deduce

$$\mathcal{R} : \left(B_{p,p}^s \right)_{\text{comp}} \to \left(F_{p,q}^{s+1/p} \right)_{\text{loc}}, \quad p > 4, \ q > 0. \tag{9.4}$$

Since $L_p^s = F_{p,2}^s \hookrightarrow B_{p,p}^s$ for $p > 2$ and $F_{p,q}^0 \hookrightarrow F_{p,2}^0 = L^p$, $q \le 2$, this implies the asserted L^p-Sobolev estimates. In order to check (9.3) we need to verify the assumptions of the proposition for the family $\{\mathcal{R}_{k,\ell}\}_{k \ge 3\ell}$.

Let $4 < p_0 < p$. By Theorem 5.1 we have (9.1a) with $A = C_p 2^{-\ell\beta}$ and $\beta < 2/p - 1/2$ if $4 < p \le 6$ and $\beta < 1/p$ if $p \ge 6$. Moreover we have (9.1b) with $B_0 = C_{p_0} 2^{-\ell\beta_0}$ and $\beta_0 < 2/p_0 - 1/2$ if $4 < p_0 \le 6$ and $\beta_0 < 1/p$ if $p_0 \ge 6$.

By integration by parts argument one has the bound

$$|R_{k,\ell}(x, y)| \le C_N \frac{2^{2k}}{(1 + 2^{k-\ell}|y' - S(x_Q, y_3)|)^N}$$

for the Schwartz kernel of $\mathcal{R}_{k,\ell}$. For a cube Q with center x_Q define

$$E_Q := \{y : |y' - S(x_Q, y_3)| \le C2^{2\ell}\text{diam}(Q)\}$$

if $\text{diam}(Q) \le 1$. If $\text{diam}(Q) \ge 1$ we let E_Q be a ball of diameter $C2^{2\ell}\text{diam}(Q)$, centered at x_Q. Assumption (9.1c) is then satisfied with the choice of $\lambda = 2^{2\ell}$ and (9.1d) holds with $B_1 = 2^{2\ell}$. The logarithmic term in (9.2) gives us an additional factor $O(\ell)$. Thus we have verified (9.3) with $\varepsilon(p) < \beta$ and (9.4) follows by summation in $\ell \ge 0$.

10 Further Results and Conjectures

In our analysis we heavily used the condition $\ell \le \lfloor k/3 \rfloor$ for the operators $\mathcal{R}_{k,\ell}$. If one is interested to relax the assumption that π_R is a fold, one needs to explore finer localizations of $\tau_1\Delta_1 + \tau_2\Delta_2$ as used by Comech in [6]. There he proves sharp L^2-Sobolev estimates under the assumption that π_L is a fold but π_R satisfies a finite-type condition of order $\mathfrak{t}$, i.e., if V_R is a kernel field for π_R then $\sum_{j=0}^{\mathfrak{t}} |V_R^j(\det \pi_R)| \ne 0$. The case $\mathfrak{t} = 1$ applies to the fold assumption on π_R. In the general finite-type situation we can show the $L^p_{\text{comp}} \to L^p_{1/p,\text{loc}}$ estimate for $p \ge 5$, and in fact in a slightly larger range.

Theorem 10.1 *Let $\mathcal{M} \subset \Omega_L \times \Omega_R$ be a four-dimensional manifold such that the projections (1.3) are submersions. Assume that the only singularities of $\pi_L : (\mathcal{N}^*\mathcal{M})' \to T^*\Omega_L$ are Whitney folds and that $\pi_R : (\mathcal{N}^*\mathcal{M})' \to T^*\Omega_R$ is of finite type $\le \mathfrak{t}$, for some $\mathfrak{t} \ge 0$. With $\mathcal{L}, \varpi$ be as in Theorem 1.1 suppose that ϖ is a submersion. Then $\mathcal{R}$ is extends to a continuous operator*

$$\mathcal{R} : L^p_{comp}(\Omega_R) \to L^p_{1/p,loc}(\Omega_L), \quad \frac{10\mathfrak{t}+2}{2\mathfrak{t}+1} < p < \infty.$$

Sketch of Proof By the L^2 estimates in [6] the operators $\mathcal{R}_{\ell,k}$ the L^2 bound in (5.3) is still valid, and all of our previous arguments apply. Hence we just need to consider the case $\ell = \lfloor k/3 \rfloor$.

The operator $\mathcal{R}_{\lfloor k/3 \rfloor, k}$, for which $|\tau_1\Delta_1 + \tau_2\Delta_2| \lesssim 2^{-k/3}$, satisfies the norm estimate $\|\mathcal{R}_{k,\lfloor k/3 \rfloor}\|_{L^2 \to L^2} \lesssim 2^{-\frac{k}{2}\frac{\mathfrak{t}+1}{2\mathfrak{t}+1}}$, a less satisfactory bound. One can show this estimate as a consequence of more refined L^2-estimates in [6]. This yields an analogue of (5.8) in the finite-type case, namely for $2 \le p \le \infty$,

$$\left(\sum_\nu \|\mathcal{R}_{k,\lfloor k/3 \rfloor}[\mathbb{1}_{\lfloor k/3 \rfloor, \nu} g_\nu]\|^p\right)^{\frac{1}{p}} \lesssim 2^{-\frac{k}{p}\frac{\mathfrak{t}+1}{2\mathfrak{t}+1} - \frac{k}{3}\left(1 - \frac{2}{p}\right)} \left(\sum_\nu \|g_\nu\|_p^p\right)^{\frac{1}{p}}.$$

Combining this with the decoupling estimate (5.6) (which remains true for $\ell = \lfloor k/3 \rfloor$) yields

$$\left\| \mathcal{R}_{k, \lfloor k/3 \rfloor} f \right\|_p \lesssim C_\varepsilon 2^{\frac{k}{3}\left(\frac{1}{2} - \frac{1}{p} + \epsilon\right)} 2^{-\frac{k}{p}\frac{t+1}{2t+1} - \frac{k}{3}\left(1 - \frac{2}{p}\right)} \| f \|_p, \quad 2 \le p \le 6,$$

i.e., $\| \mathcal{R}_{k, \lfloor k/3 \rfloor} f \|_p \lesssim 2^{-k(\alpha(p)+1/p)}$ with $\alpha(p) > 0$ for $\frac{10t+2}{2t+1} < p \le 6$. Further interpolation with the bound $\| \mathcal{R}_{k, \lfloor k/3 \rfloor} \|_{L^\infty \to L^\infty} = O(1)$ gives a similar statement for $6 \le p \le \infty$ with an $\alpha(p) > 0$ for $6 \le p < \infty$. $\qquad\square$

To improve on this result, one would have to employ finer localizations in terms of $\det \pi_L$ (which would correspond to the assumption $|\tau_1 \Delta_1 + \tau_2 \Delta_2| \approx 2^{-\ell}$ where a range of $\ell > k/3$ will depend on t). Our current arguments for the plate localization in Lemma 6.3 are not effective in that situation. Nevertheless we conjecture that the result of Theorem 10.1 remains true for all $p > 4$, and even that the assumptions on π_R can be dropped altogether in Theorem 1.1. See the discussion of model examples in Sects. 4.2 and 4.3.2.

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