BOCHNER–RIESZ MEANS ON THE HEISENBERG GROUP

DETLEF MÜLLER, LARS NIEDORF, ANDREAS SEEGER, AND BETSY STOVALL

ABSTRACT. We prove new L^p boundedness results for Bochner–Riesz means associated with the spectral decomposition of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$. Our results hold for a range $1 \le p \le p_n$ where $p_n \to 2$ as $n \to \infty$. As shown by the first named author in 1990 a Stein–Tomas type Fourier restriction theorem fails to hold on $\mathbb{H}_n$ and thus previous results based on the approach by Fefferman and Stein from the Euclidean setting only allowed to cover the cases $p = 1$ and $p = \infty$. Our results on Bochner–Riesz means follow from a more general p-sensitive spectral multiplier theorem which is the main result of this article. This is obtained as a consequence of L^p estimates for square functions associated with the Heisenberg wave operator.

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1. INTRODUCTION

Denote by $\mathbb{H}_n$ the Heisenberg group of dimension $d = 2n + 1$. On $\mathbb{H}_n$ we are using coordinates $(x, u) \in \mathbb{R}^{2n} \times \mathbb{R}$ and the group law

$$(x, u) \cdot (x', u') = (x + x', u + u' + \tfrac{1}{2}(Jx)^{\intercal} x'), \tag{1.1}$$

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where $J = J_n$ denotes the $2n \times 2n$ block-diagonal matrix $\sum_{k=1}^{n}(e_{2k}e_{2k-1}^{\intercal} - e_{2k-1}e_{2k}^{\intercal})$, i.e., the matrix with n blocks of the form $J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The purpose of this paper is to establish new spectral multiplier theorems for the *sub–Laplacian*

$$L = -\sum_{j=1}^{2n} X_j^2$$

on the Heisenberg group, one of the most basic subelliptic, non-elliptic examples of a Hörmander sum of squares operator. The vector fields $X_1, \dots, X_{2n}$ are given by

$$X_{2k-1} = \partial_{x_{2k-1}} - \tfrac{1}{2}x_{2k}\partial_u, \qquad X_{2k} = \partial_{x_{2k}} + \tfrac{1}{2}x_{2k-1}\partial_u, \qquad k = 1, \dots, n.$$

The vector fields $X_1, \dots, X_{2n}$ and $U = \partial_u$ span the Lie algebra of $\mathbb{H}_n$, and the only non-trivial commutation relations among them are

$$[X_{2k-1}, X_{2k}] = U, \qquad k = 1, \dots, n.$$

In particular, they satisfy Hörmander's bracket condition for hypoellipticity of a sum of squares operator. Moreover, the sum of squares operator L is essentially self-adjoint on $C_0^\infty(\mathbb{H}_n)$ (as is established by the methods in [45]). By the spectral theorem there is a functional calculus which defines the operator $m(\sqrt{L})$ as a bounded operator on $L^2(\mathbb{H}_n)$ for any continuous bounded function m on $[0, \infty)$. In L^p spectral multiplier theory we care about conditions on m under which the operator $m(\sqrt{L})$ extends to L^p for $p \neq 2$. Using the automorphic dilation structure

$$D_r : (x, u) \mapsto (rx, r^2u)$$

on the Heisenberg group the $L^p \to L^p$ operator norms of $m(r\sqrt{L})$ for $r > 0$ are independent of r and therefore it is natural to formulate dilation invariant assumptions for multiplier transformations.

An effective L^p-result on spectral multipliers for the standard positive Laplacian $-\sum_{k=1}^{d} \partial_{x_k}^2$ in Euclidean space $\mathbb{R}^d$, or for the Laplace–Beltrami operator $-\Delta$ on a d-dimensional compact manifold, is given by the classical Hörmander–Mikhlin multiplier theorem. One version of it states that the multiplier transformations $m(r\sqrt{-\Delta})$ extend for any $r > 0$ to L^p for $1 < p < \infty$, with operator norm independent of r, if the localized Sobolev conditions

$$\sup_{t>0} \|\eta\, m(t\cdot)\|_{L^2_\alpha} < \infty \tag{1.2}$$

are satisfied for $\alpha > d/2$; here $\eta \neq 0$ is a C^∞ function with compact support in $(0, \infty)$. The space of functions on $(0, \infty)$ defined by the finiteness of the norm in (1.2) does not depend on the choice of η (see e.g. [7]). By an interpolation argument one can also replace in (1.2) the space $L^2_{(d/2)+\varepsilon}$ with $L^q_{(d/q)+\varepsilon}$, where $\frac{1}{q} = |\frac{1}{p} - \frac{1}{2}|$ and $q > 2$.

The above Hörmander–Mikhlin-type result also holds for functions $m(\sqrt{L})$ of the sub-Laplacian on the Heisenberg group, with $d = 2n + 1$, but this is harder to prove. It was independently established by the first author and Stein [42] and by Hebisch [21], improving for Heisenberg groups earlier results on general stratified groups by Christ [13] and Mauceri–Meda [35] that involved in place of the Euclidean dimension d the larger homogeneous dimension Q (e.g. $d = 2n + 1$ for the group $\mathbb{H}_n$, but $Q = 2n + 2$).

In radial Fourier multiplier theory on Euclidean spaces and spectral multiplier theory on compact manifolds sharper p-sensitive results are known for certain $p > 1$

that cannot be obtained by interpolation between $p = 1$ and $p = 2$, and cover for suitable p-ranges essentially sharp L^p-bounds on Riesz means $(I - R^{-1}\sqrt{-\Delta})^\nu_+$ and on the closely related Bochner–Riesz means $(I+R^{-2}\Delta)^\nu_+$. On general compact manifolds such essentially optimal results for $m(\sqrt{-\Delta})$ are known for the range $1 < p \leq \frac{2(d+1)}{d+3}$ and its dual range, which corresponds to the range of the Stein–Tomas Fourier restriction theorem ([60, 58]) and its analogue by Sogge [56] on compact manifolds. Specifically, if $1 < p < \frac{2(d+1)}{d+3}$ and (1.2) holds for $\alpha > d\,|\frac{1}{p} - \frac{1}{2}|$ then $m(\sqrt{-\Delta})$ is bounded on L^p (see [54]). We remark that much better p-ranges beyond the Stein–Tomas theory are known to hold in the Euclidean case. In particular, the optimal multiplier results under condition (1.2) are directly related via [6] to results on a square function by Stein involving Bochner–Riesz means, for which the optimal results in two dimensions are due to Carbery [4], and the current best results in higher dimensions are in Gan–Oh–Wu [17]. The latter paper contains many more references.

Turning to the analogous question about p-sensitive multiplier theorems for the sub-Laplacian on the Heisenberg groups, no such optimal improvements beyond interpolation have been known for any $p \in (1, 2)$. The reason for this state of affairs can be traced back to a result by the first author in [39] which is concerned with restriction and spectral projection results for the sub-Laplacian. Specifically, it was shown in [39] that for large M

$$\big\|\mathbb{1}_{[1-M^{-1},1]}(\sqrt{L})\big\|_{L^p(\mathbb{H}_n)\to L^2(\mathbb{H}_n)} \sim M^{\frac{1}{2}-\frac{1}{p}}, \qquad 1 \leq p \leq 2, \tag{1.3}$$

while a Stein–Tomas type restriction theorem for L would imply the stronger bound $O(M^{-\frac{1}{2}})$ for some $p > 1$.

In view of this outcome there cannot be, even for compactly supported multipliers, an effective approach to sharp L^p multiplier theorems for $p > 1$ based on spectral projection bounds (1.3), such as is suggested by the method established by Fefferman [14], with simplification by Stein [15]. See also [12, 51] and then [55, 54] for results on compact manifolds, with further extensions in [11]. The first author used weaker restriction theorems on the Heisenberg group to prove estimates for Riesz means in certain mixed norm spaces [38], but no sharp results in L^p spaces have been known for $p > 1$. We note that the failure of the restriction estimate rules out a certain standard method of proof and does not disprove any sharp Bochner–Riesz result itself (contrary to what might be inferred from the title of [59]), but it shows that a different approach is needed.

Here we present such an approach and establish a satisfactory multiplier result for functions of the sub–Laplacian L in a range $1 < p < p(n)$ where $p(n) \to 2$ as $n \to \infty$. As the operators $m(\sqrt{L})$ are self-adjoint we also have $L^{p'}$ boundedness for $p' = \frac{p}{p-1}$. In what follows, keep in mind that

$$d = 2n + 1$$

denotes the topological dimension of $\mathbb{H}_n$.

We would like to point out that, at least by our approach, the p-range in our results below is determined more by the dimension $d_1 = 2n$ of the first, horizontal layer of the Heisenberg group than by the dimension d; see the discussion below. The p-range in our next theorem would then read $1 < p \leq \frac{2(d_1+2)}{d_1+5}$. We expect that an analogue of Theorem 1.1 will hold on groups of Heisenberg type in the same range (see the according discussion at the end of this section).

Theorem 1.1. *Let $n \geq 1$, and let p be in the range $1 < p \leq \frac{2(d+1)}{d+4}$, or its dual range $\frac{2(d+1)}{d-2} \leq p < \infty$. Assume that $\alpha > d\,|\frac{1}{p} - \frac{1}{2}|$. Then, for all $f \in L^p(\mathbb{H}_n)$,*

$$\|m(\sqrt{L})f\|_{L^p(\mathbb{H}_n)} \leq C_{p,\alpha,\eta} \sup_{t>0} \|\eta\, m(t\cdot)\|_{L^2_\alpha}\, \|f\|_{L^p(\mathbb{H}_n)}.$$

Note that the functions $\tau \mapsto (1-|\tau|)^\nu_+$ belong to the Sobolev space L^2_α if $\alpha < \nu + \frac{1}{2}$. This, together with standard arguments can be used to deduce L^p convergence for the Riesz means and the Bochner–Riesz means of an L^p function. Martini [32] showed that L^p boundedness for $(1-L)^\nu_+$ fails for $\nu < d\,|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$.

Thus, we get an essentially sharp result in the range $p \leq \frac{2(d+1)}{d+4}$ (and, by duality, in the corresponding dual range):

Corollary 1.2. *Let p be in the range $1 \leq p \leq \frac{2(d+1)}{d+4}$, or its dual range, and let $\nu > d\,|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. Then the Bochner–Riesz means $(1-R^{-2}L)^\nu_+$ and the Riesz means $(1-R^{-1}\sqrt{L})^\nu_+$, $R > 0$, are uniformly bounded on $L^p(\mathbb{H}_n)$.*

The essential tool in the proof of Theorem 1.1 is a square function estimate for the half wave operator $e^{it\sqrt{L}}$. This approach is motivated by the equivalence of square function bounds for Bochner–Riesz means and for derivatives of spherical averages in the Euclidean case, as demonstrated by Kaneko and Sunouchi [26].

Theorem 1.3. *Assume that I is a compact subset of $\mathbb{R}\setminus\{0\}$ and $p > \frac{2(d+1)}{d-2}$. Then for $\beta \in C^\infty_c(\mathbb{R})$ and $\lambda \geq 1$,*

$$\Big\| \Big(\int_I \big| e^{it\sqrt{L}} \beta(\tfrac{t\sqrt{L}}{\lambda}) f \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{H}_n)} \leq C_{p,\beta} \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} \|f\|_{L^p(\mathbb{H}_n)} \tag{1.4}$$

for all $f \in L^p(\mathbb{H}_n)$.

Note that the endpoint $p = \frac{2(d+1)}{d-2}$ is excluded from Theorem 1.3, but is included in Theorem 1.1 and Corollary 1.2. Nevertheless, since the regularity conditions on α and ν above are given by strict inequalities, the corresponding endpoint estimates can be recovered by interpolation with the L^2 bounds, which follow directly from the spectral theorem.

The result in Theorem 1.3 involves a better exponent than the sharp fixed time estimate

$$\big\| e^{it\sqrt{L}} \beta(\tfrac{t\sqrt{L}}{\lambda}) f \big\|_{L^p(\mathbb{H}_n)} \leq C_{\beta,s}\, \lambda^{(d-1)(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^p(\mathbb{H}_n)} \tag{1.5}$$

which holds for $2 \leq p < \infty$; this follows from a result in [41] (for previous non-endpoint results see also [43]). Analogs of (1.4) are well known in the Euclidean case and follow from corresponding $L^2 \to L^p$ inequalities for the square function. In contrast the $L^2 \to L^p$ versions for the sub-Laplacian are not effective enough to yield Theorem 1.3. The reason is that the singularities for the convolution kernel of the wave kernel are far worse than in the Euclidean case, due to the complex underlying sub-Riemannian geometry associated with the sub-Laplacian. The singular support of the convolution kernel is known to coincide with the geodesic sphere of radius $|t|$ with respect to the sub-Riemannian metric associated to L. This (together with asymptotics of the kernel in some regimes) was first described by Nachman [44]. The geodesic sphere is invariant under vertical reflections and horizontal rotations in $\mathbb{R}^{2n}$. Its $(|x|,|u|)$-profile for the case $t = 1$ is shown in Figure 1.

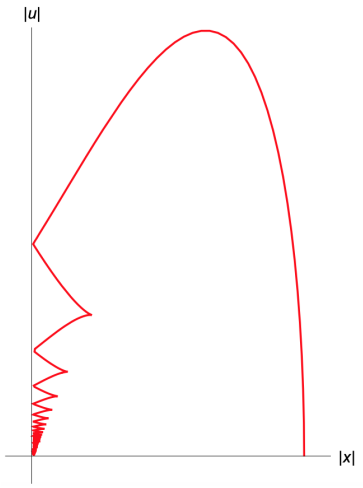


FIGURE 1. A plot of the singular support Σ_1 of the convolution kernel of $e^{i\sqrt{L}}$. The curve admits infinitely many zigzags near the origin.

Our analysis uses a more precise parametrix derived by the first and third authors in [41]; this involves a subordination argument using the Schrödinger type operators e^{itL}, and a decomposition in the joint spectrum of L and $i\partial_u$. To begin with, we note that the proof of Theorem 1.3 can easily be reduced to the case where β is supported in $[\frac{1}{2}, 2]$ (cf. Section 2). Assuming this, we have, for Schwartz functions $f \in \mathcal{S}(\mathbb{H}_n)$,

$$e^{it\sqrt{L}}\beta(\tfrac{t\sqrt{L}}{\lambda})f = T^0_{\lambda,t}f + \sum_{0<|k|\le 8\lambda}\sum_{\ell\ge 1} T^{k,\ell}_{\lambda,t}f, \tag{1.6}$$

where the operators $T^0_{\lambda,t}$, $T^{k,\ell}_{\lambda,t}$ with convolution kernels $\mathcal{K}^0_{\lambda,t}$, $\mathcal{K}^{k,\ell}_{\lambda,t}$ are defined in (4.12), (4.17). The convolution kernels $\mathcal{K}^0_{\lambda,t}$ are concentrated near the outer part in Figure 1. For the analysis of $T^{k,\ell}_{\lambda,t}$ we may assume that $k > 0$, by symmetry considerations. For $k > 0$, the convolution kernels $\sum_{\ell\ge 1}\mathcal{K}^{k,\ell}_{\lambda,t}$ are concentrated in a (Ck^{-1}, Ck^{-2}) box at distance $\sim k^{-1}$ from the horizontal plane; it corresponds to the kth zigzag. Moreover, for $\ell \ge 1$, $\mathcal{K}^{k,\ell}_{\lambda,t}$ is concentrated in a part of this box at distance $\sim 2^{-\ell}k^{-1}$ from the $|u|$-axis (these are the boxes $D^{k,\ell}$ in Figure 2 below). When acting on functions supported near the origin we may consider $T^{k,\ell}_{\lambda,t}$ as a parabolically rescaled Fourier integral operator with non-homogeneous phase (see (4.22)), with the additional complication that the phase functions become very singular for large values of ℓ. The following theorem is the key auxiliary result in the proof of Theorem 1.3.

Theorem 1.4. *Assume $p > \frac{2d}{d-2}$. Then*

$$\Big\| \Big(\int_1^2 |T^0_{\lambda,t}f|^2 dt \Big)^{\frac{1}{2}} \Big\|_p \lesssim \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} \|f\|_p. \tag{1.7}$$

Moreover, for $|k| \le 8\lambda$, $2^\ell|k| \lesssim \lambda$,

$$\Big\| \Big(\int_1^2 \big|T^{k,\ell}_{\lambda,t} f\big|^2 dt \Big)^{\frac12} \Big\|_p \lesssim \big[(2^\ell|k|)^{\frac{d+1}{p}-\frac{d}{2}} \lambda^{d(\frac12-\frac1p)-\frac12} + (2^\ell|k|^{-1}) \big] \, \|f\|_p. \tag{1.8}$$

For $2^\ell|k| \gtrsim \lambda$, there are also more elementary square function estimates which rely on L^1, or equivalently L^∞, boundedness results from [41],

$$\|T^{k,\ell}_{\lambda,t}\|_{L^\infty \to L^\infty} \lesssim (2^\ell|k|)^{-1} \lambda^{\frac12}, \tag{1.9}$$

and additional cancellations for the L^2 square function bounds; see Proposition 8.1. We note that (1.9) leads to favorable bounds in the range not covered by Theorem 1.4, namely $2^\ell|k| \gtrsim \lambda$, and indeed even in a somewhat larger p-range.

The estimates (1.8) must be summed over k, ℓ in the relevant ranges. To arrive at the conclusion (1.4) we then need the more restrictive condition $\frac{d+1}{p} - \frac{d}{2} < -1$, i.e., $p > \frac{2(d+1)}{d-2}$, which is the p-range assumed in Theorem 1.3.

On the Fourier integral methods used in the proof of Theorem 1.4. The estimates for the above mentioned Fourier integral operators rely on curvature conditions in the frequency variables formulated in [37, Sec. 2] and [37, Prop. 3.4]; we refer to this as the *rank r cone condition* where $r \le d-1$. These are homogeneous versions of the Carleson-Sjölin condition ([3, 24, 58]). For averages over curves in the plane the cone condition corresponds to the *cinematic curvature condition* which was proposed previously by Sogge in his work on maximal functions for variable averages in two dimensions [57] (for recent extensions see also [30, 9]). For wave equations associated with elliptic Laplacians, the square function estimates of the form (1.4) are consequences of an $L^2 \to L^p(L^2)$ estimate [37, Thm. 3.2], which covers classes of Fourier integral operators satisfying the maximal rank $(d-1)$ cone condition; in this case one has estimates in the Stein–Tomas range $p \ge \frac{2(d+1)}{d-1}$ associated to the space dimension d.

In contrast, for our Fourier integral operators on the Heisenberg group the rank $(d-1)$ cone condition is not satisfied; instead only a rank $d_1 - 1 = d-2$ cone condition holds (see the discussion in Appendix B), where d_1 denotes the dimension $d_1 = 2n$ of the first, horizontal layer of the Heisenberg group. While [37] covers Strichartz $L^2 \to L^p(L^p)$ estimates for lower rank cone conditions it does not cover optimal $L^2 \to L^p(L^2)$ estimates in these situations. Indeed, the proof of square function results under the same lower rank cone condition poses substantial technical difficulties as our proof of (1.7) for the best behaved family $\{T^0_{\lambda,t}\}$ already illustrates. Additional complications arise for the square functions associated with the operators $T^{k,\ell}_{\lambda,t}$, for $k, \ell \ge 1$. Note that, for each fixed $T^{k,\ell}_{\lambda,t}$, the square function estimate (1.8) holds in the Stein–Tomas range $p > \frac{2(d_1+1)}{d_1-1}$ associated with the dimension d_1. However, the constants in these estimates grow with k and ℓ, leading to the slightly smaller p-range in Theorem 1.3.

In principle, after suitable rescalings, we adapt in all cases the TT^* method from [27, 19, 37]. This uses certain variants of TT^* where we freeze one of the space variables. It turns out that in our problem, unlike in the classical applications, the standard canonical graph assumptions that lead to optimal L^2 bounds for these frozen operators are not satisfied uniformly in k and ℓ. Consequently, the resulting constants in (1.8) are less favorable for large k and ℓ than one might hope for. The lack of uniformity in k requires more sophisticated L^2 estimates, which make use of the special structure of the phase functions.

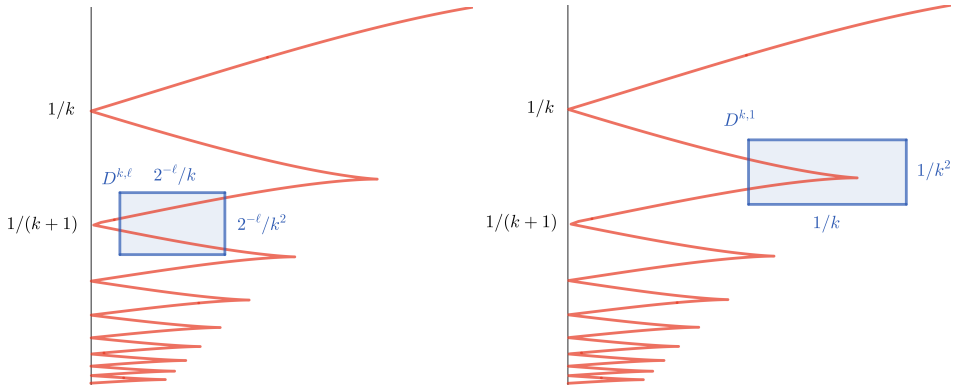


FIGURE 2. The localization regions $D^{k,\ell}$ associated with the kth zigzag of the $(|x|,|u|)$-profile of the singular support Σ_1. Case I corresponds to $\ell \geq 2$, and Case II to $\ell = 1$.

The choice and the further analysis of such frozen operators will be different in the cases $\ell \geq 2$ and $\ell = 1$ (corresponding to Case I and Case II in Figure 2, respectively). This phenomenon is linked to the different ways in which the above mentioned cone condition in [37] can be interpreted as a curvature condition. In the paper by Sogge [57] on averages over variable plane curves this condition was given in terms of a change of curvature in time (hence the use of the descriptive terminology *cinematic* curvature). However this interpretation breaks down at several points in Figure 3 where the graph has a horizontal tangent line. These points correspond to the special choices $\tau_j = j\pi + \pi/2$ in the frequency variable τ parametrizing the oscillatory integrals in (4.21), (4.22) below and are located at the top of the figure and close to the cusps. Thus there are (d_1-1)-dimensional spheres in the singular support where the tangent hyperplanes to the singular support are all horizontal. Since the affine horizontal planes will remain horizontal under application of the automorphic dilations there is no longer a change of curvature in time. Indeed, the presence of these horizontal tangent planes suggests the absence of even a lower rank cinematic phenomenon; nevertheless the rank $(d_1 - 1)$ cone condition still holds, due to a certain interplay between an actual curvature condition for the singular supports and the rotational curvature [49] implicit in the Heisenberg group law. This insight has significant consequences for our parametrizations of the associated phase functions for these operators and also for our choice of frozen variables in the analysis of various TT^* arguments. We note that a related phenomenon also occurs in the paper [50] by Roos, Srivastava and the third author, which studies certain families of κ-dimensional submanifolds of Heisenberg type groups with $\kappa < d-1$, where a cone condition of rank $d-1$ still holds.

A guide through the paper. In Section 2, we show how the square function estimates from Theorem 1.3 imply the spectral multiplier results in Theorem 1.1 and Corollary 1.2. In Section 3 we study the reduction of the proof of Theorem 1.3 to localizations based on an abstract localization result for left-invariant operators, on finite propagation speed for $k = 0$, and on further localizations which are made

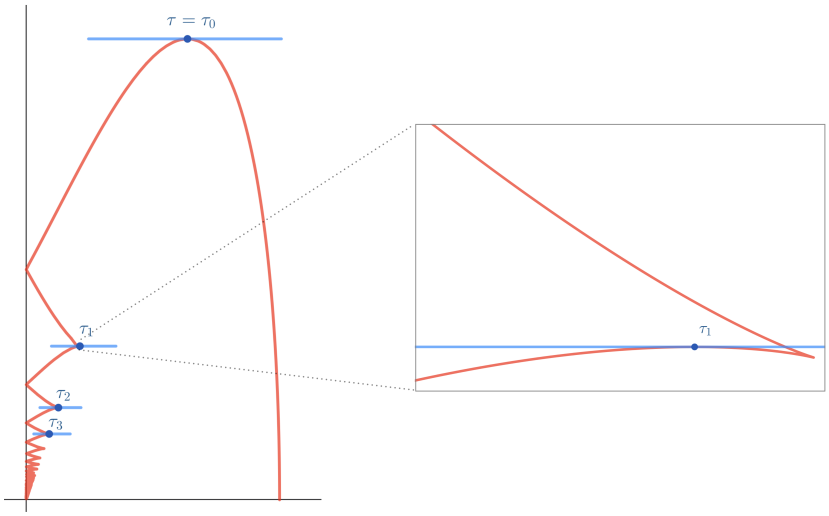


FIGURE 3. A plot indicating horizontal points of the singular support Σ_1 of the convolution kernel of $e^{i\sqrt{L}}$. Note that these horizontal points are distinct from the cusp points visible in the plot.

possible by means of error estimates for the $T^{k,\ell}_{\lambda,t}$ for $k \neq 0$. The error estimates can be found in Appendix A (see also [41] for a different treatment). For $k \neq 0$ these localizations force us to decompose the time intervals into smaller intervals of length comparable to the diameter of the $D^{k,\ell}$ in Figure 2. In Section 4 we present some preliminaries about the solution operator for the Heisenberg wave equation, including the decomposition (1.6) and further information. The proof of the square function bound (1.7) involving the operators $T^0_{\lambda,t}$ will be given in Sections 5, 6 and 7. The proof of (1.8) in the cases $k \geq 1$, $\ell \geq 1$ will be given in the remaining Sections 8–12. Appendix B contains a calculation regarding the rank $(d_1 - 1)$ cone condition.

Further directions and open problems.

Extensions to Heisenberg type groups. We expect that the methods of the current paper also give improvements to known spectral multiplier results on a Heisenberg type group G with d_2-dimensional center, where $d_2 \geq 2$. These exist for every d_2, but we note that if $d_1 = d - d_2$ is the dimension of the first, horizontal layer of the group G, then the Radon–Hurwitz formulas yield $d_2 \lesssim \log d_1$ (for more information see [1, 28]). The second author [46] proved an analogue of Theorem 1.1 in the range $p \leq \frac{2(d_2+1)}{d_2+3}$ for $\alpha > d\,|\frac{1}{p} - \frac{1}{2}|$ relying on Stein–Tomas Fourier restriction type estimates (from Casarino–Ciatti [8], Liu–Wang [31] and Chen–Ouhabaz [10], see also [47]). Note that this method does not yield multiplier results on the standard Heisenberg groups where $d_2 = 1$. It is expected that the methods in our paper can be further developed to substantially extend the p-range for general Heisenberg type groups.

Further extensions to Métivier groups. In [47], [48] the second author also extended the restriction and multiplier results to Métivier groups with higher dimensional

center (again obtaining an analog of Theorem 1.1 in the range $1 \le p \le \frac{2(d_2+1)}{d_2+3}$, $d_2 \ge 2$, at least in the case $(d_1, d_2) \notin \{(8,6),(8,7)\}$). For Métivier groups that are not of Heisenberg type the parametrix construction from [41] used in the current paper is no longer available which makes further improvements beyond this range very difficult. A new approach to describe solutions of the appropriate wave equation, with a partial parametrix, was developed by Martini and Müller in [34], relying on Fourier integral operators with complex phase. Their approach yields fixed-time estimates of the kind (1.5) (with a possible loss of λ^ε). It would be very interesting to prove new improved p-sensitive multiplier theorems in the Métivier case.

Endpoint bounds. Sharp estimates for wave multipliers on the Heisenberg group can be found in [41], and for other oscillating multipliers in [2]. It would be very interesting to find more general endpoint versions of Theorem 1.4 (covering among other things the oscillatory multiplier results) which mirror corresponding results for the Laplacian on compact manifolds [53, 29].

Improving the p-ranges. The modern powerful methods in Euclidean Fourier analysis used in Fourier restriction and radial multiplier theory beg to be used to extend the p-range in Theorem 1.3, and consequently the p-ranges in Theorem 1.1. This is an interesting direction for further research.

Local smoothing phenomena. We conjecture that for suitable ranges of $p \gg 2$ the inequality (1.4) can be strengthened by replacing the $L^2([1,2])$ norm with an $L^p([1,2])$ norm. This is analogous to Sogge's local smoothing conjecture [57] for the wave equation associated with an elliptic Laplacian. Such an inequality would also imply new L^p spectral multiplier theorems beyond the spaces defined in (1.2).

2. Consequences of the square function estimate

We first show that Theorem 1.3 implies Theorem 1.1 for multipliers that are compactly supported in $(0,\infty)$, and then provide some references for the more general situation. The results and proofs in this section and Section 3 are valid for any left-invariant homogeneous sub-Laplacian L on any Carnot group G of topological dimension d in place of $\mathbb{H}_n$.

If $T : \mathcal{S}(G) \to \mathcal{S}'(G)$ is a bounded left-invariant linear operator, then the Schwartz kernel theorem ensures the existence of a tempered distribution $k_T \in \mathcal{S}'(G)$ such that

$$T\phi = \phi * k_T, \qquad \phi \in \mathcal{S}(G), \tag{2.1}$$

where $*$ denotes the group convolution on G. We can then write

$$k_T = T\delta_0$$

(and shall often do so). In particular, by Hulanicki's result [25, Thm. 2.4] we know that $m(L)\delta_0 \in \mathcal{S}(G)$ for $m \in \mathcal{S}(\mathbb{R})$.

Proposition 2.1. *Let $p \ge 2$, $\alpha_0 > 0$, $0 < \varepsilon < 1$, and $\beta \in C_c^\infty((0,\infty))$, with $\beta \not\equiv 0$. Assume that the L^p inequality*

$$\Big\| \Big(\int_1^{1+\varepsilon} |e^{it\sqrt{L}} \beta(\tfrac{t\sqrt{L}}{\lambda}) f|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \le C_0 (1+\lambda)^{\alpha_0 - \frac{1}{2}} \|f\|_p \tag{2.2}$$

holds for all $\lambda \geq 1$ and all $f \in \mathcal{S}(G)$. Under these hypotheses, if $m : \mathbb{R} \to \mathbb{C}$ is supported in a compact subinterval I of $(0,\infty)$ and $m \in L^2_\alpha(\mathbb{R})$ for some $\alpha > \alpha_0$, then $m(\sqrt{L})$ extends to a bounded operator on $L^p(G)$ with

$$\|m(\sqrt{L})\|_{p\to p} \lesssim_{I,\alpha,\beta} C_0 \|m\|_{L^2_\alpha(\mathbb{R})}. \tag{2.3}$$

Here $m(\sqrt{L})$ on the left-hand side can also be replaced by $m(L)$.

Proof. Making $\varepsilon > 0$ in the assumption smaller if necessary we may assume that there is $r_\circ > 0$ such that $\beta(r) \neq 0$ on the interval $(r_\circ(1-\varepsilon), r_\circ(1+\varepsilon))$. We first show (2.3) under the assumption that m is supported in $(1-\frac{1}{8N}, 1+\frac{1}{8N})$ where $N > \varepsilon^{-1}$. Let $\chi \in C^\infty_{\rm c}$ with $\chi(r) = 1$ for $|r-1| \leq \frac{1}{8N}$ and χ supported in $(1-\frac{1}{6N}, 1+\frac{1}{6N})$.

Let $\ell_0 = \max\{\ell \in \mathbb{Z} : 2^\ell < r_\circ\}$. Using the Fourier inversion formula (with $\widehat{m}(\rho) = \int_{\mathbb{R}} m(r) e^{-ir\rho} dr$) we write

$$m(\sqrt{L}) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{m}(\tau)\, e^{i\tau\sqrt{L}}\, \chi(\sqrt{L})\, d\tau = u_0(\sqrt{L}) + \sum_{\pm}\sum_{\ell>\ell_0} u_{\ell,\pm}(\sqrt{L}),$$

where, with $J_0 = [-2^{\ell_0+1}, 2^{\ell_0+1}]$, and, $J_\ell^- = [-2^{\ell+1}, -2^\ell]$, and $J_\ell^+ = [2^\ell, 2^{\ell+1}]$ for $\ell > \ell_0$,

$$u_{\ell,\pm}(\sqrt{L}) = \frac{1}{2\pi}\int_{J_\ell^\pm} \widehat{m}(\tau)\, \chi(\sqrt{L}) e^{i\tau\sqrt{L}}\, d\tau.$$

The multiplier u_0 is analogously defined, replacing $J_\ell^\pm$ by $[-2^{\ell_0+1}, 2^{\ell_0+1}]$. Since u_0 is a smooth compactly supported function, the inequality

$$\|u_0(\sqrt{L})f\|_p \lesssim \|m\|_{L^2}\|f\|_p$$

follows from Hulanicki's theorem.

We shall now focus on the multipliers $u_{\ell,+}$ for $\ell > \ell_0$. We decompose

$$J_\ell^+ = \bigcup_{\nu=1}^{4N} J_{\ell,\nu},$$

where $J_{\ell,\nu} = [R_{\ell,\nu}, R_{\ell,\nu+1}]$ with $R_{\ell,\nu} = 2^\ell(1 + (\nu-1)\frac{1}{4N})$. Note that $R_{\ell,\nu}^{-1} J_{\ell,\nu} \subset [1, 1+\frac{1}{4N}]$. Use Cauchy–Schwarz and a change of variable in t to estimate

$$\begin{aligned}
&\Big|\int_{J_\ell^+} \widehat{m}(\tau)\, e^{i\tau\sqrt{L}}\, \chi(\sqrt{L})\, f\, d\tau\Big| \\
&\leq \Big(\int_{J_\ell^+} |\widehat{m}(\tau)|^2\, d\tau\Big)^{\frac12}\Big(\int_{J_\ell^+} \big|e^{i\tau\sqrt{L}}\, \chi(\sqrt{L})\, f\big|^2\, d\tau\Big)^{\frac12} \\
&\lesssim 2^{-\ell\alpha}\, \|m\|_{L^2_\alpha}\Big(\sum_{\nu=1}^{4N}\int_{J_{\ell,\nu}} \big|e^{i\tau\sqrt{L}}\, \chi(\sqrt{L})\, f\big|^2\, d\tau\Big)^{\frac12} \\
&\lesssim 2^{-\ell\alpha}\|m\|_{L^2_\alpha}\Big(\sum_{\nu=1}^{4N}\int_1^{1+\frac{1}{4N}} \big|e^{iR_{\ell,\nu}t\sqrt{L}}\, \chi(\sqrt{L})\, f\big|^2\, R_{\ell,\nu} dt\Big)^{\frac12}.
\end{aligned}$$

Let $\chi_1 \in C^\infty_{\rm c}$ be supported in $(1-\frac{3}{4N}, 1+\frac{3}{4N})$ so that $\chi_1(r) = 1$ for $|r-1| \leq \frac{1}{2N}$. If $|r-1| \leq \frac{1}{6N}$ and $|t-1| \leq \frac{1}{4N}$ then $\chi_1(tr) = 1$; hence $\chi_1(t\cdot)\chi = \chi$ for all

$t \in [1, 1+\frac{1}{4N}]$. Since $\beta(r_\circ r) \neq 0$ for $|r-1| \le N^{-1}$ we see that $\chi_2(r) = \chi_1(r)/\beta(r_\circ r)$ defines a function in C_0^∞ and we get with $g = \chi(\sqrt{L})f$

$$\int_1^{1+\frac{1}{4N}} \left|e^{iR_{\ell,\nu}t\sqrt{L}}\,\chi(\sqrt{L})\,f\right|^2 dt = \int_1^{1+\frac{1}{4N}} \left|e^{iR_{\ell,\nu}t\sqrt{L}}\,\chi_2(t\sqrt{L})\,\beta(r_\circ t\sqrt{L})\,g\right|^2 dt.$$

Since $\chi_2(\sqrt{L})\delta_0$ is a Schwartz function, we have by parabolic scaling with D_t and Hulanicki's theorem (see also [35])

$$\Big\|\Big(\int_1^{1+\frac{1}{4N}} \left|\chi_2(t\sqrt{L})F_t\right|^2 dt\Big)^{\frac12}\Big\|_p \lesssim \Big\|\Big(\int_1^{1+\frac{1}{4N}} \left|F_t\right|^2 dt\Big)^{\frac12}\Big\|_p, \quad 1 \le p \le \infty.$$

Hence, using that $R_{\ell,\nu} \sim 2^\ell$,

$$\begin{aligned}
\|u_{\ell,+}(\sqrt{L})f\|_p &\lesssim 2^{\ell(\frac12-\alpha)}\|m\|_{L^2_\alpha} \sum_{\nu=1}^{4N} \Big\|\Big(\int_1^{1+\frac{1}{4N}} |e^{iR_{\ell,\nu}t\sqrt{L}}\beta(r_\circ t\sqrt{L})\,g|^2 dt\Big)^{\frac12}\Big\|_p \\
&\lesssim 2^{\ell(\frac12-\alpha)}\|m\|_{L^2_\alpha} \sum_{\nu=1}^{4N} \Big\|\Big(\int_1^{1+\frac{1}{4N}} \left|D_{R_{\ell,\nu}^{-1}}\big[e^{it\sqrt{L}}\beta(\tfrac{r_\circ}{R_{\ell,\nu}}t\sqrt{L})\,D_{R_{\ell,\nu}}g\big]\right|^2 dt\Big)^{\frac12}\Big\|_p \\
&\lesssim \sum_{\nu=1}^{4N} C_0 2^{\ell(\frac12-\alpha)}\|m\|_{L^2_\alpha}\Big(\frac{R_{\ell,\nu}}{r_\circ}\Big)^{\alpha_0-\frac12}\|g\|_p \lesssim C_0 2^{\ell(\alpha_0-\alpha)}\|m\|_{L^2_\alpha}\|f\|_p.
\end{aligned}$$

Here we have used scaling by the dilations $D_{R_{\ell,\nu}}$, the assumption (2.2) for $\lambda = R_{\ell,\nu}/r_\circ \sim 2^\ell$, and finally $\|g\|_p = \|\chi(\sqrt{L})f\|_p \lesssim \|f\|_p$.

Since for real valued functions f we can replace $e^{it\sqrt{L}}f$ with $e^{-it\sqrt{L}}f$ the above also yields $\|u_{\ell,-}(\sqrt{L})\|_{p\to p} \lesssim C_0 2^{\ell(\alpha_0-\alpha)}\|m\|_{L^2_\alpha}$ and the proposition for m supported in $(1-\frac{1}{8N}, 1+\frac{1}{8N})$ follows after summing in ℓ. This support assumption can be removed by decomposing m into a finite sum $\sum_j m_j$ where each multiplier is supported in an interval I_j of $\mathbb{R}\setminus\{0\}$ which is of length at most $\frac{1}{8N}$. Rescaling and observing that $\|m(R\cdot)\|_{L^2_\alpha} \lesssim (1+R)^\alpha\|m\|_{L^2_\alpha}$ for $\alpha \ge 0$, $R \ge 0$, yields the general case of the proposition. Finally, by using the proved result for $m_0(\sqrt{L})$ with $m_0(r) = m(r^2)$, we get the final assertion for $m(L)$. □

Corollary 1.2 and Theorem 1.1 can now be derived from Proposition 2.1 by known arguments, as outlined in the remainder of this section. Thus in this paper we will entirely focus on the proof of Theorem 1.3.

Riesz means. To handle the Bochner-Riesz multiplier $(1-R^{-2}L)^\nu_+$, one decomposes $(1-L)^\nu_+ = m_0(L) + m_1(L)$, where m_0 is C^∞ with compact support, and m_1 is supported in $[\frac12, 1]$ and belongs to L^2_α when $\alpha < \nu + \frac12$. Then $m_0(L)$ is L^p-bounded for $1 \le p \le \infty$ by [25] and $m_1(L)$ is bounded on L^p in the range of p and ν prescribed in Corollary 1.2, and the result for $R > 0$ follows by scaling.

The Riesz multiplier $(1-R^{-1}\sqrt{L})^\nu_+$ can be similarly decomposed. Indeed $\rho \mapsto m_1(\sqrt{\rho})$ belongs to L^2_α when $\alpha < \nu + \frac12$. However, $m_0(\sqrt{L}) = h(L)$, where the mapping $\rho \mapsto h(\rho) := m_0(\sqrt{\rho})$ is no longer smooth at the origin. To address this complication, choose a smooth cut-off function χ_0 which is identically 1 near the origin and such that $m_0 = \chi_0 m_0$. By a Taylor expansion around 0, we may then decompose

$$m_0(\tau) = m_0(0)\chi_0(\tau) + \tau g(\tau),$$

where $g \in C_c^\infty$. Then $h(\rho) = m_0(\sqrt{\rho}) = m_0(0)\widetilde{\chi_0}(\rho) + \psi(\rho)$, where $\widetilde{\chi_0} \in C_c^\infty$ and and $\psi(\rho) = \sqrt{\rho} g(\sqrt{\rho})$. We note that if η is a bump function in $(1/2, 2)$, then $\psi(2^{-\ell}\cdot)\eta$ and its derivatives are $O(2^{-\ell/2})$. We again apply Hulanicki's theorem to see that $m_0(0)\widetilde{\chi_0}(L)$ is L^p-bounded for $1 \le p \le \infty$. Moreover, $\eta(L)\psi(2^{-\ell}L)$ has $L^p \to L^p$ operator norm $O(2^{-\ell/2})$, and by dilation invariance the same holds true for $\eta(2^\ell L)\psi(L)$. Choosing η appropriately we may sum in ℓ to conclude that $\psi(L)$ is bounded on $L^p(G)$ for $1 \le p \le \infty$.

Conditions of Hörmander–Mikhlin type. For multipliers which are not necessarily compactly supported (satisfying the assumptions of Theorem 1.1) one can use ideas in [5] or [51, 52] to combine a Calderón–Zygmund decomposition of $|f|^p$ with a crucial application of Littlewood–Paley theory, together with singular integral estimates in work by Hulanicki and Stein ([16, 25]) to deduce Theorem 1.1 from Proposition 2.1. We shall not give the details. Alternatively one can rely on the implementation of this argument in [11, Thm. I.19] (putting $q = \infty$ in that result).

3. First localizations

We first observe that estimate (1.4) in Theorem 1.3 can easily be reduced to the case where β is supported in $[\frac{1}{2}, 2]$. To this end, assume that β is supported in $[-R, R]$. Choose next $\chi_0, \chi_1 \in C_c^\infty(\mathbb{R})$ such that χ_0 is supported in $[-1, 1]$ and χ_1 in $[\frac{1}{2}, 2]$, and so that

$$\chi_0(\tau) + \sum_{j=0}^{\infty} \chi_1(2^{-j}\tau) = 1$$

for every $\tau \ge 0$, and decompose

$$\beta(\tau/\lambda) = \beta_0(\tau) + \sum_{1 \le 2^j \le 2R\lambda} \beta_j(\tau/2^j),$$

where $\beta_0(\tau) = \beta(\tau/\lambda)\chi_0(\tau)$ and $\beta_j(\tau) = \beta(2^j\tau/\lambda)\chi_1(\tau)$ for $1 \le 2^j \le R\lambda$.

Then

$$\beta(t\sqrt{L}/\lambda) = \psi(t^2 L) + \sum_{1 \le 2^j \le 2R\lambda} \beta_j(t\sqrt{L}/2^j),$$

where $\psi(\rho) = \beta_0(\sqrt{\rho})$.

