IMPROVED BOUNDS FOR STEIN'S SQUARE FUNCTIONS

SANGHYUK LEE KEITH M. ROGERS ANDREAS SEEGER

Abstract. We prove a weighted norm inequality for the maximal Bochner–Riesz operator and the associated square-function. This yields new $L^p(\mathbb{R}^d)$ bounds on classes of radial Fourier multipliers for $p \geq 2 + 4/d$ with $d \geq 2$, as well as space-time regularity results for the wave and Schrödinger equations.

1. Introduction

Consider the Bochner–Riesz means $\mathcal{R}^\alpha_t f$ of the Fourier integral, defined for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{R}^\alpha_t f(x) = \frac{1}{(2\pi)^d} \int_{|\xi|\leq t} \Big(1 - \frac{|\xi|^2}{t^2}\Big)^\alpha \widehat{f}(\xi) e^{\imath\langle x,\xi\rangle} d\xi,$$

where $\widehat{f}(\xi) = \int f(y) e^{-\imath y\xi} dy$. In connection with questions regarding almost everywhere summability, and in analogy to classical Littlewood–Paley functions for Poisson-integrals, Stein [35] introduced a square function defined by

$$G^\alpha f(x) = \Big(\int_0^\infty \Big| \frac{\partial}{\partial t} \mathcal{R}^\alpha_t f(x) \Big|^2 t\, dt \Big)^{1/2}.$$

One is interested in the inequality $\|G^\alpha f\|_p \lesssim \|f\|_p$, where $A \lesssim B$ denotes $A \leq CB$ with an unspecified constant independent of f. As $t\partial_t \widehat{\mathcal{R}^\alpha_t f}(\xi) = 2\alpha|\xi|^2/t^2(1 - |\xi|^2/t^2)^{\alpha-1}_+ \widehat{f}(\xi)$, one can consider the L^p problem as a question regarding the boundedness of a vector valued singular integral operator involving Riesz means of order $\alpha - 1$. It is known that L^p boundedness for $1 < p \leq 2$ holds if and only if $\alpha > d(1/p - 1/2) + 1/2$ (see [39], [20]), however the problem is more interesting for the range $p > 2$ for which the condition $\alpha > \max\{1/2, d(1/2 - 1/p)\}$ is known to be necessary and conjectured to be also sufficient. For $d = 1$ many proofs of the conjecture are known, see for example [40]. The conjecture in two dimensions was proven by Carbery [2], and partial results for $p > \frac{2(d+1)}{d-1}$, $d \geq 3$, are in [7], [31]. Here we improve on the range in dimensions $d \geq 3$.

Theorem 1.1. *Let $d \geq 2$ and $p \in [\frac{2(d+2)}{d}, \infty)$. Then*

$$\big\|G^\alpha f\big\|_p \lesssim \|f\|_p, \qquad \alpha > d\Big(\frac{1}{2} - \frac{1}{p}\Big).$$

Date: December 7, 2010.

2000 *Mathematics Subject Classification.* 42B15, 35B65.

Key words and phrases. Square functions, Bochner–Riesz means, Wave and Schrödinger semigroup, mixed norm estimates.

Supported in part by NRF grant 2009-0072531 (Korea), MICINN grant MTM2010-16518 (Spain) and NSF grant 0652890 (USA).

As in the related work by the first author [22] our main tool will be Tao's bilinear estimate [41] for the adjoint of the Fourier restriction operator. The new square function result implies the currently known sharp L^p estimates for the maximal Bochner–Riesz operator obtained by Carbery [2] and the first author [22]; however, as pointed out in [22], somewhat weaker estimates are already enough to bound the maximal function. More precisely, for a compact t-interval $I \subset (0,\infty)$, the estimate in [22] could be formulated as a variational $L^p(V_4(I))$ inequality for Riesz means of order $\lambda > \lambda_{\rm crit} := d(1/2-1/p)-1/2$, or a slightly better regularity result involving the Sobolev space $L^4_{1/4+\varepsilon}(I)$.

The L^p-estimate for the square function is significant for various reasons. Firstly, it yields regularity results for wave and Schrödinger operators which will be discussed below. Secondly, for compact $I \subset (0,\infty)$ it implies $L^p(V_2(I))$ or $L^p(L^2_{1/2+\varepsilon}(I))$ results for Bochner–Riesz means of order $\lambda > \lambda_{\rm crit}$. Thirdly, the L^p-result for G^α implies boundedness results for maximal operators associated to more general classes of radial Fourier multipliers as in [4], [12], and finally, an inequality by Carbery, Gasper and Trebels [5] relating radial multipliers and G^α yields the following sharp $L^p \to L^p$ boundedness result of Hörmander-Mikhlin type.

Corollary 1.2. *Let $d \ge 2$ and $p \in (1, \frac{2(d+2)}{d+4}] \cup [\frac{2(d+2)}{d}, \infty)$. Let φ be a nontrivial C^∞ function compactly supported in $(0,\infty)$. Then*

$$\sup_{f\in\mathcal{S}:\,\|f\|_p\le 1} \big\|\mathcal{F}^{-1}[m(|\,\cdot\,|)\widehat{f}\,]\big\|_p \lesssim \sup_{t>0} \|\varphi\, m(t\,\cdot\,)\|_{L^2_\alpha(\mathbb{R})}, \quad \alpha > d\Big|\frac{1}{p}-\frac{1}{2}\Big|.$$

Weighted inequalities. More information about G^α can be obtained by considering an L^2 weighted inequality which involves a "universal" maximal operator $\mathfrak{W}_q$ acting on the weights, and which is strong enough to imply the above L^p estimates. We initially formulate such an inequality for thin pieces of the multiplier at a time. To fix notation, let ϕ be a Schwartz function supported in $(1/2,2)$ with the property that

$$|\phi^{(\nu)}(t)| \le 1, \quad \nu = 0,\dots,d+2. \tag{1}$$

Let $0<\delta<1/2$ and define the convolution operator $S^\delta_t \equiv S^{\delta,\phi}_t$ by

$$\widehat{S^\delta_t f}(\xi) = \phi\big(\delta^{-1}\big(1-\tfrac{|\xi|^2}{t^2}\big)\big)\,\widehat{f}(\xi). \tag{2}$$

Assuming (1) we shall usually drop the superscript ϕ, as our estimates will be understood to be uniform in ϕ.

Theorem 1.3. *Let $d \ge 2$ and $q \in (1, \frac{d+2}{2})$. For $0<\delta<1/2$, there are operators $\mathfrak{W}_{q,\delta}$ defined on $L^q + L^\infty$, so that the weighted norm inequality*

$$\int_{\mathbb{R}^d}\int_0^\infty |S^\delta_t f(x)|^2 \frac{dt}{t}\, w(x)\,dx \;\lesssim\; \delta^{2-d/q} \int_{\mathbb{R}^d} |f(x)|^2\, \mathfrak{W}_{q,\delta} w(x)\,dx \tag{3}$$

holds for all $w \in L^q + L^\infty$ and the weight operators satisfy:

(i) The maximal operator defined by

$$\mathfrak{W}_q w = \sup_{0<\delta<1/2} \mathfrak{W}_{q,\delta} w \tag{4}$$

is of weak type (q,q) and bounded on L^r for $q<r\le\infty$.

(ii) If $q \in [2, \frac{d+2}{2})$, then the operators $\mathfrak{W}_{q,\delta}$ are bounded on L^q, uniformly in δ. Moreover if $q \in (1,2)$ then $\|\mathfrak{W}_{q,\delta}\|_{L^q\to L^q} \lesssim [\log(\frac{1}{\delta})]^{\frac{1}{q}-\frac{1}{2}}$.

The precise definition of the operator $\mathfrak{W}_{q,\delta}$ in (3) will be given in §2. We shall also consider local versions of (3) with the t-integral extended over a dyadic interval and for which the L^q bounds of the corresponding weight operators are independent of δ for all $q \in (1, \frac{d+2}{2})$.