Arguing in a similar way as in the arguments for the Riesz means by means of (1.3) in [42], one can show that for $1 \le t \le 2$,

$$\|e^{it\sqrt{L}}\beta_0(t\sqrt{L})\|_{L^1 \to L^1} \le C_\beta,$$

with a bound C_β controlled by the C^M-norm of β for some sufficiently large M. Moreover, assuming that estimate (1.4) holds for any β supported in $[\frac{1}{2}, 2]$, with $C_{p,\alpha,\beta}$ depending only on p, α, R and the C^M-norm of β for some sufficiently large M, we find that

$$\Big\| \Big(\int_1^2 \big| e^{it\sqrt{L}} \beta_j(\tfrac{t\sqrt{L}}{2^j}) f \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{H}_n)} \lesssim C_{p,\alpha,\beta}\, 2^{j\alpha} \, \|f\|_{L^p(\mathbb{H}_n)}.$$

Summing over all j such that $1 \le 2^j \le 2R\lambda$, we obtain (1.4).

We next show that by exploiting the finite propagation speed of solutions to the wave equation associated to the sub-Laplacian L, it suffices to prove inequality (1.4) for L^p functions f supported in a small neighborhood of 0.

The following discussions will apply to arbitrary Carnot groups equipped with a left-invariant metric satisfying the local volume growth condition specified below. Suppose that G is a Carnot group, and let $\mathfrak{g}$ be its Lie algebra. As usual, we identify G with its Lie algebra $\mathfrak{g}$ via exponential coordinates, so that the group product is given by the Baker–Campbell–Hausdorff formula. Moreover, let

$$d : G \times G \to \mathbb{R}_{\geq 0}$$

be a left-invariant metric on G, i.e., $d(wz, wz') = d(z, z')$ for all $z, z', w \in G$, and denote by $B(z, r)$ the corresponding ball of radius r centered at z, i.e.,

$$B(z, r) := \{w \in G : d(w, z) < r\}.$$

By $\overline{B}(z, r)$ we denote the corresponding closed ball, and by

$$\widetilde{B(z, r)} := B(z, 2r)$$

the doubling of $B(z, r)$. In addition to the Carnot group structure, we assume that there exists a "local homogeneous dimension" $D > 0$ such that

$$|B(z, r)| \sim r^D \qquad \text{for all } \ r \in (0, r_0], \tag{3.1}$$

where $r_0 \gg 1$ is a fixed constant (note also that $|B(z, r)| = |B(0, r)|$).

Examples. (a) Assume that $X_1, \ldots, X_d$ is a basis of the Lie algebra $\mathfrak{g}$ so that its first, horizontal layer $\mathfrak{g}_1$ is spanned by $X_1, \ldots, X_{d_1}$, and let $L = -\sum_{k=1}^{d_1} X_k^2$ be the associated sub-Laplacian (here, we identify as usually any vector $X \in \mathfrak{g}$ with the corresponding Lie derivative). For further background, we refer, e.g., to [33]. We then denote by d_{CC} the (left-invariant) Carnot–Carathéodory or sub-Riemannian distance on G associated to L. We recall that for this metric $d = d_{\mathrm{CC}}$, the volume of $B(z, r)$ is given by

$$|B(z, r)| = |B(0, r)| = |B(0, 1)|\, r^Q, \tag{3.2}$$

where $Q \geq d$ is the homogeneous dimension of G.

(b) Let d_G denote the left-invariant Riemannian metric defined by assuming that the left-invariant vector fields $X_1, \ldots, X_d$ in Example a) form an orthonormal basis of the tangent space at any point of G. Then d_G is just the sub-Riemannian metric associated to the full Laplacian $-\sum_{k=1}^{d} X_k^2$, so that $d_G \leq d_{\mathrm{CC}}$. It is easily seen that the local homogeneous dimension is here given by d.

We finally assume that X is a complex Banach space, endowed with the norm $\|\cdot\|_X$.

We begin with a general localization result.

Proposition 3.1. *(i) Fix $r_0 \gg 1$, and let $K \in L^\infty(G, X)$ be supported in a ball $\overline{B}(u_0, r)$, where u_0 lies in the center of G and $0 < r \leq r_0$, and denote by T_K the convolution operator $T_K f = f * K$, $f \in \mathcal{S}(G)$. Moreover, let $0 < \varepsilon < 1$ and $1 \leq p \leq \infty$, and assume that*

$$\|T_K f\|_{L^p(G,X)} \leq A\, \|f\|_{L^p(G)} \tag{3.3}$$

for all $f \in L^p(G)$ which are supported in $\overline{B}(0, \varepsilon r)$.

Then there is a constant C_ε which, for a given group G and metric d, depends only on ε, such that

$$\|T_K f\|_{L^p(G,X)} \leq C_\varepsilon^{1/p'} A\, \|f\|_{L^p(G)} \tag{3.4}$$

for all $f \in L^p(G)$.

(ii) More generally, if $\alpha \in \mathrm{Aut}(G)$ *is any automorphism of the Lie group* G*, then the following analogous statement holds true:*

If $\operatorname{supp} K \subset \alpha(\overline{B}(u_0, r))$*, and if* (3.3) *holds for all* $f \in L^p(G)$ *which are supported in* $\alpha(\overline{B}(0, \varepsilon r))$*, then* (3.4) *holds for all* $f \in L^p(G)$.

Proof. By Zorn's lemma, we can choose a maximal family of pairwise disjoint $\varepsilon r/2$-balls $B_i = B(z_i, \varepsilon r/2), i \in I$, in G. Then I is countable. Indeed, given any $R > 1$, if I_R denotes the set of indices $i \in I$ such that $z_i \in B(0, R)$, then

$$B_i \subset \widetilde{B(0,R)}, \quad \text{hence} \quad \sum_{i \in I_R} |B_i| \le |\widetilde{B(0,R)}|.$$

Thus, by (3.1), $(\sharp I_R)\, |B(0, \varepsilon r/2)| \le |B(0, R)|$, which shows that I_R is finite.

We may thus assume that $I = \mathbb{N}$. Moreover, the maximality of the family $(B_i)_{i \in \mathbb{N}}$ implies that the family of doublings $(\widetilde{B_i})_{i \in \mathbb{N}}$ covers G. We can therefore decompose G into the pairwise disjoint sets $D_i \subset B_i$ defined recursively by $D_0 := \widetilde{B_0}$ and $D_{i+1} := \widetilde{B_{i+1}} \setminus (\widetilde{B_0} \cup \cdots \cup \widetilde{B_i})$.

Given $f \in L^p(G)$, we may accordingly decompose $f = \sum_{i \in \mathbb{N}} f_i$, where $f_i := f \mathbb{1}_{D_i}$ satisfies $f|_{\widetilde{B_i}} = 0$.

Next, note that for any fixed $z \in G$, $q_z(f) := \|T_K f(z)\|_X$ defines a continuous semi-norm q_z on $L^p(G)$, and thus

$$q_z(f) \le \sum_{i \in \mathbb{N}} q_z(f_i).$$

Moreover,

$$T_K f_i(z) = \int_{\overline{B}(z_i, \varepsilon r)} f_i(w) K(w^{-1} z)\, dw \neq 0$$

only if there is some $w \in \overline{B}(z_i, \varepsilon r)$ such that $w^{-1} z \in \overline{B}(u_0, r)$. Then $z = w u_0 y$ for some $y \in \overline{B}(0, r)$. Noting also that $u_0^{-1} z = wy$, since u_0 is central, this implies that

$$\begin{aligned} d(z_i, u_0^{-1} z) &\le d(z_i, w) + d(w, u_0^{-1} z) = d(z_i, w) + d(w, wy) \\ &= d(z_i, w) + d(0, y) \le \varepsilon r + r < 2r. \end{aligned}$$

This implies that $\widetilde{B_i} \subset B(u_0^{-1} z, 3r)$. Thus, if I_z denotes the set of all $i \in \mathbb{N}$ such that $q_z(f_i) \neq 0$, then $\sum_{i \in I_z} |B_i| \le |B(u_0^{-1} z, 3r)| = |B(0, 3r)|$, and arguing in a similar way as before and making use of (3.1), we see that

$$\sharp I_z \le |B(0, 3r)| / |B(0, \varepsilon r/2)| \le C r^D / (\varepsilon r)^D = C \varepsilon^{-D} =: C_\varepsilon.$$

If $p < \infty$, then by Hölder's inequality, we thus get

$$q_z(f) \le \sum_{i \in I_z} q_z(f_i) \le C_\varepsilon^{1/p'} \big(\sum_{i \in \mathbb{N}} q_z(f_i)^p \big)^{1/p},$$

hence

$$\|T_K f\|_{L^p(G,X)} \le C_\varepsilon^{1/p'} \big(\sum_{i \in \mathbb{N}} \|T_K f_i\|_{L^p(G,X)}^p \big)^{1/p}.$$

But, f_i is supported in $\overline{B}(z_i, \varepsilon r)$, and since T_K is left-invariant, by means of the left-translation by z_i^{-1}, our assumption (3.3) implies that $\|T_K f_i\|_{L^p(G,X)} \le A \|f_i\|_{L^p(G)}$,

and thus

$$\|T_K f\|_{L^p(G,X)} \le C_\varepsilon^{1/p'} A \big(\sum_{i\in\mathbb{N}} \|f_i\|_{L^p(G)}^p \big)^{1/p} = C_\varepsilon^{1/p'} A \, \|f\|_{L^p(G)},$$

since the f_i's have disjoint supports. This proves (i) for $p < \infty$, and the proof for $p = \infty$ requires only the usual modifications.

To prove (ii), let $\alpha \in \mathrm{Aut}(G)$, and denote by $\alpha_* = D\alpha|_0 \in \mathrm{Aut}(\mathfrak{g})$ the associated automorphism of the Lie algebra of G (see, for example, [22]).

Then we recall that the Haar measure dw, which agrees with the Lebesgue measure on $\mathfrak{g}$, transforms under α as $d(\alpha(w)) = |\det \alpha_*| \, dw$. Thus, if we put $K_\alpha(z) = |\det \alpha_*| \, K(\alpha(z))$, then straight-forward computations show that

$$T_{K_\alpha}(f \circ \alpha) = (T_K f) \circ \alpha. \tag{3.5}$$

Moreover, if $\operatorname{supp} K \subset \alpha(\overline{B}(u_0, r))$, then $\operatorname{supp} K_\alpha \subset \overline{B}(u_0, r)$. Thus, if (3.3) holds for all $f \in L^p(G)$ which are supported in $\alpha(\overline{B}(0, \varepsilon r))$, then (3.5) implies that (3.3) holds for $T_{K\alpha}$, for all $f \in L^p(G)$ which are supported in $\overline{B}(0, \varepsilon r)$, and hence by part (i) the global estimate (3.4) will hold for $T_{K\alpha}$. Again, by (3.5), then the estimate (3.4) will also hold for T_K. □

Then we can exploit [34, Proposition 3.1] to prove the following lemma, in which we choose as metric for simplicity the Carnot–Carathéodory distance d_{CC} (the metric d_G would work as well).

Lemma 3.2. *Let $0 < \varepsilon < 1$, $p \ge 2$ and $\alpha > 0$, and suppose that $\beta \in C_{\mathrm{c}}^\infty$ is supported in $[\frac{1}{2}, 2]$. Assume that for all $\lambda \ge 1$ the following variant*

$$\Big\| \Big(\int_1^2 \big| \mathbb{1}_{\overline{B}(0,5)} e^{it\sqrt{L}} \beta(\tfrac{t\sqrt{L}}{\lambda}) f \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \le C_{p,\alpha,\beta} \lambda^\alpha \|f\|_p \tag{3.6}$$

of (1.4) holds true for all functions $f \in L^p(G)$ supported in the closed ball $\overline{B}(0, \varepsilon)$, where the constant $C_{p,\alpha,\beta}$ depends in β only on the C^M-norm of β for some sufficiently large $M \in \mathbb{N}$. Then, for all $f \in L^p(G)$ and $\lambda \ge 1$,

$$\Big\| \Big(\int_1^2 |e^{it\sqrt{L}} \beta(\tfrac{t\sqrt{L}}{\lambda}) f|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \le C_{\varepsilon,p,\alpha,\beta} \lambda^\alpha \|f\|_p, \tag{3.7}$$

where also the constant $C_{\varepsilon,p,\alpha,\beta}$ depends in β only on the C^M-norm of β for some sufficiently large $M \in \mathbb{N}$.

Proof. Denote by T_t the operator $e^{it\sqrt{L}} \beta(\frac{t\sqrt{L}}{\lambda})$. By Hulanicki's theorem, its convolution kernel $k_t = T_t \delta_0$ is a Schwartz function, so that $T_t f$ is smooth for any $f \in L^p$. Moreover, from Proposition 3.1 in [34], whose proof is based on the finite propagation speed for solutions of the wave equation (see [36]), it follows that

$$k_t = \mathbb{1}_{\overline{B}(0,4)} k_t + R_{t,\lambda} \qquad \text{for } 1 \le t \le 2,$$

where

$$\|R_{t,\lambda}\|_1 \lesssim_N \lambda^{-N} \qquad \text{for every } N \in \mathbb{N}. \tag{3.8}$$

We decompose $T_t = T_t^1 + T_t^2$, where $T_t^1 f = f * (\mathbb{1}_{\overline{B}(0,4)} k_t)$ and $T_t^2 f = f * R_{t,\lambda}$. By means of Minkowski's inequality and (3.8), we may bound

$$\Big\| \Big(\int_1^2 |T_t^2 f|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \lesssim_N \lambda^{-N} \|f\|_p \tag{3.9}$$

for every $N \in \mathbb{N}$.

As for T^1_t, let us put $X = L^2([1,2])$ and $K(z)(t) = (\mathbb{1}_{\overline{B}(0,4)} k_t)(z)$ for $z \in G$ and $t \in [1,2]$, so that K is a convolution kernel taking values in X. Then, with the notation of Proposition 3.1, $T^1_t f(z) = (f * K)(z)(t)$, so that

$$\Big(\int_1^2 |T^1_t f(z)|^2 \, dt \Big)^{\frac{1}{2}} = \|T_K f(z)\|_X.$$

Moreover, since for f supported in $B(0,\varepsilon)$, the function $f*K$ is supported in $\overline{B}(0,5)$, and since $T^1_t = T_t - T^2_t$, we see that the operator T_K satisfies the assumptions in Proposition 3.1, with $r = 4$ and $u_0 = 0$. By the proposition, this implies that, for all $f \in L^p(G)$,

$$\Big\| \Big(\int_1^2 |T^1_t f|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \le C_{\varepsilon,p,\alpha,\beta} \lambda^\alpha \|f\|_p.$$

In combination with (3.9) this implies (3.7). □

4. Preliminaries on the wave operator on $\mathbb{H}_n$ and a spectral decomposition

Recall that the Lie algebra $\mathfrak{g}$ of $\mathbb{H}_n$ admits the stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the first, horizontal layer $\mathfrak{g}_1$ is spanned by the vector fields $X_1, \dots, X_{2n}$, and the second, central layer $\mathfrak{g}_2$ by U. Then $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$, where $\mathfrak{g}_1$ has dimension $d_1 = 2n$ and $\mathfrak{g}_2$ has dimension $d_2 = 1$. If one defines dilations $D_r, r > 0$, on $\mathfrak{g}$ by setting $D_r|_{\mathfrak{g}_j} = r^j \, \mathrm{id}_{\mathfrak{g}_j}, j = 1, 2$, and correspondingly on $\mathbb{H}_n$ by setting $D_r(x,u) := (rx, r^2 u), r > 0$, then these dilations are automorphisms of the Heisenberg group.

The sub-Laplacian L on $\mathbb{H}_n$ is then homogeneous of degree 2 under these dilations, i.e., $L(f \circ D_r) = r^2 (Lf) \circ D_r$ for any $f \in \mathcal{S}(\mathbb{H}_n)$. This implies that

$$m(L)(f \circ D_r) = (m(r^2 L) f) \circ D_r. \tag{4.1}$$

If $k_{m(L)} = m(L)\delta_0$ denotes the convolution kernel of $m(L)$ as in (2.1), so that $m(L)f = f * k_{m(L)}$, then (4.1) easily implies that

$$k_{m(r^2 L)} = r^{-Q} k_{m(L)} \circ D_{r^{-1}}, \qquad r > 0, \tag{4.2}$$

where $Q = 2n + 2$ denotes the homogeneous dimension of $\mathbb{H}_n$.

4.1. The singular support and horizontal points.

Nachman [44] computed the singular support Σ_1 of the convolution kernel $\mathcal{K}_1$ of $e^{i\sqrt{L}}$ and established (non-uniform) asymptotics at generic points of Σ_1. In particular he found

$$\Sigma_1 = \{(x,u) : |x| = \rho(\tau) \text{ and } |u| = \tfrac{1}{4}\nu(\tau) \text{ for some } \tau \ge 0\}, \tag{4.3}$$

where the functions ρ and ν are given by

$$\rho(\tau) = \Big| \frac{\sin \tau}{\tau} \Big|, \qquad \nu(\tau) = \frac{1}{\tau} - \frac{\sin(2\tau)}{2\tau^2},$$

where the values at $\tau = 0$ are defined by continuous extension, so that $\rho(0) = 1$ and $\nu(0) = 0$.

The singular support Σ_1 is symmetric under rotations in $\mathbb{R}^{2n} \times \{0\}$, reflections in $\{0\} \times \mathbb{R}$, and in the $(|x|, |u|)$-plane it corresponds to the zigzag curve approaching the origin as illustrated in Figure 1 in the introduction. As pointed out in the

introduction the parameters where Σ_1 has horizontal tangent planes play a special role in our analysis. For $k \in \mathbb{N}$ and $k\pi < \tau < (k+1)\pi$, we get

$$\rho'(\tau) = (-1)^k \frac{\tau\cos\tau - \sin\tau}{\tau^2}, \qquad \nu'(\tau) = -2\frac{\cos\tau}{\tau^3}\big(\tau\cos\tau - \sin\tau\big).$$

Hence the slope is given by $\nu'(\tau)/\rho'(\tau) = (-1)^{k+1}2\frac{\cos\tau}{\tau}$, and it vanishes if and only if

$$\tau = \tau_k := k\pi + \tfrac{\pi}{2}. \tag{4.4}$$

This shows that the points with horizontal tangent plane are given by $|x| = \rho(\tau_k)$ and $|u| = \frac{1}{4}\nu(\tau_k)$, for some $k \in \mathbb{N}$ (compare Figure 3).

The paper [41] by the first and third author provides a parametrix for the wave operator by representing $e^{i\sqrt{L}}$ as a superposition of oscillatory integral operators which will be described in the next subsection. An important tool is a subordination formula which expresses wave propagators in terms of the Schrödinger propagators $e^{itL}, t > 0$, for which one has explicit expressions, up to the Fourier transform in the central variable u (cf. [18, 40]). As a consequence this allowed the authors to write down a decomposition of $\mathcal{K}_1$ and give an effective analysis for the pieces associated with the zig-zag curves in Figure 1.

In particular, for $\alpha(p) = (d-1)|\frac{1}{p} - \frac{1}{2}|$ this led to sharp fixed time $L^p \to L^p$ bounds for the operators $e^{it\sqrt{L}}(I+t^2L)^{-\alpha(p)/2}$, and for the variants $\beta(\lambda^{-1}t\sqrt{L})e^{it\sqrt{L}}$ considered here (first, for $t = 1$, and then for arbitrary t because of (4.1)).

In the next subsection, we also recall preliminaries from the joint spectral calculus of the operators L and $-iU$ which are needed to introduce this decomposition.

4.2. **Spectral calculus and a decomposition of the wave operator.** For $f \in \mathcal{S}(\mathbb{H}_n)$ and $\mu \in \mathbb{R}$, let f^μ denote the μ-section of the partial Fourier transform along the central layer $\mathfrak{g}_2$ given by

$$f^\mu(x) = \int_{\mathbb{R}} f(x,u)e^{-2\pi i\mu u}\,du, \quad x \in \mathbb{R}^{d_1} = \mathbb{R}^{2n}.$$

We extend this notation to tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. By this partial Fourier transform, the vector fields X_j are transformed into some first order differential operators X_j^μ such that $(X_jf)^\mu = X_j^\mu f^\mu$, and the sub-Laplacian L is transformed into the μ-twisted Laplacian L^μ such that

$$(Lf)^\mu = L^\mu f^\mu = -\big((X_1^\mu)^2 + \cdots + (X_{d_1}^\mu)^2\big)f^\mu.$$

The partial Fourier transform and the functional calculi of L and L^μ are compatible, that is, for any bounded Borel measurable $m : \mathbb{R} \to \mathbb{C}$ and any function $f \in L^2(\mathbb{H}_n)$,

$$(m(L)f)^\mu = m(L^\mu)f^\mu$$

for almost all $\mu \in \mathbb{R}$. Moreover, the group convolution of two suitable functions f, g on $\mathbb{H}_n$ is transformed into

$$(f * g)^\mu = f^\mu *_\mu g^\mu,$$

where $*_\mu$ is the μ-twisted convolution on $\mathbb{R}^{d_1}$ given by

$$\phi *_\mu \psi(x) = \int_{\mathbb{R}^{d_1}} \phi(x')\,\psi(x-x')\,e^{i\pi\mu\langle Jx,x'\rangle}\,dx', \quad x \in \mathbb{R}^{d_1} = \mathbb{R}^{2n}.$$

Thus, at least formally, we have

$$e^{itL^\mu} f^\mu = (e^{itL} f)^\mu = (f * (e^{itL}\delta_0))^\mu = f^\mu *_\mu (e^{itL}\delta_0)^\mu.$$

The partial Fourier transform $(e^{itL}\delta_0)^\mu = e^{itL^\mu}\delta_0$ of the Schrödinger kernel $e^{itL}\delta_0$ can be computed explicitly, see [40]: For $f \in \mathcal{S}(\mathbb{R}^{2n})$, there is a distribution $\gamma_t^\mu \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$e^{itL^\mu} f = f *_\mu \gamma_t^\mu. \tag{4.5}$$

Moreover, for all $t \in \mathbb{R}$ and $\mu \in \mathbb{R}$ such that $\sin(2\pi t\mu) \neq 0$, γ_t^μ is given by

$$\gamma_t^\mu(x) = \Big(\frac{\mu}{2\sin(2\pi t\mu)}\Big)^n e^{-i\frac{\pi}{2}\mu\cot(2\pi t\mu)|x|^2}, \quad x \in \mathbb{R}^{2n}. \tag{4.6}$$

This can also be found in [41, Proposition 3.2], together with an alternative description near $\{\mu : \sin(2\pi t\mu) = 0\}$.

The following subordination result is an immediate consequence of [41, Proposition 3.2]. Note that in [41] only the case $t = 1$ is considered, but the identity (4.7) is a consequence of a subordination formula for multipliers and thus holds when we plug in $t^2 L$ in place of L as well. Moreover, the first identity in (4.9) follows from (4.1).

Lemma 4.1. *([41, Proposition 4.1]) Let $\beta, \tilde\beta \in C_c^\infty((0,\infty))$ with $\operatorname{supp}\beta \subseteq (\frac12, 2)$ and $\tilde\beta|_{[\frac14,4]} = 1$. Then there are functions $a_{\lambda,\circ}, \rho_\lambda \in C_c^\infty(\mathbb{R})$ supported in $[\frac{1}{16}, 4]$ and $[\frac14, 4]$, respectively, such that*

$$\beta\big(\tfrac{t\sqrt{L}}{\lambda}\big) e^{it\sqrt{L}} = \tilde\beta\big(\tfrac{t^2 L}{\lambda^2}\big)\, T_{\lambda,t} + \rho_\lambda\big(\tfrac{t^2 L}{\lambda^2}\big) \tag{4.7}$$

for all $\lambda \geq 1$ and $t \in (0,\infty)$, where for all $\lambda \geq 1$,

$$T_{\lambda,t} = \lambda^{\frac12} \int_0^\infty e^{i\frac{\lambda}{4\varsigma}}\, a_{\lambda,\circ}(\varsigma)\, e^{i\varsigma t^2 L/\lambda}\, d\varsigma, \tag{4.8}$$

$$\|\rho_\lambda\big(\tfrac{t^2 L}{\lambda^2}\big)\|_{p\to p} = \|\rho_\lambda\big(\tfrac{L}{\lambda^2}\big)\|_{p\to p} \lesssim_N \lambda^{-N}, \tag{4.9}$$

and,[1] for all $N_1, N_2 \in \mathbb{N}$, and all $\lambda \geq 1$,

$$\sup_{\varsigma\in[\frac{1}{16},4]} \big|\partial_\varsigma^{N_1}\partial_\lambda^{N_2} a_{\lambda,\circ}(\varsigma)\big| \leq C(N_1,N_2)\, \|\beta\|_{C^{2(N_1+N_2+1)}}. \tag{4.10}$$

Next, we recall that the self-adjoint operators L and $-iU$ commute and thus admit a joint functional calculus. For any Borel measurable function $m : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, we can define the (possibly unbounded) operator $m(L, -iU)$ on $L^2(\mathbb{H}_n)$. The partial Fourier transform and the joint functional calculus of L and $-iU$ are compatible, that is, if m is a bounded Borel function, then, for all $f \in L^2(\mathbb{H}_n)$ and almost all $\mu \in \mathfrak{g}_2^*$,

$$(m(L,-iU)f)^\mu = m(L^\mu, 2\pi\mu) f^\mu.$$

Note that since also $-iU$ is homogeneous of degree 2 with respect to the automorphic dilations D_r, the identity (4.1) generalizes naturally to the following one for the joint spectral calculus:

$$m(L,-iU)(f\circ D_t) = (m(t^2L, t^2(-iU))f)\circ D_t, \qquad t > 0. \tag{4.11}$$

[1] There is a typo in [41, Prop 4.1] where in (33) a decay of order λ^{-N_2} is erroneously suggested (but never used).

We use the joint functional calculus to introduce a decomposition of the operator $T_{\lambda,t}$ in (4.8) in the central Fourier variable μ, which is motivated by the periodicity properties of the trigonometric functions appearing in (4.6). To this end, choose an even cut-off $\eta_0 \in C_c^\infty(\mathbb{R})$ supported in $(-\frac{5\pi}{8}, \frac{5\pi}{8})$ with $\eta_0(\tau) = 1$ for $\tau \in (-\frac{3\pi}{8}, \frac{3\pi}{8})$ such that

$$\sum_{k=-\infty}^{\infty} \eta_0(\tau - k\pi) = 1 \quad \text{for all } \tau \in \mathbb{R},$$

and define

$$T_{\lambda,t}^k = \lambda^{\frac{1}{2}} \int_0^\infty e^{i\frac{\lambda}{4\varsigma}}\, a_{\lambda,\circ}(\varsigma)\, \eta_0\Big(\frac{\varsigma t^2}{\lambda}(-iU) - k\pi\Big)\, e^{i\varsigma t^2 L/\lambda}\, d\varsigma \tag{4.12}$$

(note here that for $t = 1$ this corresponds to the according definition in (55) in [41], and for arbitrary $t > 0$ this is motivated by (4.11)).

The operator $T_{\lambda,t}$ then decomposes (compare (54) and (57) in [41]) at least formally as

$$T_{\lambda,t} = \sum_{k\in\mathbb{Z}} T_{\lambda,t}^k.$$

Each operator $T_{\lambda,t}^k$ is a convolution operator $T_{\lambda,t}^k f = f * \mathcal{K}_{\lambda,t}^k$, and by (4.6) (compare also [41, (57)][2] for the case $t = 1$), its convolution kernel is given by

$$\begin{aligned}\mathcal{K}_{\lambda,t}^k(x,u) = \lambda^{\frac{1}{2}} \int_0^\infty \int_{\mathbb{R}} e^{i\frac{\lambda}{4\varsigma}}\, a_{\lambda,\circ}(\varsigma)\, \eta_0\big(2\pi\tfrac{\varsigma t^2}{\lambda}\mu - k\pi\big)\, \Big(\frac{\mu}{2\sin(2\pi\frac{\varsigma t^2}{\lambda}\mu)}\Big)^n \\ \times\, e^{-i\frac{\pi}{2}\mu|x|^2\cot(2\pi\frac{\varsigma t^2}{\lambda}\mu)}\, e^{2\pi i\mu u}\, d\mu\, d\varsigma.\end{aligned} \tag{4.13}$$

The concrete description of the joint spectrum of L and $-iU$ also shows that for fixed λ we have

$$\widetilde{\beta}\big(\tfrac{t^2 L}{\lambda^2}\big)\, T_{\lambda,t}^k = 0 \text{ for } |k| \geq 8\lambda \tag{4.14}$$

(for $t = 1$ see [41, (56)] and the discussion leading to it, and for general $t > 0$ this follows then by (4.11)). Thus we only need to consider $T_{\lambda,t}^k$ for $|k| < 8\lambda$.

The analysis in [41] (using the previous formula for $\mathcal{K}_{\lambda,t}^k$) links the singular support of $\mathcal{K}_{\lambda,t}^0$ with the outer part in Figure 1, and for $k > 0$ the singular support of $\mathcal{K}_{\lambda,t}^{\pm k}$ with the kth zigzag curve in Figure 1. The kernels $\mathcal{K}_{\lambda,t}^k$ become more singular and less oscillatory near the vertical axis which necessitated another decomposition in [41]: With η_0 as above and for $\ell \geq 1$, let $\eta_\ell \in C_c^\infty(\mathbb{R})$ be given by

$$\eta_\ell(\tau) = \eta_0(2^{\ell-1}\tau) - \eta_0(2^\ell\tau), \qquad \tau \in \mathbb{R}. \tag{4.15}$$

Then $\eta_0(\tau) = \sum_{\ell=1}^\infty \eta_\ell(\tau)$ for all $\tau \in \mathbb{R}\setminus\{0\}$, and for $k \neq 0$ we define for any $\ell \geq 1$

$$T_{\lambda,t}^{k,\ell} = \lambda^{\frac{1}{2}} \int_0^\infty e^{i\frac{\lambda}{4\varsigma}}\, a_{\lambda,\circ}(\varsigma)\, \eta_\ell\Big(\frac{\varsigma t^2}{\lambda}(-iU) - k\pi\Big)\, e^{i\varsigma t^2 L/\lambda}\, d\varsigma. \tag{4.16}$$

[2]Note that in contrast to [41, (57)], where each factor except for the last depends only on $|\mu|$, here we must allow for negative $k \in \mathbb{Z}$, and correspondingly replace each instance of $|\mu|$ by μ.

This defines a convolution operator $f \mapsto T^{k,\ell}_{\lambda,t} f = f * \mathcal{K}^{k,\ell}_{\lambda,t}$, whose convolution kernel is given by

$$\mathcal{K}^{k,\ell}_{\lambda,t}(x,u) = \lambda^{\frac{1}{2}} \int_0^\infty \int_{\mathbb{R}} e^{i\frac{\lambda}{4\varsigma}}\, a_{\lambda,\circ}(\varsigma)\, \eta_\ell\big(2\pi \tfrac{\varsigma t^2}{\lambda}\mu - k\pi\big) \Big(\frac{\mu}{2\sin(2\pi\frac{\varsigma t^2}{\lambda}\mu)}\Big)^n \times e^{-i\frac{\pi}{2}\mu|x|^2\cot(2\pi\frac{\varsigma t^2}{\lambda}\mu)}\, e^{2\pi i\mu u}\, d\mu\, d\varsigma, \tag{4.17}$$

and we get the decomposition $T^k_{\lambda,t} f = \sum_{\ell=1}^\infty T^{k,\ell}_{\lambda,t} f$, with convergence at least in the sense of distributions (indeed by [41] also in L^1).

To summarize, we have decomposed

$$\beta\big(\tfrac{t\sqrt{L}}{\lambda}\big) e^{it\sqrt{L}} f = \widetilde{\beta}\big(\tfrac{t^2 L}{\lambda^2}\big)\, T^0_{\lambda,t} f + \sum_{0<|k|<8\lambda} \sum_{\ell\geq 1} \widetilde{\beta}\big(\tfrac{t^2 L}{\lambda^2}\big)\, T^{k,\ell}_{\lambda,t} f + \rho_\lambda\big(\tfrac{t^2 L}{\lambda^2}\big) f, \tag{4.18}$$

with $T^0_{\lambda,t} f = f * \mathcal{K}^0_{\lambda,t}$ and $T^{k,\ell}_{\lambda,t} f = f * \mathcal{K}^{k,\ell}_{\lambda,t}$.

Here, by (4.9), the error term $\rho_\lambda\big(\frac{t^2 L}{\lambda^2}\big)$ can be ignored in the sequel.

Finally, the following proposition provides concrete representations of the convolution kernels $\mathcal{K}^0_{\lambda,t}$ and $\mathcal{K}^{k,\ell}_{\lambda,t}$. Let

$$\Phi(x,u,t,\sigma,\tau) = \sigma(t^2 - |x|^2 g(\tau) + 4u\tau), \text{ with } g(\tau) = \tau\cot\tau, \tag{4.19}$$

and

$$a_\lambda(\sigma) = (2\pi^{n+1})^{-1}\, a_{\lambda,\circ}(\tfrac{1}{4\sigma})\, \sigma^{n-1}. \tag{4.20}$$

Note that a_λ is a C^∞_{c} function supported in $(\frac{1}{16}, 4)$.

Proposition 4.2. *Let Φ and a_λ be as in (4.19) and (4.20). Then for $k = 0$*

$$\mathcal{K}^0_{\lambda,t}(x,u) = \lambda^{n+\frac{3}{2}} t^{-2n} \int_{\mathbb{R}} \int_0^\infty e^{i\lambda\Phi(x,u,t,\sigma,\tau)}\, a_\lambda(t^2\sigma)\, \eta_0(\tau) \Big(\frac{\tau}{\sin\tau}\Big)^n d\sigma\, d\tau, \tag{4.21}$$

and for $k \neq 0$, $\ell \geq 1$

$$\mathcal{K}^{k,\ell}_{\lambda,t}(x,u) = \lambda^{n+\frac{3}{2}} t^{-2n} \int_{\mathbb{R}} \int_0^\infty e^{i\lambda\Phi(x,u,t,\sigma,\tau)}\, a_\lambda(t^2\sigma)\, \eta_\ell(\tau - k\pi) \Big(\frac{\tau}{\sin\tau}\Big)^n d\sigma\, d\tau. \tag{4.22}$$

Proof. To prove (4.21), we change variables in (4.13) from ς to σ via $1/(4\varsigma) = t^2\sigma$, so that $\varsigma = 1/(4t^2\sigma)$, and then from μ to τ by setting $2\pi\varsigma t^2\mu/\lambda = \tau$, so that $\mu = 2\lambda\sigma\tau/\pi$. The Jacobian of the total change of variables is $-\lambda/(2\pi\sigma t^2)$, and identity (4.21) follows by straight-forward computations from (4.13). (4.22) is derived in the same way from (4.17). □

Remark 4.3. In the following sections, we will typically suppress the case $k < 0$ and state estimates only for $k > 0$. The corresponding bounds for negative k can be recovered directly from the positive case. Indeed, since χ_0 was chosen to be even, each η_ℓ is even as well. Thus, replacing τ by $-\tau$ in (4.22) yields

$$\mathcal{K}^{-k,\ell}_{\lambda,t}(x,u) = \mathcal{K}^{k,\ell}_{\lambda,t}(x,-u) \quad \text{for } k > 0.$$

Moreover, the involution $\theta(x,u) := (x_1, -x_2, \dots, x_{2n-1}, -x_{2n}, -u)$ is an automorphism of $\mathbb{H}_n$, and (4.19) is radially symmetric in the x variable. In combination, this implies that

$$T^{-k,\ell}_{\lambda,t} f = \big(T^{k,\ell}_{\lambda,t}(f\circ\theta)\big)\circ\theta.$$

5. The case $k = 0$: Preliminaries

We begin by stating a localization result for $\mathbb{H}_n$ which is an almost immediate consequence of Lemma 3.2, since the Euclidean topology and the sub-Riemannian topology on $\mathbb{H}_n$ are the same, and since the Euclidean distance is even bounded by the sub-Riemannian distance. The Euclidean ball of radius r centered at z will be denoted by $B_r(z)$, i.e.,

$$B_r(z) = \{w \in \mathbb{H}_n : |w - z| < r\}$$

where $|\cdot|$ denotes the Euclidean norm.

We introduce a smooth bump function χ supported in the Euclidean ball $B_1(0) \subset \mathbb{H}_n$ such that $\chi \equiv 1$ on $B_{\frac{1}{2}}(0)$, and put $\chi_\varepsilon(w) := \chi(w/\varepsilon)$ for $\varepsilon > 0$.

Proposition 5.1. *Let $0 < \varepsilon < 1$, $p \geq 2$, $\alpha > 0$, and let β be a smooth bump function supported in $[\frac{1}{2}, 2]$. Suppose that for every $(x^0, u^0) \in \mathbb{H}_n$ with $x^0 \in \mathbb{R}^{2n}$ of the form $x^0 = (|x^0|, 0, \dots, 0)$, with $|x^0| < 10$ and every $f \in \mathcal{S}(\mathbb{H}_n)$, we have the estimate*

$$\Big\| \Big(\int_1^2 \Big| \tilde{\chi}_\varepsilon \, e^{it\sqrt{L}} \, \beta\Big(\frac{t\sqrt{L}}{\lambda}\Big) \, (\chi_\varepsilon f) \Big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \leq C(\varepsilon, \alpha, p) \lambda^\alpha \, \|f\|_2, \tag{5.1}$$

with $\tilde{\chi}_\varepsilon(x, u) := \chi_\varepsilon(x - x^0, u - u^0)$. Then the square function estimate (3.6) holds.

Proof. Choose $\varepsilon' > 0$ so small that $B(0, \varepsilon') \subset B_\varepsilon(0)$. Then, for f supported in $B(0, \varepsilon')$, $\|f\|_2 \leq C_{\varepsilon'} \|f\|_p$ by Hölder's inequality, and thus (5.1) implies (3.6), with ε' in place of ε.

Finally, it suffices to consider only $x^0 = (|x^0|, 0, \dots, 0)$ because the unitary group $U(n)$, which acts transitively on $\mathbb{C}^n$, acts by automorphisms of $\mathbb{H}_n$ that leave the center fixed. The sub-Laplacian L is invariant under this action, and hence so is $\beta(\frac{t\sqrt{L}}{\lambda})$. □

We will exploit this for $k = 0$ from the decomposition of Proposition 4.2.

Now let $\varepsilon \in (0, 10^{-4})$ and (x^0, u^0) with $|(x^0, u^0)| < 10$ and $x^0 = (|x^0|, 0, \dots, 0)$ be fixed, and define $\chi_\varepsilon, \tilde{\chi}_\varepsilon$ as above. The following proposition treats the case $k = 0$.

Proposition 5.2. *If $p \geq \frac{2(d_1+1)}{d_1-1}$ and $\alpha \geq d\,(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$, then for $\varepsilon > 0$ sufficiently small,*

$$\Big\| \Big(\int_1^2 \big| \tilde{\chi}_\varepsilon \, T^0_{\lambda,t}(\chi_\varepsilon f) \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \leq C(a_\lambda, \varepsilon, \alpha, p) \lambda^\alpha \, \|f\|_2 \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d), \tag{5.2}$$

where the constant $C(a_\lambda, \varepsilon, \alpha, p)$ depends in a_λ only on the C^M-norm of a_λ for some sufficiently large $M \in \mathbb{N}$.

To prepare the proof of Proposition 5.2, recall from Lemma 4.2 that the convolution kernel $\mathcal{K}^0_{\lambda,s}$ of $T^0_{\lambda,t}$ is given by

$$\mathcal{K}^0_{\lambda,t}(x, u) = \lambda^{n+\frac{3}{2}} t^{-2n} \int_{\mathbb{R}} \int_0^\infty e^{i\lambda\Phi(x,u,t,\sigma,\tau)} \, a_\lambda(t^2\sigma) \, \eta_0(\tau) \, \Big(\frac{\tau}{\sin\tau}\Big)^n d\sigma \, d\tau.$$

Since the factor t^{-2n} is irrelevant for the estimate (5.2), we drop it in the sequel. Moreover, it will be convenient to replace t by

$$s = t^2 \in [1, 4].$$

Moreover, since $\operatorname{supp}\eta_0 \subset [-\frac{5}{8}\pi, \frac{5}{8}\pi]$, the factor $(\frac{\tau}{\sin\tau})^n$ is smooth and can be included into the amplitude by putting $\tilde\eta_0(\tau) := \eta_0(\tau)\,(\frac{\tau}{\sin\tau})^n$.

We shall therefore rather work with the convolution kernel

$$K^0_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_{\mathbb{R}} \int_0^\infty e^{i\lambda\tilde\Phi(x,u,s,\sigma,\tau)}\, a_\lambda(s\sigma)\, \tilde\eta_0(\tau)\, d\sigma\, d\tau, \tag{5.3}$$

where

$$\tilde\Phi(x,u,s,\sigma,\tau) = \sigma(s - |x|^2 g(\tau) + 4u\tau), \quad \text{with } g(\tau) = \tau\cot\tau, \tag{5.4}$$

and denote the corresponding convolution operator by $\tilde T^0_{\lambda,s} f = f * K^0_{\lambda,s}$.

We then define $T^{0,\varepsilon}_\lambda : \mathcal{S}(\mathbb{R}^d) \to L^0(\mathbb{R}^{d+1})$ as the operator given by

$$(T^{0,\varepsilon}_\lambda f)(x,u,s) := \tilde\chi_\varepsilon(x,u)\big(\tilde T^0_{\lambda,s}(\chi_\varepsilon f)\big)(x,u), \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{5.5}$$

We then need to prove estimates of the form

$$\Big\| \Big(\int_1^4 \big|T^{0,\varepsilon}_\lambda f\big|^2\, ds\Big)^{\frac12} \Big\|_p \lesssim \lambda^\alpha \|f\|_2.$$

By decomposing the s-interval $[1,4]$ into smaller intervals of length $\varepsilon \in (0,1)$ and applying suitable automorphic scalings for each of them, we may actually reduce to smaller s-intervals $[1, 1+\varepsilon]$. Accordingly, we always assume that $s \in [1, 1+\varepsilon]$ in the definition of $T^{0,\varepsilon}_\lambda f$.

Similarly, by means of a partition of unity consisting of smooth cut-off functions supported in intervals of length ε we may further assume that a_λ is supported in such a shorter interval. Thus, we may reduce to proving the following:

Suppose that $p \ge \frac{2(d_1+1)}{d_1-1}$ and $\alpha \ge d\,(\frac12 - \frac1p) - \frac12$. If the constant $\varepsilon > 0$ is sufficiently small, and if a_λ is supported in an interval of length ε contained in $(\frac{1}{16}, 4)$, then we can bound

$$\Big\| \Big(\int_1^{1+\varepsilon} \big|T^{0,\varepsilon}_\lambda f\big|^2\, ds\Big)^{\frac12} \Big\|_p \le C(a_\lambda,\varepsilon,\alpha,p)\lambda^\alpha \|f\|_2 \quad \textit{for all } f \in \mathcal{S}(\mathbb{R}^d), \tag{5.6}$$

where the constant $C(a_\lambda,\varepsilon,\alpha,p)$ depends in a_λ only on the C^M-norm of a_λ for some sufficiently large $M \in \mathbb{N}$.

Recall that the point (x^0,u^0) satisfies $|(x^0,u^0)| < 10$. For $|x^0| \ll 1$, we obtain rapid decay in λ by exploiting that the phase of the associated integral kernel is non-stationary in at least one of the parameters σ or τ.

Lemma 5.3. *If $|x^0| \le \frac{1}{100}$, then*

$$\Big\| \Big(\int_1^{1+\varepsilon} \big|T^{0,\varepsilon}_\lambda f\big|^2\, ds\Big)^{\frac12} \Big\|_p \lesssim_N \lambda^{-N} \|f\|_2 \quad \textit{for all } f \in \mathcal{S}(\mathbb{R}^d).$$

Proof. Recall that $\varepsilon \le \frac{1}{10^4}$. Since $T^{0,\varepsilon}_\lambda f$ is supported on $\operatorname{supp}\tilde\chi_\varepsilon$, we have

$$\Big\| \Big(\int_1^{1+\varepsilon} \big|T^{0,\varepsilon}_\lambda f\big|^2\, ds\Big)^{\frac12} \Big\|_p \lesssim \Big\| \Big(\int_1^{1+\varepsilon} \big|T^{0,\varepsilon}_\lambda f\big|^2\, ds\Big)^{\frac12} \Big\|_\infty,$$

so it suffices to show uniformly in (x,u,s) that

$$|T^{0,\varepsilon}_\lambda f(x,u,s)| \lesssim_N \lambda^{-N} \|f\|_2. \tag{5.7}$$

By (5.5), we have

$$\begin{aligned}(T_\lambda^{0,\varepsilon} f)(z,s) &= \tilde\chi_\varepsilon(z)\big((\chi_\varepsilon f) * K^0_{\lambda,s}\big)(z)\\ &= \tilde\chi_\varepsilon(z)\int_{\mathbb{H}_n} \chi_\varepsilon(w)\, K^0_{\lambda,s}\big(w^{-1}z\big)\, f(w)\, dw,\end{aligned}$$

where, by (5.3) and (5.4), when writing $z=(x,u)$ and $w=(y,v)$, the phase of the associated integral kernel is given by

$$\begin{aligned}\tilde\Phi(w^{-1}z,s,\sigma,\tau) &= \tilde\Phi(x-y,u-v+\tfrac12\langle Jx,y\rangle,s,\sigma,\tau)\\ &= \sigma\left(s-|x-y|^2\, g(\tau)+4\,(u-v+\tfrac12\langle Jx,y\rangle)\,\tau\right),\end{aligned}$$

where $|(x-x^0,u-u^0)|<\varepsilon$ on the support of $\tilde\chi_\varepsilon$ and $|(y,v)|\le\varepsilon$ for $(y,v)\in\operatorname{supp}\chi_\varepsilon$.

Assume that $|x^0|\le\frac{1}{100}$. We distinguish the cases $|(x,u)|<\frac{1}{10}$ and $|(x,u)|\ge\frac{1}{10}$.

Suppose that $|(x,u)|<\frac{1}{10}$. Then since $|g(\tau)|=|\tau\cot\tau|\le 1$ in the above integrals, we have

$$\big||x-y|^2\, g(\tau)\big|\le(\tfrac{1}{10}+\varepsilon)^2<\tfrac{2}{100}$$

and also

$$|4\,(u-v+\tfrac12\langle Jx,y\rangle)|\,|\tau|\le(4\,(\tfrac{1}{10}+\varepsilon)+2\,\tfrac{1}{10}\,\varepsilon)\,\tfrac58\pi<\tfrac{3}{10}\pi<\tfrac{95}{100}.$$

Thus, since $s\in[1,1+\varepsilon]$, we see that $|\partial_\sigma(\tilde\Phi(w^{-1}z,s,\sigma,\tau))|\ge\frac{3}{100}$ and we can integrate by parts. Combining this with an application of Cauchy–Schwarz in the integral above shows (5.7) for $|(x,u)|<\frac{1}{10}$.

Next, suppose that $|(x,u)|\ge\frac{1}{10}$. We have $|x|<\varepsilon+\frac{1}{100}$ for $(x,u)\in\operatorname{supp}\tilde\chi_\varepsilon$, which then implies $|u|\ge|(x,u)|-|x|\ge\frac{1}{10}-|x|\ge\frac{1}{20}$. Hence,

$$|4\,(u-v+\tfrac12\langle Jx,y\rangle)|\ge\tfrac15-4\varepsilon-2\,(\varepsilon+\tfrac{1}{100})\,\varepsilon\ge\tfrac{1}{10}.$$

Moreover, since $g'(\tau)=\cot\tau-\tau/(\sin\tau)^2=(\sin2\tau-2\tau)/(1-\cos2\tau)$ and $g''(\tau)=2\,(\tau\cot\tau-1)/(\sin\tau)^2\le0$, we get for $|\tau|\le\frac58\pi$ that

$$|g'(\tau)|\le g'(-\tfrac58\pi)=\frac{\sin\frac{\pi}{4}+\frac54\pi}{1+\cos\frac{\pi}{4}}=\frac{\sqrt2+\frac52\pi}{2+\sqrt2}\le3.$$

Thus, we have $|x-y|^2\,|g'(\tau)|\le3\,(2\varepsilon+\frac{1}{100})^2\le\frac{1}{20}$ for $(y,v)\in\operatorname{supp}\chi_\varepsilon$. Hence, we obtain $|\partial_\tau(\tilde\Phi(w^{-1}z,s,\sigma,\tau))|\ge\frac{\sigma}{20}$, and we can argue again via integration by parts and Cauchy–Schwarz. □

To prove (5.6), we shall assume from now on that $|x^0|\ge\frac{1}{100}$, where we may assume that $x^0=(|x^0|,0,\dots,0)$ since rotations in the x variable are automorphisms of $\mathbb{H}_n$. In particular, if $|x-x^0|<\varepsilon<10^{-4}$ and $|x-y|<\varepsilon$, we see that $|x_1-y_1|\sim1$ and $|x_j-y_j|\ll1$ for $j=2,\dots,d_1$.

5.1. **Decomposition into the non-horizontal and horizontal part.** Recall from the discussion preceding (5.3) that

$$\operatorname{supp}\tilde\eta_0\subset[-\tfrac58\pi,\tfrac58\pi],$$

and that $\pm\frac{\pi}{2}\in\operatorname{supp}\tilde\eta_0$. The parameters $\tau=\pm\frac{\pi}{2}$ correspond to horizontal points on Σ_s, that is, points at which the tangent plane to Σ_s is horizontal. The heuristic discussion in the Introduction therefore suggests decomposing

$$\tilde\eta_0=\eta_{0,\mathsf{nh}}+\eta_{0,\mathsf{h}}+\eta_{0,\check{\mathsf{h}}}$$

into smooth bump functions $\eta_{0,\mathsf{nh}}, \eta_{0,\mathsf{h}}$ and $\eta_{0,\check{\mathsf{h}}}$ such that

$$\operatorname{supp}\eta_{0,\mathsf{nh}} \subset [-\tfrac{3}{8}\pi, \tfrac{3}{8}\pi] \quad \text{and} \quad \operatorname{supp}\eta_{0,\mathsf{h}} \subset [\tfrac{3}{16}\pi, \tfrac{5}{8}\pi],$$

where $\eta_{0,\check{\mathsf{h}}}(\tau) := \eta_{0,\mathsf{h}}(-\tau)$. Accordingly, we decompose

$$T_\lambda^{0,\varepsilon} = T_\lambda^{0,\mathsf{nh},\varepsilon} + T_\lambda^{0,\mathsf{h},\varepsilon} + T_\lambda^{0,\check{\mathsf{h}},\varepsilon}, \tag{5.8}$$

where the non-horizontal part $T_\lambda^{0,\mathsf{nh},\varepsilon}$ is defined like $T_\lambda^{0,\varepsilon}$, only with $\tilde\eta_0$ in (5.3) replaced by $\eta_{0,\mathsf{nh}}$, and the other two summands are defined analogously.

By symmetry, estimates for $T_\lambda^{0,\check{\mathsf{h}},\varepsilon}$ can be reduced to estimates for $T_\lambda^{0,\mathsf{h},\varepsilon}$, so that it suffices to work with $T_\lambda^{0,\mathsf{nh},\varepsilon}$ and $T_\lambda^{0,\mathsf{h},\varepsilon}$.

5.2. **The TT^* argument for the operators $T_\lambda^{0,\varepsilon}$.** To prove that

$$T_\lambda^{0,\varepsilon} : L^2_{(x,u)} \to L^p_{(x,u)}(L^2_s)$$

is bounded, we first decompose it as in (5.8). We then apply the Stein–Tomas TT^* argument separately to the non-horizontal and horizontal parts, proving

$$L^{p'}_{(x,u)}(L^2_s) \to L^p_{(x,u)}(L^2_s)$$

estimates for the operators $T_\lambda^{0,\mathsf{nh},\varepsilon}(T_\lambda^{0,\mathsf{nh},\varepsilon})^*$ and $T_\lambda^{0,\mathsf{h},\varepsilon}(T_\lambda^{0,\mathsf{h},\varepsilon})^*$. We first record the corresponding kernel formula for the former operator.

Lemma 5.4. *Assume that $\varepsilon > 0$ is sufficiently small. Then, for $s, s' \in [1, 1+\varepsilon]$, the integral kernel $Q_\lambda^{0,\mathsf{nh},\varepsilon}$ of $T_\lambda^{0,\mathsf{nh},\varepsilon}(T_\lambda^{0,\mathsf{nh},\varepsilon})^*$ is given by*

$$\begin{aligned} Q_\lambda^{0,\mathsf{nh},\varepsilon}(x,u,s,x',u',s') &= \lambda^{d+1}\,\chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0) \\ &\times \int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{d_1}} e^{i\lambda\Psi}\, a_\lambda(s\sigma)\,\overline{a_\lambda(s'\sigma')}\,\eta_{0,\mathsf{nh}}(\tau')\,\chi_\varepsilon(y)\,h_\lambda\,dy\,d\sigma\,d\sigma'\,d\tau', \end{aligned} \tag{5.9}$$

where $h_\lambda = h_\lambda(\sigma,\sigma',\tau',x,y,u)$ is smooth and such that each partial derivative of h_λ is uniformly bounded, if $\sigma,\sigma' \sim 1$, $|\tau'|, |y| \lesssim 1$ and $\lambda \gg 1$, and where the phase function Ψ is given by

$$\begin{aligned} \Psi &= \sigma s - \sigma' s' - |x-y|^2\,\sigma\, g(\tfrac{\sigma'}{\sigma}\tau') \\ &\quad + |x'-y|^2\,\sigma' g(\tau') + 4\sigma'\tau'(u-u'+\tfrac{1}{2}\langle J(x-x'),y\rangle), \end{aligned} \tag{5.10}$$

where we recall that $g(\tau) = \tau\cot\tau$.

Remark 5.5. Correspondingly, the integral kernel $Q_\lambda^{0,\mathsf{h},\varepsilon}$ of $T_\lambda^{0,\mathsf{h},\varepsilon}(T_\lambda^{0,\mathsf{h},\varepsilon})^*$ is given by the analogous formula, with each occurrence of $\eta_{0,\mathsf{nh}}$ replaced by $\eta_{0,\mathsf{h}}$.

Proof. Recall that a_λ in (5.3) is supported in an interval of length ε contained in $(\frac{1}{16}, 4)$ and that $\eta_{0,\mathsf{nh}}$ is supported in $[-\frac{3}{8}\pi, \frac{3}{8}\pi]$. We put $z^0 = (x^0, u^0)$, and let $z = (x,u), z' = (x',u') \in \mathbb{R}^d$. Then, for $\varphi \in \mathcal{S}(\mathbb{H}_n)$,

$$\begin{aligned} (T_\lambda^{0,\mathsf{nh},\varepsilon}\varphi)(z,s) &= \tilde\chi_\varepsilon(z)\big((\chi_\varepsilon\varphi) * K_{\lambda,s}^{0,\mathsf{nh}}\big)(z) \\ &= \tilde\chi_\varepsilon(z)\int_{\mathbb{H}_n} \chi_\varepsilon(w)\,K_{\lambda,s}^{0,\mathsf{nh}}\big((w)^{-1}z\big)\,\varphi(w)\,dw, \\ \big((T_\lambda^{0,\mathsf{nh},\varepsilon})^* f\big)(w) &= \chi_\varepsilon(w)\int_1^{1+\varepsilon}\int_{\mathbb{H}_n} \tilde\chi_\varepsilon(z')\,\overline{K_{\lambda,s'}^{0,\mathsf{nh}}}\big(w^{-1}z'\big) f(z',s')\,dz'\,ds', \end{aligned}$$

where $K^{0,\mathsf{nh}}_{\lambda,s}$ is defined as $K^0_{\lambda,s}$ in (5.3), but with $\tilde\eta_0$ replaced by $\eta_{0,\mathsf{nh}}$. We obtain

$$\big(T^{0,\mathsf{nh},\varepsilon}_\lambda (T^{0,\mathsf{nh},\varepsilon}_\lambda)^* f\big)(z,s) = \int_1^{1+\varepsilon} \int_{\mathbb{H}_n} Q^{0,\mathsf{nh},\varepsilon}_\lambda(z,s,z',s')\, f(z',s')\, dz'\, ds',$$

where

$$Q^{0,\mathsf{nh},\varepsilon}_\lambda(z,s,z',s') = \tilde\chi_\varepsilon(z)\,\tilde\chi_\varepsilon(z') \int_{\mathbb{H}_n} K^{0,\mathsf{nh}}_{\lambda,s}\big(w^{-1}z\big)\,\overline{K^{0,\mathsf{nh}}_{\lambda,s'}}\big(w^{-1}z'\big)\,\chi^2_\varepsilon(w)\,dw.$$

Moreover, by (5.3) and (5.4),

$$\begin{aligned}
&K^{0,\mathsf{nh}}_{\lambda,s}\big((y,v)^{-1}(x,u)\big)\,\overline{K^{0,\mathsf{nh}}_{\lambda,s'}}\big((y,v)^{-1}(x',u')\big)\\
&= K^{0,\mathsf{nh}}_{\lambda,s}\big(x-y,u-v+\tfrac12\langle Jx,y\rangle\big)\,\overline{K^{0,\mathsf{nh}}_{\lambda,s'}}\big(x'-y,u'-v+\tfrac12\langle Jx',y\rangle\big)\big)\\
&= \lambda^{d+2}\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}}\int_0^\infty e^{i\lambda\Psi_0}\,a_\lambda(s\sigma)\,\eta_{0,\mathsf{nh}}(\tau)\,\overline{a_\lambda(s'\sigma')}\,\eta_{0,\mathsf{nh}}(\tau')d\sigma\,d\tau\,d\sigma'\,d\tau',
\end{aligned}$$

and

$$\begin{aligned}
\Psi_0 = \sigma\big(s-|x-y|^2\,g(\tau)+4\tau(u-v+\tfrac12\langle Jx,y\rangle)\big)\\
-\sigma'\big(s'-|x'-y|^2\,g(\tau')+4\tau'(u'-v+\tfrac12\langle Jx',y\rangle)\big).
\end{aligned}$$

Recall that $w=(y,v)$. We may assume that $\chi(w)$ equals a product of a smooth bump in y and a smooth bump in v. To help keep expressions more compact, we incorporate this assumption by a slight abuse of notation, writing $\chi^2_\varepsilon(w) = \chi_\varepsilon(y)\chi_\varepsilon(v)$. With this assumption, the only term depending on v in the previous identity for $Q^{0,\mathsf{nh}\varepsilon}_\lambda(z,s,z',s')$ is $e^{-i\lambda 4(\sigma\tau-\sigma'\tau')v}\chi_\varepsilon(v)$. Since

$$\int_{\mathbb{R}} e^{-i\lambda 4(\sigma\tau-\sigma'\tau')v}\chi_\varepsilon(v)\,dv = \widehat{\chi_\varepsilon}\big(4\lambda(\sigma\tau-\sigma'\tau')\big),$$

we may simplify the previous expression for $Q^{0,\mathsf{nh},\varepsilon}_\lambda$ to

$$\begin{aligned}
Q^{0,\mathsf{nh}\varepsilon}_\lambda(z,s,z',s') = \tilde\chi_\varepsilon(z)\,\tilde\chi_\varepsilon(z')\,\lambda^{d+2}\\
\times\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}}\int_0^\infty e^{i\lambda\Psi_1}\chi_\varepsilon(y)\,\widehat{\chi_\varepsilon}\big(4\lambda(\sigma\tau-\sigma'\tau')\big)\\
\times a_\lambda(s\sigma)\,\eta_{0,\mathsf{nh}}(\tau)\,\overline{a_\lambda(s'\sigma')}\,\eta_{0,\mathsf{nh}}(\tau')\,d\sigma\,d\tau\,d\sigma'\,d\tau'\,dy,
\end{aligned}$$

where

$$\begin{aligned}
\Psi_1 = \sigma\big(s-|x-y|^2\,g(\tau)+4\tau(u+\tfrac12\langle Jx,y\rangle)\big)\\
-\sigma'\big(s'-|x'-y|^2\,g(\tau')+4\tau'(u'+\tfrac12\langle Jx',y\rangle)\big).
\end{aligned}$$

We change variables from τ to ϑ, defined by

$$\tau = \tfrac{\sigma'}{\sigma}\tau' + \vartheta, \quad \text{i.e., } \ \sigma\tau-\sigma'\tau' = \sigma\vartheta.$$

Next, we decompose Ψ_1 as $\Psi_1 = \Psi_2+\Psi_3$, where

$$\Psi_2 = \sigma\,|x-y|^2\,\big(g(\tfrac{\sigma'}{\sigma}\tau') - g(\tfrac{\sigma'}{\sigma}\tau'+\vartheta)\big) + 4\sigma\vartheta(u+\tfrac12\langle Jx,y\rangle),$$

and

$$\begin{aligned}
\Psi_3 = \sigma\big(s-|x-y|^2\,g(\tfrac{\sigma'}{\sigma}\tau')+4\tfrac{\sigma'}{\sigma}\tau'(u+\tfrac12\langle Jx,y\rangle)\big)\\
-\sigma'\big(s'-|x'-y|^2\,g(\tau')+4\tau'(u'+\tfrac12\langle Jx',y\rangle)\big).
\end{aligned}$$

Note that if we choose $\varepsilon > 0$ sufficiently small in (5.6), then, due to our support assumption on a_λ, we may assume that σ'/σ is very close to 1, so that the argument $\frac{\sigma'}{\sigma}\tau'$ will still stay away from singularities of g.

Let h_λ be given by

$$h_\lambda(\sigma,\sigma',\tau',x,y,u) = \lambda \int_{\mathbb{R}} e^{i\lambda\Psi_2}\, \eta_{0,\mathsf{nh}}(\tfrac{\sigma'}{\sigma}\tau'+\vartheta)\, \widehat{\chi_\varepsilon}(4\lambda\sigma\vartheta)\, d\vartheta.$$

Then h_λ is smooth, and each partial derivative of h_λ is uniformly bounded, if $\sigma,\sigma' \sim 1$, $|\tau'|, |y| \lesssim 1$ and $\lambda \gg 1$. Indeed, for $|\vartheta| \gg 1$, this smoothness is clear since $\widehat{\chi_\varepsilon} \in \mathcal{S}$, and for $|\vartheta| \lesssim 1$, this can be seen by writing $g(\frac{\sigma'}{\sigma}\tau') - g(\frac{\sigma'}{\sigma}\tau'+\vartheta) = \vartheta\, G(\sigma,\sigma',\tau',\vartheta)$ and scaling in ϑ by λ^{-1}.

Thus, the previous expression for $Q_\lambda^{0,\mathsf{nh}\varepsilon}(z,s,z',s')$ becomes

$$\begin{aligned} Q_\lambda^{0,\mathsf{nh},\varepsilon}(z,s,z',s') &= \tilde\chi_\varepsilon(z)\,\tilde\chi_\varepsilon(z')\,\lambda^{d+1} \\ &\quad\times \int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{d_1}} e^{i\lambda\Psi_3}\, a_\lambda(s\sigma)\,\overline{a_\lambda(s'\sigma')}\,\eta_{0,\mathsf{nh}}(\tau')\,\chi_\varepsilon(y)\,h_\lambda\, dy\, d\sigma\, d\sigma'\, d\tau'. \end{aligned}$$

This finishes the proof of Lemma 5.4, since $\Psi = \Psi_3$. □

6. The non-horizontal part

In this section, we prove Proposition 5.2 with $T_{\lambda,t}^{0,\mathsf{nh}}$ in place of $T_{\lambda,t}^0$.

6.1. **The key estimates for the non-horizontal part.** Given $x, x' \in \mathbb{R}^{d_1}$, we split coordinates as $x = (x_1, \underline{x})$ and $x' = (x_1', \underline{x}')$. For fixed $x_1 \in \mathbb{R}$, let $T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be the operator given by

$$(T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon}\varphi)(\underline{x},u,s) = (T_\lambda^{0,\mathsf{nh},\varepsilon}\varphi)(x_1,\underline{x},u,s).$$

If we write $f(x',u',s') = f_{x_1'}(\underline{x}',u',s')$, then

$$\big(T_\lambda^{0,\mathsf{nh},\varepsilon}(T_\lambda^{0,\mathsf{nh},\varepsilon})^* f\big)(x,u,s) = \int_{\mathbb{R}} \big(T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon}(T_{\lambda,x_1'}^{0,\mathsf{nh},\varepsilon})^*\, f_{x_1'}\big)(\underline{x},u,s)\, dx_1'. \tag{6.1}$$

Let $Q_{\lambda,x_1,x_1'}^{0,\mathsf{nh},\varepsilon}$ denote the integral kernel of the operator $T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon}(T_{\lambda,x_1'}^{0,\mathsf{nh},\varepsilon})^*$. Note that

$$Q_{\lambda,x_1,x_1'}^{0,\mathsf{nh},\varepsilon}(\underline{x},u,s,\underline{x}',u',s') = Q_\lambda^{0,\mathsf{nh},\varepsilon}(x,u,s,x',u',s'). \tag{6.2}$$

In the subsequent discussions, we shall always assume the constant $\varepsilon > 0$ to be sufficiently small, usually without further mentioning.

Proposition 6.1. *We have*

$$\|T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon}(T_{\lambda,x_1'}^{0,\mathsf{nh},\varepsilon})^*\|_{L^1_{(\underline{x},u)}(L^2_s)\to L^\infty_{(\underline{x},u)}(L^2_s)} \lesssim \lambda^{d-1}\,\langle\lambda\,|x_1'-x_1|\rangle^{-\frac{d-2}{2}},$$

with control of the implicit constant in this estimate as in Proposition 5.2.

Proposition 6.2. *We have*

$$\|T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon}(T_{\lambda,x_1'}^{0,\mathsf{nh},\varepsilon})^*\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim 1,$$

with control of the implicit constant in this estimate as in Proposition 5.2.

We postpone the proofs of Propositions 6.1 and 6.2 for the moment and show how they imply the estimates in Proposition 5.2 for $T_\lambda^{0,\mathsf{nh},\varepsilon}$.

Proof of Proposition 5.2 for $T_\lambda^{0,\mathsf{nh},\varepsilon}$. Since $p>2$, using (6.1) and Minkowski's inequality, we obtain

$$\|T_\lambda^{0,\mathsf{nh},\varepsilon}(T_\lambda^{0,\mathsf{nh},\varepsilon})^* f\|_{L^p_{(x,u)}(L^2_s)} \le \Big\| \int_{\mathbb{R}} \big\|T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon}(T_{\lambda,x_1'}^{0,\mathsf{nh},\varepsilon})^*\, f_{x_1'}\big\|_{L^p_{(\underline{x},u)}(L^2_s)}\, dx_1' \Big\|_{L^p_{x_1}}.$$

Interpolating the estimates of Propositions 6.1 and 6.2, we get

$$\|T_{\lambda,x_1}^{0,\mathsf{nh},\varepsilon}(T_{\lambda,x_1'}^{0,\mathsf{nh},\varepsilon})^* f_{x_1'}\|_{L^p_{(\underline{x},u)}(L^2_s)} \lesssim \lambda^{(d-1)(1-\frac{2}{p})}\, \langle \lambda\,|x_1'-x_1|\rangle^{-\gamma_p}\, \|f_{x_1'}\|_{L^{p'}_{(\underline{x},u)}(L^2_s)},$$

where $\gamma_p = \frac{d-2}{2}(1-\frac{2}{p})$.

Assume first that $p=\frac{2d}{d-2}$. Then $(d-1)(1-\frac{2}{p})-\gamma_p = \frac{d}{2}(1-\frac{2}{p}) = 1$, and if we choose q so that $1+\frac{1}{p}=\frac{1}{q}+\frac{1}{p'}$, then $q=\frac{p}{2}>1$, and so $q\gamma_p = \frac{d-2}{2}(\frac{p}{2}-1)=1$. Moreover, $2\big(d(\frac12-\frac1p)-\frac12\big)=1$.

Thus, applying Hardy–Littlewood–Sobolev's inequality, we get

$$\begin{aligned}\lambda^{(d-1)(1-\frac{2}{p})} \Big\| \int_{\mathbb{R}} \langle \lambda\,|x_1'-x_1|\rangle^{-\gamma_p}\, \|f_{x_1'}\|_{L^{p'}_{(\underline{x},u)}(L^2_s)}\, dx_1' \Big\|_{L^p_{x_1}} \\ \lesssim \lambda \|f\|_{L^{p'}_{(x,u)}(L^2_s)} = \lambda^{2\left(d(\frac12-\frac1p)-\frac12\right)} \|f\|_{L^{p'}_{(x,u)}(L^2_s)},\end{aligned}$$

hence

$$\|T_\lambda^{0,\mathsf{nh},\varepsilon}\|_{L^2_{(x,u)}\to L^p_{(x,u)}(L^2_s)} \lesssim \lambda^{d(\frac12-\frac1p)-\frac12},$$

which proves (5.6) for this critical Lebesgue exponent p, with $\alpha = d(\frac12-\frac1p)-\frac12$.

For $p=\infty$, since we may assume that $|x|, |x'| \lesssim 1$, and since $\langle \lambda\,|x_1'-x_1|\rangle^{-\gamma_p} \le 1$, we can trivially estimate

$$\lambda^{(d-1)(1-\frac{2}{p})} \Big\| \int_{\mathbb{R}} \langle \lambda\,|x_1'-x_1|\rangle^{-\gamma_p}\, \|f_{x_1'}\|_{L^{p'}_{(\underline{x},u)}(L^2_s)}\, dx_1' \Big\|_{L^p_{x_1}} \lesssim \lambda^{d-1} \|f\|_{L^1_{(x,u)}(L^2_s)},$$

so that (5.6) holds as well.

Then (5.6) follows by interpolation, which completes the proof of Proposition 5.2 with $T_\lambda^{0,\mathsf{nh},\varepsilon}$ in place of $T_\lambda^{0,\varepsilon}$. □

6.2. **Pointwise estimates.** In this section, we prove Proposition 6.1. Recall from Lemma 5.4 that the integral kernel $Q_\lambda^{0,\mathsf{nh},\varepsilon}$ of $T_\lambda^{0,\mathsf{nh},\varepsilon}(T_\lambda^{0,\mathsf{nh},\varepsilon})^*$ is given by

$$\begin{aligned}Q_\lambda^{0,\mathsf{nh},\varepsilon}(x,u,s,x',u',s') &= \lambda^{d+1}\, \chi_\varepsilon(x-x^0,u-u^0)\, \chi_\varepsilon(x'-x^0,u'-u^0) \\ &\quad \times \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{d_1}} e^{i\lambda\Psi}\, a_\lambda(s\sigma)\, \overline{a_\lambda(s'\sigma')}\, \chi_\varepsilon(y)\, h_\lambda\, dy\, d\sigma\, d\sigma'\, \eta_{0,\mathsf{nh}}(\tau')\, d\tau',\end{aligned}$$

where $\eta_{0,\mathsf{nh}}$ is supported in $[-\frac38\pi, \frac38\pi]$ with phase

$$\begin{aligned}\Psi = \sigma s - \sigma' s' - |x-y|^2\, \sigma'\tau' \cot(\tfrac{\sigma'}{\sigma}\tau') \\ + |x'-y|^2\, \sigma'\tau' \cot\tau' + 4\sigma'\tau'(u-u'+\tfrac12\langle J(x-x'),y\rangle).\end{aligned}$$

The localizations given by the cut-offs χ_ε allow us to assume in the subsequent discussions that $|(x,u)-(x^0,u^0)| \le \varepsilon$, $|(x',u')-(x^0,u^0)| \le \varepsilon$, and $|y| \le \varepsilon$. We recall that $|(x^0,u^0)| \le 10$, $|x^0| \ge \frac{1}{100}$, and $x^0 = (|x^0|,0,\dots,0)$.

Proposition 6.1 will follow immediately from the next proposition by means of Schur's test, since $|\Delta x| \ge |\Delta x_1|$.

Proposition 6.3. *If the constant $\varepsilon > 0$ is sufficiently small, then for all $s \in [1, 1+\varepsilon]$,*

$$\int_1^{1+\varepsilon} |Q_\lambda^{0,\mathsf{nh},\varepsilon}(x,u,s,x',u',s')|\, ds' \le C(\varepsilon, a_\lambda)\, \lambda^{d-1} \langle \lambda\, |\Delta x| \rangle^{-\frac{d_1-1}{2}}, \tag{6.3}$$

and for all $s' \in [1, 1+\varepsilon]$,

$$\int_1^{1+\varepsilon} |Q_\lambda^{0,\mathsf{nh},\varepsilon}(x,u,s,x',u',s')|\, ds \le C(\varepsilon, a_\lambda)\, \lambda^{d-1} \langle \lambda\, |\Delta x| \rangle^{-\frac{d_1-1}{2}}, \tag{6.4}$$

uniformly in (x,u) and (x',u'), where the constant $C(\varepsilon, a_\lambda)$ can be chosen to be independent of (x^0,u^0) and to depend only on the C^M-norm of a_λ for some sufficiently large M.

We will prove Proposition 6.3 by showing the following.

Lemma 6.4. *We can decompose $Q_\lambda^{0,\mathsf{nh},\varepsilon} = Q_{1,\lambda}^\varepsilon + E_\lambda^\varepsilon$, where the main term $Q_{1,\lambda}^\varepsilon$ can be estimated pointwise by*

$$\begin{aligned} |Q_{1,\lambda}^\varepsilon(x,u,s,x',u',s')| &\le C_N(\varepsilon, a_\lambda)\, \lambda^d \langle \lambda\, |\Delta x| \rangle^{-\frac{d_1-1}{2}} \\ &\quad \times \int_{\mathbb{R}} \big\langle \lambda\, (\Delta s - q(s, \Delta x, \Delta w, \tau')) \big\rangle^{-N} \eta_{0,\mathsf{nh}}(\tau')\, d\tau' \end{aligned} \tag{6.5}$$

for any $N \in \mathbb{N}$, where

$$\Delta x = x - x',\ \Delta w = u - u' + \frac{1}{2}\langle J\Delta x, x\rangle \text{ and } \Delta s = s - s',$$

and where q is a smooth function satisfying estimates of the form $\partial_s^j q = \mathcal{O}(\varepsilon)$ for every $j \in \mathbb{N}$. Here, the constant $C_N(\varepsilon, a_\lambda)$ can be chosen to be independent of τ' and to depend, for a given $N \in \mathbb{N}$, in a_λ only on the C^M-norm of a_λ for some sufficiently large M (which may depend on N.)

For every $N \in \mathbb{N}$, for all $s \in [1, 1+\varepsilon]$,

$$\int_1^{1+\varepsilon} |E_\lambda^\varepsilon(x,u,s,x',u',s')|\, ds' \le C_N(\varepsilon, a_\lambda)\, \lambda^d \langle \lambda\, |\Delta x| \rangle^{-N},$$

and for all $s' \in [1, 1+\varepsilon]$,

$$\int_1^{1+\varepsilon} |E_\lambda^\varepsilon(x,u,s,x',u',s')|\, ds \le C_N(\varepsilon, a_\lambda)\, \lambda^d \langle \lambda\, |\Delta x| \rangle^{-N}.$$

Before the prove this lemma, we show how Proposition 6.3 is derived from it.

Proof of Proposition 6.3. We need to show that the main term $Q_{1,\lambda}^\varepsilon$ will satisfy estimates of the form (6.3) and (6.4). For $N \ge 2$, we clearly have

$$\int_1^{1+\varepsilon} \big\langle \lambda\, (\Delta s - q(s, \Delta x, \Delta w, \tau')) \big\rangle^{-N} ds' \lesssim \lambda^{-1}.$$

Noting that the properties of the function q imply that

$$\partial_s\big(\Delta s - q(s, \Delta x, \Delta w, \tau')\big) = 1 + \mathcal{O}(\varepsilon) \sim 1,$$

the inverse function theorem allows to see that the estimate

$$\int_1^{1+\varepsilon} \big\langle \lambda\, (\Delta s - q(s, \Delta x, \Delta w, \tau')) \big\rangle^{-N} ds \lesssim \lambda^{-1}$$

holds as well. The estimates (6.3) and (6.4) are immediate consequences. □

The rest of this subsection will be devoted to the proof of Lemma 6.4.

6.2.1. *Freezing τ'.* We freeze $\tau' \in \operatorname{supp} \eta_{0,\mathsf{nh}}$, and consider the oscillatory integral

$$Q^{0,\varepsilon}_{\lambda,\tau'}(x,u,s,x',u',s') := \lambda^{d+1} \int_{\sigma'\sim 1}\int_{\sigma\sim 1}\int_{\mathbb{R}^{d_1}} e^{i\lambda\Psi}\, a_\lambda(s\sigma)\,\overline{a_\lambda(s'\sigma')} \times \chi_\varepsilon(y)\, h_\lambda\, dy\, d\sigma\, d\sigma' \tag{6.6}$$

(by writing $\sigma\sim 1$, etc., we wish to remind that a_λ is supported in an interval of length ε contained in $(\frac{1}{16},4)$). Note also that

$$Q^{0,\mathsf{nh},\varepsilon}_{\lambda}(x,u,s,x',u',s') = \chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0) \times \int_{\mathbb{R}} Q^{0,\varepsilon}_{\lambda,\tau'}(x,u,s,x',u',s')\,\eta_{0,\mathsf{nh}}(\tau')\,d\tau'.$$

6.2.2. *Changes of variables and stationary phase for a spherical integration.* First, by a change of variables from y to $y'=y-x$, (6.6) becomes

$$\lambda^{d+1} \int_{\sigma'\sim 1}\int_{\sigma\sim 1}\int_{\mathbb{R}^{d_1}} e^{i\lambda\Psi_1}\, a_\lambda(s\sigma)\,\overline{a_\lambda(s'\sigma')}\,\chi_\varepsilon(y'+x)\, h_{1,\lambda}\, dy'\, d\sigma\, d\sigma', \tag{6.7}$$

where $h_{1,\lambda}$ is a slightly modified version of h_λ sharing the essential properties with h_λ, with phase function

$$\Psi_1 = \sigma s - \sigma' s' - |y'|^2\,\sigma'\tau'\cot(\tfrac{\sigma'}{\sigma}\tau') + |y'+\Delta x|^2\,\sigma'\tau'\cot\tau' + 4\sigma'\tau'(\Delta w + \tfrac{1}{2}\langle J\Delta x, y'\rangle).$$

Next, we use polar coordinates in y', that is, $y'=\rho\,\omega$, where $\rho\in(0,\infty)$ satisfies $\rho\sim 1$, since $|y'+x^0|<2\varepsilon$, and where ω lies in the unit sphere S^{d_1-1} of $\mathbb{R}^{d_1}$. Denoting by $d\omega$ the surface measure of S^{d_1-1}, by this change of coordinates (6.7) equals

$$\lambda^{d+1} \int_{\sigma'\sim 1}\int_{\sigma\sim 1}\int_{\rho\sim 1}\int_{S^{d_1-1}} e^{i\lambda\Psi_2}\, a_\lambda(s\sigma)\,\overline{a_\lambda(s'\sigma')}\,\chi_\varepsilon(\rho\omega+x)\, h_{2,\lambda}\, d\omega\, d\rho\, d\sigma\, d\sigma',$$

with a function $h_{2,\lambda}$ having analogous properties in these new coordinates to those of $h_{1,\lambda}$, and where

$$\Psi_2 = \sigma s-\sigma' s'+\rho^2\big(\sigma'\tau'\cot\tau'-\sigma'\tau'\cot(\tfrac{\sigma'}{\sigma}\tau')\big)+2\rho\,\big\langle\omega,\Delta x\,\sigma'\tau'\cot\tau'+\sigma'\tau' J\Delta x\big\rangle + |\Delta x|^2\sigma'\tau'\cot\tau'+4\sigma'\tau'\Delta w.$$

Note that the factor $\chi_\varepsilon(\rho\omega+x)$ localizes the integration in ω to an $\mathcal{O}(\varepsilon)$- neighborhood of the point $-x^0/|x^0|$ in S^{d_1-1}.

Finally, we change variables from σ to β such that

$$\frac{\sigma'}{\sigma} = 1+|\Delta x|\,\beta.$$

Note that since a_λ is supported in an interval of length ε within $(\frac{1}{16},4)$ and $s,s'\in[1,1+\varepsilon]$, we may assume that $|\Delta x|\,|\beta|\lesssim\varepsilon$, which will allow us to add a cut-off factor $\chi_\varepsilon(|\Delta x|\,\beta)$ to the amplitude.

We can then write (6.7) as

$$\lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1}\int\int_{\rho\sim 1}\int_{S^{d_1-1}} e^{i\lambda\Psi_3}\,\overline{a_\lambda(s'\sigma')} \times \chi_\varepsilon(\rho\omega+x)\,\chi_\varepsilon(|\Delta x|\,\beta)\,\tilde a_\lambda\,h_{3,\lambda}\,d\omega\,d\rho\,d\beta\,d\sigma', \tag{6.8}$$

where $\tilde a_\lambda$ stands for $a_\lambda(s\frac{\sigma'}{1+|\Delta x|\beta})$, with $h_{3,\lambda}$ sharing the essential properties with h_λ, and with phase

$$\begin{aligned}\Psi_3 &= \sigma'\Big[\Delta s - \frac{s\,|\Delta x|\,\beta}{1+|\Delta x|\,\beta} + \rho^2\psi_2 + 2\rho\,\big\langle \omega, \Delta x\,\tau'\cot\tau' + \tau' J\Delta x\big\rangle \\ &\qquad\qquad + |\Delta x|^2\,\tau'\cot\tau' + 4\tau'\Delta w\Big] \\ &= \sigma'\Big[\Delta s - q(s,\Delta x,\Delta w,\tau',\omega,\rho,|\Delta x|\beta)\Big].\end{aligned}$$

Here, we have set

$$\psi_2 = -\tau'\big(\cot\big((1+|\Delta x|\,\beta)\tau'\big) - \cot\tau'\big).$$

Note that

$$\psi_2 \sim \Big(\frac{\tau'}{\sin\tau'}\Big)^2\,|\Delta x|\,\beta \sim |\Delta x|\,\beta \quad\text{and}\quad |\big\langle\omega,\Delta x\,\tau'\cot\tau'+\tau' J\Delta x\big\rangle| \lesssim |\Delta x|.$$

Let us denote by $E^{\varepsilon,2}_{\lambda,\tau'}$ the contribution of the region where $|\beta|\gg 1$ to (6.8). For β in this region, we see that we can integrate by parts in ρ in order to gain negative powers $\langle\lambda|\Delta x||\beta|\rangle^{-N}$. Applying subsequently integrations by parts in σ', we finally see that we may write

$$\begin{aligned}E^{\varepsilon,2}_{\lambda,\tau'} = \lambda^d \int_{\sigma'\sim 1}\int_{|\beta|\gg 1}\int_{\rho\sim 1}\int_{S^{d_1-1}} &\big\langle\lambda\,(\Delta s - q(s,\Delta x,\Delta w,\tau',\omega,\rho,|\Delta x|\beta))\big\rangle^{-N} \\ &\times (\lambda|\Delta x|)\,\langle\lambda|\Delta x||\beta|\rangle^{-N}\,e^{i\lambda\Psi_3}\,\overline{a_\lambda(s'\sigma')} \\ &\times \chi_\varepsilon(\rho\omega+x)\,\chi_\varepsilon(|\Delta x|\,\beta)\,\tilde a_\lambda\,h_{N,\lambda}\,d\omega\,d\rho\,d\beta\,d\sigma'\end{aligned}$$

Observing that also here $\partial_s^j q = \mathcal{O}(\varepsilon)$ for every $j\in\mathbb{N}$, we can then essentially argue as before for the main term $Q^\varepsilon_{1,\lambda}$ to prove that the contribution by the error term $E^{\varepsilon,2}_{\lambda,\tau'}$ even satisfies the stronger error term estimates, with the right-hand sides in (6.3) and (6.4) replaced by $C_N(\varepsilon,a_\lambda)\,\lambda^d\,\langle\lambda\,|\Delta x|\rangle^{-N}$. Note to this end that for $N\ge 2$

$$\int(\lambda|\Delta x|)\langle\lambda|\Delta x||\beta|\rangle^{-N}\,d\beta \lesssim 1,$$

so that we obtain these estimates when $\lambda|\Delta x|\le 1$, and when $\lambda|\Delta x|>1$, then

$$(\lambda|\Delta x|)\langle\lambda|\Delta x||\beta|\rangle^{-N} \lesssim (\lambda|\Delta x|)^{1-N}\,|\beta|^{-N},$$

so that the integral in $\beta\gg 1$ is finite, and we also gain any factor $\langle\lambda|\Delta x|\rangle^{1-N}$.

We are thus reduced to considering the contribution by the region where $|\beta|\lesssim 1$, i.e.,

$$Q^{\varepsilon,2}_{\lambda,\tau'} = \lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1}\int_{|\beta|\lesssim 1}\int_{\rho\sim 1}\int_{S^{d_1-1}} e^{i\lambda\Psi_3}\,\overline{a_\lambda(s'\sigma')} \times \chi_\varepsilon(\rho\omega+x)\,\chi_\varepsilon(|\Delta x|\,\beta)\,\tilde a_\lambda\,h_{3,\lambda}\,d\omega\,d\rho\,d\beta\,d\sigma'.$$

We examine the term in the phase Ψ_3 involving the inner product with ω. Using that J is skew-symmetric with $J^2 = -\mathrm{id}_{\mathbb{R}^{d_1}}$, we get

$$\begin{aligned}\big|\Delta x\,\sigma'\tau'\cot\tau' + \sigma'\tau' J\Delta x\big| &= \sqrt{\left(|\Delta x|\,\sigma'\tau'\cot\tau'\right)^2 + |\sigma'\tau'|^2\,|\Delta x|^2}\\ &= \sigma'\,|\Delta x|\,\frac{\tau'}{\sin\tau'} \sim |\Delta x|.\end{aligned}$$

Thus, performing stationary phase in ω when $|\underline{\Delta x}| \lesssim \varepsilon|\Delta x_1|$, and N integrations by parts in ω along S^{d_1-1} when $|\underline{\Delta x}| \gg \varepsilon|\Delta x_1|$, we find that (6.8) is the sum of two corresponding terms

$$Q^{\varepsilon,2}_{\lambda,\tau'} = Q^{\varepsilon,3}_{\lambda,\tau'} + E^{\varepsilon,3}_{\lambda,\tau'}. \tag{6.9}$$

Here, the first term is given by

$$Q^{\varepsilon,3}_{\lambda,\tau'} = \lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1}\int_{|\beta|\lesssim 1}\int_{\rho\sim 1}\langle\lambda\Delta x\rangle^{-\frac{d_1-1}{2}}\,e^{i\lambda\Psi_4^{\pm}}\,\tilde a_\lambda\,\overline{a_\lambda(s'\sigma')}\,h_{4,\lambda}\,d\rho\,d\beta\,d\sigma',$$

where the function $h_{4,\lambda}$ has similar properties as its predecessors and in addition localizes to the region where $|\underline{\Delta x}| \lesssim \varepsilon|\Delta x_1|$, and the phase is given by

$$\Psi_4^{\pm} = \sigma'\Big[\Delta s - \frac{s\,|\Delta x|\,\beta}{1+|\Delta x|\,\beta} + \psi^{\pm} + |\Delta x|^2\,\tau'\cot\tau' + 4\tau'\Delta w\Big],$$

where $\psi^{\pm} = \rho^2\,\psi_2 \pm 2\rho\,\psi_1$ collects all terms in the phase depending on ρ, with

$$\psi_2 = -\tau'\big(\cot\big((1+|\Delta x|\,\beta)\tau'\big) - \cot\tau'\big) \quad\text{and}\quad \psi_1 = \frac{\tau'}{\sin\tau'}\,|\Delta x|\,.$$

The sign in this phase depends on whether Δx_1 is positive, or negative. Moreover, with a slight abuse of notation, the factor $\langle\lambda\Delta x\rangle^{-\frac{d_1-1}{2}}$ actually denotes a symbol of order $-\frac{d_1-1}{2}$ in $\rho(\Delta x\,\sigma'\tau'\cot\tau' + \sigma'\tau' J\Delta x)$.