For the square-function G^α and the maximal Bochner–Riesz operator we obtain

Corollary 1.4. *Let $d \geq 2$ and $q \in (1, \frac{d+2}{2})$, and let $\mathfrak{W}_q$ be as in (4). Then*

$$\int_{\mathbb{R}^d} \bigl|G^\alpha f(x)\bigr|^2 w(x)\,dx \lesssim \int |f(x)|^2\, \mathfrak{W}_q w(x)\,dx, \quad \alpha > \frac{d}{2q}\,. \tag{5}$$

Moreover,

$$\int_{\mathbb{R}^d} \sup_{t>0} \bigl|\mathcal{R}_t^\lambda f(x)\bigr|^2 w(x)\,dx \lesssim \int |f(x)|^2\, \mathfrak{W}_q w(x)\,dx, \quad \lambda > \frac{d-q}{2q}\,. \tag{6}$$

The weighted inequalities are motivated by one of Stein's problems in [37]. It was asked whether the operator defining the weight on the right hand side of (3) could be chosen to be a Nikodym maximal operator (see also Córdoba [8] for a related question). This seems currently unknown. For the range $q \in (1, \frac{d+1}{2}]$, Christ [7] proved (3) with the operator $\widetilde{\mathfrak{W}}_q = (M|w|^q)^{1/q}$ where M is the Hardy–Littlewood maximal operator. In two dimensions, Carbery [3] proved (3), with a necessary loss of a power of $\log(1/\delta)$, and with an operator W_2 in place of $\mathfrak{W}_2$, such that W_2 is bounded on $L^r(\mathbb{R}^2)$ for $r \in [2,4]$. The extension of that result with the operator bounded for $r \in [2,\infty]$ was established by Carbery and the third author [6].

To deduce inequality (5) one splits the multiplier into a part near the origin and a part near the unit sphere. The part near the origin is dealt with by the standard weighted norm inequality for singular integrals in [11]. One decomposes the part near the unit sphere into smooth multipliers supported on thin annuli of width $\delta = 2^{-j}$, applies Theorem 1.3 and sums a geometric series. The maximal inequality (6) follows from (5) by well-known arguments in ([38, §VII.5]) together with a weighted norm inequality for the Hardy–Littlewood maximal function ([15]).

If we take $q = (p/2)'$ in (5) then by duality and an application of the Marcinkiewicz interpolation theorem one obtains Theorem 1.1 for $p > 2 + 4/d$. Interpolation with an L^2 inequality yields the result also for $p = 2 + 4/d$. Theorem 1.3 also implies a sharp result for the square-functions generated by S_t^δ which is stated in Corollary 4.2 below.

Wave and Schrödinger operators. One can apply L^p bounds for variants of the Bochner–Riesz square-function to obtain regularity results for spherical means and solutions of the wave and Schrödinger equations. This application is suggested by a theorem of Kaneko and Sunouchi [21] relating G^α to another square function which was introduced by Stein in his study of spherical maximal operators (*cf.* [36]). Define the spherical mean of order $\beta > 0$ by

$$\mathcal{A}_t^\beta f(x) = \frac{\Gamma(\frac{d+2\beta}{2})}{\pi^{d/2}\Gamma(\beta)} \int_{|y|\leq t} \frac{1}{t^d}\Bigl(1 - \frac{|y|^2}{t^2}\Bigr)^{\beta-1} f(x-y)\,dy;$$

for smaller values of β the definition can be extended by analytic continuation. In [21] an application of Plancherel's theorem with respect to t is used to show that G^α is pointwise equivalent with a square function generated by spherical means, namely, for $\alpha > 0$,

$$G^\alpha f(x) \approx \Bigl(\int_0^\infty \Bigl|\frac{\partial}{\partial t}\mathcal{A}_t^{\alpha-\frac{d-2}{2}} f(x)\Bigr|^2 t\,dt\Bigr)^{1/2},$$

for all Schwartz functions f. Here we shall not use this equivalence explicitly but prove a closely related sharp $L^p(L^2)$ regularity result for solutions of the wave and Schrödinger equations with initial data in L^p Sobolev spaces. In order to formulate a unifying result, for $a \in (0,\infty)$, we let $U_t^a f$ denote the solution to the initial value problem $\imath\partial_t u + (-\Delta_x)^{a/2} u = 0$ with $u(\,\cdot\,,0) = f$;

$$U_t^a f = \exp(\imath t(-\Delta)^{a/2}) f. \tag{7}$$

The case $a = 2$ corresponds to the Schrödinger equation and the case $a = 1$ to a wave equation.