Note that

$$\psi_2 \sim \Big(\frac{\tau'}{\sin\tau'}\Big)^2\,|\Delta x|\,\beta \sim |\Delta x|\,\beta.$$

The second term in (6.9) is again an error term, which for any $N \in \mathbb{N}$ can be written as

$$\begin{aligned}E^{\varepsilon,3}_{\lambda,\tau'} = \lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1}\int_{|\beta|\lesssim 1}\int_{\rho\sim 1}\int_{S^{d_1-1}}&\langle\lambda\Delta x\rangle^{-N}\,e^{i\lambda\Psi_3}\,\overline{a_\lambda(s'\sigma')}\,\chi_\varepsilon(\rho\omega+x)\\ &\times\ \tilde a_\lambda\,h_{N,\lambda}\,d\omega\,d\rho\,d\beta\,d\sigma',\end{aligned}$$

where $h_{N,\lambda}$ localizes to where $|\underline{\Delta x}| \gg \varepsilon|\Delta x_1|$. After integrations by parts in σ', its contribution can be handled very much in the same way as we handled the one by $E^{\varepsilon,2}_{\lambda,\tau'}$ before.

We shall thus continue with the main term $Q^{\varepsilon,3}_{\lambda,\tau'}$.

Let us first examine again the term $\psi^{\pm}$ in the phase function $\Psi_4^{\pm}$ involving ρ. If $|\beta| \ll 1$, then

$$|\partial_\rho\psi^{\pm}| = |2\rho\psi_2 \pm 2\psi_1| \sim |\Delta x|.$$

Thus, again by integration by parts in ρ, we may again gain any factors $\langle\lambda\Delta x\rangle^{-N}$, and so also the contributions by the region where $|\beta| \ll 1$ can be handled as an error term in a similar way as before.

Thus, we are left with

$$Q^{\varepsilon,4}_{\lambda,\tau'} = \lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1}\int_{|\beta|\sim 1}\int_{\rho\sim 1}\langle\lambda\Delta x\rangle^{-\frac{d_1-1}{2}}\,e^{i\lambda\Psi_4^{\pm}}\,\tilde a_\lambda\,\overline{a_\lambda(s'\sigma')}\,h_{4,\lambda}\,d\rho\,d\beta\,d\sigma'.$$

6.2.3. *Stationary phase for the integration in ρ.* Since the critical point of the phase $\psi^{\pm} = \rho^2\,\psi_2 \pm 2\rho\,\psi_1$ in ρ is given by $\rho_c = \mp\psi_1/\psi_2$, observing that $\rho_c \sim \mp\beta^{-1}$, where $|\beta| \sim 1$, we see that we may assume that $\psi^{\pm}$ has a critical point if and only if $\mp\beta \sim 1$.

For the following discussion, we shall consider the phase Ψ_4^- (the cases of the phase Ψ_4^+ can be treated analogously).

If $-\beta \sim 1$, we can then again integrate by parts in ρ and see that the corresponding contribution can be treated as an error term as before. Thus, we shall assume in the sequel that $\beta \sim 1$, and that ψ^- does have a critical point for any such β.

We can then insert a suitable additional smooth cut-off functions $\chi_1(\beta)$ which localizes to the region where $\beta \sim 1$ that we have devised, i.e., we may assume that

$$Q^{\varepsilon,4}_{\lambda,\tau'} = \lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1}\int_{\beta\sim 1}\int_{\rho\sim 1}\langle\lambda\Delta x\rangle^{-\frac{d_1-1}{2}}\,e^{i\lambda\Psi_4^-}\,\tilde a_\lambda\,\overline{a_\lambda(s'\sigma')} \\ \times h_{4,\lambda}\,\chi_1(\beta)d\rho\,d\beta\,d\sigma'.$$

Using various trigonometric identities, we get

$$\begin{aligned}\psi^-(\rho)|_{\rho=\rho_c} = -\frac{\psi_1^2}{\psi_2} &= \frac{|\Delta x|^2\,\tau'}{(\sin^2\tau')\left(\cot\left((1+|\Delta x|\,\beta)\tau'\right)-\cot\tau'\right)}\\ &= -\,|\Delta x|^2\,\tau'(\cot\tau' + \cot(|\Delta x|\,\beta\tau')).\end{aligned}$$

Since $\partial_\rho^2\psi^- = 2\psi_2 \sim |\Delta x|$, applying stationary phase in ρ we get

(6.10)
$$Q^{\varepsilon,4}_{\lambda,\tau'} = \lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1}\int_{|\beta|\sim 1}\langle\lambda\Delta x\rangle^{-\frac{d_1}{2}}\,e^{i\lambda\Psi_5}\,\tilde a_\lambda\,\overline{a_\lambda(s'\sigma')}\,h_{5,\lambda}\,\chi_1(\beta)\,d\beta\,d\sigma',$$

where

$$\Psi_5 = \sigma'\Big[\Delta s - \frac{s\,|\Delta x|\,\beta}{1+|\Delta x|\,\beta} - |\Delta x|^2\,\tau'\cot(|\Delta x|\,\beta\tau') + 4\tau'\Delta w\Big].$$

6.2.4. *Stationary phase for the integration in β.* Consider the function

$$F(s,\beta,a,b) = \frac{s\beta}{1+a\beta} + \frac{g(b\,\beta)}{\beta},$$

for $|a|, |b| \lesssim \varepsilon \ll 1$ and $\beta \sim 1$, where again $g(\tau) = \tau\cot\tau$. Then

$$\Psi_5 = \sigma'\Big[\Delta s - |\Delta x|\,F\big(s,\beta,|\Delta x|,|\Delta x|\tau'\big) + 4\tau'\Delta w\Big]. \tag{6.11}$$

Note that for $a = |\Delta x|$ and $b = |\Delta x|\tau'$ the conditions $|a|, |b| \lesssim \varepsilon \ll 1$ are indeed satisfied, since we are assuming that $|\Delta x| \lesssim \varepsilon$. If $a = b = 0$ and $s = 1$, then $F(1,\beta,0,0) = \beta + \frac{1}{\beta}$, so $F_\beta(1,\beta,0,0) = 1 - \frac{1}{\beta^2}$ and $F_{\beta\beta}(1,\beta,0,0) = 2\frac{1}{\beta^3}$. Then $\beta_c(1,0,0) := 1$ is the unique critical point. Thus, by the implicit function theorem, for a, b and $s-1$ sufficiently small, F has a unique critical point $\beta_c(s,a,b)$ which depends smoothly on s, a, b.

Let us put $h(s,a,b) = F(s,\beta_c(s,a,b),a,b)$. Then h is smooth, and we see by (6.11) that if we let $a = |\Delta x|$ and $b = |\Delta x|\tau'$ and assume ε to be sufficiently small, then

$$\Psi_6 = \Psi_5\big|_{\beta=\beta_c(|\Delta x|,|\Delta x|\tau')} = \sigma'\Big[\Delta s - q(s,\Delta x,\Delta w,\tau')\Big], \tag{6.12}$$

where

$$q(s,\Delta x,\Delta w,\tau') = |\Delta x|\, h(s,|\Delta x|,\, |\Delta x|\tau') - 4\tau'\Delta w.$$

Note that $\partial_s^j q = \mathcal{O}(\varepsilon)$ for every $j \in \mathbb{N}$, since $|\Delta x|, |\Delta w| \lesssim \varepsilon$. Hence, returning to the oscillatory integral in (6.10), after applying the method of stationary phase in β, we obtain

$$Q^{\varepsilon,4}_{\lambda,\tau'} = \lambda^d(\lambda|\Delta x|)\int_{\sigma'\sim 1} \langle \lambda\Delta x\rangle^{-\frac{d_1+1}{2}}\, e^{i\lambda\Psi_6}\, \tilde{a}_\lambda\, \overline{a_\lambda(s'\sigma')}\, h_{6,\lambda}\, d\sigma'.$$

Integrating N-times by parts in σ', we see that the contribution by $Q^{\varepsilon,4}_{\lambda,\tau'}$ gives our main term $Q^{\varepsilon}_{1,\lambda}$ in (6.5). Note also that the asserted control of the constants $C(\varepsilon, a_\lambda)$ in Proposition 6.3 is evident from our proof.

This completes the proof of Lemma 6.4, hence also that of Proposition 6.3 and Proposition 6.1. □

6.3. L^2 **estimates.** We proceed by proving Proposition 6.2, that is, by showing that the operator

$$T^{0,\mathsf{nh},\varepsilon}_{\lambda,x_1}(T^{0,\mathsf{nh},\varepsilon}_{\lambda,x_1'})^* : L^2 \to L^2$$

is uniformly bounded. Recalling (6.2), Proposition 6.2 will then be a direct consequence of the following (see [23], [20]).

Lemma 6.5. *The phase function* Ψ *in* (5.10) *satisfies*

$$\left|\det\begin{pmatrix} \Psi_{(\underline{x},u,s),(\underline{x}',u',s')} & \Psi_{(\underline{x},u,s),(\sigma,\sigma',\tau',y)} \\ \Psi_{(\sigma,\sigma',\tau',y),(\underline{x}',u',s')} & \Psi_{(\sigma,\sigma',\tau',y),(\sigma,\sigma',\tau',y)}\end{pmatrix}\right| \sim 1$$

on the support of the oscillatory integral in (5.9).

Proof. Recall that the phase function Ψ of Lemma 5.4 is given by

$$\begin{aligned}\Psi = \sigma s - \sigma' s' - |x-y|^2\,\sigma'\tau'\cot(\tfrac{\sigma'}{\sigma}\tau') \\ + |x'-y|^2\,\sigma'\tau'\cot\tau' + 4\sigma'\tau'(u-u'+\tfrac{1}{2}\langle J(x-x'),y\rangle).\end{aligned}$$

Thus, the Monge-Ampère matrix

$$\begin{pmatrix} \Psi_{(\underline{x},u,s),(\underline{x}',u',s')} & \Psi_{(\underline{x},u,s),(\sigma,\sigma',\tau',y)} \\ \Psi_{(\sigma,\sigma',\tau',y),(\underline{x}',u',s')} & \Psi_{(\sigma,\sigma',\tau',y),(\sigma,\sigma',\tau',y)}\end{pmatrix}$$

is given by

$$\begin{pmatrix} 0 & 0 & 0 & \Psi_{\underline{x}\sigma} & \Psi_{\underline{x}\sigma'} & \Psi_{\underline{x}\tau'} & \Psi_{\underline{x}y} \\ 0 & 0 & 0 & 0 & 4\tau' & 4\sigma' & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi_{\sigma\sigma} & \Psi_{\sigma\sigma'} & \Psi_{\sigma\tau'} & \Psi_{\sigma y} \\ \Psi_{\sigma'\underline{x}'} & -4\tau' & -1 & \Psi_{\sigma'\sigma} & \Psi_{\sigma'\sigma'} & \Psi_{\sigma'\tau'} & \Psi_{\sigma' y} \\ \Psi_{\tau'\underline{x}'} & -4\sigma' & 0 & \Psi_{\tau'\sigma} & \Psi_{\tau'\sigma'} & \Psi_{\tau'\tau'} & \Psi_{\tau' y} \\ \Psi_{y\underline{x}'} & 0 & 0 & \Psi_{y\sigma} & \Psi_{y\sigma'} & \Psi_{y\tau'} & \Psi_{yy} \end{pmatrix}.$$

Using first the s-row and the s'-column and then the u'-column for cofactor expansion, we see that the modulus of the determinant of the above matrix equals $4\sigma'$ times the modulus of

$$\begin{vmatrix} 0 & \Psi_{\underline{x}\sigma'} & \Psi_{\underline{x}\tau'} & \Psi_{\underline{x}y} \\ 0 & 4\tau' & 4\sigma' & 0 \\ 0 & \Psi_{\sigma\sigma'} & \Psi_{\sigma\tau'} & \Psi_{\sigma y} \\ \Psi_{y\underline{x}'} & \Psi_{y\sigma'} & \Psi_{y\tau'} & \Psi_{yy} \end{vmatrix}$$

Note that $\sigma'\,\Psi_{\underline{x}\sigma'} = \Psi_{\underline{x}\tau'}\,\tau'$ and $\sigma'\,\Psi_{\sigma\sigma'} = \Psi_{\sigma\tau'}\,\tau'$. Thus, multiplying first the σ'-column by σ', and subtracting then τ' times the τ'-column from the σ'-column, we get that the determinant above is comparable to

$$\begin{vmatrix} 0 & 0 & \Psi_{\underline{x}\tau'} & \Psi_{\underline{x}y} \\ 0 & 0 & 4\sigma' & 0 \\ 0 & 0 & \Psi_{\sigma\tau'} & \Psi_{\sigma y} \\ \Psi_{y\underline{x}'} & \sigma'\,\Psi_{y\sigma'} - \Psi_{y\tau'}\,\tau' & \Psi_{y\tau'} & \Psi_{yy} \end{vmatrix}.$$

Using the u-row for cofactor expansion, the modulus of this determinant equals $4\sigma'$ times the modulus of

$$\begin{vmatrix} 0 & 0 & \Psi_{\underline{x}y} \\ 0 & 0 & \Psi_{\sigma y} \\ \Psi_{y\underline{x}'} & \sigma'\,\Psi_{y\sigma'} - \Psi_{y\tau'}\,\tau' & \Psi_{yy} \end{vmatrix}.$$

Let

$$h(\tau') = g(\tau') - \tau' g'(\tau') = (\tau')^2/(\sin\tau')^2, \tag{6.13}$$

and note that $h(\tau') \sim 1$ for $|\tau'| \le 5\pi/8$. Then

$$\sigma'\,\Psi_{y\sigma'} - \Psi_{y\tau'}\,\tau' = -2\sigma'\,(x'-y)\,h(\tau').$$

Recall that $|x_1' - y_1| \sim 1$ and $|\underline{x}' - \underline{y}| \ll 1$. Thus, subtracting $\frac{x_j'-y_j}{x_1'-y_1}$ times the y_1-row from the y_j-row for all $j \in \{2,\dots,d_1\}$, the above determinant equals

$$\begin{vmatrix} 0 & 0 & \Psi_{\underline{x}y} \\ 0 & 0 & \Psi_{\sigma y} \\ \Psi_{y_1\underline{x}'} & -2\sigma'\,(x_1'-y_1)\,h(\tau') & \Psi_{y_1 y} \\ \big(\Psi_{y_j\underline{x}'} - \frac{x_j'-y_j}{x_1'-y_1}\Psi_{y_1\underline{x}'}\big)_{j\ge 2} & 0 & \big(\Psi_{y_j y} - \frac{x_j'-y_j}{x_1'-y_1}\Psi_{y_1 y}\big)_{j\ge 2} \end{vmatrix}.$$

Using the middle column in this block matrix for cofactor expansion, the modulus of this determinant equals $2\sigma'\,|x_1'-y_1|\,h(\tau')$ times the modulus of

$$\begin{vmatrix} 0 & \Psi_{\underline{x}y} \\ 0 & \Psi_{\sigma y} \\ \big(\Psi_{y_j\underline{x}'} - \frac{x_j'-y_j}{x_1'-y_1}\Psi_{y_1\underline{x}'}\big)_{j\ge 2} & \big(\Psi_{y_j y} - \frac{x_j'-y_j}{x_1'-y_1}\Psi_{y_1 y}\big)_{j\ge 2} \end{vmatrix}.$$

Note that $\Psi_{y_j y_k} = 0$ for $j \ne k$. Thus, the above determinant equals

$$\begin{vmatrix} 0 & \Psi_{\underline{x}y_1} & \big(\Psi_{\underline{x}y_k}\big)_{k\ge 2}^{\intercal} \\ 0 & \Psi_{\sigma y_1} & \big(\Psi_{\sigma y_k}\big)_{k\ge 2}^{\intercal} \\ \big(\Psi_{y_j\underline{x}'} - \frac{x_j'-y_j}{x_1'-y_1}\Psi_{y_1\underline{x}'}\big)_{j\ge 2} & \big(-\frac{x_j'-y_j}{x_1'-y_1}\Psi_{y_1 y_1}\big)_{j\ge 2} & \big(\Psi_{y_j y_k}\big)_{j,k\ge 2} \end{vmatrix}.$$

since $\Psi_{y_1 y_k} = 0$ for $k \ge 2$. Moreover,

$$\Psi_{\sigma y} = 2\,(x-y)\,h(\tfrac{\sigma'}{\sigma}\tau').$$

Thus, using that $|x_1 - y_1| \sim 1$ and $|\underline{x} - \underline{y}| \ll 1$, subtracting $\frac{x_k-y_k}{x_1-y_1}$ times the y_1-column from the y_k-column for all $k \in \{2,\dots,d_1\}$, and using then cofactor expansion in the σ-row, the above determinant equals $2\,|x_1-y_1|\,h(\frac{\sigma'}{\sigma}\tau')$ times

$$\begin{vmatrix} 0 & \big(\Psi_{\underline{x}y_k} - \frac{x_k-y_k}{x_1-y_1}\Psi_{\underline{x}y_1}\big)_{k\ge 2}^{\intercal} \\ \big(\Psi_{y_j\underline{x}'} - \frac{x_j'-y_j}{x_1'-y_1}\Psi_{y_1\underline{x}'}\big)_{j\ge 2} & \big(\Psi_{y_j y_k} + \frac{x_j'-y_j}{x_1'-y_1}\frac{x_k-y_k}{x_1-y_1}\Psi_{y_1 y_1}\big)_{j,k\ge 2} \end{vmatrix}.$$

Note that all these matrices are $(d_1-1)\times(d_1-1)$ matrices. Let A be the matrix in the lower left corner and B upper right corner.

Then the modulus of the determinant of the matrix above equals $|\det A|\,|\det B|$. Note that

$$\Psi_{\underline{y}\underline{x}'} = -2\sigma'\big(g(\tau')I_{d_1-1} + \tau'\,\underline{J}\big) \quad \text{and} \quad \Psi_{\underline{x}\underline{y}} = 2\sigma g\big(\tfrac{\sigma'}{\sigma}\tau'\big)I_{d_1-1} - 2\sigma'\tau'\,\underline{J},$$

where the matrix $\underline{J}$ is skew-symmetric. Since $|\tau'| \le \frac{3}{8}\pi$ on the support of $\eta_{0,\mathsf{nh}}$ and σ'/σ is sufficiently close to 1, we have $g(\tau') \sim 1$ and $g(\frac{\sigma'}{\sigma}\tau') \sim 1$. Thus,

$$|\Psi_{\underline{y}\underline{x}'}v| \sim |v| \quad \text{and} \quad |\Psi_{\underline{x}\underline{y}}v| \sim |v| \quad \text{for all } v \in \mathbb{R}^{d_1-1}.$$

Since $|\frac{x_j'-y_j}{x_1'-y_1}| \ll 1$ and $|\frac{x_j-y_j}{x_1-y_1}| \ll 1$ for $j \ge 2$, we get

$$|Av| \sim |v| \quad \text{and} \quad |Bv| \sim |v| \quad \text{for all } v \in \mathbb{R}^{d_1-1}.$$

In particular, both A and B are invertible with determinant comparable to 1. □

Remark 6.6. If $\tau' = \frac{\pi}{2}$, then $g(\tau') = 0$. Moreover, the matrix $\underline{J} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \underline{\underline{J}} \end{smallmatrix}\right)$ (where $\underline{\underline{J}} = J_{n-1}$) is degenerate. Thus the last step of the proof would break down if we were to apply the same arguments to the horizontal part of $T_\lambda^{0,\varepsilon}$.

In analogy to what we shall do later for the horizontal part of the operator $\breve{T}_\lambda^{k,1,\varepsilon}$ in Proposition 12.4, we should not freeze the variable x_1 here, but x_2.

7. The horizontal part

Our goal in this section is to prove Proposition 5.2 with $T_{\lambda,t}^{0,\mathsf{h}}$ in place of $T_{\lambda,t}^{0}$. In conjunction with Section 6, this completes the proof of Proposition 5.2.

For the horizontal part $T_\lambda^{0,\mathsf{h},\varepsilon}$, we have to replace the kernel $K_{\lambda,s}^0$ in (5.3) by

$$K_{\lambda,s}^{0,\mathsf{h}}(x,u) := \lambda^{n+\frac{3}{2}} \int_{\mathbb{R}} \int_0^\infty e^{i\lambda\tilde{\Phi}(x,u,s,\sigma,\tau)}\, a_\lambda(s\sigma)\, \eta_{0,\mathsf{h}}(\tau)\, d\sigma\, d\tau,$$

where $\operatorname{supp}\eta_{0,\mathsf{h}} \subset [\frac{3}{16}\pi, \frac{5}{8}\pi]$, with phase as in (5.4), that is,

$$\tilde{\Phi}(x,u,s,\sigma,\tau) = \sigma\big(s - \tau(|x|^2\cot(\tau) - 4u)\big).$$

For the pointwise estimates, one easily checks that the arguments in the proof of Proposition 6.3 remain valid for $|\tau|, |\tau'| \le \frac{5}{8}\pi$, so the proposition also holds for the kernel $Q_\lambda^{0,\mathsf{h},\varepsilon}$ in place of $Q_\lambda^{0,\mathsf{nh},\varepsilon}$.

On the other hand, for the L^2 estimates, shifting coordinates $\tau = \tilde{\tau} + \frac{\pi}{2}$, so that $\tilde{\tau} \in [-\frac{5}{16}\pi, \frac{1}{8}\pi] \subset [-\frac{3}{8}\pi, \frac{3}{8}\pi]$, we arrive at the phase

$$\breve{\Phi} = \sigma\big(s + (\tfrac{\pi}{2} + \tilde{\tau})(|x|^2\tan(\tilde{\tau}) + 4u)\big). \tag{7.1}$$

Up to a translation and a constant rescaling in the u-variable, this phase agrees with the phase

$$\sigma\,\tilde{\Phi}^- = \sigma\Big[\delta^{-1} - 1 + s + \big(k\pi - \tfrac{1}{2}\pi + \tilde{\tau}\big)\Big(\delta|x|^2\tan(\tilde{\tau}) + \frac{u}{k} + 4\delta^{-1}u_c(\tau_{k,1})\Big)\Big]$$

in the oscillatory integral (12.1), if we set $k = 1$ and hence $\delta = 1$. Thus, freezing the variable x_2, the L^2 estimates from Proposition 12.4 include the estimate for $T_{\lambda,x_2}^{0,\mathsf{h},\varepsilon}(T_{\lambda,x_2'}^{0,\mathsf{h},\varepsilon})^*$ as a special case when $\delta = 1$.

Alternatively, for the L^2 estimate, we may also follow the arguments of the proof of Lemma 6.5. In what follows we set $\tilde{x} = (x_1, x_3, \dots, x_{d_1})$ and $\tilde{x}' =$

$(x_1', x_3', \dots, x_{d_1}')$, while keeping the previous notation for $\underline{y} = (y_2, \dots, y_{d_1})$. For the L^2 estimate for $T^{0,\mathsf{h},\varepsilon}_{\lambda,x_2}(T^{0,\mathsf{h},\varepsilon}_{\lambda,x_2'})^*$ we need to consider the phase function Ψ in (5.10), but with x_2, x_2' frozen, that is,

$$\Psi(\tilde{x}, u, s, \tilde{x}', u', s', \sigma, \sigma', \tau', y) = \sigma s - \sigma' s' - |x-y|^2 \sigma'\tau' \cot(\tfrac{\sigma'}{\sigma}\tau') \\ + |x'-y|^2 \sigma'\tau' \cot \tau' + 4\sigma'\tau'(u - u' + \tfrac{1}{2}\langle J(x-x'), y\rangle).$$

We need to verify

$$\left| \det \begin{pmatrix} \Psi_{(\tilde{x},u,s),(\tilde{x}',u',s')} & \Psi_{(\tilde{x},u,s),(\sigma,\sigma',\tau',y)} \\ \Psi_{(\sigma,\sigma',\tau',y),(\tilde{x}',u',s')} & \Psi_{(\sigma,\sigma',\tau',y),(\sigma,\sigma',\tau',y)} \end{pmatrix} \right| \sim 1$$

The same matrix and determinant manipulations as in the previous section, with $\underline{x}, \underline{x}'$ replaced by $\tilde{x}, \tilde{x}'$, reduce matters to verifying that, for $\tau' \in \operatorname{supp} \eta_{0,\mathsf{h}}$, the modulus of the determinant

$$\left| \begin{matrix} 0 & \big(\Psi_{\tilde{x}y_k} - \frac{x_k - y_k}{x_1 - y_1}\Psi_{\tilde{x}y_1}\big)_{k\ge 2}^{\intercal} \\ \big(\Psi_{y_j\tilde{x}'} - \frac{x_j' - y_j}{x_1' - y_1}\Psi_{y_1\tilde{x}'}\big)_{j\ge 2} & \big(\Psi_{y_jy_k} + \frac{x_j' - y_j}{x_1' - y_1}\frac{x_k - y_k}{x_1 - y_1}\Psi_{y_1y_1}\big)_{j,k\ge 2} \end{matrix} \right|$$

is bounded below.

Since, as before, $|x_k - y_k| \ll |x_1 - y_1|$ and $|x_k' - y_k| \ll |x_1' - y_1|$ for $k \ge 2$, it suffices to verify that the $(d_1-1)\times(d_1-1)$ matrices $\Psi_{\underline{y}\tilde{x}'}$ and $\Psi_{\tilde{x}\underline{y}}$ are uniformly invertible. We compute

$$\Psi_{\underline{y}\tilde{x}'} = -2\sigma'\big(g(\tau')A + \tau' B\big),$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & I_{d_1-2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{J} \end{pmatrix},$$

and where $\tilde{J} = J_{n-1}$ is a $(d_1-2)\times(d_1-2)$ invertible skew-symmetric matrix. Clearly B is invertible and since in our present case $|\sigma' g(\tau')| \ll 1$ we get that $\Psi_{\underline{y}\tilde{x}'}$ is also invertible. The same argument shows that $\Psi_{\tilde{x}\underline{y}}$ is also invertible.

Altogether, in either way, we see that the "frozen" operators $T^{0,\mathsf{h},\varepsilon}_{\lambda,x_2}$ satisfy the analogous estimates of Propositions 6.1 and 6.2, only with x_1, x_1' replaced by x_2, x_2'. Following from thereon the previous arguments for the non-horizontal part, we thus complete the proof of Proposition 5.2.

8. The case $k \ge 1$: Preliminary reductions

In this section, we first treat the range $2^\ell k \gg \lambda$ using estimates from [41]. In remaining the range $2^\ell k \lesssim \lambda$, we reduce the proof of Theorem 1.3 to the square function estimate (8.10) in Proposition 8.4.

Recall from (4.18) that we have the decomposition

$$\beta\big(\tfrac{t\sqrt{L}}{\lambda}\big)e^{it\sqrt{L}}f = \widetilde{\beta}\big(\tfrac{t^2L}{\lambda^2}\big)\, T^0_{\lambda,t} f + \sum_{0<|k|<8\lambda} \sum_{\ell\ge 1} \widetilde{\beta}\big(\tfrac{t^2L}{\lambda^2}\big)\, T^{k,\ell}_{\lambda,t} f + \rho_\lambda\big(\tfrac{t^2L}{\lambda^2}\big) f.$$

We shall now turn to estimating the contributions of the operators $T^{k,\ell}_{\lambda,t} f = f * \mathcal{K}^{k,\ell}_{\lambda,t}$ to (1.4) in Theorem 1.3. In view of Remark 4.3, we may and shall assume that $k > 0$ in what follows.

8.1. **The contributions by the $T^{k,\ell}_{\lambda,t}$ with $2^\ell k \gg \lambda$.** We consider estimates for the case $2^\ell k \gg \lambda$, with $k \le 8\lambda$, for which we are not expected to use curvature and oscillation in a significant way.

In this subsection, we establish estimates dual to the square function estimates in Theorem 1.3. Note that the conjugate exponent of $\frac{2d_1}{d_1+1}$ is $\frac{2d_1}{d_1-1}$.

Proposition 8.1. *For $1 \le p < \frac{2d_1}{d_1+1}$ and all $\ell \ge 1$,*

$$\Big\| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big\|_p \lesssim 2^{-\frac{\ell}{4}} \lambda^{d(\frac{1}{p}-\frac12)-\frac12} \Big\| \Big(\int_1^2 |f_t|^2 \, dt \Big)^{\frac12} \Big\|_p,$$

where $\mathcal{I}_{\lambda,\ell} := \{k > 0 : 10^5\lambda \le 2^\ell k \text{ and } k \le 8\lambda\}$.

Proof. We interpolate between L^1 and L^2. For the L^1 bounds, we use Lemmas 8.5 and 8.6 of [41] (see also Eq. (100) there), which yield

$$\|\lambda^{-\frac{d-1}{2}} T^{k,\ell}_{\lambda,t}\|_{L^1\to L^1} \lesssim \lambda^{-\frac{d_1-1}{2}} (2^\ell k)^{-1} \quad \text{for } k \in \mathcal{I}_{\lambda,\ell}.$$

Thus, for $k \in \mathcal{I}_{\lambda,\ell}$,

$$\Big\| \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big\|_1 \lesssim \lambda^{\frac12} (2^\ell k)^{-1} \Big\| \int_1^2 |f_t| \, dt \Big\|_1,$$

and therefore, for $K \le \lambda$,

$$\Big\| \sum_{k\sim K} \Big| \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big| \Big\|_1 \lesssim \lambda^{\frac12} 2^{-\ell} \Big\| \int_1^2 |f_t| \, dt \Big\|_1.$$

Note that $10^5 2^{-\ell}\lambda \le k \le 8\lambda$ for $k \in \mathcal{I}_{\lambda,\ell}$. Hence

$$\Big\| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \Big| \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big| \Big\|_1 \lesssim C(\lambda,\ell) \Big\| \int_1^2 |f_t| \, dt \Big\|_1, \tag{8.1}$$

with constant

$$C(\lambda,\ell) = \begin{cases} \lambda^{\frac12} 2^{-\ell}(\ell+1) & \text{if } 2^\ell \le \lambda, \\ \lambda^{\frac12} \log(\lambda+1) 2^{-\ell} & \text{if } 2^\ell \ge \lambda. \end{cases}$$

We proceed with the L^2 estimates. Note that

$$\begin{aligned} T^{k,\ell}_{\lambda,t} &= \lambda^{\frac12} \int_0^\infty e^{i\frac{\lambda}{4\varsigma}} \, a_{\lambda,\circ}(\varsigma) \, \eta_\ell\Big(\frac{\varsigma t^2}{\lambda}(-iU) - k\pi\Big) \, e^{i\varsigma t^2 L/\lambda} \, d\varsigma \\ &= \lambda^{\frac12} \int_0^\infty e^{i\frac{\lambda t^2}{4s}} \, a_{\lambda,\circ}(st^{-2}) \, \eta_\ell(\lambda^{-1}s(-iU) - k\pi) \, e^{isL/\lambda} \, t^{-2} \, ds. \end{aligned}$$

Thus,

$$\int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt = \int \eta_\ell(\lambda^{-1}s(-iU) - k\pi) \, e^{isL/\lambda} \, G_s \, ds,$$

where

$$G_s = \lambda^{\frac12} \int e^{i\frac{\lambda t^2}{4s}} \, a_{\lambda,\circ}(st^{-2}) \, t^{-2} \, f_t \, dt.$$

Note that $|\frac{\partial^2}{\partial s \partial t}(t^2 s^{-1}/4)| \sim 1$ for $s, t \sim 1$. Thus, by standard oscillatory L^2 integral estimates, we have

$$\Big(\int_J |G_s(x,u)|^2 \, ds \Big)^{\frac12} \lesssim \Big(\int_1^2 |f_t(x,u)|^2 \, dt \Big)^{\frac12}. \tag{8.2}$$

Note that the s integral is extended over a compact interval $J \subset [\frac{1}{8}, 32]$. Using Plancherel in the central variable u, we get

$$\Big\| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big\|_2 = \Big(\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}} \Big| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_1^2 (T^{k,\ell}_{\lambda,t} f_t)^\mu(x)\, dt \Big|^2 d\mu\, dx \Big)^{\frac12}.$$

Note that for any $x \in \mathbb{R}^{d_1}$,

$$\int_1^2 (T^{k,\ell}_{\lambda,t} f_t)^\mu(x)\, dt = \int_J \eta_\ell(2\pi s\tfrac{\mu}{\lambda} - k\pi)(e^{isL/\lambda} G_s)^\mu(x)\, ds.$$

Since $2^\ell \ge 10^5\lambda/k \ge 10^5/8 \gg 1$, we have $2^{-\ell} \ll 1$. For fixed μ, the condition $\eta_\ell(2\pi s\frac{\mu}{\lambda} - k\pi) \neq 0$ implies that $|2\pi s\mu - \lambda k\pi| \le \lambda 2^{1-\ell}$, and since $1/8 \le s \le 32$, this implies that

$$1 \le k \sim \frac{|\mu|}{\lambda}.$$

Moreover, since $|s - \frac{\lambda}{\mu}\frac{k}{2}| \le \frac{\lambda}{|\mu|}2^{-\ell}$, where $2^{-\ell} \ll \frac12$, we see that for fixed μ and x, the functions

$$g^{\mu,x}_k(s) = \eta_\ell(2\pi s\tfrac{\mu}{\lambda} - k\pi)(e^{isL/\lambda} G_s)^\mu(x)$$

are supported in pairwise disjoint intervals of length $\frac{\lambda}{|\mu|}2^{-\ell}$. And, since $k \sim |\mu|/\lambda$, we see that $\sum_{k\in\mathcal{I}_{\lambda,\ell}} g^{\mu,x}_k$ is supported in a set of total volume bounded by a constant times $(\frac{|\mu|}{\lambda})\frac{\lambda}{|\mu|}2^{-\ell} \sim 2^{-\ell}$. Thus, by Cauchy–Schwarz,

$$\begin{aligned}\Big| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_1^2 (T^{k,\ell}_{\lambda,t} f_t)^\mu(x)\, dt \Big| &= \Big| \int_J \sum_{k\in\mathcal{I}_{\lambda,\ell}} g^{\mu,x}_k(s)\, ds \Big| \\ &\lesssim 2^{-\frac{\ell}{2}} \Big(\int_J \Big| \sum_{k\in\mathcal{I}_{\lambda,\ell}} g^{\mu,x}_k(s) \Big|^2 ds \Big)^{\frac12} \lesssim 2^{-\frac{\ell}{2}} \Big(\int_J \sum_{k\in\mathcal{I}_{\lambda,\ell}} |g^{\mu,x}_k(s)|^2\, ds \Big)^{\frac12}.\end{aligned}$$

We thus obtain

$$\begin{aligned}&\Big\| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big\|_2 \\ &\lesssim 2^{-\frac{\ell}{2}} \Big(\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}} \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_J \Big| \eta_\ell(2\pi s\tfrac{\mu}{\lambda} - k\pi)(e^{isL/\lambda} G_s)^\mu(x) \Big|^2 ds\, d\mu\, dx \Big)^{\frac12} \\ &\lesssim 2^{-\frac{\ell}{2}} \Big(\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}} \int_J \Big| (e^{isL/\lambda} G_s)^\mu(x) \Big|^2 ds\, d\mu\, dx \Big)^{\frac12},\end{aligned}$$

since $\sum_{k\in\mathcal{I}_{\lambda,\ell}} |\eta_\ell(2\pi s\frac{\mu}{\lambda} - k\pi)|^2 \lesssim 1$. Hence, again by Plancherel, since $e^{isL/\lambda}$ is a unitary operator on L^2,

$$\Big\| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big\|_2 \lesssim 2^{-\frac{\ell}{2}} \Big\| \Big(\int_J |G_s|^2\, ds \Big)^{\frac12} \Big\|_2 \lesssim 2^{-\frac{\ell}{2}} \Big\| \Big(\int_1^2 |f_t|^2\, dt \Big)^{\frac12} \Big\|_2, \tag{8.3}$$

where we have applied (8.2) in the last inequality.

Finally, for $1 \le p \le 2$, by interpolation between (8.1) and (8.3), we get

$$\Big\| \sum_{k\in\mathcal{I}_{\lambda,\ell}} \int_1^2 T^{k,\ell}_{\lambda,t} f_t \, dt \Big\|_p \lesssim \lambda^{\frac{1}{p}-\frac{1}{2}}\, 2^{-\frac{\ell}{p}} \times \max\big\{(\ell+1)^{\frac{1}{p}}, (\log\lambda)^{\frac{1}{p}}\big\} \, \Big\| \Big(\int_1^2 |f_t|^p \, dt \Big)^{\frac{1}{p}} \Big\|_p .$$

Now

$$\Big(\int_1^2 |f_t|^p \, dt \Big)^{\frac{1}{p}} \le \Big(\int_1^2 |f_t|^2 \, dt \Big)^{\frac{1}{2}}$$

and since $\frac{1}{p} - \frac{1}{2} < d\,(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ if and only if $p < \frac{2d_1}{d_1+1}$ we have established Proposition 8.1. □

Note that the estimates in Proposition 8.1 can be summed over all $\ell > 0$, so that the contributions of these k and ℓ to the left-hand side of (1.4) in Theorem 1.3 are fine.

8.2. **Support reduction of the convolution kernel of $T^{k,\ell}_{\lambda,t}$.** We are left with the contributions of the operators $T^{k,\ell}_{\lambda,t}$ with $0 < k \le 8\lambda$ and $2^\ell k \lesssim \lambda$. By Proposition 4.2, for $k \neq 0$ and $\ell \ge 1$, the convolution kernel of $T^{k,\ell}_{\lambda,t}$ is given by

$$\mathcal{K}^{k,\ell}_{\lambda,t}(x,u) = \lambda^{n+\frac{3}{2}}\, t^{-2n} \int_0^\infty \int_0^\infty e^{i\lambda\Phi(x,u,t,\sigma,\tau)}\, a_\lambda(t^2\sigma)\, \eta_\ell(\tau - k\pi) \Big(\frac{\tau}{\sin\tau}\Big)^n \, d\sigma\, d\tau,$$

where a_λ is supported in $(\frac{1}{16}, 4)$, and

$$\Phi(x,u,t,\sigma,\tau) = \sigma(t^2 - |x|^2 g(\tau) + 4u\tau), \text{ with } g(\tau) = \tau\cot\tau.$$

As in the case $k = 0$, in order to simplify the notation slightly, we will drop the factor t^{-2n}, replace t by

$$s := t^2 \in [1,4],$$

and work from here on with the convolution kernel

$$K^{k,\ell}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty \int_0^\infty e^{i\lambda\tilde\Phi(x,u,s,\sigma,\tau)}\, a_\lambda(s\sigma)\, \eta_\ell(\tau - k\pi) \Big(\frac{\tau}{\sin\tau}\Big)^n d\sigma\, d\tau \tag{8.4}$$

in place of $\mathcal{K}^{k,\ell}_{\lambda,t}$, where, as in (5.4),

$$\tilde\Phi(x,u,s,\sigma,\tau) = \sigma(s - |x|^2 g(\tau) + 4u\tau). \tag{8.5}$$

Correspondingly, we shall estimate the operator $\tilde T^{k,\ell}_{\lambda,s}$ defined by

$$\tilde T^{k,\ell}_{\lambda,s} f = f * K^{k,\ell}_{\lambda,s}$$

in place of $T^{k,\ell}_{\lambda,t}$. Note that

$$\Big\| \Big(\int_1^2 \big| T^{k,\ell}_{\lambda,t} f \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \sim \Big\| \Big(\int_1^4 \big| \tilde T^{k,\ell}_{\lambda,s} f \big|^2 \, ds \Big)^{\frac{1}{2}} \Big\|_p . \tag{8.6}$$

Assume now that $k \ge 1$ and $\ell \ge 1$. In order to prepare our decay estimates for fixed k and ℓ, we compute the critical points of the phase function $\tilde\Phi$. We have

$$\nabla^{\mathsf{T}}_{\sigma,\tau} \tilde\Phi(x,u,s,\sigma,\tau) = \begin{pmatrix} s - |x|^2 g(\tau) + 4\tau u \\ \sigma(-|x|^2 g'(\tau) + 4u) \end{pmatrix} = A_{\sigma,\tau} \begin{pmatrix} |x|^2 \\ u \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix},$$

where

$$A_{\sigma,\tau} := \begin{pmatrix} -g(\tau) & 4\tau \\ -\sigma g'(\tau) & 4\sigma \end{pmatrix}.$$

Note that

$$A_{\sigma,\tau}^{-1} = \frac{1}{\sigma\tau^2} \begin{pmatrix} -\sigma \sin^2\tau & \tau \sin^2\tau \\ \frac{1}{4}\sigma(\tau - \sin\tau\cos\tau) & \frac{1}{4}\tau\sin\tau\cos\tau \end{pmatrix}. \tag{8.7}$$

We have $g'(\tau) = \cot\tau - \tau/\sin^2\tau$. Thus, $\nabla_{\sigma,\tau}\tilde\Phi(x,u,s,\sigma,\tau) = 0$ if and only if

$$\begin{pmatrix} |x|^2 \\ u \end{pmatrix} = A_{\sigma,\tau}^{-1} \begin{pmatrix} -s \\ 0 \end{pmatrix} = s \begin{pmatrix} \sin^2\tau/\tau^2 \\ -\frac{1}{4}(\tau - \sin\tau\cos\tau)/\tau^2 \end{pmatrix}.$$

Recall that the factor $\eta_\ell(\tau - k\pi)$ localizes to the union $J_{k,\ell}$ of two intervals on which $|\tau - k\pi| \sim 2^{-\ell}$. Given $\tau \in J_{k,\ell}$, we write

$$|x|_c(s,\tau) := \sqrt{s}\,\Big|\frac{\sin\tau}{\tau}\Big| \quad \text{and} \quad u_c(s,\tau) := -s\,\frac{\tau - \sin\tau\cos\tau}{4\tau^2}. \tag{8.8}$$

Moreover, we write

$$\Delta x^2(s,\tau) := |x|^2 - |x|_c^2(s,\tau) \quad \text{and} \quad \Delta u(s,\tau) := u - u_c(s,\tau).$$

Then

$$\begin{aligned} \nabla_{\sigma,\tau}^{\mathsf T}\tilde\Phi(x,u,s,\sigma,\tau) &= A_{\sigma,\tau}\begin{pmatrix} |x|^2 - |x|_c^2(s,\tau) \\ u - u_c(s,\tau) \end{pmatrix} \\ &= A_{\sigma,\tau}\begin{pmatrix} \Delta x^2(s,\tau) \\ \Delta u(s,\tau) \end{pmatrix}. \end{aligned} \tag{8.9}$$

We fix any point $\tau_{k,\ell} \in J_{k,\ell}$. For $\tau \in J_{k,\ell}$, we have

$$|x|_c(s,\tau) \sim 2^{-\ell}/k \quad \text{and} \quad |u_c(s,\tau)| \sim 1/k.$$

Moreover,

$$|u_c(s,\tau) - u_c(s,\tau_{k,\ell})| \lesssim \frac{2^{-\ell}}{k^2} \quad \text{and} \quad \big||x|_c^2(s,\tau) - |x|_c^2(s,\tau_{k,\ell})\big| \lesssim \Big(\frac{2^{-\ell}}{k}\Big)^2.$$

This suggest the following to hold: $K_{\lambda,s}^{k,\ell}$ is essentially supported in a "hollow cylinder" of the form

$$||x| - |x|_c(s,\tau_{k,\ell})| \lesssim \frac{2^{-\ell}}{k} \quad \text{and} \quad |u - u_c(s,\tau_{k,\ell})| \lesssim \frac{2^{-\ell}}{k^2}.$$

We will make this more precise in the next proposition, whose proof will be given in Appendix A.

Proposition 8.2. *There is some constant $R_0 > 0$ such that for any $s \in [1,4]$ the following holds: Let $k \ge 1$ and $\ell \ge 1$, and let $\tau_{k,\ell}$ be in the support of $\eta_\ell(\tau - k\pi)$. Denote by $E_s^{k,\ell}$ the set of all $(x,u) \in \mathbb{R}^{d_1} \times \mathbb{R}$ such that*

$$|x| \le 10^{-1}\frac{2^{-\ell}}{k}, \quad \text{or} \quad |x| \ge \sqrt{10}\,\frac{2^{-\ell}}{k}, \quad \text{or} \quad |u - u_c(s,\tau_{k,\ell})| \ge R_0\frac{2^{-\ell}}{k^2}.$$

Then, for any $N \in \mathbb{N}$,

$$\int_{E_s^{k,\ell}} |K_{\lambda,s}^{k,\ell}(x,u)|\,d(x,u) \le C_N\Big(\frac{\lambda}{2^\ell k}\Big)^{-N}\Big(\frac{2^\ell}{k}\Big)^{\frac{1}{2}}.$$

Let us put $u_c(\tau) = u_c(1,\tau)$, so that $u_c(s,\tau) = s\,u_c(\tau)$.

Corollary 8.3. *There is some constant $R_0 > 0$ such that for any $s \in [1, 1 + \frac{2^{-\ell}}{k}]$ the following holds: If $E^{k,\ell}$ denotes the set of all $(x,u) \in \mathbb{R}^{d_1} \times \mathbb{R}$ such that*

$$|x| \le 10^{-1} \frac{2^{-\ell}}{k}, \quad \text{or} \quad |x| \ge \sqrt{10}\, \frac{2^{-\ell}}{k}, \quad \text{or} \quad |u - u_c(\tau_{k,\ell})| \ge R_0 \frac{2^{-\ell}}{k^2},$$

then, for any $N \in \mathbb{N}$,

$$\int_{E^{k,\ell}} |K^{k,\ell}_{\lambda,s}(x,u)|\, d(x,u) \le C_N \Big(\frac{\lambda}{2^\ell k}\Big)^{-N} \Big(\frac{2^\ell}{k}\Big)^{\frac12}.$$

Proof. Since $|u_c(\tau)| \sim 1/k$, we see that for $s \in [1, 1+\frac{2^{-\ell}}{k}]$ we have $|u_c(s,\tau) - u_c(\tau)| \lesssim 2^{-\ell} k^{-1} k^{-1} = 2^{-\ell} k^{-2}$, so that, for some constant $C > 0$,

$$|u - u_c(s, \tau_{k,\ell})| \ge |u - u_c(\tau_{k,\ell})| - C \frac{2^{-\ell}}{k^2}.$$

The asserted estimate follows thus from the one in Proposition 8.2, if we increase the constant R_0 in a suitable way. □

8.3. **Reduction to shorter time intervals.** Assume again that $k, \ell \ge 1$ are such that $k \le 8\lambda$ and $2^\ell k \lesssim \lambda$. For k and ℓ fixed, we shall use in the sequel the abbreviation

$$\delta = \delta(k,\ell) := 2^{1-\ell}/k.$$

In view of Corollary 8.3, we shall next reduce the square function estimates for $\tilde T^{k,\ell}_{\lambda,s}$ to estimates over shorter s-intervals of length at most δ.

Proposition 8.4. *Assume there is some $\varepsilon \in (0,1)$ such that for all $p \ge \frac{2(d_1+1)}{d_1-1}$ and all k, ℓ as above the following estimate*

$$\Big\| \Big(\int_1^{1+\varepsilon\delta(k,\ell)} \big| \tilde T^{k,\ell}_{\lambda,s} f \big|^2 \, ds \Big)^{\frac12} \Big\|_p \le C(\varepsilon, a_\lambda, p)\, \delta(k,\ell)^{\frac12} \times \Big[\lambda^{d(\frac12 - \frac1p) - \frac12} \, (2^\ell k)^{-d(\frac12-\frac1p)+\frac1p} + \Big(\frac{2^\ell}{k}\Big)^{\frac12} \Big] \, \|f\|_p \tag{8.10}$$

holds for all $f \in \mathcal{S}(\mathbb{R}^d)$, where the constant $C(\varepsilon, a_\lambda, p)$ depends in a_λ only on the C^M-norm of a_λ for some sufficiently large $M \in \mathbb{N}$.

Then for $p \ge \frac{2(d_1+2)}{d_1-1}$ and $\alpha > d\,(\frac12 - \frac1p) - \frac12$, or $p > \frac{2(d_1+2)}{d_1-1}$ and $\alpha \ge d\,(\frac12 - \frac1p) - \frac12$, the following estimate

$$\Big\| \Big(\int_1^2 \Big| \sum_{1 \le k \le 8\lambda,\, 2^\ell k \lesssim \lambda} T^{k,\ell}_{\lambda,t} f \Big|^2 dt \Big)^{\frac12} \Big\|_p \le C_\varepsilon\, C(\varepsilon, a_\lambda, p)\, \lambda^\alpha \, \|f\|_p \tag{8.11}$$

holds true, which completes the proof of Theorem 1.3.

Proof. As a first step, we decompose the time interval $[1,2]$ in (8.11) into about $1/\varepsilon$ intervals of length $\sim \varepsilon$. Exploiting homogeneity, by means of automorphic scalings, we may then reduce each of the corresponding integrals in t to the interval $[1, \sqrt{1+\varepsilon}]$, which corresponds to the s-interval $[1, 1+\varepsilon]$.

Given k and ℓ, we next decompose the s-interval $[1, 1+\varepsilon]$ into disjoint subintervals

$$I_j = [s_j, s_{j+1}], \qquad s_1 = 1, \qquad s_{N+1} = 1 + \varepsilon,$$

with $\delta = \delta(k,\ell)$, where $s_{j+1}/s_j = 1 + \varepsilon\delta$ for $j = 1, \dots, N-1$ and $s_{N+1}/s_N \le 1 + \varepsilon\delta$. Then $s_j \in [1, 1+\varepsilon]$ for all $j \in \{1, \dots, N\}$. Moreover, $s_j^{-1} I_j \subset [1, 1+\varepsilon\delta]$ and $|I_j| \lesssim \varepsilon\delta$. Note that $N \sim \delta^{-1}$ with a constant that depends on ε.

Using Minkowski's inequality, we obtain

$$\Big\|\Big(\int_1^{1+\varepsilon}\big|\tilde T^{k,\ell}_{\lambda,s}f\big|^2\,ds\Big)^{\frac12}\Big\|_p=\Big\|\Big(\sum_{j=1}^N\int_{I_j}\big|\tilde T^{k,\ell}_{\lambda,s}f\big|^2\,ds\Big)^{\frac12}\Big\|_p$$
$$\le\Big(\sum_{j=1}^N\Big\|\Big(\int_{I_j}\big|\tilde T^{k,\ell}_{\lambda,s}f\big|^2\,ds\Big)^{\frac12}\Big\|_p^2\Big)^{\frac12}.$$

For a given j, making the change of variables $s=s_j\tilde s$ and putting $r_j=s_{j+1}/s_j\le 1+\varepsilon\delta$, we have

$$\Big(\int_{I_j}\big|\tilde T^{k,\ell}_{\lambda,s}f\big|^2\,ds\Big)^{\frac12}=s_j^{\frac12}\Big(\int_1^{r_j}\big|\tilde T^{k,\ell}_{\lambda,s_j\tilde s}f\big|^2\,d\tilde s\Big)^{\frac12}.$$

Since $s_j\in[1,1+\varepsilon]$, the factor $s_j^{\frac12}\sim 1$ is harmless. The phase function $\tilde\Phi$ is homogeneous of degree 2 with respect to the parabolic rescaling $(s,x,u)\mapsto(\alpha^2 s,\alpha x,\alpha^2 u)$. Combining the automorphic rescaling $D_{\sqrt{s_j}}$ with the change of variables $\tilde\sigma=s_j\sigma$, we obtain

$$K^{k,\ell}_{\lambda,s_j\tilde s}\big(D_{\sqrt{s_j}}(x,u)\big)=s_j^{-1}K^{k,\ell}_{\lambda,\tilde s}(x,u).$$

Since $s_j\in[1,1+\varepsilon]$, all resulting scaling factors are harmless. Extending the $\tilde s$-integral from $[1,r_j]$ to $[1,1+\varepsilon\delta]$, and using $N^{\frac12}\sim\delta^{-\frac12}$, we obtain

$$\|T^{k,\ell}_{\lambda,t}\|_{L^p\to L^p(L^2([1,2]))}\le C_\varepsilon\,\delta^{-\frac12}\|\tilde T^{k,\ell}_{\lambda,s}\|_{L^p\to L^p(L^2([1,1+\varepsilon\delta]))}.$$

Hence, by (8.10),

$$\|T^{k,\ell}_{\lambda,t}\|_{L^p\to L^p(L^2([1,2]))}\ \le C_\varepsilon\,C(\varepsilon,a_\lambda,p)\Big[\lambda^{d(\frac12-\frac1p)-\frac12}\,(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p}+\Big(\frac{2^\ell}{k}\Big)^{\frac12}\Big].$$

Let us denote by $p_1=\frac{2(d_1+1)}{d_1-1}=\frac{2d}{d-2}$ the Stein–Tomas exponent associated with d_1. Then $d\,(\frac12-\frac1{p_1})=1$, and thus if $p\ge p_1$ and $\alpha>d\,(\frac12-\frac1p)-\frac12$, then $\alpha>\frac12$. Thus, for $2^\ell k\lesssim\lambda$, we may estimate

$$\Big(\frac{2^\ell}{k}\Big)^{\frac12}\lesssim\Big(\frac{2^\ell}{k}\Big)^{\frac12}(2^\ell k)^{-\alpha}\lambda^\alpha=2^{-\ell(\alpha-\frac12)}k^{-(\alpha+\frac12)}\lambda^\alpha,$$

so we can bound the sum of the second terms $(\frac{2^\ell}{k})^{\frac12}$ over all k and ℓ with $2^\ell k\lesssim\lambda$ by a constant times λ^α. Similarly, if $p>p_1$ and $\alpha\ge d\,(\frac12-\frac1p)-\frac12$, then again $\alpha>\frac12$, and the summation in k and ℓ leads to the same bound of order λ^α.

Under the stronger assumption $p>\frac{2(d_1+2)}{d_1-1}=\frac{2(d+1)}{d-2}$ and $\alpha\ge d\,(\frac12-\frac1p)-\frac12$, we even have $-d\,(\frac12-\frac1p)+\frac1p<-1$, so we are able to sum the first term in k and ℓ since

$$\lambda^{d(\frac12-\frac1p)-\frac12}\,(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p}\le\lambda^\alpha\,(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p}.$$

Similarly, if $p\ge\frac{2(d_1+2)}{d_1-1}$ and $\alpha>d\,(\frac12-\frac1p)-\frac12$, then

$$\lambda^{d(\frac12-\frac1p)-\frac12}\,(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p}\le\lambda^\alpha\lambda^{-\varepsilon_p}\,(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p}$$
$$\le\lambda^\alpha(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p-\varepsilon_p}$$

for some $\varepsilon_p>0$, so we can again sum in k and ℓ.

Thus, (8.11) follows by applying Minkowski's inequality. □

9. Reductions to unit scales

In this section, we localize the kernels $K^{k,\ell}_{\lambda,s}$ to the regions $D^{k,\ell}$ below and reduce the square function estimate (8.10) of Proposition 8.4 to the rescaled square function estimate (9.16) in Proposition 9.3.

Assume again that $2^\ell k \lesssim \lambda$. For k,ℓ fixed, let us denote by $D^{k,\ell}$ the complement of the set $E^{k,\ell}$ from Corollary 8.3, which is given by

$$D^{k,\ell} = \Big\{(x,u) : 10^{-1}\frac{2^{-\ell}}{k} < |x| < \sqrt{10}\,\frac{2^{-\ell}}{k} \quad \text{and} \quad |u-u_c(\tau_{k,\ell})| < R_0\frac{2^{-\ell}}{k^2}\Big\}, \tag{9.1}$$

and let $\dot K^{k,\ell}_{\lambda,s} = \mathbb{1}_{D^{k,\ell}}\, K^{k,\ell}_{\lambda,s}$ denote the according localization of $K^{k,\ell}_{\lambda,s}$, with associated convolution operator $\dot T^{k,\ell}_{\lambda,s} f = f * \dot K^{k,\ell}_{\lambda,s}$. Then decompose

$$\tilde T^{k,\ell}_{\lambda,s} = \dot T^{k,\ell}_{\lambda,s} + R^{k,\ell}_{\lambda,s}, \tag{9.2}$$

where $R^{k,\ell}_{\lambda,s} f = f * \big(\mathbb{1}_{E^{k,\ell}}\, K^{k,\ell}_{\lambda,s}\big)$. By Corollary 8.3, we have that for $s \in [1, 1+\frac{2^{-\ell}}{k}]$ and $2^\ell k \lesssim \lambda$

$$\|\mathbb{1}_{E^{k,\ell}}\, K^{k,\ell}_{\lambda,s}\|_{L^1} \le C\Big(\frac{2^\ell}{k}\Big)^{\frac12}.$$

Then Minkowski's inequality shows that the contribution by the operator $R^{k,\ell}_{\lambda,s}$ to the square function estimate (8.10) in Proposition 8.4 adds the second term $\delta(k,\ell)^{\frac12}(\frac{2^\ell}{k})^{\frac12}$.

We are thus left with estimating the contributions by the operators $\dot T^{k,\ell}_{\lambda,s}$. The goal of this and the next two sections is to prove the following

Proposition 9.1. *If $\varepsilon > 0$ is sufficiently small and $p \ge \frac{2(d_1+1)}{d_1-1}$, then, for all $k \ge 1$, $\ell \ge 1$,*

$$\begin{aligned}\Big\|\Big(\int_1^{1+\varepsilon\delta(k,\ell)} \big|\dot T^{k,\ell}_{\lambda,s} f\big|^2\, ds\Big)^{\frac12}\Big\|_p &\le C(\varepsilon, a_\lambda, p)\,\delta(k,\ell)^{\frac12} \\ &\quad\times\Big[\lambda^{d(\frac12-\frac1p)-\frac12}\,(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p} + \Big(\frac{2^\ell}{k}\Big)^{\frac12}\Big]\,\|f\|_p\end{aligned} \tag{9.3}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$, where the constant $C(\varepsilon, a_\lambda, p)$ depends in a_λ only on the C^M-norm of a_λ for some sufficiently large $M \in \mathbb{N}$.

By Proposition 8.4, this will then complete the proof of Theorem 1.3.

9.1. Secondary localizations for the kernels $K^{k,\ell}_{\lambda,s}$.

We fix again a smooth bump function χ supported in the Euclidean ball $B_1(0) \subset \mathbb{R}^{d_1}\times\mathbb{R}$ such that $\chi \equiv 1$ on $B_{\frac12}(0)$. For $\varepsilon > 0$, we set

$$\chi^{k,\ell}_\varepsilon(x,u) = \chi\Big(\frac{k2^\ell}{\varepsilon}x, \frac{k^2 2^\ell}{\varepsilon}u\Big).$$

Proposition 9.2. *Let $\varepsilon > 0$ be sufficiently small and $p \ge \frac{2(d_1+1)}{d_1-1}$. Suppose that, for every $k,\ell \ge 1$, every $(x^0,u^0) \in \mathbb{H}_n$ such that $x^0 = (|x^0|,0,\dots,0)$ and*

$$100^{-1}\frac{2^{-\ell}}{k} \le |x^0| \le 100\,\frac{2^{-\ell}}{k} \quad \text{and} \quad |u^0 - u_c(\tau_{k,\ell})| \le 10R_0\frac{2^{-\ell}}{k^2}, \tag{9.4}$$

and every $f \in \mathcal{S}(\mathbb{H}_n)$, we have the estimate

$$\Big\| \Big(\int_1^{1+\varepsilon\delta(k,\ell)} \big| \tilde\chi_\varepsilon^{k,\ell}\, \tilde T_{\lambda,s}^{k,\ell}(\chi_\varepsilon^{k,\ell} f)\big|^2\, ds\Big)^{\frac12} \Big\|_p \le C(a_\lambda,\varepsilon,p)\,\delta(k,\ell)^{\frac12} \times \lambda^{d(\frac12-\frac1p)-\frac12}(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p}\|f\|_p, \tag{9.5}$$

where $\tilde\chi_\varepsilon^{k,\ell}(x,u) := \chi_\varepsilon^{k,\ell}(x-x^0,u-u^0)$, and where the constant $C(a_\lambda,\varepsilon,p)$ is uniform in k,ℓ and (x^0,u^0) and depends on a_λ only through its C^M-norm for some sufficiently large $M\in\mathbb{N}$. Then the square function estimate (9.3) *holds.*

Proof. This is again an easy consequence of Proposition 3.1. Assume that (9.5) holds true. As in (9.2), we decompose again

$$\tilde T_{\lambda,s}^{k,\ell} = \dot T_{\lambda,s}^{k,\ell} + R_{\lambda,s}^{k,\ell}.$$

Since $\dot T_{\lambda,s}^{k,\ell} = \tilde T_{\lambda,s}^{k,\ell} - R_{\lambda,s}^{k,\ell}$, Corollary 8.3 and (9.5) imply that

$$\Big\| \Big(\int_1^{1+\varepsilon\delta(k,\ell)} \big| \tilde\chi_\varepsilon^{k,\ell}\, \dot T_{\lambda,s}^{k,\ell}(\chi_\varepsilon^{k,\ell} f)\big|^2\, ds\Big)^{\frac12} \Big\|_p \le C(a_\lambda,\varepsilon,p)\,\delta(k,\ell)^{\frac12} \times \Big[\lambda^{d(\frac12-\frac1p)-\frac12}(2^\ell k)^{-d(\frac12-\frac1p)+\frac1p} + \Big(\frac{2^\ell}{k}\Big)^{\frac12}\Big]\|f\|_p.$$

Note first that if we fix a sufficiently large constant $R\gg 1$, then for $k=1$ the set $D^{1,\ell}$ is contained in the Euclidean ball $B_{R2^{-\ell}}(z_{1,\ell}^0)$ centered at the point $z_{1,\ell}^0=(0,u_c(\tau_{1,\ell}))$, which lies in the center of the Heisenberg group $G=\mathbb{H}_n$. Since the Euclidean distance and the left-invariant Riemannian distance d_G are equivalent on bounded sets, we may assume that $D^{1,\ell}\subset B(z_{1,\ell}^0,R2^{-\ell})$ after increasing R if necessary, where $B(z,r)$ denotes the ball of radius r centered at z with respect to the distance d_G.

For general $k\ge 1$, let $\alpha = D_{k^{-1}}$ be the automorphic dilation given by $\alpha(x,u)=(k^{-1}x,k^{-2}u)$. Then we see analogously that $D^{k,\ell}\subset \alpha\big(B(z_{k,\ell}^0,R2^{-\ell})\big)$, where the point $z_{k,\ell}^0=(0,k^2u_c(\tau_{k,\ell}))$ is again central in $\mathbb{H}_n$. Thus, by Proposition 3.1 (ii), in order to prove (9.3), it suffices to show this only for f supported in $\alpha(B(0,\varepsilon 2^{-\ell}))$. After adjusting ε by a fixed constant, it therefore suffices to assume that f is supported in

$$B_\varepsilon^{k,\ell} = \Big\{(x,u): |x|\le \varepsilon\frac{2^{-\ell}}{k}, |u|\le \varepsilon\frac{2^{-\ell}}{k^2}\Big\}.$$

But then, if ε is chosen sufficiently small, it is easy to see that $\dot T_{\lambda,s}^{k,\ell} f = f * \dot K_{\lambda,s}^{k,\ell}$ is supported in the group product set $B_\varepsilon^{k,\ell} D^{k,\ell}$, which is contained in

$$\tilde D^{k,\ell} = \Big\{(x,u): \frac{1}{50}\frac{2^{-\ell}}{k} < |x| \le 50\frac{2^{-\ell}}{k} \quad\text{and}\quad |u-u_c(\tau_{k,\ell})| < 5R_0\frac{2^{-\ell}}{k^2}\Big\}.$$

The latter is clear for $k=1$, and follows for general $k\ge1$ again by scaling with the dilation $D_{k^{-1}}$.

Since we can cover the set $\tilde D^{k,\ell}$ by $\mathcal{O}(\varepsilon^{-d})$ Euclidean translates of the set $B_\varepsilon^{k,\ell}$, we see that the estimate in (9.5) will imply estimate (9.3), where for the given ε the constant in (9.3) will be of the form $C(\varepsilon,a_\lambda,p)=C_\varepsilon\, C(a_\lambda,\varepsilon,p)$. Arguing as in the proof of Proposition 5.1, after rotation we may assume that $x_0=(|x_0|,0,\dots,0)$. □

In analogy with the operator $T_\lambda^{0,\varepsilon}$, we shall here define the operator $T_\lambda^{k,\ell,\varepsilon}$ by

$$(T_\lambda^{k,\ell,\varepsilon} f)(x,u,s) := \tilde\chi_\varepsilon^{k,\ell}\big(\tilde T_{\lambda,s}^{k,\ell}(\chi_\varepsilon^{k,\ell} f)\big)(x,u), \quad f\in\mathcal{S}(\mathbb{R}^d), \tag{9.6}$$

where we recall that $\tilde{T}^{k,\ell}_{\lambda,s} f = f * K^{k,\ell}_{\lambda,s}$.

Arguing in a similar way as for the case $k = 0$, by means of further localizations we may then reduce the proof of Proposition 9.2 to showing the following:

Suppose that $p \geq \frac{2(d_1+1)}{d_1-1}$. If the constant $\varepsilon > 0$ is sufficiently small, and if a_λ is supported in an interval of length ε contained in $(\frac{1}{16}, 4)$, then we can bound

$$\Big\| \Big(\int_1^{1+\varepsilon\delta(k,\ell)} \big| T^{k,\ell,\varepsilon}_\lambda f \big|^2 \, ds \Big)^{\frac{1}{2}} \Big\|_p \leq C(a_\lambda, \varepsilon, p)\, \delta(k,\ell)^{\frac{1}{2}}\, \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} \times (2^\ell k)^{-d(\frac{1}{2}-\frac{1}{p})+\frac{1}{p}}\, \|f\|_p \tag{9.7}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$, where the constant $C(a_\lambda, \varepsilon, p)$ depends in a_λ only on the C^M-norm of a_λ for some sufficiently large $M \in \mathbb{N}$.

To prove (9.7), we shall normalize the amplitude functions and pass to unit scales where all variables are of unit size by means of a couple of coordinate changes. We first concentrate on the convolution kernel $K^{k,\ell}_{\lambda,s}$ before reinstalling the cut-offs in (9.6). Recall from (8.4) that

$$K^{k,\ell}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty \int_0^\infty e^{i\lambda\tilde{\Phi}(x,u,s,\sigma,\tau)}\, a_\lambda(s\sigma)\, \eta_\ell(\tau - k\pi) \Big(\frac{\tau}{\sin\tau} \Big)^n \, d\sigma\, d\tau,$$

where

$$\tilde{\Phi}(x,u,s,\sigma,\tau) = \sigma(s - |x|^2 g(\tau) + 4u\tau).$$

Recalling (4.15), we see that if $\eta_\ell(\tau - k\pi) \neq 0$, then

$$\tau \in \big[(k - 2^{-\ell}\, \tfrac{5}{4})\pi, (k - 2^{-\ell}\, \tfrac{3}{8})\pi \big] \cup \big[(k + 2^{-\ell}\, \tfrac{3}{8})\pi, (k + 2^{-\ell}\, \tfrac{5}{4})\pi \big]. \tag{9.8}$$

First, changing coordinates via $\tau = k\pi + 2^{1-\ell}\, \tilde{\tau}$, where $\tilde{\tau} \in \mathbb{R}$ with $|\tilde{\tau}| \sim 1$, and observing that

$$g(k\pi + 2^{1-\ell}\, \tilde{\tau}) = (k\pi + 2^{1-\ell}\, \tilde{\tau}) \cot(2^{1-\ell}\, \tilde{\tau}),$$

we see that the above expression equals

$$\lambda^{n+\frac{3}{2}}\, 2^{1-\ell} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda\sigma\tilde{\Phi}_1}\, a_\lambda(s\sigma)\, \eta_1(\tilde{\tau}) \Big(\frac{k\pi + 2^{1-\ell}\, \tilde{\tau}}{(-1)^k \sin(2^{1-\ell}\, \tilde{\tau})} \Big)^n \, d\tilde{\tau}\, d\sigma,$$

where the phase function $\tilde{\Phi}_1$ is given by

$$\tilde{\Phi}_1 = s - \big(k\pi + 2^{1-\ell}\, \tilde{\tau} \big) \big(|x|^2 \cot(2^{1-\ell}\, \tilde{\tau}) - 4u \big).$$

Expanding $(k\pi + 2^{1-\ell}\, \tilde{\tau})^n$, we see that in place of $K^{k,\ell}_{\lambda,s}$, it is sufficient to bound convolution operators with kernels of the form

$$\tilde{K}^{k,\ell}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}}\, k^n\, 2^{-\ell} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda\sigma\tilde{\Phi}_1}\, a_\lambda(s\sigma)\, \tilde{\eta}_1(\tilde{\tau}) \sin(2^{1-\ell}\, \tilde{\tau})^{-n}\, d\tilde{\tau}\, d\sigma, \tag{9.9}$$

where $\tilde{\eta}_1$ is an adapted smooth cutoff satisfying the same support conditions as η_1. In what follows, we drop the tilde and write η_1 in place of $\tilde{\eta}_1$ for simplicity.

9.2. **Translation in u and rescaling in space.** Next, we change coordinates x and u by passing to new coordinates $\breve{x}$ and $\breve{u}$ via

$$x = \frac{2^{1-\ell}}{k}\breve{x}, \quad 4u = 4u_c(\tau_{k,\ell}) + \frac{2^{1-\ell}}{k^2}\breve{u},$$

where $u_c(\tau_{k,\ell})$ is defined as in (8.8), and, recalling (9.8), we choose

$$\tau_{k,\ell} := (k + 2^{-\ell-1})\pi \in \operatorname{supp}\eta_\ell.$$

Recall that $u_c(\tau_{k,\ell}) \sim 1/k$. Then, if (x,u) lies in the set $D^{k,\ell}$ defined in (9.1), we have $|\breve{x}| \sim 1$ and $|\breve{u}| \lesssim 1$. We define

$$\tilde{\tilde{K}}^{k,\ell}_{\lambda,s}(\breve{x},\breve{u}) := 2^{(1-\ell)d}k^{-(d+1)}\tilde{K}^{k,\ell}_{\lambda,s}\Big(\frac{2^{1-\ell}}{k}\breve{x}, u_c(\tau_{k,\ell}) + \frac{2^{1-\ell}}{4k^2}\breve{u}\Big), \tag{9.10}$$

Up to the harmless fixed rescaling of the central variable by a factor of 4, the above change of coordinates is the composition of a translation by a central element in $\mathbb{H}_n$, a parabolic scaling by $D_{1/k}$ and an isotropic scaling by $r = 2^{1-\ell}$.

Now, if K is any convolution kernel on $\mathbb{H}_n$, we first observe that the norms of the operators of right-convolution with K, and with any translate of K by a central element of $\mathbb{H}_n$, are the same on L^p. Similarly, since $D_{1/k}$ is an automorphism of $\mathbb{H}_n$, the same holds true for the right convolution with K, and the right-convolution with the rescaled kernel $k^{-(d+1)}K \circ D_{1/k}$.

For the isotropic scaling $(x,u) \mapsto (rx, ru)$, the same is not true, however. We here need to pass to a group $\mathbb{H}_{n,r}$ which is only isomorphic to $\mathbb{H}_n$: The underlying manifold of $\mathbb{H}_{n,r}$ is again $\mathbb{R}^{2n} \times \mathbb{R}$, but the group law in $\mathbb{H}_{n,r}$ is given by

$$(x,u)\cdot(x',u') = (x+x', u+u' + \tfrac{1}{2}(rJx)^{\intercal}x'),$$

i.e., we replace the skew matrix J from (1.1) by rJ. The mapping $(x,u) \mapsto (rx,ru)$ defines then an isomorphism from $\mathbb{H}_{n,r}$ onto $\mathbb{H}_n$. Correspondingly, if $f *_r g$ denotes the convolution of two functions f and g on $\mathbb{H}_{n,r}$, one easily computes that

$$f(r\,\cdot) *_r (r^d K(r\,\cdot)) = (f * K)(r\,\cdot).$$

This implies that the operator of right-convolution with K on $\mathbb{H}_n$ and the operator of right-convolution with $r^d K(r\,\cdot)$ on $\mathbb{H}_{n,r}$ have the same norm on L^p.

Altogether, the convolution operators $f \mapsto f * \tilde{K}^{k,\ell}_{\lambda,s}$ on $\mathbb{H}_n$ and $f \mapsto f *_r \tilde{\tilde{K}}^{k,\ell}_{\lambda,s}$ on $\mathbb{H}_{n,r}$ have the same L^p operator norms, up to the harmless fixed rescaling of the central variable by a factor of 4.

9.3. **Rescaling in time.** Fix $k,\ell \geq 1$ and put

$$\delta = \delta(k,\ell) = 2^{1-\ell}k^{-1}.$$

Since in (9.7) we assume that $s \in [1, 1+\varepsilon\delta]$, we finally rescale the variable s by setting $s = 1 + \delta(\breve{s}-1)$, so that $\breve{s} \in [1, 1+\varepsilon]$. For the square function estimate for the operator $T^{k,\ell,\varepsilon}_\lambda$ this scaling leads to a gain by the factor $\delta^{\frac{1}{2}}$, i.e.,

$$\Big(\int_1^{1+\varepsilon\delta} \big|(T^{k,\ell,\varepsilon}_\lambda f)(\cdot,s)\big|^2\,ds\Big)^{\frac{1}{2}} = \delta^{\frac{1}{2}}\Big(\int_1^{1+\varepsilon} \big(T^{k,\ell,\varepsilon}_\lambda f)(\cdot, 1+\delta(\breve{s}-1))\big|^2\,d\breve{s}\Big)^{\frac{1}{2}}. \tag{9.11}$$

The same gain will appear for the kernels $\tilde{K}^{k,\ell}_{\lambda,s}$ and $\tilde{\tilde{K}}^{k,\ell}_{\lambda,s}$ when they are used in place of $K^{k,\ell}_{\lambda,s}$ herein.

Gathering all the factors from (9.9) and (9.10), we observe that

$$\lambda^{n+\frac{3}{2}}\, k^n\, 2^{-\ell}\, 2^{(1-\ell)d}\, k^{-(d+1)} = 2^{n-1}\, \delta^{\frac{1}{2}}\, 2^{-n\ell}\, \lambda_*^{n+\frac{3}{2}}, \tag{9.12}$$

where $\lambda_* := \lambda 2^{1-\ell} k^{-1} = \lambda\delta \gtrsim 1$.

Thus, to summarize, in the new coordinates we are led to consider convolution kernels of the form

(9.13)

$$\breve{K}^{k,\ell}_{\lambda_*,\breve{s}}(\breve{x},\breve{u}) := \lambda_*^{n+\frac{3}{2}}\, 2^{-n\ell} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda_*\sigma\breve{\Phi}_0}\, \tilde{a}_{\lambda,\delta}(\sigma,\breve{s})\, \eta_1(\tilde{\tau})\, \sin(2^{1-\ell}\,\tilde{\tau})^{-n}\, d\tilde{\tau}\, d\sigma,$$

with phase

$$\breve{\Phi}_0 = \delta^{-1} - 1 + \breve{s} - \big(k\pi + 2^{1-\ell}\,\tilde{\tau}\big)\Big(\delta|\breve{x}|^2 \cot(2^{1-\ell}\,\tilde{\tau}) - \frac{\breve{u}}{k} - 4\delta^{-1} u_c(\tau_{k,\ell})\Big),$$

where $\breve{s}$ is assumed to be in $[1, 1+\varepsilon]$, and

$$\tilde{a}_{\lambda,\delta}(\sigma,\breve{s}) = a_\lambda\big((1+\delta(\breve{s}-1))\sigma\big). \tag{9.14}$$

Note also that by (9.12) the kernel $\tilde{\breve{K}}^{k,\ell}_{\lambda,1+\delta(\breve{s}-1)}$ corresponds to the kernel $\delta^{\frac{1}{2}}\breve{K}^{k,\ell}_{\lambda_*,\breve{s}}$ in the rescaled coordinates.

Now, similar to (9.6), we define the operators $\breve{T}^{k,\ell,\varepsilon}_{\lambda_*}$ by

$$(\breve{T}^{k,\ell,\varepsilon}_{\lambda_*} f)(\breve{x},\breve{u},\breve{s}) := \tilde{\chi}_\varepsilon\big((\chi_\varepsilon f) *_{2^{1-\ell}} \breve{K}^{k,\ell}_{\lambda_*,\breve{s}}\big)(\breve{x},\breve{u}), \quad f \in \mathcal{S}(\mathbb{R}^d), \tag{9.15}$$

where $\chi_\varepsilon := \chi^{1,0}_\varepsilon$ is supported in the Euclidean ball of radius $\varepsilon > 0$ centered at the origin, and where $\tilde{\chi}_\varepsilon(\breve{x},\breve{u}) := \chi_\varepsilon(\breve{x} - x^0, \breve{u} - u^0)$. Here, we may again assume that $x_0 = (|x_0|, 0, \dots, 0)$ with $|x_0| \sim 1$ and $|u_0| \lesssim 1$.

To prove (9.7), we will establish the following square function estimates:

Proposition 9.3. *Suppose that $p \geq \frac{2d}{d-2}$. If the constant $\varepsilon > 0$ is sufficiently small, then, for all $k \geq 1$, $\ell \geq 1$, we can bound*

$$\Big\| \Big(\int_1^{1+\varepsilon} \big|\breve{T}^{k,\ell,\varepsilon}_{\lambda_*} f\big|^2\, d\breve{s} \Big)^{\frac{1}{2}} \Big\|_p \leq C(a_\lambda,\varepsilon,p)\, \delta^{-\frac{1}{p}}\, \lambda_*^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\, \|f\|_2 \tag{9.16}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$, where the constant $C(a_\lambda,\varepsilon,p)$ depends in a_λ only on the C^M-norm of a_λ for some sufficiently large $M \in \mathbb{N}$.

Note that the estimate (9.16) is stronger than (9.7), since the right-hand side involves $\|f\|_2$ rather than $\|f\|_p$. By Hölder's inequality, exploiting the localization in (9.15), we can immediately pass from $\|f\|_2$ to $\|f\|_p$ in this estimate, after absorbing a constant depending only on ε. Moreover, note that if (9.16) holds, then

$$\begin{aligned}
\Big\| \Big(\int_1^{1+\varepsilon} \big|\tilde{\chi}_\varepsilon\big((\chi_\varepsilon f) *_{2^{1-\ell}} \tilde{\breve{K}}^{k,\ell}_{\lambda,1+\delta(\breve{s}-1)}\big)\big|^2\, d\breve{s} \Big)^{\frac{1}{2}} \Big\|_p & \\
&\lesssim \delta^{\frac{1}{2}}\, \|\breve{T}^{k,\ell,\varepsilon}_{\lambda_*}\|_{L^p\to L^p(L^2([1,1+\varepsilon]))}\, \|f\|_p \\
&\lesssim \delta^{\frac{1}{2}}\, \delta^{-\frac{1}{p}}\, \lambda_*^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\, \|f\|_p \\
&\lesssim \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\, (2^\ell k)^{-d(\frac{1}{2}-\frac{1}{p})+\frac{1}{p}}\, \|f\|_p.
\end{aligned}$$

Our previous discussion, keeping (9.11) in mind, shows that this estimate, in turn, implies the estimate (9.7). The remaining part of this and the following two sections will therefore be devoted to the proof of Proposition 9.3. To that end, with a slight abuse of notation, we shall rename λ_* back to λ, and $(\breve{x},\breve{u},\breve{s})$ back to

(x,u,s). From here on, we shall have to distinguish between Case I where $\ell \geq 2$, and Case II where $\ell = 1$, since these cases exhibit a quite different geometric behavior, as explained in the Introduction.

10. The key estimates for the non-horizontal parts given by $\ell \geq 2$

As in the case $k = 0$, we apply the Stein-Tomas TT^* argument, so that the square function (9.16) of Proposition 9.3 will be a consequence of the estimates in Propositions 10.2 and 10.3.

Recall from (9.13) that, in the renamed coordinates,

$$\breve{K}^{k,\ell}_{\lambda,s}(x,u) := \lambda^{n+\frac{3}{2}}\, 2^{-n\ell} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda\sigma\breve{\Phi}_0}\, \tilde{a}_{\lambda,\delta}(\sigma,s)\, \eta_1(\tau) \sin(2^{1-\ell}\,\tau)^{-n}\, d\tau\, d\sigma,$$

with phase (recall that $\delta = \delta(k,\ell) = 2^{1-\ell}k^{-1}$)

$$\breve{\Phi}_0 = \delta^{-1} - 1 + s - \big(k\pi + 2^{1-\ell}\,\tau\big)\Big(\delta|x|^2 \cot(2^{1-\ell}\,\tau) - \frac{u}{k} - 4\delta^{-1}u_c(\tau_{k,\ell})\Big).$$

Here, s is assumed to be in $|1, 1+\varepsilon]$.

Next, note that if $\ell \geq 2$ and $\tau \in \operatorname{supp}\eta_1$, then (compare also (9.8))

$$2^{1-\ell}\frac{3}{16}\pi \leq |2^{1-\ell}\,\tau| \leq 2^{1-\ell}\frac{5}{8}\pi \leq \frac{5}{16}\pi < \frac{\pi}{2},$$

so that we may change coordinates via

$$\breve{\tau} = 2^{\ell-1}\tan(2^{1-\ell}\,\tau), \quad \text{i.e.,} \quad 2^{1-\ell}\tau = \arctan(2^{1-\ell}\breve{\tau}).$$

Observe that for $\ell = 1$ this change of coordinates is not possible, since then $2^{1-\ell}\tau = \tau$ can become $\frac{\pi}{2}$ and we may run into a singularity of the tangent.

Using $\sin(\arctan\alpha) = \alpha(1+\alpha^2)^{-\frac{1}{2}}$, with $\alpha = 2^{1-\ell}\breve{\tau}$, we may rewrite

$$\breve{K}^{k,\ell}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda\sigma\breve{\Phi}_1}\, \tilde{a}_{\lambda,\delta}(\sigma,s)\, \breve{\eta}_{1,\ell}(\breve{\tau})\, d\breve{\tau}\, d\sigma,$$

with phase

$$\breve{\Phi}_1 = \delta^{-1} - 1 + s - \Big(\pi + \frac{\arctan(2^{1-\ell}\breve{\tau})}{k}\Big)\Big(\frac{|x|^2}{\breve{\tau}} - u - 2^{\ell+1}k^2u_c(\tau_{k,\ell})\Big),$$

and

$$\breve{\eta}_{1,\ell}(\breve{\tau}) := 2^{-n}\,\eta_{1,\ell}\big(2^{\ell-1}\arctan(2^{1-\ell}\breve{\tau})\big)\frac{(1+(2^{1-\ell}\breve{\tau})^2)^{\frac{n}{2}-1}}{\breve{\tau}^n}.$$

Note that the function $\breve{\eta}_{1,\ell}$ has a similar support as η_1, so that $|\breve{\tau}| \sim 1$ on its support, and has uniform bounds on its derivatives, independently of $\ell \geq 2$.

Finally, changing coordinates via $\breve{\tau} = 1/\tau$ and setting

$$\varphi_\ell(\tau) := 2^{\ell-1}\arctan\Big(\frac{2^{1-\ell}}{\tau}\Big),$$

we may write

$$\breve{K}^{k,\ell}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda\sigma\breve{\Phi}}\, \tilde{a}_{\lambda,\delta}(\sigma,s)\, \breve{\eta}_{1,\ell}(\tau)\, d\tau\, d\sigma, \tag{10.1}$$

with a slightly modified function $\breve{\eta}_{1,\ell}$ and phase

$$\breve{\Phi} := \delta^{-1} - 1 + s - (\pi + \delta\varphi_\ell(\tau))\big(\tau|x|^2 - u - 2^{\ell+1}k^2u_c(\tau_{k,\ell})\big). \tag{10.2}$$

Note also that every derivative of φ_ℓ is uniformly bounded on the support of $\breve{\eta}_{1,\ell}$, independently of $\ell \geq 2$.

10.1. **The TT^* argument for the operators $\breve{T}^{k,\ell,\varepsilon}_\lambda$.** Analogous to the case $k=0$, we will prove $L^{p'}_{(x,u)}(L^2_s) \to L^p_{(x,u)}(L^2_s)$ estimates for the operator $\breve{T}^{k,\ell,\varepsilon}_\lambda (\breve{T}^{k,\ell,\varepsilon}_\lambda)^*$.

Lemma 10.1. *The integral kernel $\breve{Q}^{k,\ell,\varepsilon}_\lambda$ of $\breve{T}^{k,\ell,\varepsilon}_\lambda (\breve{T}^{k,\ell,\varepsilon}_\lambda)^*$ is given by*

$$\begin{aligned}(10.3)\quad \breve{Q}^{k,\ell,\varepsilon}_\lambda(x,u,s,x',u',s') &= \lambda^{d+1}\,\chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0)\\ &\times \int_{\eta\sim 1}\int_{|\tau'|\sim 1}\int_{|\tau|\sim 1}\int_{\mathbb{R}^{d_1}} e^{i\lambda\eta\Psi} a(\eta,\tau,\tau',y,s,x',u',s')\,dy\,d\tau\,d\tau' d\eta,\end{aligned}$$

with phase function

$$\begin{aligned}(10.4)\quad \Psi = \frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell(\tau')} - \tau|x-y|^2 + \tau'|x'-y|^2\\ + \big(u-u'+\tfrac12\langle 2^{1-\ell}J(x-x'),y\rangle\big)\end{aligned}$$

and full amplitude

$$\begin{aligned}(10.5)\quad a(\eta,\tau,\tau',y,s,x',u',s') &= a_{1,\lambda,\delta}(\eta,\tau,s)\,\chi_\varepsilon(y)\,\breve{\eta}_{1,\ell}(\tau)\,\breve{\eta}_{1,\ell}(\tau')\\ &\times \frac{h_\lambda(\eta,\tau,\tau',s',x',u',y)}{(\pi+\delta\varphi_\ell(\tau))(\pi+\delta\varphi_\ell(\tau'))},\end{aligned}$$

where

$$(10.6)\qquad a_{1,\lambda,\delta}(\eta,\tau,s) = a_\lambda\Big(\big(1+\delta(s-1)\big)\frac{\eta}{\pi+\delta\varphi_\ell(\tau)}\Big)$$

and

$$(10.7)\qquad h_\lambda(\eta,\tau,\tau',s',x',u',y) = \int_{\mathbb{R}} e^{i\lambda\xi\Psi_2}\lambda\,\widehat{\chi_\varepsilon}(\lambda\xi)\,\overline{a_{1,\lambda,\delta}(\eta-\xi,\tau',s')}\,d\xi,$$

with

$$(10.8)\qquad \Psi_2 = \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell(\tau')} - \tau'\,|x'-y|^2 + 2^{\ell+1}k^2u_c(\tau_{k,\ell}) + u' + \tfrac12\langle 2^{1-\ell}Jx',y\rangle.$$

The full amplitude a and all its partial derivatives are uniformly bounded on the support of the integral, with bounds controlled by finitely many derivatives of a_λ.