Theorem 1.5. *Let $d \geq 2$, $p \in (\frac{2(d+2)}{d}, \infty)$, $a \in (0,\infty)$, and let I be compact time interval. Then*

$$\Big\| \Big(\int_I |U_t^a f|^2 \, dt \Big)^{1/2} \Big\|_p \lesssim \|f\|_{L^p_s}, \quad \frac{s}{a} = d\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{2}. \tag{8}$$

In fact we will prove a refinement of this for initial data in the Besov space $B^p_{s,p}$ (which contains L^p_s for $p \geq 2$).

One can also consider the same regularity problem in the mixed norm space $L^p(L^q(I))$ with $q \in (2,\infty]$. For this range the analogy between the wave and Schrödinger equation breaks down (and some endpoint versions of the deeper 'local smoothing' result for the wave equation are currently available only in four and higher dimensions, *cf.* [19]). However, for $a \in (0,1) \cup (1,\infty)$ and $d \geq 2$, one can deduce sharp $B^p_{s,p} \to L^p(L^q(I))$ estimates with $s = ad(1/2 - 1/p) - a/q$, in the range $p \in (2 + 4/d, \infty)$. This follows from a combination of Theorem 1.5 and the result in Appendix A. Moreover, one can, for a limited range of q, obtain further estimates for $p > 2 + 4/(d+1)$ and $d \geq 1$, essentially by interpolation with results in [28]; in dimensions $d \geq 2$ this currently requires the restriction $a > 1$. These $L^p(L^q)$-estimates are stated in §6, and a further, more substantial improvement for $d = 2$ will be considered in [24].

This paper. In §2 we give the definitions of the weight operators and establish the bounds which were asserted in Theorem 1.3. In §3 we prove some $L^2 \to L^p$ estimates for radial convolution operators and prepare for the proof of the weighted inequalities. These are established in §4. In §5 we prove $L^p(L^2)$ estimates for wave and Schrödinger equations and in §6 we discuss some $L^p(L^q)$ bounds for $q > 2$. Appendix A contains auxiliary results on combining inequalities for frequency localized operators.

Some notational references. To avoid unwieldy formulas we will sometimes shorten the notation for products involving a complex conjugate and use, given two complex terms $\mathcal{E}_1$ and $\mathcal{E}_2$, the expression $\prod^*_{i=1,2}[\mathcal{E}_i] = \mathcal{E}_1\overline{\mathcal{E}_2}$. For convolution operators given by Fourier multipliers $a(\xi)$ we occasionally use the symbol notation $a(D)f := \mathcal{F}^{-1}[a\widehat{f}\,]$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. By P_n, $\widetilde{P}_n$ we denote dyadic frequency cutoff operators which localize to frequencies of size $\approx 2^n$, so that $P_n\widetilde{P}_n = P_n$, see §2 for the precise definition.

2. Definition and Boundedness of the Weight Operators

Definition of $\mathfrak{W}_{q,\delta}$. We assume throughout this section that $d \geq 2$. The definition of $\mathfrak{W}_{q,\delta} w$ in (16) involves a suitably damped Nikodym maximal function and another (more important) maximal function acting on functions with Fourier transform supported away from the origin.