Proof. Let us again set $r := 2^{1-\ell}$. Arguing in an analogous way as in the proof of Lemma 5.4, the integral kernel $\breve{Q}^{k,\ell,\varepsilon}_\lambda(x,u,s,x',u',s')$ is given by

$$\begin{aligned}&\tilde\chi_\varepsilon(x,u)\,\tilde\chi_\varepsilon(x',u')\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}} \breve{K}^{k,\ell}_{\lambda,s}\big(x-y,u-v+\tfrac12\langle rJx,y\rangle\big)\\ &\qquad\times\overline{\breve{K}^{k,\ell}_{\lambda,s'}\big(x'-y,u'-v+\tfrac12\langle rJx',y\rangle\big)}\,\chi_\varepsilon(y)\,\chi_\varepsilon(v)\,dv\,dy,\end{aligned}$$

with slightly modified functions χ_ε. Assuming in the sequel the localizations given by $\tilde\chi_\varepsilon(x,u)\,\tilde\chi_\varepsilon(x',u')$, we shall suppress these localization factors from here on. The above then equals

$$\begin{aligned}&\lambda^{d+2}\int_{\sigma'\sim1}\int_{\sigma\sim1}\int_{|\tau'|\sim1}\int_{|\tau|\sim1}\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}} e^{i\lambda\breve\Psi}\,\tilde a_{\lambda,\delta}(\sigma,s)\,\overline{\tilde a_{\lambda,\delta}(\sigma',s')}\,\breve\eta_{1,\ell}(\tau)\\ &\qquad\times\breve\eta_{1,\ell}(\tau')\,\chi_\varepsilon(y)\,\chi_\varepsilon(v)\,dv\,dy\,d\tau\,d\tau'\,d\sigma\,d\sigma',\end{aligned}$$

where the phase function $\breve{\Psi}$ is given by

$$\breve{\Psi} = \sigma\,\breve{\Phi}(\tau, x-y, u-v+\tfrac{1}{2}\langle rJx, y\rangle, s) - \sigma'\breve{\Phi}(\tau', x'-y, u'-v+\tfrac{1}{2}\langle rJx', y\rangle, s'). \tag{10.9}$$

Using (10.2), and changing coordinates from σ and σ' to

$$\eta = \sigma(\pi + \delta\varphi_\ell(\tau)) \quad \text{and} \quad \eta' = \sigma'(\pi + \delta\varphi_\ell(\tau')),$$

we see that $\eta \sim 1$ and $\eta' \sim 1$, and (10.9) becomes

$$\eta\Big(\frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \tau|x-y|^2 + \big(2^{\ell+1}k^2u_c(\tau_{k,\ell}) + u - v + \tfrac{1}{2}\langle rJx, y\rangle\big)\Big) - \eta'\Big(\frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell(\tau')} - \tau'\,|x'-y|^2 + \big(2^{\ell+1}k^2u_c(\tau_{k,\ell}) + u' - v + \tfrac{1}{2}\langle rJx', y\rangle\big)\Big). \tag{10.10}$$

Note that the terms containing v sum up to $(\eta'-\eta)v$. We may therefore first carry out the integration in v, obtaining

$$\int_{\mathbb{R}} e^{i\lambda(\eta'-\eta)v}\chi_\varepsilon(v)\,dv = \widehat{\chi_\varepsilon}\,(\lambda(\eta-\eta'))\,.$$

This yields a strong localization to $|\eta-\eta'| \lesssim \lambda^{-1}$. We next change coordinates from η' to ξ such that $\eta - \eta' = \xi$. Then the phase function in (10.10) becomes $\breve{\Psi} = \eta\Psi + \xi\Psi_2$, with Ψ and Ψ_2 given by (10.4) and (10.8), respectively.

Hence, after carrying out the integration in v and changing coordinates from η' to ξ, we obtain the oscillatory integral (10.3), with phase function given by (10.4) and full amplitude given by (10.5)–(10.7).

Finally, for the uniform bounds for the full amplitude and all its partial derivatives, note that $\pi + \delta\varphi_\ell(\tau') \sim 1$ and that all derivatives of $\delta^{-1}/(\pi+\delta\varphi_\ell(\tau'))$ with respect to τ' are $\mathcal{O}(1)$, uniformly in δ. Moreover, the factor $\lambda\widehat{\chi}_\varepsilon(\lambda\xi)$ localizes effectively to $|\xi| \lesssim \lambda^{-1}$. □

Given $x, x' \in \mathbb{R}^{d_1}$, we split coordinates as $x = (x_1, \underline{x})$ and $x' = (x_1', \underline{x}')$. For fixed $x_1 \in \mathbb{R}$, let $\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1}$ be the operator given by

$$(\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1}\varphi)(\underline{x}, u, s) = (\breve{T}^{k,\ell,\varepsilon}_{\lambda}\varphi)(x, u, s).$$

If we write $f(x', u', s') = f_{x_1'}(\underline{x}', u', s')$, then

$$\big(\breve{T}^{k,\ell,\varepsilon}_{\lambda}(\breve{T}^{k,\ell,\varepsilon}_{\lambda})^* f\big)(x,u,s) = \int_{\mathbb{R}} \big(\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1}(\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1'})^*\, f_{x_1'}\big)(\underline{x}, u, s)\,dx_1'.$$

Note that the integral kernel $\breve{Q}^{k,\ell,\varepsilon}_{\lambda,x_1,x_1'}$ of $\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1}(\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1'})^*$ is given by $\breve{Q}^{k,\ell,\varepsilon}_{\lambda}(x_1, \cdot, x_1', \cdot)$.

In analogy with Propositions 6.1 and 6.2 we shall here prove the following propositions:

Proposition 10.2. *We have*

$$\|\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1}(\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1'})^*\|_{L^1_{(\underline{x},u)}(L^2_s)\to L^\infty_{(\underline{x},u)}(L^2_s)} \lesssim \lambda^{d-1}\,\langle\lambda\,|x_1'-x_1|\rangle^{-\frac{d_1-1}{2}}.$$

Proposition 10.3. *We have*

$$\|\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1}(\breve{T}^{k,\ell,\varepsilon}_{\lambda,x_1'})^*\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim \delta^{-1}.$$

The square function estimate in Proposition 9.3 follows from these two propositions by the same arguments as in the proof of Proposition 5.2. The proofs of Propositions 10.2 and 10.3 will be given in Sections 11.1 and 11.2.

11. Proof of the key estimates for $\ell \geq 2$

In this section, we prove Propositions 10.2 and 10.3.

11.1. **Pointwise estimates for $\ell \geq 2$.** We follow essentially the scheme of the proof of Proposition 6.1. By Schur's test, Proposition 10.2 will again be an immediate consequence of the next analogue of Proposition 6.3.

Proposition 11.1. *If the constant $\varepsilon > 0$ is sufficiently small, then for all $s \in [1, 1+\varepsilon]$,*

$$\int_1^{1+\varepsilon} |\breve{Q}_\lambda^{k,\ell,\varepsilon}(x,u,s,x',u',s')|\, ds' \leq C(\varepsilon, a_\lambda)\, \lambda^{d-1}\, \langle \lambda\, |\Delta x| \rangle^{-\frac{d_1-1}{2}}, \tag{11.1}$$

and for all $s' \in [1, 1+\varepsilon]$,

$$\int_1^{1+\varepsilon} |\breve{Q}_\lambda^{k,\ell,\varepsilon}(x,u,s,x',u',s')|\, ds \leq C(\varepsilon, a_\lambda)\, \lambda^{d-1}\, \langle \lambda\, |\Delta x| \rangle^{-\frac{d_1-1}{2}}, \tag{11.2}$$

uniformly in (x,u) and (x',u'), $k \geq 1$, and $\ell \geq 2$, where the constant $C(\varepsilon, a_\lambda)$ can be chosen to be independent of (x^0, u^0) and to depend only on the C^M-norm of a_λ for some sufficiently large M.

We will prove Proposition 11.1 by showing the following.

Lemma 11.2. *We can decompose $\breve{Q}_\lambda^{k,\ell,\varepsilon} = \breve{Q}_{1,\lambda}^{k,\ell,\varepsilon} + E_\lambda^{k,\ell,\varepsilon}$, where the main term $\breve{Q}_{1,\lambda}^{k,\ell,\varepsilon}$ can be estimated pointwise by*

$$\begin{aligned} |\breve{Q}_{1,\lambda}^{k,\ell,\varepsilon}(x,u,s,x',u',s')| &\leq C_N(\varepsilon, a_\lambda)\, \lambda^d\, \langle \lambda\, |\Delta x| \rangle^{-\frac{d_1-1}{2}} \\ &\quad \times \int_{\mathbb{R}} \big\langle \lambda\, (\Delta s - q(s', \Delta x, \Delta w_\ell, \tau)) \big\rangle^{-N} \chi_1(\tau)\, d\tau \end{aligned} \tag{11.3}$$

for any $N \in \mathbb{N}$, where χ_1 localizes to $|\tau| \sim 1$, and where, with $r = 2^{1-\ell}$,

$$\Delta x = x - x',\ \Delta w_\ell = u - u' + \tfrac{1}{2}\langle rJ\Delta x, x\rangle \text{ and } \Delta s = s - s'.$$

Here q is a smooth function satisfying estimates of the form $\partial_s^j q = \mathcal{O}(\varepsilon)$ for every $j \in \mathbb{N}$, and the constant $C_N(\varepsilon, a_\lambda)$ can be chosen to be independent of τ and to depend, for a given $N \in \mathbb{N}$, on a_λ only through the C^M-norm of a_λ for some sufficiently large M (which may depend on N).

For every $N \in \mathbb{N}$, for all $s \in [1, 1+\varepsilon]$,

$$\int_1^{1+\varepsilon} |E_\lambda^{k,\ell,\varepsilon}(x,u,s,x',u',s')|\, ds' \leq C_N(\varepsilon, a_\lambda)\, \lambda^d\, \langle \lambda\, |\Delta x| \rangle^{-N},$$

and for all $s' \in [1, 1+\varepsilon]$,

$$\int_1^{1+\varepsilon} |E_\lambda^{k,\ell,\varepsilon}(x,u,s,x',u',s')|\, ds \leq C_N(\varepsilon, a_\lambda)\, \lambda^d\, \langle \lambda\, |\Delta x| \rangle^{-N},$$

uniformly in (x,u) and (x',u'), $k \geq 1$, and $\ell \geq 2$.

The rest of this section will be devoted to the proof of Lemma 11.2, which will be kept a bit more sketchy compared to the proof of Proposition 6.3 and will concentrate mostly on differences compared to the arguments used for Proposition 6.3.

11.1.1. *Freezing τ.* By Lemma 10.1, we have

$$\breve{Q}^{k,\ell,\varepsilon}_{\lambda} = \int_{|\tau|\sim 1} \breve{Q}^{k,\ell,\varepsilon}_{\lambda,\tau}\, d\tau,$$

where (suppressing again the localizing factors $\chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0)$)

$$\breve{Q}^{k,\ell,\varepsilon}_{\lambda,\tau} = \lambda^{d+1} \int_{\eta\sim 1}\int_{|\tau'|\sim 1}\int_{\mathbb{R}^{d_1}} e^{i\lambda\eta\Psi} a(\eta,\tau,\tau',y,s,x',u',s')\, dy\, d\tau' d\eta, \tag{11.4}$$

with phase Ψ given by (10.4). In the sequel, we assume that τ with $|\tau|\sim 1$ is frozen.

11.1.2. *Changes of variables and stationary phase for a spherical integration.* Let us again use the abbreviation $r = 2^{1-\ell}$. In the oscillatory integral (11.4), we first change coordinates from y to $y' = y - x$. Since $|x|\sim 1$, we then have $|y'|\sim 1$. We use polar coordinates $y' = \rho\,\omega$, where $\rho\in(0,\infty)$ satisfies $\rho\sim 1$, and where $\omega\in S^{d_1-1}$. Then the phase function Ψ in (10.4) changes to

$$\Psi_1 = \frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell(\tau')} + (\tau'-\tau)\rho^2 + \tau'|\Delta x|^2 + \rho\,\langle\theta,\omega\rangle + \Delta w_\ell,$$

where

$$\theta = 2\tau'\Delta x + \tfrac{r}{2} J\Delta x.$$

Since $J^2 = -I_{d_1}$ and J is skew-symmetric, we have

$$|\theta|^2 = \Big(4\,(\tau')^2 + \frac{r^2}{4}\Big)|\Delta x|^2 \sim |\Delta x|^2.$$

Note also that the factor $\chi_\varepsilon(\rho\omega+x)$ localizes the integration in ω to an $\mathcal{O}(\varepsilon)$ neighborhood of the point $-x^0/|x^0|$ in S^{d_1-1}. Absorbing the polar-coordinate Jacobian and the cutoffs into a new full amplitude a_1, we obtain an oscillatory integral of the form

$$\lambda^{d+1}\int_{\eta\sim1}\int_{|\tau'|\sim1}\int_{\rho\sim1}\int_{S^{d_1-1}} e^{i\lambda\eta\Psi_1} a_1\, d\omega\, d\rho\, d\tau' d\eta.$$

Next, we change coordinates from τ' to β via

$$\frac{\tau'}{\tau} = 1 + |\Delta x|\beta,$$

so that $\tau'-\tau = \beta\tau|\Delta x|$. This turns (11.4) into an oscillatory integral of the form

$$\lambda^{d+1}|\Delta x|\int_{\eta\sim1}\int_{S^{d_1-1}}\int_{\rho\sim1}\int_{\mathbb{R}} e^{i\lambda\eta\Psi_2} a_2(\eta,\tau,\beta,s,s',x,x',u',\rho,\omega) \times \chi_\varepsilon(|\Delta x|\beta)\, d\beta\, d\rho\, d\omega\, d\eta, \tag{11.5}$$

where the Jacobian factor has been absorbed into a_2, and

$$\Psi_2 = \frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell\,(\tau+\beta\tau|\Delta x|)} + \beta\tau\,|\Delta x|\,\rho^2 + \rho\,\langle\tilde\theta,\omega\rangle + \tau\,(1+\beta|\Delta x|)\,|\Delta x|^2 + \Delta w_\ell,$$

with

$$\tilde\theta = 2\tau\,(1+\beta|\Delta x|)\,\Delta x + \tfrac{r}{2} J\Delta x.$$

We write

$$\Psi_2 = \frac{s}{\pi+\delta\varphi_\ell(\tau)} - \frac{s'}{\pi+\delta\varphi_\ell(\tau+\beta\tau|\Delta x|)} + q_\ell(\tau,\beta|\Delta x|,\rho,\omega),$$

where

$$q_\ell(\tau,\beta|\Delta x|,\rho,\omega) = (1-\delta)\,\frac{\varphi_\ell(\tau+\beta\tau|\Delta x|)-\varphi_\ell(\tau)}{(\pi+\delta\varphi_\ell(\tau+\beta\tau|\Delta x|))(\pi+\delta\varphi_\ell(\tau))} + \beta\tau\,|\Delta x|\,\rho^2 \\ + \rho\,\langle\tilde\theta,\omega\rangle + \tau\,(1+\beta|\Delta x|)\,|\Delta x|^2 + \Delta w_\ell.$$

The amplitude a_2 and all its partial derivatives are uniformly bounded on the support of the integral, with the same type of uniformity as the amplitude a in Lemma 10.1.

Let us denote by $E^{\varepsilon,2}_{\lambda,\tau}$ the contribution of the region where $|\beta|\gg 1$ to (11.5). For β in this region, we can integrate by parts in ρ in order to gain negative powers $\langle\lambda|\Delta x||\beta|\rangle^{-N}$. Applying subsequently integrations by parts in η, we may write

$$E^{\varepsilon,2}_{\lambda,\tau} = \lambda^d \int_{\eta\sim 1}\int_{S^{d_1-1}}\int_{\rho\sim 1}\int_{|\beta|\gg 1} e^{i\lambda\eta\Psi_2}\big\langle\lambda\,\Psi_2\big\rangle^{-N}(\lambda|\Delta x|)\langle\lambda|\Delta x||\beta|\rangle^{-N} \\ \times a_{2,N}\,\chi_\varepsilon(|\Delta x|\,\beta)\,d\beta\,d\rho\,d\omega\,d\eta,$$

where $a_{2,N}$ is another amplitude that satisfies the corresponding uniform symbol bounds. From here on, we can argue as for the error term $E^{\varepsilon,2}_{\lambda,\tau'}$ in the proof of Proposition 6.3 to see that $E^{\varepsilon,2}_{\lambda,\tau}$ is an error term which gives a contribution to $E^{k,\ell,\varepsilon}_\lambda$. Note here that, in contrast to the situation in the latter proposition, only the term

$$\frac{s}{\pi+\delta\varphi_\ell(\tau)} - \frac{s'}{\pi+\delta\varphi_\ell(\tau+\beta\tau|\Delta x|)}$$

depends on s and s', so after freezing the frequency variables τ,β,ω,ρ, the estimates (11.1) and (11.2) become clear.

We are thus reduced to considering the contribution by the region where $|\beta|\lesssim 1$, that is, an oscillatory integral of the form

$$Q^{\varepsilon,2}_{\lambda,\tau} = \lambda^d(\lambda|\Delta x|)\int_{\eta\sim 1}\int_{S^{d_1-1}}\int_{\rho\sim 1}\int_{|\beta|\lesssim 1} e^{i\lambda\eta\Psi_2}\,\tilde a_2\,\chi_\varepsilon(|\Delta x|\,\beta)\,d\beta\,d\rho\,d\omega\,d\eta,$$

where, as often in what follows, we have suppressed the dependence of $Q^{\varepsilon,2}_{\lambda,\tau}$ on k,ℓ.

Next, performing stationary phase in ω when $|\underline{\Delta x}|\lesssim\varepsilon|\Delta x_1|$, and N integrations by parts in ω along S^{d_1-1} when $|\underline{\Delta x}|\gg\varepsilon|\Delta x_1|$, we find that

$$Q^{\varepsilon,2}_{\lambda,\tau} = Q^{\varepsilon,3}_{\lambda,\tau} + E^{\varepsilon,3}_{\lambda,\tau}.$$

Here, the first term is given by

$$Q^{\varepsilon,3}_{\lambda,\tau} = \lambda^d(\lambda|\Delta x|)\int_{\eta\sim 1}\int_{\rho\sim 1}\int_{|\beta|\lesssim 1}\langle\lambda\eta\rho|\Delta x|\rangle^{-\frac{d_1-1}{2}} e^{i\lambda\eta\Psi_3^\pm}\,a_3 \\ \times\chi_\varepsilon(|\Delta x|\,\beta)\,d\beta\,d\rho\,d\eta, \tag{11.6}$$

where a_3 is an amplitude satisfying the same type of uniform symbol bounds and in addition localizes to the region where $|\underline{\Delta x}|\lesssim\varepsilon|\Delta x_1|$, and the phase is given by

$$\Psi_3^\pm = \frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell\,(\tau+\beta\tau|\Delta x|)} + \beta\tau\,|\Delta x|\,\rho^2 \\ \pm|\tilde\theta|\rho + \tau\,(1+\beta|\Delta x|)\,|\Delta x|^2 + \Delta w_\ell.$$

The second term $E^{\varepsilon,3}_{\lambda,\tau}$ corresponding to the region where $|\underline{\Delta x}| \gg \varepsilon|\Delta x_1|$ is again an error term, which can be handled in a similar way as the one in (6.9) by means of integrations by parts in ω and η. It thus again just contributes to $E^{k,\ell,\varepsilon}_{\lambda}$.

11.1.3. *Stationary phase for the integration in ρ.* Note that

$$\psi^{\pm} = \beta\tau\,|\Delta x|\,\rho^2 \pm |\tilde\theta|\rho = \rho^2\psi_2 + \rho\psi_1$$

collects all terms of the phase $\Psi_3^{\pm}$ which depend on ρ. Here, $|\psi_2| \sim \beta|\Delta x|$, and, as we have seen, $|\psi_1| \sim |\tilde\theta| \sim |\Delta x|$.

Thus, if $|\beta| \ll 1$, then $|\partial_\rho\psi^{\pm}| \sim |\Delta x|$, so we can again integrate by parts in ρ in order to gain any factors $\langle\lambda\Delta x\rangle^{-N}$. This shows that also the contributions from the region where $|\beta| \ll 1$ can be handled as an error term in a similar way as before.

We are thus left with an oscillatory integral of the form

$$Q^{\varepsilon,4}_{\lambda,\tau} = \lambda^d(\lambda|\Delta x|)\int_{\eta\sim1}\int_{\rho\sim1}\int_{|\beta|\sim1}\langle\lambda\eta\rho|\Delta x|\rangle^{-\frac{d_1-1}{2}} e^{i\lambda\eta\Psi_3^{\pm}}\tilde a_3\,d\beta\,d\rho\,d\eta. \tag{11.7}$$

Note that

$$\partial_\rho\psi^{\pm} = 2\beta\tau\,|\Delta x|\,\rho \pm |\tilde\theta|.$$

Thus, if $|\beta| \sim 1$, then $\psi^{\pm}$ has a unique critical point $\rho_c \sim 1$ in ρ if and only if $\mp\beta \sim 1$. For the following discussion, we shall consider the phase Ψ_3^-; the case of the phase Ψ_3^+ can be treated analogously.

If $-\beta \sim 1$, we can then again integrate by parts in ρ and see that the corresponding contribution can be treated as an error term as before. Thus, we shall assume in the sequel that $\beta \sim 1$, and that ψ^- does have a critical point for any such β, which is then clearly given by

$$\rho_c = \frac{|\tilde\theta|}{2\beta\tau|\Delta x|} = \Big(4\tau^2(1+\beta|\Delta x|)^2 + \tfrac{r^2}{4}\Big)^{\frac12}/(2\beta\tau).$$

Note also that $|\partial_\rho^2\psi^-| \sim |\Delta x|$. Thus, applying stationary phase in ρ, we are reduced to the oscillatory integral

$$Q^{\varepsilon,4}_{\lambda,\tau} = \lambda^d(\lambda|\Delta x|)\int_{\eta\sim1}\int_{\beta\sim1}\langle\lambda|\Delta x|\rangle^{-\frac{d_1}{2}} e^{i\lambda\eta\Psi_4}a_4\,d\beta\,d\eta, \tag{11.8}$$

where

$$\Psi_4 = \frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell\left(\tau+\beta\tau|\Delta x|\right)} - \frac{4\tau^2(1+\beta|\Delta x|)^2+\frac{r^2}{4}}{4\beta\tau}\,|\Delta x| + \tau\,(1+\beta|\Delta x|)\,|\Delta x|^2 + \Delta w_\ell.$$

As in Section 6.2, the factor $\langle\lambda|\Delta x|\rangle^{-\frac{d_1}{2}}$ in the integral is here to be interpreted as a shorthand for a symbol of order $-\frac{d_1}{2}$ in $\lambda|\Delta x|$.

11.1.4. *Stationary phase for the integration in β.* Note first that

$$\partial_\beta\Psi_4 = H(\beta, s', \tau, |\Delta x|)\,|\Delta x|,$$

where

$$H = \frac{1+\delta(s'-1)}{\big(\pi+\delta\varphi_\ell(\tau+\beta\tau|\Delta x|)\big)^2}\,\tau\varphi_\ell'\left(\tau+\beta\tau|\Delta x|\right) + \frac{4\tau^2(1+\beta|\Delta x|)^2+\frac{r^2}{4}}{4\beta^2\tau} + \mathcal{O}(|\Delta x|).$$

For fixed τ_0 and $s'=1, |\Delta x|=0$, we obtain

$$H(\beta,1,\tau_0,0) = \frac{\tau_0\varphi_\ell'(\tau_0)}{(\pi+\delta\varphi_\ell(\tau_0))^2} + \frac{4\tau_0^2+\frac{r^2}{4}}{4\beta^2\tau_0}.$$

This function has a unique zero $\beta_c(1,\tau_0,0)\sim 1$. Moreover,

$$\partial_\beta H(\beta,1,\tau_0,0) = -2\frac{4\tau_0^2+\frac{r^2}{4}}{4\beta^3\tau_0}, \qquad |\partial_\beta H(\beta,1,\tau_0,0)|\sim 1.$$

By the implicit function theorem, we thus see that for $|s'-1|$, $|\Delta x|$, and $|\tau-\tau_0|$ sufficiently small, there is a unique zero

$$\beta_c(s',\tau,|\Delta x|)\sim 1$$

of $H(\beta,s',\tau,|\Delta x|)$ in β, which is then also the unique critical point of Ψ_4 in β.

Note that by means of a decomposition into intervals of length ε, we may indeed reduce to assuming that τ lies in an interval $[\tau_0,\tau_0+\varepsilon]$. Thus, if we assume that ε is sufficiently small, the existence of the unique critical point $\beta_c(s',\tau,|\Delta x|)\sim 1$ of $H(\beta,s',\tau,|\Delta x|)$ is guaranteed. Moreover,

$$\big|\partial_\beta^2\Psi_4\big| \sim |\Delta x|$$

at this critical point. Applying the method of stationary phase in β, we then obtain an oscillatory integral of the form

$$Q^{\varepsilon,4}_{\lambda,\tau} = \lambda^d(\lambda|\Delta x|)\int_{\eta\sim 1}\langle\lambda|\Delta x|\rangle^{-\frac{d_1+1}{2}}e^{i\lambda\eta\Psi_5}a_5\,d\eta, \tag{11.9}$$

with phase $\Psi_5=\Psi_4|_{\beta=\beta_c}$. Since

$$\begin{aligned}
&\frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell(\tau+\beta_c\tau|\Delta x|)}\\
&= \frac{1}{\pi+\delta\varphi_\ell(\tau)}\left[s-s'+(1-\delta+\delta s')\,\frac{\varphi_\ell(\tau+\beta_c\tau|\Delta x|)-\varphi_\ell(\tau)}{\pi+\delta\varphi_\ell(\tau+\beta_c\tau|\Delta x|)}\right]\\
&= \frac{1}{\pi+\delta\varphi_\ell(\tau)}\Big(\Delta s+\mathcal{O}(|\Delta x|)\Big),
\end{aligned}$$

we see that

$$\Psi_5 = \frac{1}{\pi+\delta\varphi_\ell(\tau)}\left(\Delta s - q(s',\Delta x,\Delta w_\ell,\tau)\right),$$

where

$$q=\mathcal{O}\big(|\Delta x|+|\Delta w_\ell|\big)=\mathcal{O}(\varepsilon), \qquad \partial_{s'}^j q=\mathcal{O}(|\Delta x|), \quad j\geq 1.$$

By means of integrations by parts in η, it now follows easily that the main term

$$\breve{Q}^{k,\ell,\varepsilon}_{1,\lambda} = \int_{|\tau|\sim 1}\breve{Q}^{\varepsilon,4}_{\lambda,\tau}\,d\tau$$

satisfies estimates of the form (11.3). □

11.2. **Refined L^2 estimates for $\ell \geq 2$.** In this section we shall prove Proposition 10.3. Recall from Lemma 10.1 that the integral kernel $\breve{Q}_\lambda^{k,\ell,\varepsilon}$ of $\breve{T}_\lambda^{k,\ell,\varepsilon}(\breve{T}_\lambda^{k,\ell,\varepsilon})^*$ can be written as

$$\breve{Q}_\lambda^{k,\ell,\varepsilon}(x,u,s,x',u',s') = \lambda^{d+1}\,\chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0) \\ \times \int_{\eta\sim 1}\int_{|\tau'|\sim 1}\int_{|\tau|\sim 1}\int_{\mathbb{R}^{d_1}} e^{i\lambda\eta\Psi} a(\eta,\tau,\tau',y,s,x',u',s')\,dy\,d\tau\,d\tau'\,d\eta,$$

with phase function (recall that $r = 2^{1-\ell}$)

$$\Psi = \frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell(\tau')} - \tau|x-y|^2 + \tau'|x'-y|^2 \\ + \big(u-u'+\tfrac12\langle rJ(x-x'),y\rangle\big)$$

and amplitude a given by (10.5), which comes with the factors

$$a_{1,\lambda,\delta}(\eta,\tau,s) = a_\lambda\Big(\big(1+\delta(s-1)\big)\frac{\eta}{\pi+\delta\varphi_\ell(\tau)}\Big)$$

and

$$h_\lambda(\eta,\tau,\tau',s',x',u',y) = \int_{\mathbb{R}} e^{i\lambda\xi\Psi_2}\lambda\,\widehat{\chi}_\varepsilon(\lambda\xi)\,\overline{a_{1,\lambda,\delta}(\eta-\xi,\tau',s')}\,d\xi.$$

Here Ψ_2 is given by (10.8). The function h_λ and all its partial derivatives are uniformly bounded on the support of the integral, with bounds controlled by finitely many derivatives of a_λ.

Regretfully, compared to the proof of Proposition 6.2, it turns out that here the Monge–Ampère determinant of Ψ will only be of order $\mathcal{O}(\delta^2)$. To address this problem, we shall need to apply a more sophisticated argument, which exploits the special structure of the phase Ψ and which is based on a rescaling argument in the variable s, in which s will be replaced by $\delta^{-1}s$.

For later use, let us write the full amplitude in the form

$$a(\eta,\tau,\tau',y,s,x',u',s') = b(\eta,\tau,\tau',y,s)\,h_\lambda(\eta,\tau,\tau',s',x',u',y),$$

where

$$b(\eta,\tau,\tau',y,s) := a_{1,\lambda,\delta}(\eta,\tau,s)\,\chi_\varepsilon(y)\,\frac{\breve{\eta}_{1,\ell}(\tau)\,\breve{\eta}_{1,\ell}(\tau')}{(\pi+\delta\varphi_\ell(\tau))(\pi+\delta\varphi_\ell(\tau'))}. \tag{11.10}$$

Note that the scaling in s turns $a_{1,\lambda,\delta}(\eta,\tau,s)$ into

$$\breve{a}_{1,\lambda,\delta}(\eta,\tau,s) = a_\lambda\Big((1+s-\delta)\frac{\eta}{\pi+\delta\varphi_\ell(\tau)}\Big). \tag{11.11}$$

Thus, all derivatives of the rescaled amplitude factor

$$\breve{b}(\eta,\tau,\tau',y,s) := b(\eta,\tau,\tau',y,\delta^{-1}s)$$

are still uniformly bounded. Regretfully, the same is not immediately true for the rescaled function h_λ.

11.2.1. *Freezing ξ in the defining integral for h_λ.* Write

$$\Psi_2 = \psi(s') + u' + \dot\Psi_2, \qquad \psi(s') = \frac{\delta^{-1} - 1 + s'}{\pi},$$

where

$$\dot\Psi_2 = -\frac{1+\delta(s'-1)}{\pi+\delta\varphi_\ell(\tau')}\frac{\varphi_\ell(\tau')}{\pi} - \tau'\,|x'-y|^2 + 2^{\ell+1}k^2 u_c(\tau_{k,\ell}) + \tfrac{1}{2}\langle rJx', y\rangle.$$

We may then write

$$h_\lambda(\eta,\tau,\tau',s',x',u',y) = \int_{\mathbb{R}} e^{i\lambda\xi(\psi(s')+u')} H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y)\,d\xi, \tag{11.12}$$

where

$$H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y) := e^{i\lambda\xi\dot\Psi_2}\,\lambda\,\widehat{\chi}_\varepsilon(\lambda\xi)\,\overline{a_{1,\lambda,\delta}(\eta-\xi,\tau',s')}.$$

Let us also put

$$\breve H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y) := H_{\lambda,\xi}(\eta,\tau,\tau',\delta^{-1}s',x',y).$$

Then, for every $M\in\mathbb{N}$, we have

$$\int_{\mathbb{R}} \|\breve H_{\lambda,\xi}\|_{C^M}\,d\xi \lesssim_M 1, \tag{11.13}$$

uniformly in λ, where the C^M norm is taken in $(\eta,\tau,\tau',s',x',y)$, with $s'\sim\delta$.

For frozen ξ (and ε), let us then define

$$\begin{aligned}\breve Q^{k,\ell}_{\lambda,\xi}(x,u,s,x',u',s') &:= \lambda^{d+1}\,\chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0)\\ &\times\int_{\eta\sim1}\int_{|\tau'|\sim1}\int_{|\tau|\sim1}\int_{\mathbb{R}^{d_1}} e^{i\lambda\eta\Psi}\,b(\eta,\tau,\tau',y,s)\\ &\qquad\times H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y)\,dy\,d\tau\,d\tau'\,d\eta,\end{aligned} \tag{11.14}$$

so that

$$\breve Q^{k,\ell,\varepsilon}_{\lambda}(x,u,s,x',u',s') = \int_{\mathbb{R}} \breve Q^{k,\ell}_{\lambda,\xi}(x,u,s,x',u',s')\,e^{i\lambda\xi(\psi(s')+u')}\,d\xi.$$

For fixed ξ and x_1, x_1' frozen, let $S^{k,\ell}_{\lambda,\xi,x_1,x_1'}$ be the operator whose kernel is obtained from (11.14) by replacing the two outer localization factors by $\chi_\varepsilon(x-x^0)\,\chi_\varepsilon(x'-x^0)$. The operators $S^{k,\ell}_{\lambda,\xi,x_1,x_1'}$ and $e^{i\lambda\xi(\psi(s')+u')}\,S^{k,\ell}_{\lambda,\xi,x_1,x_1'}$ have the same operator norm on L^2. Thus, if

$$\|S^{k,\ell}_{\lambda,\xi,x_1,x_1'}\|_{L^2\to L^2} \lesssim \delta^{-1}\|\breve H_{\lambda,\xi}\|_{C^M}, \tag{11.15}$$

uniformly in ξ,λ, etc., for some $M\in\mathbb{N}$, then we can bound

$$\|\breve T^{k,\ell,\varepsilon}_{\lambda,x_1}(\breve T^{k,\ell,\varepsilon}_{\lambda,x_1'})^*\|_{L^2\to L^2} \lesssim \delta^{-1}\int_{\mathbb{R}}\|\breve H_{\lambda,\xi}\|_{C^M}\,d\xi \lesssim \delta^{-1},$$

which will prove Proposition 10.3.

11.2.2. *Exploiting the partial Fourier transform in u.* Note the summand $u-u'$ in the phase function Ψ, and let us change coordinates from η to $\mu=\lambda\eta$, so that $|\mu|\sim\lambda$. Then

$$\int_{\mathbb{R}} f(\underline{x}',u',s')\,e^{i\lambda\eta(u-u')}\,du' = \mathcal{F}_u f(\underline{x}',\lambda\eta,s')\,e^{i\lambda\eta u} = f^\mu(\underline{x}',s')\,e^{i\mu u}$$

Here, $\mathcal{F}_u$ denotes the partial Fourier transform in u, in which we now have suppressed the factor 2π, as also for f^μ.

For given ξ and μ, let $S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ be the integral operator given by

$$S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\,g(\underline{x},s) = \int_{\mathbb{R}^{d_1-1}}\int_0^\infty q^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s')\,g(\underline{x}',s')\,d\underline{x}'\,ds',$$

with integral kernel

$$\begin{aligned} q^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s') &= \chi_\varepsilon(x-x^0)\,\chi_\varepsilon(x'-x^0)\\ &\quad\times\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^{d_1}} e^{i\mu\Theta}\,b(\tfrac{\mu}{\lambda},\tau,\tau',y,s)\,H_{\lambda,\xi}(\tfrac{\mu}{\lambda},\tau,\tau',s',x',y)\,dy\,d\tau\,d\tau'.\end{aligned}$$

The phase Θ is given by

$$\Theta = \frac{\delta^{-1}-1+s}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}-1+s'}{\pi+\delta\varphi_\ell(\tau')} - \tau|x-y|^2+\tau'|x'-y|^2+\tfrac12\langle rJ\Delta x,y\rangle.$$

Then, by the change of coordinates from η to μ, we see that

$$\big(S^{k,\ell}_{\lambda,\xi,x_1,x_1'}f\big)(\underline{x},u,s) = \lambda^d\int_{\mu\sim\lambda}\big(S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}f^\mu\big)(\underline{x},s)\,e^{i\mu u}\,d\mu.$$

Thus, by Plancherel in u,

$$\big\|S^{k,\ell}_{\lambda,\xi,x_1,x_1'}f\big\|_2 \sim \lambda^d\Big(\int_{\mu\sim\lambda}\big\|S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}f^\mu\big\|_2^2\,d\mu\Big)^{\frac12}.$$

Suppose we can show that

$$\|S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}g\|_2 \lesssim \delta^{-1}\lambda^{-d}\|\breve H_{\lambda,\xi}\|_{C^M}\,\|g\|_2, \tag{11.16}$$

uniformly for $\mu\sim\lambda$. Then, again by Plancherel,

$$\begin{aligned}\big\|S^{k,\ell}_{\lambda,\xi,x_1,x_1'}f\big\|_2 &\lesssim \lambda^d\Big(\int_{\mu\sim\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^{d_1-1}}\delta^{-2}\,\lambda^{-2d}\,\|\breve H_{\lambda,\xi}\|^2_{C^M}\,|f^\mu(\underline{x},s)|^2\,d\underline{x}\,ds\,d\mu\Big)^{\frac12}\\ &\sim \delta^{-1}\,\|\breve H_{\lambda,\xi}\|_{C^M}\|f\|_2,\end{aligned}$$

which would prove (11.15), hence also Proposition 10.3.

11.2.3. *Scaling in s.* We are thus left with showing estimate (11.16). To this end, let $\breve S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ denote the scaled operator

$$\breve S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\,g(\underline{x},s) = \int_{\mathbb{R}^{d_1-1}}\int_0^\infty \breve q^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s')\,g(\underline{x}',s')\,d\underline{x}'\,ds',$$

with integral kernel

$$\breve{q}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s') := q^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},\delta^{-1}s,\underline{x}',\delta^{-1}s')$$
$$= \chi_\varepsilon(x-x^0)\,\chi_\varepsilon(x'-x^0)\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^{d_1}} e^{i\mu\breve{\Theta}}\,\breve{b}(\tfrac{\mu}{\lambda},\tau,\tau',y,s)$$
$$\times\,\breve{H}_{\lambda,\xi}(\tfrac{\mu}{\lambda},\tau,\tau',s',x',y)\;dy\,d\tau\,d\tau'.$$

This kernel is obtained from $q^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ by scaling s and s' by the factor δ^{-1}.

The scaled phase function $\breve{\Theta}$ is given by

$$\breve{\Theta} = \frac{\delta^{-1}(s+1)-1}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}(s'+1)-1}{\pi+\delta\varphi_\ell(\tau')} - \tau|x-y|^2+\tau'|x'-y|^2+\tfrac{1}{2}\,\langle rJ\Delta x,y\rangle\,.$$

Since

$$\frac{\delta^{-1}(s+1)-1}{\pi+\delta\varphi_\ell(\tau)} - \frac{\delta^{-1}(s+1)-1}{\pi} = -\frac{(1-\delta+s)\varphi_\ell(\tau)}{\pi(\pi+\delta\varphi_\ell(\tau))},$$

we may write

$$\breve{\Theta} = \frac{s-s'}{\delta\pi} + \frac{(1-\delta+s')\varphi_\ell(\tau')}{\pi(\pi+\delta\varphi_\ell(\tau'))} - \frac{(1-\delta+s)\varphi_\ell(\tau)}{\pi(\pi+\delta\varphi_\ell(\tau))}$$
$$-\tau|x-y|^2+\tau'|x'-y|^2+\tfrac{1}{2}\,\langle rJ\Delta x,y\rangle\,.$$

In order to remove the first summand in this phase, which has a possibly large coefficient δ^{-1}, we pass to the modified operator $\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ defined by

$$\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\,g(\underline{x},s) = e^{-i\frac{\mu s}{\delta\pi}}\,\breve{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\big(e^{i\frac{\mu s'}{\delta\pi}}g\big)(\underline{x},s). \tag{11.17}$$

The operator $\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ is defined in the same way as $\breve{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$, but with the phase $\breve{\Theta}$ replaced by the phase $\tilde{\Theta}$ given by

$$\tilde{\Theta} = \frac{(1-\delta+s')\varphi_\ell(\tau')}{\pi(\pi+\delta\varphi_\ell(\tau'))} - \frac{(1-\delta+s)\varphi_\ell(\tau)}{\pi(\pi+\delta\varphi_\ell(\tau))} - \tau|x-y|^2+\tau'|x'-y|^2+\tfrac{1}{2}\,\langle rJ\Delta x,y\rangle\,.$$

Thus,

$$\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\,g(\underline{x},s) = \int_{\mathbb{R}^{d_1-1}}\int_0^\infty \tilde{q}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s')\,g(\underline{x}',s')\,ds'\,d\underline{x}'.$$

with

$$\tilde{q}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s') = \chi_\varepsilon(x-x^0)\,\chi_\varepsilon(x'-x^0)$$
$$\times\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^{d_1}} e^{i\mu\tilde{\Theta}}\,\breve{b}(\tfrac{\mu}{\lambda},\tau,\tau',y,s)\,\breve{H}_{\lambda,\xi}(\tfrac{\mu}{\lambda},\tau,\tau',s',x',y)\,dy\,d\tau\,d\tau'.$$

Clearly, $\breve{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ and $\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ have the same operator norms on L^2, i.e.,

$$\|\breve{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\|_{2\to2} = \|\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\|_{2\to2}.$$

By means of the next lemma, we shall conclude that

$$\|S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\|_{2\to2} \lesssim \delta^{-1}\|\breve{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\|_{2\to2}. \tag{11.18}$$

This reduces the estimate for $\|S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\|_{2\to2}$ to that for $\|\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\|_{2\to2}$.

Estimate (11.18) will follow easily from the following general scaling result.

Lemma 11.3. *Suppose $K = K(s,s')$ is a square integrable integral kernel taking values in the space of bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, i.e., $K \in L^2(\mathbb{R}\times\mathbb{R}, \mathcal{B}(\mathcal{H}))$. Let A be any measurable subset of $\mathbb{R}$, and let $0<\delta<1$. Define the integral operators T and $\breve{T}$ from $L^2(\mathbb{R},\mathcal{H})$ to $L^2(\mathbb{R},\mathcal{H})$ by*

$$Tf(s) = \mathbb{1}_A(s)\int_{\mathbb{R}} K(s,s')\,(\mathbb{1}_A\, f)(s')\,ds'$$

and

$$\breve{T}f(s) = \int_{\mathbb{R}} K\Big(\frac{s}{\delta},\frac{s'}{\delta}\Big)\, f(s')\,ds',$$

for $f\in L^2(\mathbb{R},\mathcal{H})$. Then

$$\|T\|_{L^2(\mathbb{R},\mathcal{H})\to L^2(\mathbb{R},\mathcal{H})} \le \delta^{-1}\,\|\breve{T}\|_{L^2(\mathbb{R},\mathcal{H})\to L^2(\mathbb{R},\mathcal{H})}.$$

Proof. For $f\in L^2(\mathbb{R},\mathcal{H})$, put $f_\delta(s) = f(s/\delta)$. Then

$$\begin{aligned}(Tf)_\delta(s) &= \mathbb{1}_A\Big(\frac{s}{\delta}\Big)\int_{\mathbb{R}} K\Big(\frac{s}{\delta},s'\Big)(\mathbb{1}_A\, f)(s')\,ds'\\ &= \delta^{-1}\,\mathbb{1}_A\Big(\frac{s}{\delta}\Big)\int_{\mathbb{R}} K\Big(\frac{s}{\delta},\frac{s'}{\delta}\Big)(\mathbb{1}_A\, f)\Big(\frac{s'}{\delta}\Big)\,ds'\\ &= \delta^{-1}\,\mathbb{1}_A\Big(\frac{s}{\delta}\Big)\,\breve{T}\big((\mathbb{1}_A)_\delta\, f_\delta\big)(s).\end{aligned}$$

Thus,

$$\|(Tf)_\delta\|_{L^2(\mathbb{R},\mathcal{H})} \le \delta^{-1}\,\|\breve{T}\|_{L^2(\mathbb{R},\mathcal{H})\to L^2(\mathbb{R},\mathcal{H})}\,\|f_\delta\|_{L^2(\mathbb{R},\mathcal{H})},$$

from which our asserted estimate is immediate. □

To prove (11.18), let $\mathcal{H} = L^2(\mathbb{R}^{d_1-1})$, and let $K(s,s')$ be the operator on $\mathcal{H}$ with integral kernel

$$\big(K(s,s')\big)(\underline{x},\underline{x}') := q^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s').$$

Then the operator $S^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ corresponds to the operator-valued integral kernel

$$\mathbb{1}_{[1,1+\varepsilon]}(s)\,K(s,s')\,\mathbb{1}_{[1,1+\varepsilon]}(s').$$

Here, recall that we had implicitly always assumed that $s,s'\in[1,1+\varepsilon]$, but suppressed the localizing factors $\mathbb{1}_{[1,1+\varepsilon]}(s)$ and $\mathbb{1}_{[1,1+\varepsilon]}(s')$ for a while. Therefore, choosing $A=[1,1+\varepsilon]$ in Lemma 11.3, we see that estimate (11.18) is indeed an immediate consequence of Lemma 11.3.