Let $0 < \delta_\circ < 1$, let θ be a unit vector in $\mathbb{R}^d$ and let

$$R^\theta_{\delta_\circ,t} = \{y \in \mathbb{R}^d : |\langle y,\theta\rangle| \le t, |y - \langle y,\theta\rangle\theta| \le t\delta_\circ\}.$$

The Nikodym maximal function associated to tubes with eccentricity $\delta_\circ < 1$ is defined by

$$\mathfrak{M}_{\delta_\circ} g(x) = \sup_{\theta\in S^{d-1}} \sup_{t>0} \frac{1}{|R^\theta_{\delta_\circ,t}|} \int_{R^\theta_{\delta_\circ,t}} |g(x+y)|dy. \tag{9}$$

The definition of our second maximal operator is reminiscent of the "grand maximal function" in the theory of Hardy spaces ([16]) as it involves the supremum over convolutions with kernels in a suitably normalized and rescaled class of Schwartz functions; this is necessary to exploit cancellations. We let $\mathcal{N} \ge d+3$ be a large positive integer and let $\mathcal{S}(\mathcal{N})$ be the set of all Schwartz functions ψ for which

$$\|\!|\psi|\!\|_{\mathcal{N}} := \max_{|\alpha|\le\mathcal{N}} (1+|x|)^{\mathcal{N}} |\partial_x^\alpha \psi(x)| \le 1. \tag{10}$$

The number $\mathcal{N}$ will be fixed throughout the paper and constants in inequalities will depend on $\mathcal{N}$ (one may want to choose $\mathcal{N} = d+3$).

For $j \ge 0$ and $\theta \in S^{d-1}$ let $\ell_{\theta,j}$ be the unique linear transformation defined by

$$\ell_{\theta,j}(\theta) = 2^j\theta, \qquad \ell_{\theta,j}(y) = y \ \text{ if } \ \langle\theta, y\rangle = 0.$$

Then $\det \ell_{\theta,j} = 2^j$. Let $\mathcal{S}^{\theta,j}_n$ be the set of all Ψ for which $2^{-nd}2^j\Psi(\ell_{\theta,j}2^{-n}\cdot)$ belongs to $\mathcal{S}(\mathcal{N})$. Typical examples of functions in $\mathcal{S}^{\theta,j}_n$ are L^1 normalized bump functions essentially supported on a tube with long side in the direction of θ and of length 2^{j-n} and width 2^{-n}. We define a maximal function which involves convolutions for Ψ in the classes $\mathcal{S}^{\theta,j}_n$ and in our application it is crucial that these convolutions will be acting on functions with cancellation, namely with frequency support in annuli. Let

$$\mathcal{M}^{\theta,j}_n g(x) = \sup_{\Psi\in\mathcal{S}^{\theta,j}_n} |\Psi * g(x)|$$

and set, for $\tau > 0$,

$$\mathcal{K}^{\theta,j}_\tau(x) := \frac{2^{-j}\tau^d}{(1+|\ell^{-1}_{\theta,j}(\tau x)|)^{d+1}}. \tag{11}$$

For future reference note that

$$|\Psi(x)| \lesssim \frac{2^{kd}2^{-j(d+1)}}{(1+2^{k-2j}|\langle x,\theta\rangle| + 2^{k-j}|x-\langle x,\theta\rangle\theta|)^{d+3}} \lesssim \mathcal{K}^{\theta,j}_{2^{k-j}}(x) \ \text{ if } \Psi \in \mathcal{S}^{\theta,j}_{k-j}\,. \tag{12}$$

We use the dyadic frequency cutoff operator P_n defined by

$$\widehat{P_n f}(\xi) = \chi(2^{-n}|\xi|)\widehat{f}(\xi) \tag{13}$$

where $\chi \in C^\infty$ is nonnegative and supported in $(5/8, 15/8)$ so that $\sum_{k\in\mathbb{Z}} \chi(2^{-k}t) = 1$ for all $t > 0$. Set

$$\mathcal{W}^j_{q,\delta,k} g(x) := \Big(\sup_\theta \mathcal{K}^{\theta,j}_{2^{k+j}\delta} * \big|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g\big|^q(x)\Big)^{1/q}, \tag{14}$$

$$\mathcal{W}^j_{q,\delta} g(x) := \sup_{k\in\mathbb{Z}} \mathcal{W}^j_{q,\delta,k} g(x)\,. \tag{15}$$