Finally, to estimate the operator $\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$, recall that its integral kernel is given by the oscillatory integral

$$\begin{aligned}\tilde{q}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}(\underline{x},s,\underline{x}',s') &= \chi_\varepsilon(x-x^0)\,\chi_\varepsilon(x'-x^0)\\ &\quad\times\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^{d_1}} e^{i\mu\tilde{\Theta}}\,\breve{b}(\tfrac{\mu}{\lambda},\tau,\tau',y,s)\,\breve{H}_{\lambda,\xi}(\tfrac{\mu}{\lambda},\tau,\tau',s',x',y)\,dy\,d\tau\,d\tau'.\end{aligned}$$

The scaled amplitude

$$\breve{b}(\tfrac{\mu}{\lambda},\tau,\tau',y,s)\,\breve{H}_{\lambda,\xi}(\tfrac{\mu}{\lambda},\tau,\tau',s',x',y)$$

is now smooth, with uniform bounds on the derivatives of the first factor and with $\breve{H}_{\lambda,\xi}$ satisfying (11.13). The corresponding phase $\tilde{\Theta}$ is smooth and of order $\mathcal{O}(1)$, and the same is true of all its derivatives.

Lemma 11.4. *If $\varepsilon > 0$ is chosen sufficiently small, then*

$$\left| \det \begin{pmatrix} \tilde{\Theta}_{(\underline{x},s),(\underline{x}',s')} & \tilde{\Theta}_{(\underline{x},s),(\tau,\tau',y)} \\ \tilde{\Theta}_{(\tau,\tau',y),(\underline{x}',s')} & \tilde{\Theta}_{(\tau,\tau',y),(\tau,\tau',y)} \end{pmatrix} \right| \sim 1,$$

uniformly in $k \geq 1$ and $\ell \geq 2$.

Proof. We compute that

$$\begin{pmatrix} \tilde{\Theta}_{(\underline{x},s),(\underline{x}',s')} & \tilde{\Theta}_{(\underline{x},s),(\tau,\tau',y)} \\ \tilde{\Theta}_{(\tau,\tau',y),(\underline{x}',s')} & \tilde{\Theta}_{(\tau,\tau',y),(\tau,\tau',y)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \tilde{\Theta}_{\underline{x}\tau} & 0 & \tilde{\Theta}_{\underline{x}y} \\ 0 & 0 & \tilde{\Theta}_{s\tau} & 0 & 0 \\ 0 & 0 & \tilde{\Theta}_{\tau\tau} & 0 & \tilde{\Theta}_{\tau y} \\ \tilde{\Theta}_{\tau'\underline{x}'} & \tilde{\Theta}_{\tau's'} & 0 & \tilde{\Theta}_{\tau'\tau'} & \tilde{\Theta}_{\tau'y} \\ \tilde{\Theta}_{y\underline{x}'} & 0 & \tilde{\Theta}_{y\tau} & \tilde{\Theta}_{y\tau'} & \tilde{\Theta}_{yy} \end{pmatrix}.$$

Recall that $r = 2^{1-\ell}$ and $\varphi_\ell'(\tau) = -1/(\tau^2 + r^2)$. Moreover,

$$\tilde{\Theta}_{s\tau} = -\frac{\varphi_\ell'(\tau)}{\big(\pi + \delta\varphi_\ell(\tau)\big)^2},$$

so that $|\tilde{\Theta}_{s\tau}| \sim 1$, and similarly $|\tilde{\Theta}_{\tau's'}| \sim 1$. Using the s'-column and then the s-row for cofactor expansions, we thus see that the modulus of the determinant of the matrix above equals $|\tilde{\Theta}_{s\tau}|\,|\tilde{\Theta}_{\tau's'}| \sim 1$ times the modulus of

$$\begin{vmatrix} 0 & 0 & \tilde{\Theta}_{\underline{x}y} \\ 0 & 0 & \tilde{\Theta}_{\tau y} \\ \tilde{\Theta}_{y\underline{x}'} & \tilde{\Theta}_{y\tau'} & \tilde{\Theta}_{yy} \end{vmatrix}.$$

The modulus of the above determinant equals $|\det(\tilde{\Theta}_{y,(\underline{x}',\tau')})|\,|\det(\tilde{\Theta}_{(\underline{x},\tau),y})|$. Due to symmetry, it suffices to show that the latter determinant is comparable to 1.

We write $\underline{x} = (x_2, \underline{\underline{x}})$ and $y = (y_1, y_2, \underline{\underline{y}})$, and present J as a block matrix with respect to the coordinate blocks $(y_1, y_2, \underline{\underline{y}})$ in the form

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \underline{\underline{J}} \end{pmatrix},$$

where $\underline{\underline{J}} = J_{n-1}$. It will be convenient to rather consider the matrix $E := \tilde{\Theta}_{(\tau,\underline{x}),y}$ in place of $\tilde{\Theta}_{(\underline{x},\tau),y}$, which, up to sign, has the same determinant. Then

$$E = \begin{pmatrix} 2(x_1 - y_1) & 2(x_2 - y_2) & 2(\underline{\underline{x}} - \underline{\underline{y}}) \\ -\frac{r}{2} & 2\tau & 0 \\ 0 & 0 & 2\tau I_{d_1-2} - \frac{r}{2}\underline{\underline{J}} \end{pmatrix}.$$

Since $|x_1 - y_1| \sim 1$ and $|\underline{x} - \underline{y}| \lesssim \varepsilon$, assuming that ε is sufficiently small we see that

$$|\det E| \sim \begin{vmatrix} 2\tau & 0 \\ 0 & 2\tau I_{d_1-2} - \frac{r}{2}\underline{\underline{J}} \end{vmatrix}.$$

Here $|\tau| \sim 1$, and since $\underline{\underline{J}}$ is skew-symmetric, we see that

$$|\det E| \sim |2\tau|\,|\det(2\tau I_{d_1-2} - \tfrac{r}{2}\underline{\underline{J}})| \sim 1 \cdot 1.$$

Hence $|\det \tilde{\Theta}_{(\underline{x},\tau),y}| \sim 1$. Consequently, the Monge–Ampère matrix of $\tilde{\Theta}$ also has determinant comparable to 1. □

Since the oscillatory integral operator $\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}$ acts on functions of d_1 space coordinates and is defined by means of d_1+2 frequency variables, Lemma 11.4 implies that for a sufficiently large N,

$$\|\tilde{S}^{k,\ell}_{\lambda,\xi,\mu,x_1,x_1'}\|_{2\to 2} \lesssim \lambda^{-\frac{1}{2}\left(d_1+(d_1+2)\right)}\, \|\breve{H}_{\lambda,\xi}\|_{C^N} = \lambda^{-d}|\breve{H}_{\lambda,\xi}\|_{C^N},$$

uniformly for $k\geq 1$, $\ell\geq 2$, and $|\mu|\sim\lambda$ (see [23] and [20]).

In combination with (11.18), this implies estimate (11.16). The proof of Proposition 10.3 is thus complete.

12. Estimates for the horizontal parts given by $\ell = 1$

In this case, we have $\delta=\delta(k,1)=k^{-1}$. Ignoring a factor 2^{-n}, (9.13) shows that in the renamed coordinates we have

$$\breve{K}^{k,1}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty\int_{\mathbb{R}} e^{i\lambda\sigma\breve{\Phi}_0}\, \tilde{a}_{\lambda,\delta}(\sigma,s)\, \eta_1(\tau)\sin(\tau)^{-n}\, d\tau\, d\sigma,$$

with phase

$$\breve{\Phi}_0 = \delta^{-1} - 1 + s - (k\pi+\tau)\left(\delta|x|^2\cot(\tau) - \frac{u}{k} - 4\delta^{-1}u_c(\tau_{k,1})\right).$$

Here, s is assumed to be in $|1,1+\varepsilon]$. Recall that

$$\operatorname{supp}\eta_1 \subset \big[-\tfrac{5}{8}\pi, -\tfrac{3}{16}\pi\big]\cup\big[\tfrac{3}{16}\pi,\tfrac{5}{8}\pi\big].$$

Hence, the factor $(\sin\tau)^{-n}$ is harmless and can be hidden in the amplitude function η_1 by slightly modifying this function. Note that $\pm\frac{\pi}{2}\in\pm\big[\frac{3}{16}\pi,\frac{5}{8}\pi\big]$, and recall that the parameters $\tau=\pm\frac{\pi}{2}$ (i.e., $k\pi\pm\frac{1}{2}\pi$ in the original expression for $K^{k,1}_{\lambda,s}$) correspond to horizontal points. Thus, in the sense of Subsection 5.1, the contribution of each of the subintervals $\pm\big[\frac{3}{16}\pi,\frac{5}{8}\pi\big]$ to $\breve{K}^{k,1}_{\lambda,s}$ defines a "horizontal part".

Remark 12.1. Even though this seems to suggest that for any k there are two different horizontal points associated with the kth "zigzag" of Σ_s, there is in fact only one such horizontal point. Recall to this end that the cut-offs $\eta_0(\cdot-k)$ and $\eta_0(\cdot-(k+1))$ overlap, and note that $k\pi+\frac{\pi}{2}=(k+1)\pi-\frac{\pi}{2}$.

We next decompose $\eta_1=\eta_+ + \eta_-$, where the smooth bump functions $\eta_\pm$ are such that $\operatorname{supp}\eta_\pm\subset\pm[\frac{3}{16}\pi,\frac{5}{8}\pi]$ and decompose accordingly

$$\breve{K}^{k,1}_{\lambda,s} = \breve{K}^{k,+}_{\lambda,s} + \breve{K}^{k,-}_{\lambda,s},$$

where

$$\breve{K}^{k,\pm}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty\int_{\mathbb{R}} e^{i\lambda\sigma\breve{\Phi}_0}\, \tilde{a}_{\lambda,\delta}(\sigma,s)\, \eta_\pm(\tau)\, d\tau\, d\sigma.$$

The corresponding contributions to $\breve{T}^{k,1,\varepsilon}_\lambda$ are denoted by $\breve{T}^{k,\pm,\varepsilon}_\lambda$.

Changing coordinates via $\tau=\pm\frac{\pi}{2}+\tilde{\tau}$, so that $\cot(\tau)=-\tan(\tilde{\tau})$, and setting $\tilde{\eta}_\pm(\tilde{\tau}):=\eta_\pm(\pm\frac{\pi}{2}+\tilde{\tau})$, we may rewrite

$$\breve{K}^{k,\pm}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty\int_{\mathbb{R}} e^{i\lambda\sigma\tilde{\Phi}^\pm}\, \tilde{a}_{\lambda,\delta}(\sigma,s)\, \tilde{\eta}_\pm(\tilde{\tau})\, d\tilde{\tau}\, d\sigma, \tag{12.1}$$

with phase

$$\tilde{\Phi}^\pm = \delta^{-1} - 1 + s + \big(k\pi\pm\tfrac{1}{2}\pi+\tilde{\tau}\big)\left(\delta|x|^2\tan(\tilde{\tau}) + \frac{u}{k} + 4\delta^{-1}u_c(\tau_{k,1})\right), \tag{12.2}$$

where

$$\operatorname{supp}\tilde{\eta}_\pm\subset\pm\big[-\tfrac{5}{16}\pi,\tfrac{1}{8}\pi\big]\subset(-\tfrac{3}{8}\pi,\tfrac{3}{8}\pi)$$

and $|\pm\frac{1}{2}\pi+\tilde\tau|\le\frac{5}{8}\pi$ on the support.

Finally, changing coordinates from $\tilde\tau$ to $\tau=\tan(\tilde\tau)$, and putting here

$$\varphi_1(\tau):=\arctan\tau,$$

we may write $\breve K^{k,\pm}_{\lambda,s}$ in a very similar way as in (10.1) as

$$\breve K^{k,\pm}_{\lambda,s}(x,u)=\lambda^{n+\frac32}\int_0^\infty\!\!\int_{\mathbb R} e^{i\lambda\sigma\breve\Phi^\pm}\,\tilde a_{\lambda,\delta}(\sigma,s)\,\breve\eta_\pm(\tau)\,d\tau\,d\sigma, \tag{12.3}$$

with phase

$$\breve\Phi^\pm:=\delta^{-1}-1+s+(\pi\pm\delta\tfrac{\pi}{2}+\delta\varphi_1(\tau))\big(\tau|x|^2+u+4k^2u_c(\tau_{k,1})\big). \tag{12.4}$$

Note that here

$$\operatorname{supp}\breve\eta_\pm\subset\tan\big([-\tfrac38\pi,\tfrac38\pi]\big), \tag{12.5}$$

so that $|\varphi_1(\tau)|\le\frac38\pi<\frac{\pi}{2}\le\pi\pm\delta\frac{\pi}{2}$ on $\operatorname{supp}\breve\eta_\pm$, and

$$\varphi_1'(\tau)\sim1\quad\text{and}\quad|\varphi_1^{(j)}(\tau)|\lesssim1\text{ for }j\ge2.$$

To simplify the notation, let us put $\pi_\delta:=\pi\pm\delta\frac{\pi}{2}$, so that

$$\breve K^{k,\pm}_{\lambda,s}(x,u)=\breve K^{k}_{\lambda,s}(x,u):=\lambda^{n+\frac32}\int_0^\infty\!\!\int_{\mathbb R} e^{i\lambda\sigma\breve\Phi}\,\tilde a_{\lambda,\delta}(\sigma,s)\,\breve\eta_0(\tau)\,d\tau\,d\sigma,$$

with phase

$$\breve\Phi:=\delta^{-1}-1+s+(\pi_\delta+\delta\varphi_1(\tau))\big(\tau|x|^2+u+4k^2u_c(\tau_{k,1})\big)$$

and $\breve\eta_0:=\breve\eta_\pm$. The form of the oscillatory integral $\breve K^k_{\lambda,s}$ and its phase are thus very similar to (10.1) and (10.2), except that here the amplitude is no longer supported where $|\tau|\sim1$, but satisfies (12.5) and thus includes $\tau=0$. However, the Jacobian factor arising from the change of variables $\tau=\tan(\tilde\tau)$ is comparable to 1 on the support of $\breve\eta_0$.

Even though $\breve K^{k,1}_{\lambda,s}$ is in fact the sum of the two terms corresponding to the signs $\pm$, with a slight abuse of notation we shall from here on simply assume that $\breve K^{k,1}_{\lambda,s}=\breve K^{k}_{\lambda,s}$, with the understanding that the following arguments are applied separately to each fixed sign. Note also that for $\ell=1$ we are still working on the Heisenberg group $\mathbb H_n$, since here $2^{1-\ell}=1$, so that we did not perform an isotropic scaling but only a parabolic, hence automorphic, scaling by $1/k$.

We next look at the analogue of the TT^* argument from Subsection 10.1:

Lemma 12.2. *Let $\delta:=k^{-1}$. The integral kernel $\breve Q^{k,1,\varepsilon}_\lambda$ of $\breve T^{k,1,\varepsilon}_\lambda(\breve T^{k,1,\varepsilon}_\lambda)^*$ is given by*

$$\begin{aligned}\breve Q^{k,1,\varepsilon}_\lambda(x,u,s,x',u',s')&=\lambda^{d+1}\,\chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0)\\&\quad\times\int_{\eta\sim1}\int_{|\tau'|\lesssim1}\int_{|\tau|\lesssim1}\int_{\mathbb R^{d_1}} e^{i\lambda\eta\Psi}a(\eta,\tau,\tau',y,s,x',u',s')\,dy\,d\tau\,d\tau'\,d\eta,\end{aligned} \tag{12.6}$$

with phase function

$$\begin{aligned}\Psi=\frac{\delta^{-1}-1+s}{\pi_\delta+\delta\varphi_1(\tau)}-\frac{\delta^{-1}-1+s'}{\pi_\delta+\delta\varphi_1(\tau')}+\tau|x-y|^2-\tau'|x'-y|^2\\+(u-u'+\tfrac12\langle J(x-x'),y\rangle)\end{aligned} \tag{12.7}$$

and amplitude

$$a(\eta,\tau,\tau',y,s,x',u',s') = a_{1,\lambda,\delta}(\eta,\tau,s)\,\chi_\varepsilon(y)\,\breve\eta_0(\tau)\,\breve\eta_0(\tau') \times \frac{h_\lambda(\eta,\tau,\tau',s',x',u',y)}{\big(\pi_\delta+\delta\varphi_1(\tau)\big)\big(\pi_\delta+\delta\varphi_1(\tau')\big)}, \tag{12.8}$$

where

$$a_{1,\lambda,\delta}(\eta,\tau,s) = a_\lambda\Big(\big(1+\delta(s-1)\big)\frac{\eta}{\pi_\delta+\delta\varphi_1(\tau)}\Big) \tag{12.9}$$

and

$$h_\lambda(\eta,\tau,\tau',s',x',u',y) = \int_{\mathbb{R}} e^{i\lambda\xi\Psi_2}\lambda\,\widehat{\chi_\varepsilon}(\lambda\xi)\,\overline{a_{1,\lambda,\delta}(\eta-\xi,\tau',s')}\,d\xi, \tag{12.10}$$

with

$$\Psi_2 = \frac{\delta^{-1}-1+s'}{\pi_\delta+\delta\varphi_1(\tau')} + \tau'|x'-y|^2 + 4k^2u_c(\tau_{k,1}) + u' + \tfrac12\langle Jx',y\rangle. \tag{12.11}$$

The amplitudes a and h_λ, together with all their partial derivatives, are uniformly bounded on the support of the integral, with bounds depending only on finitely many derivatives of a_λ.

We shall skip the proof, since it is obtained by following almost verbatim the proof of Lemma 10.1, up to the indicated minor modifications.

Next, in contrast to the case $\ell \ge 2$, we shall here freeze the variable x_2. Thus, given $x, x' \in \mathbb{R}^{d_1}$, we set

$$\underline{x} := (x_1, x_3, \dots, x_{d_1})$$

and define, for fixed $x_2 \in \mathbb{R}$, the operator $\breve T^{k,1,\varepsilon}_{\lambda,x_2}$ by

$$(\breve T^{k,1,\varepsilon}_{\lambda,x_2}\varphi)(\underline{x},u,s) = (\breve T^{k,1,\varepsilon}_{\lambda}\varphi)(x,u,s).$$

If we write $f(x',u',s') = f_{x_2'}(\underline{x}',u',s')$, then

$$\big(\breve T^{k,1,\varepsilon}_{\lambda}(\breve T^{k,1,\varepsilon}_{\lambda})^* f\big)(x,u,s) = \int_{\mathbb{R}} \big(\breve T^{k,1,\varepsilon}_{\lambda,x_2}(\breve T^{k,1,\varepsilon}_{\lambda,x_2'})^*\, f_{x_2'}\big)(\underline{x},u,s)\,dx_2'.$$

Note that the integral kernel $\breve Q^{k,1,\varepsilon}_{\lambda,x_2,x_2'}$ of $\breve T^{k,1,\varepsilon}_{\lambda,x_2}(\breve T^{k,1,\varepsilon}_{\lambda,x_2'})^*$ is given by $\breve Q^{k,1,\varepsilon}_{\lambda}(x_2,\cdot,x_2',\cdot)$.

In analogy with Propositions 10.2 and 10.3, we shall here prove the following propositions:

Proposition 12.3. *We have*

$$\|\breve T^{k,1,\varepsilon}_{\lambda,x_2}(\breve T^{k,1,\varepsilon}_{\lambda,x_2'})^*\|_{L^1_{(\underline{x},u)}(L^2_s)\to L^\infty_{(\underline{x},u)}(L^2_s)} \lesssim \lambda^{d-1}\,\langle\lambda\,|x_2'-x_2|\rangle^{-\frac{d_1-1}{2}}.$$

Proposition 12.4. *We have*

$$\|\breve T^{k,1,\varepsilon}_{\lambda,x_2}(\breve T^{k,1,\varepsilon}_{\lambda,x_2'})^*\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim \delta^{-1}.$$

The square function estimate in Proposition 9.3 follows from these two propositions by the same arguments as in the proof of Proposition 5.2. The proofs of Propositions 12.3 and 12.4 will be given in Sections 12.1 and 12.2.

12.1. **Pointwise estimates for $k \geq 1$ and $\ell = 1$.** We follow again the strategy of the proofs of Propositions 6.1 and 10.2 by establishing analogous pointwise estimates as in Proposition 11.1. Since the following arguments are very similar to those of Sections 6.2 and 11.1, we keep it again a bit more sketchy and mainly focus on the main contributions of the oscillatory integral.

12.1.1. *Freezing τ'.* By Lemma 12.2, we have

$$\breve{Q}_\lambda^{k,1,\varepsilon} = \int_{|\tau'|\lesssim 1} \breve{Q}_{\lambda,\tau'}^{k,1,\varepsilon}\, d\tau',$$

where, suppressing again the localizing factors $\chi_\varepsilon(x-x^0, u-u^0)\,\chi_\varepsilon(x'-x^0, u'-u^0)$,

$$\breve{Q}_{\lambda,\tau'}^{k,1,\varepsilon} = \lambda^{d+1} \int_{\eta\sim 1} \int_{|\tau|\lesssim 1} \int_{\mathbb{R}^{d_1}} e^{i\lambda\eta\Psi} a(\eta,\tau,\tau',y,s,x',u',s')\, dy\, d\tau\, d\eta,$$

with phase function given by (12.7), that is,

$$\Psi = \frac{\delta^{-1}-1+s}{\pi_\delta+\delta\varphi_1(\tau)} - \frac{\delta^{-1}-1+s'}{\pi_\delta+\delta\varphi_1(\tau')} + \tau\,|x-y|^2 - \tau'|x'-y|^2 + (u-u'+\tfrac{1}{2}\langle J(x-x'),y\rangle), \tag{12.12}$$

where $\delta = k^{-1}$ and $\varphi_1(\tau) = \arctan\tau$. In the following, we assume that τ' is frozen.

12.1.2. *Changing variables and stationary phase for a spherical integration.* In the oscillatory integral above, we change variables $y = x + y'$ and write $y' = \rho\omega$, where $\rho \sim 1$ and $\omega \in S^{d_1-1}$. Then, using $|x-y|^2 = |y'|^2 = \rho^2$ and

$$|x'-y|^2 = |\Delta x + y'|^2 = |\Delta x|^2 + \rho^2 + 2\rho\langle \Delta x, \omega\rangle, \qquad \Delta x = x - x',$$

the phase (12.12) becomes

$$\Psi_1 = \frac{\delta^{-1}-1+s}{\pi_\delta+\delta\varphi_1(\tau)} - \frac{\delta^{-1}-1+s'}{\pi_\delta+\delta\varphi_1(\tau')} + (\tau-\tau')\rho^2 - \tau'|\Delta x|^2 + \rho\langle\theta,\omega\rangle + \Delta w, \tag{12.13}$$

where $\theta = \frac{1}{2}J\Delta x - 2\tau'\Delta x$ and $\Delta w = u - u' + \frac{1}{2}\langle J\Delta x, x\rangle$. Note that

$$|\theta|^2 = \left(\tfrac{1}{4} + 4\,(\tau')^2\right)|\Delta x|^2 \sim |\Delta x|^2. \tag{12.14}$$

If we absorb the Jacobian factor ρ^{d_1-1} and all cutoffs into a new amplitude function a_1, we are reduced to an oscillatory integral of the form

$$\lambda^{d+1} \int_{\eta\sim 1} \int_{|\tau|\lesssim 1} \int_{\rho\sim 1} \int_{S^{d_1-1}} e^{i\lambda\eta\Psi_1}\, a_1\, d\omega\, d\rho\, d\tau\, d\eta. \tag{12.15}$$

We now change coordinates from τ to β by

$$\tau = \tau' + \beta|\Delta x|.$$

Note that $|\beta\Delta x| \lesssim 1$ on the support of the oscillatory integral above. Hence, we are reduced to an oscillatory integral of the form

$$\lambda^{d+1}\,|\Delta x| \int_{\eta\sim 1} \int_{\rho\sim 1} \int_{S^{d_1-1}} \int_{\mathbb{R}} e^{i\lambda\eta\Psi_2}\, a_2\, \chi_0(\beta|\Delta x|)\, d\beta\, d\omega\, d\rho\, d\eta, \tag{12.16}$$

where χ_0 is a smooth cutoff supported where $|\beta\Delta x| \lesssim 1$, and Ψ_2 is given by

$$\Psi_2 = \frac{\delta^{-1}-1+s}{\pi_\delta + \delta\varphi_1(\tau'+\beta|\Delta x|)} - \frac{\delta^{-1}-1+s'}{\pi_\delta+\delta\varphi_1(\tau')} + \beta\,|\Delta x|\,\rho^2 - \tau'|\Delta x|^2 + \rho\langle\theta,\omega\rangle + \Delta w. \tag{12.17}$$

Note that the vector θ is independent of β. For the region where $|\beta| \gg 1$, we can integrate by parts in ρ, so this part of the integral is an error term. The main contribution is given by the region where $|\beta| \lesssim 1$. Thus, since also $|\Delta x| \lesssim \varepsilon$, the integral (12.16) reduces to

$$\lambda^{d+1}\,|\Delta x| \int_{\eta\sim 1}\int_{\rho\sim 1}\int_{S^{d_1-1}}\int_{|\beta|\lesssim 1} e^{i\lambda\eta\Psi_2}\,\tilde a_2\,\chi_\varepsilon(\beta|\Delta x|)\,d\beta\,d\omega\,d\rho\,d\eta. \tag{12.18}$$

Next, we focus on the integration in ω. Recall that $y = x + y'$ and $y \in \operatorname{supp}\chi_\varepsilon$ in our original coordinates. Thus, we have $|\rho\omega + x| \lesssim \varepsilon$, which localizes ω to an $O(\varepsilon)$ neighborhood of $-x^0/|x^0|$, whose first component dominates all others. Writing $\theta = (\theta_1, \underline{\theta})$, for the region where $|\underline{\theta}| \gg \varepsilon\,|\theta_1(\tau')|$, we can argue via integration by parts, which only contributes to the error term. For $|\underline{\theta}| \lesssim \varepsilon\,|\theta_1(\tau')|$, we perform stationary phase in ω which reduces (12.17) to an oscillatory integral of the form

$$\lambda^{d+1}\,|\Delta x| \int_{\eta\sim 1}\int_{\rho\sim 1}\int_{|\beta|\lesssim 1} \langle\lambda\eta\rho|\theta|\rangle^{-\frac{d_1-1}{2}}\, e^{i\lambda\eta\Psi_3^\pm}\,a_3\,\chi_\varepsilon(\beta|\Delta x|)\,d\beta\,d\rho\,d\eta, \tag{12.19}$$

where the factor $\langle\lambda\eta\rho|\theta|\rangle^{-\frac{d_1-1}{2}}$ actually denotes a symbol of order $-\frac{d_1-1}{2}$, a_3 is a smooth amplitude satisfying the same symbol bounds as $\tilde a_2$, and

$$\Psi_3^\pm = \frac{\delta^{-1}-1+s}{\pi_\delta + \delta\varphi_1(\tau'+\beta|\Delta x|)} - \frac{\delta^{-1}-1+s'}{\pi_\delta+\delta\varphi_1(\tau')} + \beta\,|\Delta x|\,\rho^2 - \tau'|\Delta x|^2 \pm \rho|\theta| + \Delta w, \tag{12.20}$$

where the sign $\pm$ depends on the stationary point $\pm\theta/|\theta|$ which is contained in the $\mathcal{O}(\varepsilon)$ cap.

12.1.3. *Stationary phase for the integration in ρ.* Next, note that

$$\psi^\pm = \beta\,|\Delta x|\,\rho^2 \pm |\theta|\,\rho = \rho^2\,\psi_2 + \rho\,\psi_1^\pm$$

collects all the terms in (12.20) that depend on ρ. Here $|\psi_2| \sim |\beta\Delta x|$, and, by (12.14), we have $|\psi_1^\pm| \sim |\theta| \sim |\Delta x|$. Thus, if $|\beta| \ll 1$, then, for $\rho \sim 1$, we have $|\partial_\rho\psi^\pm| \gtrsim |\Delta x|$, and we can integrate by parts in ρ, whence this region again only contributes to the error term. Hence (12.19) reduces to an integral of the form

$$\lambda^d(\lambda|\Delta x|)\int_{\eta\sim 1}\int_{\rho\sim 1}\int_{|\beta|\sim 1}\langle\lambda|\Delta x|\rangle^{-\frac{d_1-1}{2}} e^{i\lambda\eta\Psi_3^\pm}\,\tilde a_3\,d\beta\,d\rho\,d\eta, \tag{12.21}$$

where $\tilde a_3$ satisfies the same symbol bounds as a_3. Next, consider again

$$\partial_\rho\psi^\pm = 2\beta|\Delta x|\rho \pm |\theta|.$$

Since $|\beta| \sim 1$, $\psi^\pm$ has a unique critical point $\rho_c \sim 1$ in ρ if and only if $\mp\beta \sim 1$. In the following, we consider only the phase Ψ_3^- (the case of Ψ_3^+ can be treated analogously). If $\beta \sim -1$, then ψ^- has no critical point on $\rho \sim 1$ and we can again integrate by parts in ρ, showing that the corresponding contribution is an error

term. For the main contribution where $\beta \sim 1$, the function ψ^- does have a critical point for any such β, which is given by

$$\rho_c = \frac{|\theta|}{2\beta|\Delta x|} \sim 1.$$

Moreover, note that $|\partial_\rho^2 \psi^-| = 2\beta|\Delta x| \sim |\Delta x|$ for $\beta \sim 1$. Thus, applying stationary phase in ρ reduces us to the oscillatory integral

$$\lambda^d (\lambda|\Delta x|) \int_{\eta\sim 1} \int_{\beta \sim 1} \langle \lambda|\Delta x|\rangle^{-\frac{d_1}{2}} e^{i\lambda\eta\Psi_4}\, a_4\, d\beta\, d\eta, \tag{12.22}$$

where a_4 is a smooth amplitude satisfying the same symbol bounds as $\tilde{a}_3$, and

$$\Psi_4 = \frac{\delta^{-1} - 1 + s}{\pi_\delta + \delta\varphi_1(\tau' + \beta|\Delta x|)} - \frac{\delta^{-1} - 1 + s'}{\pi_\delta + \delta\varphi_1(\tau')} - \tau'|\Delta x|^2 - \frac{|\theta|^2}{4\beta|\Delta x|} + \Delta w. \tag{12.23}$$

12.1.4. *Stationary phase for the integration in β.* Note that

$$\partial_\beta \Psi_4 = |\Delta x|\, H(\beta, s, \tau', |\Delta x|),$$

where, recalling that $|\theta|^2 = (\frac{1}{4} + 4\,(\tau')^2)|\Delta x|^2$ by (12.14),

$$H = -(\delta^{-1} - 1 + s)\, \frac{\delta\varphi_1'(\tau' + \beta|\Delta x|)}{(\pi_\delta + \delta\varphi_1(\tau' + \beta|\Delta x|))^2} + \frac{\frac{1}{4} + 4\,(\tau')^2}{4\beta^2}.$$

Thus, for fixed τ_0' and $s = 1$, $|\Delta x| = 0$, we obtain

$$H(\beta, 1, \tau_0', 0) = -\frac{\frac{1}{1+(\tau_0)^2}}{(\pi_\delta + \delta \arctan \tau_0')^2} + \frac{\frac{1}{4} + 4\,(\tau_0')^2}{4\beta^2}.$$

Note that this function has a unique zero $\beta_c(1, \tau_0', 0) \sim 1$. On the other hand,

$$\partial_\beta H = -\frac{\frac{1}{4} + 4\,(\tau')^2}{2\beta^3} + \mathcal{O}(|\Delta x|).$$

Thus, $|\partial_\beta H| \sim 1$ for $|\beta| \sim 1$, and the implicit function theorem implies that for $|s-1|$, $|\Delta x|$, and $|\tau' - \tau_0'|$ sufficiently small, there is a unique zero $\beta_c = \beta_c(s, \tau', |\Delta x|)$ for H. Note that we can assume that τ' lies in an interval $[\tau_0', \tau_0' + \varepsilon]$ by decomposing the τ' support into $\mathcal{O}(\varepsilon^{-1})$ many subintervals.

Since $\partial_\beta^2 \Psi_4(\beta) \sim |\Delta x|$ for $|\beta| \sim 1$, applying stationary phase in β, we see that (12.22) reduces to an oscillatory integral of the form

$$\lambda^d (\lambda|\Delta x|) \int_{\eta\sim 1} \langle \lambda|\Delta x|\rangle^{-\frac{d_1+1}{2}} e^{i\lambda\eta\Psi_5}\, a_5\, d\eta, \tag{12.24}$$

where a_5 is a smooth amplitude satisfying the same symbol bounds as a_4, and the new phase is given by $\Psi_5 = \Psi_4|_{\beta=\beta_c}$.

Finally, note that

$$\frac{\delta^{-1} - 1 + s}{\pi_\delta + \delta\varphi_1(\tau' + \beta_c|\Delta x|)} - \frac{\delta^{-1} - 1 + s'}{\pi_\delta + \delta\varphi_1(\tau')} = \frac{\Delta s}{\pi_\delta + \delta\varphi_1(\tau')} + \mathcal{O}(|\Delta x|).$$

Thus, integration by parts in η shows that we arrive at an analogous estimate to that in Proposition 11.1 for the main contribution of the oscillatory integral.

12.2. **Refined L^2 estimates for $k \geq 1$ and $\ell = 1$.** In this section we prove Proposition 12.4. We shall closely follow the arguments from Section 11.2.

12.2.1. *Freezing ξ in the defining integral for h_λ.* Write again

$$\Psi_2 = \psi(s') + u' + \dot\Psi_2, \qquad \psi(s') = \frac{\delta^{-1} - 1 + s'}{\pi_\delta},$$

where now

$$\dot\Psi_2 = -\frac{1+\delta(s'-1)}{\pi_\delta + \delta\varphi_1(\tau')}\frac{\varphi_1(\tau')}{\pi_\delta} + \tau'|x'-y|^2 + 4k^2 u_c(\tau_{k,1}) + \tfrac12\langle Jx', y\rangle.$$

Writing the amplitude in (12.8) as

$$a(\eta,\tau,\tau',y,s,x',u',s') = b(\eta,\tau,\tau',y,s)\, h_\lambda(\eta,\tau,\tau',s',x',u',y),$$

where

$$b(\eta,\tau,\tau',y,s) := a_{1,\lambda,\delta}(\eta,\tau,s)\,\chi_\varepsilon(y)\,\frac{\breve\eta_0(\tau)\,\breve\eta_0(\tau')}{\big(\pi_\delta+\delta\varphi_1(\tau)\big)\big(\pi_\delta+\delta\varphi_1(\tau')\big)},$$

we decompose

$$h_\lambda(\eta,\tau,\tau',s',x',u',y) = \int_{\mathbb{R}} e^{i\lambda\xi(\psi(s')+u')} H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y)\,d\xi,$$

where

$$H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y) := e^{i\lambda\xi\dot\Psi_2}\,\lambda\,\widehat{\chi}_\varepsilon(\lambda\xi)\,\overline{a_{1,\lambda,\delta}(\eta-\xi,\tau',s')}.$$

For frozen ξ (and ε), we define

$$\begin{aligned}(12.25)\quad \breve Q^{k,1}_{\lambda,\xi}(x,u,s,x',u',s') &:= \lambda^{d+1}\,\chi_\varepsilon(x-x^0,u-u^0)\,\chi_\varepsilon(x'-x^0,u'-u^0)\\ &\times\int_{\eta\sim1}\int_{|\tau'|\lesssim1}\int_{|\tau|\lesssim1}\int_{\mathbb{R}^{d_1}} e^{i\lambda\eta\Psi}\,b(\eta,\tau,\tau',y,s)\\ &\qquad\times H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y)\,dy\,d\tau\,d\tau'\,d\eta.\end{aligned}$$

Arguing as in the case $\ell\ge 2$, in order to prove Proposition 12.4 it will then suffice to bound the operator $S^{k,1}_{\lambda,\xi,x_2,x_2'}$ with integral kernel $\breve Q^{k,1}_{\lambda,\xi,x_2,x_2'}$ by

$$(12.26)\qquad \|S^{k,1}_{\lambda,\xi,x_2,x_2'}\|_{L^2\to L^2} \lesssim \delta^{-1}\|\breve H_{\lambda,\xi}\|_{C^M}$$

for some $M\in\mathbb{N}$, uniformly in ξ,λ, etc., where again

$$\breve H_{\lambda,\xi}(\eta,\tau,\tau',s',x',y) := H_{\lambda,\xi}(\eta,\tau,\tau',\delta^{-1}s',x',y).$$

12.2.2. *Exploiting the partial Fourier transform in u.* Following the arguments in Section 11.2, for fixed ξ and μ of size $\mu\sim\lambda$, we define $S^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}$ to be the integral operator given by

$$S^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}\,g(\underline{x},s) = \int_{\mathbb{R}^{d_1-1}}\int_0^\infty q^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}(\underline{x},s,\underline{x}',s')\,g(\underline{x}',s')\,d\underline{x}'\,ds',$$

with integral kernel

$$\begin{aligned}q^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}(\underline{x},s,\underline{x}',s') &= \chi_\varepsilon(x-x^0)\,\chi_\varepsilon(x'-x^0)\\ &\times\int_{|\tau'|\lesssim1}\int_{|\tau|\lesssim1}\int_{\mathbb{R}^{d_1}} e^{i\mu\Theta}\,b(\tfrac{\mu}{\lambda},\tau,\tau',y,s)\\ &\qquad\times H_{\lambda,\xi}(\tfrac{\mu}{\lambda},\tau,\tau',s',x',y)\,dy\,d\tau\,d\tau',\end{aligned}$$

where the phase Θ is given by

$$\Theta = \frac{\delta^{-1}-1+s}{\pi_\delta+\delta\varphi_1(\tau)} - \frac{\delta^{-1}-1+s'}{\pi_\delta+\delta\varphi_1(\tau')} + \tau|x-y|^2 - \tau'|x'-y|^2 + \tfrac12\langle J\Delta x, y\rangle.$$

12.2.3. *Scaling in s.* As before, let $\breve{S}^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}$ denote the integral operator whose integral kernel is obtained from that of $S^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}$ by replacing s and s' with $\delta^{-1}s$ and $\delta^{-1}s'$, respectively. Its phase can be written as

$$\breve{\Theta} = \frac{s-s'}{\delta\pi_\delta} + \frac{(1-\delta+s')\varphi_1(\tau')}{\pi_\delta(\pi_\delta+\delta\varphi_1(\tau'))} - \frac{(1-\delta+s)\varphi_1(\tau)}{\pi_\delta(\pi_\delta+\delta\varphi_1(\tau))} + \tau|x-y|^2 - \tau'|x'-y|^2 + \tfrac{1}{2}\langle J\Delta x, y\rangle.$$

By replacing this phase by the one in which the first term is removed, i.e., by

$$\tilde{\Theta} = \frac{(1-\delta+s')\varphi_1(\tau')}{\pi_\delta(\pi_\delta+\delta\varphi_1(\tau'))} - \frac{(1-\delta+s)\varphi_1(\tau)}{\pi_\delta(\pi_\delta+\delta\varphi_1(\tau))} + \tau|x-y|^2 - \tau'|x'-y|^2 + \tfrac{1}{2}\langle J\Delta x, y\rangle,$$

we obtain the related operator $\tilde{S}^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}$.

By means of a similar conjugation trick as in (11.17) and the rescaling trick based on Lemma 11.3 in Subsection 11.2.3, we can finally reduce the proof of estimate (12.26) to showing

$$\|\tilde{S}^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}\|_{2\to 2} \lesssim \lambda^{-d}\|\breve{H}_{\lambda,\xi}\|_{C^M}, \tag{12.27}$$

uniformly in ξ and $\mu \sim \lambda$.

The scaled amplitude of both operators $\breve{S}^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}$ and $\tilde{S}^{k,1}_{\lambda,\xi,\mu,x_2,x_2'}$ is given by

$$\breve{b}(\tfrac{\mu}{\lambda},\tau,\tau',y,s)\,\breve{H}_{\lambda,\xi}(\tfrac{\mu}{\lambda},\tau,\tau',s',x',y),$$

where $\breve{b}(\eta,\tau,\tau',y,s) := b(\eta,\tau,\tau',y,\delta^{-1}s)$. The amplitude is smooth, with uniform bounds on the derivatives of $\breve{b}$, while $\breve{H}_{\lambda,\xi}$ satisfies the estimates in (11.13). The phase $\tilde{\Theta}$ is smooth and of order $\mathcal{O}(1)$, and the same is true of all its derivatives.

Thus, the proof of estimate (12.27) will be an immediate consequence of the following analogue of Lemma 11.4.

Lemma 12.5. *If $\varepsilon > 0$ is chosen sufficiently small, then*

$$\left|\det\begin{pmatrix}\tilde{\Theta}_{(\underline{x},s),(\underline{x}',s')} & \tilde{\Theta}_{(\underline{x},s),(\tau,\tau',y)}\\ \tilde{\Theta}_{(\tau,\tau',y),(\underline{x}',s')} & \tilde{\Theta}_{(\tau,\tau',y),(\tau,\tau',y)}\end{pmatrix}\right| \sim 1,$$

uniformly in $k \geq 1$.

Proof. The matrix above equals

$$\begin{pmatrix} 0 & 0 & \tilde{\Theta}_{\underline{x}\tau} & 0 & \tilde{\Theta}_{\underline{x}y}\\ 0 & 0 & \tilde{\Theta}_{s\tau} & 0 & 0\\ 0 & 0 & \tilde{\Theta}_{\tau\tau} & 0 & \tilde{\Theta}_{\tau y}\\ \tilde{\Theta}_{\tau'\underline{x}'} & \tilde{\Theta}_{\tau's'} & 0 & \tilde{\Theta}_{\tau'\tau'} & \tilde{\Theta}_{\tau'y}\\ \tilde{\Theta}_{y\underline{x}'} & 0 & \tilde{\Theta}_{y\tau} & \tilde{\Theta}_{y\tau'} & \tilde{\Theta}_{yy}\end{pmatrix}.$$

Note that $\varphi_1'(\tau) = 1/(1+\tau^2) \sim 1$ on the support of $\breve{\eta}_0$. We have

$$\tilde{\Theta}_{s\tau} = -\frac{\varphi_1'(\tau)}{(\pi_\delta+\delta\varphi_1(\tau))^2}, \qquad \tilde{\Theta}_{\tau's'} = \frac{\varphi_1'(\tau')}{(\pi_\delta+\delta\varphi_1(\tau'))^2},$$

and therefore $|\tilde\Theta_{s\tau}| \sim 1$ and $|\tilde\Theta_{\tau' s'}| \sim 1$. By cofactor expansion, as in the first part of the proof of Lemma 11.4, we are reduced to considering the matrix

$$\begin{pmatrix} 0 & 0 & \tilde\Theta_{\underline{x}y} \\ 0 & 0 & \tilde\Theta_{\tau y} \\ \tilde\Theta_{y\underline{x}'} & \tilde\Theta_{y\tau'} & \tilde\Theta_{yy} \end{pmatrix}.$$

By symmetry, we are thus reduced to showing that

$$|\det(\tilde\Theta_{(\underline{x},\tau),y})| \sim 1.$$

To this end, it will be convenient to flip the first two coordinates of x and write $x = (x_2, x_1, \underline{\underline{x}})$ and $\underline{x} = (x_1, \underline{\underline{x}})$, and accordingly $y = (y_2, y_1, \underline{\underline{y}})$, and to write J as a block matrix with respect to the coordinate blocks $(y_2, y_1, \underline{\underline{y}})$ in the form

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \underline{\underline{J}} \end{pmatrix},$$

where $\underline{\underline{J}} = J_{n-1}$. It will be convenient to consider the matrix $E := \tilde\Theta_{(\tau,\underline{x}),y}$ in place of $\tilde\Theta_{(\underline{x},\tau),y}$, which, up to sign, has the same determinant. Then

$$E = \begin{pmatrix} 2(y_2 - x_2) & 2(y_1 - x_1) & 2(\underline{\underline{y}} - \underline{\underline{x}}) \\ \frac{1}{2} & -2\tau & 0 \\ 0 & 0 & -2\tau I_{d_1-2} - \frac{1}{2}\underline{\underline{J}} \end{pmatrix}.$$

Here, $|\tau| \lesssim 1$, and moreover $|x_1 - y_1| \sim 1, |x_2 - y_2| \lesssim \varepsilon$ and $|\underline{\underline{x}} - \underline{\underline{y}}| \lesssim \varepsilon$. Hence, for $\varepsilon > 0$ sufficiently small,

$$|\det E| \sim 1 \cdot |\det(-2\tau I_{d_1-2} - \tfrac{1}{2}\underline{\underline{J}})| \sim 1,$$

since $\underline{\underline{J}}$ is skew-symmetric and $\underline{\underline{J}}^2 = -I_{d_1-2}$. Hence $|\det \tilde\Theta_{(\underline{x},\tau),y}| \sim 1$. □

Appendix A. Proof of the decay estimates in Proposition 8.2

In this section, we shall give the proof of Proposition 8.2. To this end, recall from Section 8.2 that the critical points of the phase

$$\tilde\Phi(x, u, s, \sigma, \tau) = \sigma(s - |x|^2 g(\tau) + 4u\tau)$$

in (8.5) are given by

$$|x|_c(s,\tau) := \sqrt{s}\left|\frac{\sin\tau}{\tau}\right| \quad \text{and} \quad u_c(s,\tau) := -s\,\frac{\tau - \sin\tau\cos\tau}{4\tau^2},$$

where the factor $\eta_\ell(\tau - k\pi)$ localizes to the union $J_{k,\ell}$ of two intervals on which $|\tau - k\pi| \sim 2^{-\ell}$.

We assume that $s \in [1,4]$, and set

$$\Delta x^2(s,\tau) := |x|^2 - |x|_c^2(s,\tau) \quad \text{and} \quad \Delta u(s,\tau) := u - u_c(s,\tau).$$

Recall finally that $\tau_{k,\ell} \in J_{k,\ell}$ is fixed.

Lemma A.1. *Suppose that $k, \ell \geq 1$ and $s \in [1,4]$, Then the following statements hold:*

(1) If $|x|^2 \leq \frac{1}{100}(\frac{2^{-\ell}}{k})^2$ or $|x|^2 \geq 10\,(\frac{2^{-\ell}}{k})^2$, then

$$|\Delta x^2(s,\tau)| \gtrsim (\tfrac{2^{-\ell}}{k})^2 \quad \textit{for all } \tau \in J_{k,\ell}.$$

(2) There is some constant $R_0 > 0$ such that, if $|\Delta u(s,\tau_{k,\ell})| \geq R_0 \frac{2^{-\ell}}{k^2}$, then

$$|\Delta u(s,\tau)| \gtrsim \tfrac{2^{-\ell}}{k^2} \quad \textit{for all } \tau \in J_{k,\ell}.$$

Proof. Recall that $|u_c(s,\tau) - u_c(s,\tau_{k,\ell})| \lesssim \frac{2^{-\ell}}{k^2}$ for $\tau \in J_{k,\ell}$. Thus, the second part of the lemma follows immediately since

$$|\Delta u(s,\tau)| \geq |\Delta u(s,\tau_{k,\ell})| - |u_c(s,\tau) - u_c(s,\tau_{k,\ell})| \gtrsim \tfrac{2^{-\ell}}{k^2}$$

if $|\Delta u(s,\tau_{k,\ell})| \geq R_0 \frac{2^{-\ell}}{k^2}$ for sufficiently large $R_0 > 0$. To show the first part of the lemma, recall that

$$\Delta x^2(s,\tau) := |x|^2 - |x|_c^2(s,\tau) = |x|^2 - s\,\frac{\sin^2\tau}{\tau^2}.$$

Thus, we need to bound $(\sin^2\tau)/\tau^2$ from above and below.

Recall from the discussion preceding (4.16) that $\eta_\ell(\tau) = \eta_0(2^{\ell-1}\tau) - \eta_0(2^\ell\tau)$ for $\ell \geq 1$, where η_0 is supported in $(-\frac{5\pi}{8}, \frac{5\pi}{8})$ with $\eta_0(\tau) = 1$ for $\tau \in (-\frac{3\pi}{8}, \frac{3\pi}{8})$. Hence, if $\tau \in J_{k,l}$, we can write $\tau = (k \pm 2^{-\ell}\alpha)\pi$ for some $\frac{3}{8} \leq \alpha < \frac{5}{4}$.

We distinguish the cases $\ell = 1$ and $\ell \geq 2$. For $\ell = 1$ obverse that for $\tau \in J_{k,l}$

$$\frac{\sin^2\tau}{\tau^2} = \frac{\sin^2(\frac{1}{2}\alpha\pi)}{((k \pm \frac{1}{2}\alpha)\pi)^2} \leq \frac{1}{k^2}\frac{1}{((1-\frac{\alpha}{2})\pi)^2} \leq \frac{64}{9\pi^2}\frac{1}{k^2} < \frac{1}{k^2}.$$

On the other hand,

$$\frac{\sin^2\tau}{\tau^2} \geq \frac{1}{k^2}\frac{\sin^2(\frac{1}{2}\alpha\pi)}{((1+\frac{5}{8})\pi)^2} \leq \frac{1}{k^2}\frac{\sin^2(\frac{1}{2}\alpha\pi)}{((1+\frac{5}{8})\pi)^2}\bigg|_{\alpha=\frac{3}{8}} > 0.011\,\frac{1}{k^2}$$

Now suppose that $k \geq 1$ and $\ell \geq 2$. Using the Taylor series expansion $\sin^2\tau = (1-\cos(2\tau))/2 = \tau^2 - \tau^4/3 \pm \cdots$, we see that

$$\frac{\sin^2\tau}{\tau^2} = \frac{\sin^2(2^{-\ell}\alpha\pi)}{((k \pm 2^{-\ell}\alpha)\pi)^2} \sim \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\alpha^2}{(1 \pm \frac{2^{-\ell}}{k}\alpha)^2}.$$

The error of this approximation is sufficiently small: Note that

$$\Big(\frac{d}{d\tau}\Big)^4 \sin^2\tau = \Big(\frac{d}{d\tau}\Big)^4 \frac{1-\cos(2\tau)}{2} = 8\cos(2\tau).$$

Hence $|\sin^2\tau - \tau^2| \leq 8\,|\tau|^4/4! = |\tau|^4/3$ for all $\tau \in \mathbb{R}$. Thus

$$\begin{aligned}\Big|\frac{\sin^2(2^{-\ell}\alpha\pi)}{((k \pm 2^{-\ell}\alpha)\pi)^2} - \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\alpha^2}{(1 \pm \frac{2^{-\ell}}{k}\alpha)^2}\Big| &\leq \frac{\frac{1}{3}\left(2^{-\ell}\alpha\pi\right)^4}{((k \pm 2^{-\ell}\alpha)\pi)^2} \\ &= \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\frac{1}{3}(2^{-\ell}\pi)^2\alpha^4}{(1 \pm \frac{2^{-\ell}}{k}\alpha)^2}.\end{aligned}$$

Since $k \geq 1$ and $\ell \geq 2$, we obtain

$$\begin{aligned}\frac{\sin^2\tau}{\tau^2} &\leq \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\alpha^2 + \frac{1}{3}(2^{-\ell}\pi)^2\alpha^4}{(1 \pm \frac{2^{-\ell}}{k}\alpha)^2} \leq \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\alpha^2 + \frac{1}{3}\left(\frac{\pi}{4}\right)^2\alpha^4}{(1-\frac{1}{4}\alpha)^2} \\ &= \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\alpha^2 + \frac{1}{48}\pi^2\alpha^4}{(1-\frac{1}{4}\alpha)^2} \leq \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\alpha^2 + \frac{1}{48}\pi^2\alpha^4}{(1-\frac{1}{4}\alpha)^2}\bigg|_{\alpha=\frac{5}{4}} < 1.092\,\Big(\frac{2^{-\ell}}{k}\Big)^2,\end{aligned}$$

since $\alpha \mapsto 1/(1-\frac{1}{4}\alpha)^2$ and $\alpha \mapsto \alpha^2 + \frac{1}{48}\pi^2\alpha^4$ are increasing functions.

On the other hand,

$$\frac{\sin^2\tau}{\tau^2} \geq \Big(\frac{2^{-\ell}}{k}\Big)^2 \frac{\alpha^2 - \frac{1}{3}(2^{-\ell}\pi)^2\alpha^4}{(1 \pm \frac{2^{-\ell}}{k}\alpha)^2}.$$

Note that the enumerator on the right hand side above is positive. Hence

$$\frac{\alpha^2 - \frac{1}{3}(2^{-\ell}\pi)^2\alpha^4}{(1 \pm \frac{2^{-\ell}}{k}\alpha)^2} \geq \frac{\alpha^2 - \frac{1}{48}\pi^2\alpha^4}{(1+\frac{1}{4}\alpha)^2} = \frac{\alpha}{(1+\frac{1}{4}\alpha)^2}\left(\alpha - \frac{1}{48}\pi^2\alpha^3\right) =: R(\alpha).$$

Recall that $\frac{3}{8} \leq \alpha < \frac{5}{4}$. Note that

$$\frac{d}{d\alpha}\Big(\frac{\alpha}{(1+\frac{1}{4}\alpha)^2}\Big) = \frac{(1+\frac{1}{4}\alpha)^2 - \alpha(1+\frac{1}{4}\alpha)}{(1+\frac{1}{4}\alpha)^4} = \frac{1-\frac{1}{4}\alpha}{(1+\frac{1}{4}\alpha)^3} > 0,$$

$$\frac{d}{d\alpha}\Big(\alpha - \frac{1}{48}\pi^2\alpha^3\Big) = 1 - \frac{1}{16}\pi^2\alpha^2 \geq 1 - \frac{1}{16}\pi^2\Big(\frac{5}{4}\Big)^2 = 0.03\ldots > 0.$$

Hence $R(\alpha) \geq R\left(\frac{3}{16}\right) > 0.028$. Altogether, we have

$$0.028\left(\frac{2^{-\ell}}{k}\right)^2 < \frac{\sin^2\tau}{\tau^2} < 1.092\left(\frac{2^{-\ell}}{k}\right)^2 \quad \text{for all } \tau \in J_{k,\ell}.$$

This finishes the proof since $\Delta x^2(s,\tau) = |x|^2 - |x|_c^2(s,\tau) = |x|^2 - s\,(\sin^2\tau)/\tau^2$. □

Recall next from (8.9) that

$$\nabla^{\mathsf{T}}_{\sigma,\tau}\tilde\Phi(x,u,s,\sigma,\tau) = A_{\sigma,\tau}\begin{pmatrix}\Delta x^2(s,\tau)\\ \Delta u(s,\tau)\end{pmatrix}.$$

We want to integrate by parts to obtain rapid decay in terms of $\lambda\Delta x^2(s,\tau)$ and $\lambda\Delta u(s,\tau)$.

To that end, observe that if we choose $a = (a_1, a_2)$ so that $aA_{\sigma,\tau} = (1,0)$, then $(a\nabla^{\mathsf{T}}_{\sigma,\tau})\tilde\Phi(x,u,s,\sigma,\tau) = \Delta x^2(s,\tau)$, hence $(a\nabla^{\mathsf{T}}_{\sigma,\tau})\,e^{i\lambda\tilde\Phi} = i\lambda\Delta x^2(s,\tau)\,e^{i\lambda\tilde\Phi}$. From (8.7) we obtain

$$a = \Big(-\frac{\sin^2\tau}{\tau^2}, \frac{\sin^2\tau}{\sigma\tau}\Big)$$

Thus, if we define the vector field

$$V := -\frac{\sin^2\tau}{\tau^2}\partial_\sigma + \frac{\sin^2\tau}{\sigma\tau}\partial_\tau,$$

then

$$V(e^{i\lambda\tilde\Phi}) = i\lambda\Delta x^2(s,\tau)e^{i\lambda\tilde\Phi}. \tag{A.1}$$

Similarly, if we choose $b = (b_1, b_2)$ so that $bA_{\sigma,\tau} = (0,1)$, then

$$(b\nabla^{\mathsf{T}}_{\sigma,\tau})\tilde\Phi(x,u,s,\sigma,\tau) = \Delta u(s,\tau),$$

hence $(b\nabla^{\mathsf{T}}_{\sigma,\tau})\,e^{i\lambda\tilde\Phi} = i\lambda\Delta u(s,\tau)\,e^{i\lambda\tilde\Phi}$. From (8.7) we obtain

$$b = \frac{1}{\sigma\tau^2}\Big(\frac{1}{4}\sigma(\tau - \sin\tau\cos\tau), \frac{1}{4}\tau\sin\tau\cos\tau\Big).$$

Thus, if we define the vector field

$$W := \varphi(\tau)\partial_\sigma + \psi(\sigma,\tau)\partial_\tau,$$

where

$$\varphi(\tau) := \frac{\tau - \sin\tau\cos\tau}{\tau^2} \quad \text{and} \quad \psi(\sigma,\tau) := \frac{\sin\tau\cos\tau}{\sigma\tau},$$

then

$$W(e^{i\lambda\tilde{\Phi}}) = 4i\lambda\Delta u(s,\tau)e^{i\lambda\tilde{\Phi}}. \tag{A.2}$$

Recall also from (8.4) that

$$K^{k,\ell}_{\lambda,s}(x,u) = \lambda^{n+\frac{3}{2}} \int_0^\infty \int_0^\infty e^{i\lambda\tilde{\Phi}(x,u,s,\sigma,\tau)}\, a_\lambda(s\sigma)\, \eta_\ell(\tau - k\pi) \Big(\frac{\tau}{\sin\tau}\Big)^n \, d\sigma\, d\tau.$$

Lemma A.2. *For every $N, M \in \mathbb{N}$, the kernel $K^{k,\ell}_{\lambda,s}$ can be written as a finite series*

$$K^{k,\ell}_{\lambda,s} = \sum_{\alpha\in\mathbb{N}^2\times\mathbb{Z}\times\mathbb{N}^6} K^{k,\ell}_{\lambda,s,N,M,\alpha}$$

of summands of the form

$$K^{k,\ell}_{\lambda,s,N,M,\alpha}(x,u) = \lambda^{n+\frac{3}{2}-N-M} \int_0^\infty \int_0^\infty e^{i\lambda\tilde{\Phi}}\, \partial_\sigma^{\alpha_1}(a_\lambda(s\sigma))\, \partial_\tau^{\alpha_2}(\eta_\ell(\tau-k\pi)) \\ \times \Big(\frac{\tau}{\sin\tau}\Big)^n \frac{\sigma^{\alpha_3}\,(\sin\tau)^{\alpha_5}\, P_{\alpha_6}(\cos\tau,\sin\tau)}{\tau^{\alpha_4}\,(\Delta x^2(s,\tau))^{\alpha_7}\,(\Delta u(s,\tau))^{\alpha_8}}\, s^{\alpha_9}\, d\sigma\, d\tau,$$

where $P_{\alpha_6} \in \mathbb{Z}[X,Y]$ is a polynomial with integer coefficients and $\alpha = (\alpha_1,\dots,\alpha_9) \in \mathbb{N}^2\times\mathbb{Z}\times\mathbb{N}^6$ satisfies the relations

- $\alpha_4 - 2\alpha_7 - 2\alpha_8 \geq -N - M$,
- $\alpha_5 - \alpha_2 - 2\alpha_7 - \alpha_8 \geq -N - M$,
- $\alpha_7 \geq N$ *and* $\alpha_8 \geq M$.

Moreover, if $N = 0$, then $\alpha_7 = 0$, and if $M = 0$, then $\alpha_8 = 0$.

Proof. We do induction over $|N| + |M|$. For $|N| + |M| = 0$ we have $K^{k,\ell}_{\lambda,s} = K^{k,\ell}_{\lambda,\alpha}$ if $P_{\alpha_6} = 1$ and $\alpha_j = 0$ for all j.

Now suppose that the statement holds for the tuple (N,M). We will show that the statement then also holds for $(N+1,M)$ and $(N,M+1)$. We use the notation

$$K^{k,\ell}_{\lambda,s,N,M,\alpha}[E](x,u) = \lambda^{n+\frac{3}{2}-N-M} \int_0^\infty \int_0^\infty E(x,u,s,\tau,\sigma)\, \partial_\sigma^{\alpha_1}(a_\lambda(s\sigma)) \\ \times \partial_\tau^{\alpha_2}(\eta_\ell(\tau-k\pi)) \Big(\frac{\tau}{\sin\tau}\Big)^n \frac{\sigma^{\alpha_3}\,(\sin\tau)^{\alpha_5}\, P_{\alpha_6}(\cos\tau,\sin\tau)}{\tau^{\alpha_4}\,(\Delta x^2(s,\tau))^{\alpha_7}\,(\Delta u(s,\tau))^{\alpha_8}}\, s^{\alpha_9}\, d\sigma\, d\tau.$$

With this notation, $K_{\lambda,s,N,M,\alpha} = K^{k,\ell}_{\lambda,s,N,M,\alpha}[e^{i\lambda\tilde{\Phi}}]$. By (A.1), this yields

$$K^{k,\ell}_{\lambda,s,N,M,\alpha}[i\, e^{i\lambda\tilde{\Phi}}] = K^{k,\ell}_{\lambda,s,N+1,M,\alpha+e_7}[V(e^{i\lambda\tilde{\Phi}})] \\ = -K^{k,\ell}_{\lambda,s,N+1,M,\alpha+2e_4+2e_5+e_7}[\partial_\sigma(e^{i\lambda\tilde{\Phi}})] + K^{k,\ell}_{\lambda,s,N+1,M,\alpha-e_3+e_4+2e_5+e_7}[\partial_\tau(e^{i\lambda\tilde{\Phi}})].$$

When integrating by parts with respect to σ in the first integral and employing the Leibniz rule, we see that ∂_σ either hits $\partial_\sigma^{\alpha_1}(a_\lambda(s\sigma))$, which increases α_1, or σ^{α_3}, which decreases α_3. By induction hypothesis,

$$(\alpha_4+2) - 2(\alpha_7+1) - 2\alpha_8 > -(N+1) - M, \\ (\alpha_5+2) - \alpha_2 - 2(\alpha_7+1) - \alpha_8 > -(N+1) - M.$$

For the second integral, when integrating by parts is τ and employing the Leibniz rule again, we make the following observations:

(i) When ∂_τ hits $\partial_\tau^{\alpha_2}(\eta_\ell(\tau-k\pi))$, then α_2 is additionally increased by 1. However, the induction hypothesis still implies

$$(\alpha_5+2)-(\alpha_2+1)-2(\alpha_7+1)-2\alpha_8\ge -(N+1)-M.$$

(ii) When ∂_τ hits τ^n or $1/\tau^{\alpha_4}$, then α_4 is additionally increased by 1. This additional gain is not relevant for our purposes.

(iii) When ∂_τ hits $1/(\sin\tau)^n$ or $(\sin\tau)^{\alpha_5}$, then α_5 changes additionally by -1, but α_2 is unchanged, and

$$(\alpha_5+1)-\alpha_2-2(\alpha_7+1)-\alpha_8\ge -(N+1)-M.$$

(iv) When ∂_τ hits $P_{\alpha_6}(\cos\tau,\sin\tau)$, we just pass to another polynomial.

(v) When ∂_τ hits $(\Delta x^2(s,\tau))^{-(\alpha_7+1)}$, we get

$$\partial_\tau(\Delta x^2(s,\tau))^{-(\alpha_7+1)}=(\alpha_7+1)(\Delta x^2(s,\tau))^{-(\alpha_7+1)}\partial_\tau\Big(s\,\frac{\sin^2\tau}{\tau^2}\Big).$$

Thus α_7 is increased in total by 2. If ∂_τ hits the enumerator of the factor $\sin^2\tau/\tau^2$, α_4 and α_5 are both increased in total by 3, and if ∂_τ hits the denominator of the factor $\sin^2\tau/\tau^2$, then both α_4 and α_5 are increased in total by 4. Note that

$$(\alpha_4+3)-2(\alpha_7+2)-2\alpha_8\ge -(N+1)-M,$$
$$(\alpha_5+3)-\alpha_2-2(\alpha_7+2)-\alpha_8\ge -(N+1)-M,$$

and if we increase both α_4 and α_5 by 4, we even get strict inequalities.

(vi) When ∂_τ hits $(\Delta u(s,\tau))^{-\alpha_8}$, we get

$$\partial_\tau\,(\Delta u(s,\tau))^{-\alpha_8}=\alpha_8\,(\Delta u(s,\tau))^{-(\alpha_8+1)}\,\partial_\tau\Big(s\,\frac{\tau-\sin\tau\cos\tau}{4\tau^2}\Big).$$

Similar as in (v), α_8 is increased by 1, while the factor $\partial_\tau((\tau-\sin\tau\cos\tau)/\tau^2)$ leads to two summands where α_4 or α_5 is additionally increased. If ∂_τ hits $1/\tau$, or the enumerator of $-\sin\tau\cos\tau/\tau^2$, then α_4 is increased in total by 3. Note here that

$$(\alpha_4+3)-2(\alpha_7+1)-2(\alpha_8+1)\ge -(N+1)-M,$$
$$(\alpha_5+2)-\alpha_2-2(\alpha_7+1)-(\alpha_8+1)\ge -(N+1)-M.$$

If ∂_τ hits the denominator of $-\sin\tau\cos\tau/\tau^2$, then α_4 is increased in total by 4, and we even get strict inequalities.

This verifies the statement for $(N+1,M)$. We can argue similarly for $(N,M+1)$ by using (A.2). This yields

$$\begin{aligned}K^{k,\ell}_{\lambda,s,N,M,\alpha}[4i\,e^{i\lambda\tilde\Phi}]&=K_{\lambda,s,N,M+1,\alpha+e_8}[W(e^{i\lambda\tilde\Phi})]\\&=K_{\lambda,s,N,M+1,\alpha+e_4+e_8}[\partial_\sigma(e^{i\lambda\tilde\Phi})]-K_{\lambda,s,N,M+1,\beta+2e_4+e_5+e_8}[\partial_\sigma(e^{i\lambda\tilde\Phi})]\\&\qquad+K_{\lambda,s,N,M+1,\gamma-e_3+e_4+e_5+e_8}[\partial_\tau(e^{i\lambda\tilde\Phi})]\end{aligned}$$

with some new polynomials $P_{\beta_6},P_{\gamma_6}\in\mathbb{Z}[X,Y]$ and $\alpha_j=\beta_j=\gamma_j$ for all $j\neq 6$. For the first two summands, observe that

$$(\alpha_4+1)-2\alpha_7-2(\alpha_8+1)\ge -N-(M+1),$$
$$(\alpha_5+1)-\alpha_2-2\alpha_7-(\alpha_8+1)>-N-(M+1).$$

For the third summand, we can argue similarly to (i)–(iv) above. For case (v), observe that

$$(\alpha_4+3)-2(\alpha_7+1)-2(\alpha_8+1)\geq -N-(M+1),$$
$$(\alpha_5+2)-\alpha_2-2(\alpha_7+1)-(\alpha_8+1)\geq -N-(M+1).$$

For case (vi), observe that

$$(\alpha_4+3)-2\alpha_7-2(\alpha_8+2)\geq -N-(M+1),$$
$$(\alpha_5+1)-\alpha_2-2\alpha_7-(\alpha_8+2)\geq -N-(M+1).$$

Altogether, we have verified the statement also for $(N, M+1)$, which finishes the induction. □

Proof of Proposition 8.2. We use polar coordinates with $\rho=|x|$. Note that $K^{k,l}_{\lambda,s}$ and $K^{k,\ell}_{\lambda,s,N,M,\alpha}$ are radial symmetric in x. In the following, we write $K^{k,l}_{\lambda,s}(x,u)=\mathcal{K}^{k,l}_{\lambda,s}(\rho,u)$ and $K^{k,\ell}_{\lambda,s,N,M,\alpha}(x,u)=\mathcal{K}^{k,\ell}_{\lambda,s,N,M,\alpha}(\rho,u)$. Let

$$I^{k,\ell}_{\lambda,1}:=\{\rho\in(0,\infty):\tfrac{1}{100}(\tfrac{2^{-\ell}}{k})^2<\rho^2<10\,(\tfrac{2^{-\ell}}{k})^2\}$$
$$I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell}):=\{u\in\mathbb{R}:|\Delta u(s,\tau_{k,\ell})|<R_0\tfrac{2^{-\ell}}{k^2}\}.$$

Moreover, let

$$E^{k,\ell}_{\lambda,1}(s,\tau_{k,\ell}):=\{(\rho,u)\in(0,\infty)\times\mathbb{R}:\rho\notin I^{k,\ell}_{\lambda,1}\text{ and }u\in I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell})\},$$
$$E^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell}):=\{(\rho,u)\in(0,\infty)\times\mathbb{R}:\rho\notin I^{k,\ell}_{\lambda,1}\text{ and }u\notin I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell})\},$$
$$E^{k,\ell}_{\lambda,3}(s,\tau_{k,\ell}):=\{(\rho,u)\in(0,\infty)\times\mathbb{R}:\rho\in I^{k,\ell}_{\lambda,1}\text{ and }u\notin I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell})\}.$$

It suffices to prove the bound

$$\iint_{E^{k,\ell}_{\lambda,j}(s,\tau_{k,\ell})}|\mathcal{K}^{k,\ell}_{\lambda,s}(\rho,u)|\,\rho^{d_1-1}d\rho\,du\leq C_N\Big(\frac{\lambda}{2^\ell k}\Big)^{-N}\Big(\frac{2^\ell}{k}\Big)^{\frac12}\quad\text{for all } j\in\{1,2,3\}.$$

Suppose that $j\in\{1,2,3\}$ is fixed. Let $N,M\in\mathbb{N}$. We will choose N and M later depending on j. By Lemma A.2, given $N,M\in\mathbb{N}$, we may write the kernel $\mathcal{K}^{k,\ell}_{\lambda,s}$ as a finite series

$$\mathcal{K}^{k,\ell}_{\lambda,s}=\sum_{\alpha\in\mathbb{N}^2\times\mathbb{Z}\times\mathbb{N}^6}\mathcal{K}^{k,\ell}_{\lambda,s,N,M,\alpha},$$

where $P_{\alpha_6}\in\mathbb{Z}[X,Y]$ and $\alpha=(\alpha_1,\dots,\alpha_7)\in\mathbb{N}^2\times\mathbb{Z}\times\mathbb{N}^6$ satisfies

- $\alpha_4-2\alpha_7-2\alpha_8\geq -N-M$,
- $\alpha_5-\alpha_2-2\alpha_7-\alpha_8\geq -N-M$,
- $\alpha_7\geq N$ and $\alpha_8\geq M$.

Clearly, it suffices to bound each of the summands $\mathcal{K}^{k,\ell}_{\lambda,s,N,M,\alpha}$ separately. We have

$$\begin{aligned}(\text{A.3})\quad&\iint_{E^{k,\ell}_{\lambda,j}(s,\tau_{k,\ell})}|\mathcal{K}^{k,\ell}_{\lambda,s,N,M,\alpha}(\rho,u)|\,\rho^{d_1-1}d\rho\,du\\&=\lambda^{n+\frac32-N-M}\iint_{E^{k,\ell}_{\lambda,j}(s,\tau_{k,\ell})}\Big|\int_0^\infty\int_0^\infty e^{i\lambda\tilde\Phi}\partial_\sigma^{\alpha_1}(a_\lambda(s\sigma))\partial_\tau^{\alpha_2}(\eta_\ell(\tau-k\pi))\Big(\frac{\tau}{\sin\tau}\Big)^n\\&\qquad\times\frac{\sigma^{\alpha_3}(\sin\tau)^{\alpha_5}P_{\alpha_6}(\cos\tau,\sin\tau)}{\tau^{\alpha_4}(\Delta\rho^2(s,\tau))^{\alpha_7}(\Delta u(s,\tau))^{\alpha_8}}s^{\alpha_9}\,d\sigma\,d\tau\Big|\,\rho^{d_1-1}d\rho\,du.\end{aligned}$$

Let

$$\tilde I^{k,\ell}_{\lambda,1}(s,\tau) := \{\rho\in(0,\infty) : \big|\Delta\rho^2(s,\tau)\big| \lesssim (\tfrac{2^{-\ell}}{k})^2\},$$
$$\tilde I^{k,\ell}_{\lambda,2}(s,\tau) := \{u\in\mathbb{R} : |\Delta u(s,\tau)| \lesssim \tfrac{2^{-\ell}}{k^2}\}.$$

By Lemma A.1, we have $(I^{k,\ell}_{\lambda,1})^c \subseteq (\tilde I^{k,\ell}_{\lambda,1}(s,\tau))^c$ and $(I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell}))^c \subseteq (\tilde I^{k,\ell}_{\lambda,2}(s,\tau))^c$. For $\tau\in J_{k,\ell}$ let

$$\tilde E^{k,\ell}_{\lambda,1}(s,\tau) := \{(\rho,u)\in(0,\infty)\times\mathbb{R} : \rho\notin \tilde I^{k,\ell}_{\lambda,1}(s,\tau) \text{ and } u\in \tilde I^{k,\ell}_{\lambda,2}(s,\tau)\},$$
$$\tilde E^{k,\ell}_{\lambda,2}(s,\tau) := \{(\rho,u)\in(0,\infty)\times\mathbb{R} : \rho\notin \tilde I^{k,\ell}_{\lambda,1}(s,\tau) \text{ and } u\notin \tilde I^{k,\ell}_{\lambda,2}(s,\tau)\},$$
$$\tilde E^{k,\ell}_{\lambda,3}(s,\tau) := \{(\rho,u)\in(0,\infty)\times\mathbb{R} : \rho^2\sim(\tfrac{2^{-\ell}}{k})^2 \text{ and } u\notin \tilde I^{k,\ell}_{\lambda,2}(s,\tau)\}.$$

Note that $\rho\in I^{k,\ell}_{\lambda,1}$ implies $\rho^2\sim(\frac{2^{-\ell}}{k})^2$. On the other hand, if $u\in I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell})$, then

$$|\Delta u(s,\tau)| \le |\Delta u(s,\tau_{k,\ell})| + |u_c(s,\tau)-u_c(s,\tau_{k,\ell})| \lesssim \tfrac{2^{-\ell}}{k^2}.$$

Thus $I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell})\subseteq \tilde I^{k,\ell}_{\lambda,2}(s,\tau)$, hence $E^{k,\ell}_{\lambda,1}(s,\tau_{k,\ell})\subseteq \tilde E^{k,\ell}_{\lambda,1}(s,\tau)$.

Similarly, if $u\notin I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell})$, then by choosing R_0 sufficiently large, we have

$$|\Delta u(s,\tau)| \ge |\Delta u(s,\tau_{k,\ell})| - |u_c(s,\tau)-u_c(s,\tau_{k,\ell})| \gtrsim \tfrac{2^{-\ell}}{k^2},$$

hence $(I^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell}))^c\subseteq(\tilde I^{k,\ell}_{\lambda,2}(s,\tau))^c$, hence $E^{k,\ell}_{\lambda,2}(s,\tau_{k,\ell})\subseteq \tilde E^{k,\ell}_{\lambda,2}(s,\tau)$.

In a similar way, we see that $E^{k,\ell}_{\lambda,3}(s,\tau_{k,\ell})\subseteq \tilde E^{k,\ell}_{\lambda,3}(s,\tau)$.

Hence, for every $j\in\{1,2,3\}$,

$$E^{k,\ell}_{\lambda,j}(s,\tau_{k,\ell})\subseteq \tilde E^{k,\ell}_{\lambda,j}(s,\tau).$$

Using Fubini's theorem, we may bound the term in (A.3) by

$$\lambda^{n+\frac32-N-M}\int_{|\tau-k\pi|\sim 2^\ell}\int_{\sigma\sim1}\iint_{\tilde E^{k,\ell}_{\lambda,j}(s,\tau)} \frac{(2^\ell)^{\alpha_2}(2^\ell k)^n(2^{-\ell})^{\alpha_5}}{k^{\alpha_4}\,|\Delta\rho^2(s,\tau)|^{\alpha_7}\,|\Delta u(s,\tau)|^{\alpha_8}} \times \rho^{d_1-1}d\rho\,du\,d\sigma\,d\tau.$$

Recall that $\Delta\rho^2(s,\tau)=\rho^2-\rho_c^2(s,\tau)$ with $\rho_c^2(s,\tau)\sim(\frac{2^{-\ell}}{k})^2$, and $\Delta u(s,\tau)=u-u_c(s,\tau)$ with $u_c(s,\tau)\sim k^{-1}$.

Thus, if we choose N,M large enough, since $\alpha_7\ge N$ and $\alpha_8\ge M$, we obtain via homogeneity

$$\int_{|\Delta\rho^2(s,\tau)|\gtrsim(\frac{2^{-\ell}}{k})^2} |\Delta\rho^2(s,\tau)|^{-\alpha_7}\rho^{d_1-1}d\rho \sim \Big(\frac{2^{-\ell}}{k}\Big)^{-2\alpha_7+d_1},$$

and

$$\int_{|\Delta u(\tau)|\gtrsim\frac{2^{-\ell}}{k^2}} |\Delta u(s,\tau)|^{-\alpha_8}du \sim \Big(\frac{2^{-\ell}}{k^2}\Big)^{-\alpha_8+1}.$$

Suppose $j=1$. In that case, we choose $M=0$ and N sufficiently large. By Lemma A.2, we then have $\alpha_8=0$, $\alpha_4-2\alpha_7\ge -N$ and $\alpha_5-\alpha_2-2\alpha_7\ge -N$. Thus, the above term is bounded by

$$\lambda^{n+\frac32-N}\frac{(2^\ell)^{\alpha_2}(2^\ell k)^n(2^{-\ell})^{\alpha_5}}{k^{\alpha_4}}\Big(\frac{2^{-\ell}}{k}\Big)^{-2\alpha_7+d_1}\frac{2^{-\ell}}{k^2}$$
$$=\lambda^{n+\frac32-N}\frac{(2^{-\ell})^{\alpha_5-\alpha_2-2\alpha_7+1+d_1-n}}{k^{\alpha_4-2\alpha_7+d_1-n+2}} \le \Big(\frac{\lambda}{2^\ell k}\Big)^{n+\frac32-N}\Big(\frac{2^\ell}{k}\Big)^{\frac12}.$$

Suppose $j = 2$. In that case, we choose N and M sufficiently large. Since $\alpha_4 - 2\alpha_7 - 2\alpha_8 \geq -N - M$ and $\alpha_5 - \alpha_2 - 2\alpha_7 - \alpha_8 \geq -N - M$, we obtain the bound

$$\begin{aligned}&\lambda^{n+\frac{3}{2}-N-M}\,\frac{(2^{\ell})^{\alpha_2}(2^{\ell}k)^n(2^{-\ell})^{\alpha_5}}{k^{\alpha_4}}\Big(\frac{2^{-\ell}}{k}\Big)^{-2\alpha_7+d_1}\Big(\frac{2^{-\ell}}{k^2}\Big)^{-\alpha_8+1}\\ &=\lambda^{n+\frac{3}{2}-N-M}\,\frac{(2^{-\ell})^{\alpha_5-\alpha_2-2\alpha_7-\alpha_8+1+d_1-n}}{k^{\alpha_4-2\alpha_7-2\alpha_8+d_1-n+2}}\leq\Big(\frac{\lambda}{2^{\ell}k}\Big)^{n+\frac{3}{2}-N-M}\Big(\frac{2^{\ell}}{k}\Big)^{\frac{1}{2}}.\end{aligned}$$

Suppose $j = 3$. In that case, we choose $N = 0$ and M sufficiently large. By Lemma A.2, we have $\alpha_7 = 0$, $\alpha_4 - 2\alpha_8 \geq -M$ and $\alpha_5 - \alpha_2 - \alpha_8 \geq -M$. Thus, we obtain the bound

$$\begin{aligned}&\lambda^{n+\frac{3}{2}-M}\,\frac{(2^{\ell})^{\alpha_2}(2^{\ell}k)^n(2^{-\ell})^{\alpha_5}}{k^{\alpha_4}}\Big(\frac{2^{-\ell}}{k}\Big)^{d_1}\Big(\frac{2^{-\ell}}{k^2}\Big)^{-\alpha_8+1}\\ &=\lambda^{n+\frac{3}{2}-M}\,\frac{(2^{-\ell})^{\alpha_5-\alpha_2-\alpha_8+1+d_1-n}}{k^{\alpha_4-2\alpha_8+d_1-n+2}}\leq\Big(\frac{\lambda}{2^{\ell}k}\Big)^{n+\frac{3}{2}-M}\Big(\frac{2^{\ell}}{k}\Big)^{\frac{1}{2}}.\end{aligned}$$

This finishes the proof of Proposition 8.2. □

Appendix B. The rank $d_1 - 1$ cone condition on Heisenberg groups

Recall from Proposition 4.2 that the convolution kernels $\mathcal{K}^0_{\lambda,t}(x,u)$ and $\mathcal{K}^{k,\ell}_{\lambda,t}(x,u)$ are oscillatory integrals, with common phase

$$\Phi(x,u,t,\sigma,\tau) = \sigma\big(t^2 - |x|^2 g(\tau) + 4u\tau\big), \quad \text{with } g(\tau) = \tau\cot\tau,$$

where $\sigma \sim 1$ and $\tau \in \mathbb{R}$. Thus, the phase Ψ of the integral kernel of the corresponding convolution operators is given by

$$\begin{aligned}\Psi(x,u,y,v,t,\sigma,\tau) &:= \Phi((y,v)^{-1}(x,u),t,\sigma,\tau)\\ &= \Phi(x-y,u-v+\tfrac{1}{2}\langle Jx,y\rangle,t,\sigma,\tau).\end{aligned}$$

Thus,

$$\Psi(x,u,y,v,t,\sigma,\tau) = \sigma\Big(t^2 - |x-y|^2 g(\tau) + 4\big(u-v+\tfrac{1}{2}\langle Jx,y\rangle\big)\tau\Big).$$

Let

$$A_\Psi := \{(x,u,y,v,t,\sigma,\tau) : \Psi'_{\sigma,\tau}(x,u,y,v,t,\sigma,\tau) = 0,\ \sigma\sim 1\},$$

and set

$$\Gamma := \{(x,u,t,\Psi'_x,\Psi'_u,\Psi'_t) : (x,u,y,v,t,\sigma,\tau)\in A_\Psi\}.$$

For (x,u,t) fixed, with, say, $t > 0$, let $\Gamma_{(x,u,t)}$ be the corresponding set

$$\Gamma_{(x,u,t)} := \{(\Psi'_x,\Psi'_u,\Psi'_t) : (x,u,y,v,t,\sigma,\tau)\in A_\Psi\}.$$

We want to show the following.

Lemma B.1. *Let $(x,u)\in\mathbb{H}_n$ and $t > 0$. Then $\Gamma_{(x,u,t)}$ is an open part of a conic surface which has $d_1 - 1 = 2n-1$ non-vanishing principal curvatures at every point.*

Proof. In view of the left-invariance of the sub-Laplacian L, it suffices to prove this at the point $(x,u) = (0,0)$. We compute the critical points of Ψ with respect to the frequency variables σ,τ at $(x,u) = (0,0)$. Since $\sigma > 0$, these are given by the solutions to the equations

$$t^2 - |y|^2 g(\tau) - 4v\tau = 0, \tag{B.1}$$

$$-|y|^2 g'(\tau) - 4v = 0. \tag{B.2}$$

Moreover,

$$\Psi'_x = 2\sigma\big(g(\tau)y - \tau Jy\big), \qquad \Psi'_u = 4\sigma\tau, \qquad \Psi'_t = 2\sigma t.$$

By (B.2), $4v = -|y|^2 g'(\tau)$, so by (B.1) and (6.13),

$$0 = t^2 - |y|^2\big(g(\tau) - \tau g'(\tau)\big) = t^2 - |y|^2 h(\tau) = t^2 - |y|^2\Big(\frac{\tau}{\sin\tau}\Big)^2.$$

Hence, using polar coordinates $y = \rho\,\omega$, where $\rho := |y|$ and $\omega \in S^{d_1-1}$, we obtain

$$y = t\,\Big|\frac{\sin\tau}{\tau}\Big|\,\omega. \tag{B.3}$$

This equation should be solved as a smooth function of ρ. Since the mapping $\tau \mapsto (\sin\tau)/\tau$ is by far not injective, we can find such solutions $\tau = q_{\tau_*}(\rho)$ only locally near a fixed point τ_* (with the exception of a discrete set of points τ_* at which the derivative of $\tau \mapsto (\sin\tau)/\tau$ vanishes).

Then,

$$4v = -|y|^2 g'(\tau) = -t^2\Big(\frac{\sin\tau}{\tau}\Big)^2 g'(\tau).$$

Thus, if $(\sin\tau)/\tau > 0$, say, using (B.3), we obtain

$$\Psi'_x = 2\sigma t\,(\cos(\tau)\,\omega - \sin(\tau)\,J\omega) = 2\sigma t\,e^{-\tau J}\omega.$$

So this local part of $\Gamma_{(0,0,t)}$ is of the form

$$\begin{aligned}\Gamma^{\tau_*}_{(0,0,t)} &= \Big\{2\sigma\Big(t\,e^{-q_{\tau_*}(|y|)J}\,\frac{y}{|y|}, 2q_{\tau_*}(|y|), t\Big) : a_* < |y| < b_*\Big\}\\ &= \Big\{2\sigma\Big(t\frac{y}{|y|}, 2q_{\tau_*}(|y|), t\Big) : a_* < |y| < b_*\Big\},\end{aligned}$$

where a_*, b_* depend on τ_*. In particular, $\Gamma^{\tau_*}_{(0,0,t)}$ is an open subset of the cone

$$\tilde\Gamma := \big\{\sigma\big(t\omega, \tau, t\big) : \sigma > 0, \tau \in \mathbb{R}, \omega \in S^{d_1-1}\big\} \subset \mathbb{R}^d \times \mathbb{R}. \tag{B.4}$$

The section of this cone at $\sigma = 1$ is given by $tS^{d_1-1} \times \mathbb{R} \times \{t\}$, so that it has exactly $d_1 - 1$ non-vanishing principal curvatures at every point. □

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D. Müller, Mathematisches Seminar, C.A.-Universität Kiel, Heinrich-Hecht-Platz 6, 24118 Kiel, Germany
Email address: mueller@math.uni-kiel.de

L. Niedorf, Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, USA
Email address: niedorf@wisc.edu

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, USA
Email address: seeger@math.wisc.edu

B. Stovall, Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, USA
Email address: stovall@math.wisc.edu