Next, fix $s \in (1,q)$ and define the maximal operator $\mathfrak{W}_{q,\delta}$ by

$$\mathfrak{W}_{q,\delta} w := \sum_{1\le 2^{2j}<\delta^{-1}} 2^{-2j(\frac{d}{q}-1)} \big(M|M\circ\mathcal{W}^j_{q,\delta} w|^s\big)^{1/s} + \delta^{\frac{d}{q}-1}\big(M|M\circ\mathfrak{M}_{\sqrt{\delta}} w|^s\big)^{1/s} \tag{16}$$

where M is the Hardy–Littlewood maximal operator. We also recall from the statement of Theorem 1.3 the definition $\mathfrak{W}_q w = \sup_{0<\delta<1/2} \mathfrak{W}_{q,\delta} w$.

Boundedness. In the proofs we will frequently use a dyadic frequency cutoff $\widetilde{P}_n$ which reproduces P_n and is similarly defined. That is to say, $\widehat{\widetilde{P}_n f}(\xi) = \widetilde{\chi}(2^{-n}|\xi|)\widehat{f}(\xi)$ where $\widetilde{\chi}$ is supported in $(1/2, 2)$ and has the property $\widetilde{\chi}(s) = 1$ for $s \in [5/8, 15/8]$. Then $P_n \widetilde{P}_n = P_n$.

It is obvious that the operators $\mathcal{W}^j_{q,\delta,k}$, $\mathcal{W}^j_{q,\delta}$, $\mathfrak{M}_{\sqrt{\delta}}$ are bounded on L^∞. For the L^q boundedness we state

Proposition 2.1. *(i) For $1 \le q \le \infty$, the operator $\mathcal{W}^j_{q,\delta,k}$ satisfies*

$$\sup_{\delta,k} \sup_{\|w\|_q \le 1} \|\mathcal{W}^j_{q,\delta,k} w\|_q \lesssim \begin{cases} 2^{j\frac{d-2}{q}} & \text{if } \ 2 \le q \le \infty \\ 2^{j(\frac{d}{q}-1)} & \text{if } \ 1 \le q < 2. \end{cases}$$

(ii) For $1 < q \le \infty$, the operator $\mathcal{W}^j_{q,\delta}$ satisfies

$$\sup_{\|w\|_q \le 1} \|\mathcal{W}^j_{q,\delta} w\|_q \lesssim \begin{cases} 2^{j\frac{d-2}{q}} & \text{if } \ 2 \le q \le \infty \\ 2^{j(\frac{d}{q}-1)} [\log(\frac{1}{\delta})]^{1/q-1/2} & \text{if } \ 1 < q < 2. \end{cases}$$

Moreover for $q = 1$ the operator $\mathcal{W}^j_{1,\delta}$ maps the Hardy space H^1 to L^1 with operator norm $\lesssim 2^{j(d-1)}[\log(\frac{1}{\delta})]^{1/2}$.
(iii) We also have the weak type (q, q) estimate

$$\sup_{\|w\|_q \le 1} \Big\| \sup_{0<\delta<1/2} \mathcal{W}^j_{\delta,q} w \Big\|_{L^{q,\infty}} \lesssim \begin{cases} 2^{j\frac{d-2}{q}} & \text{if } \ 2 \le q < \infty \\ 2^{j(\frac{d}{q}-1)} (1+j)^{\frac{1}{q}-\frac{1}{2}} & \text{if } \ 1 < q < 2. \end{cases}$$

The proposition implies statements (i) and (ii) of Theorem 1.3. Clearly the operators $\mathfrak{W}_{q,\delta}$ and $\mathfrak{W}_q$ are bounded on L^∞ if $q < d$. The L^q bound for the first (main) term in (16) is immediate from Proposition 2.1 since $\frac{d-2}{q} - 2(\frac{d}{q} - 1) < 0$ iff $q < \frac{d+2}{2}$. For the second term in (16) we use standard non-endpoint L^q bounds for the Nikodym maximal operator (see [8], [9], [1], [26]). Namely $\mathfrak{M}_{\sqrt{\delta}}$ is bounded on L^q with operator norm $\le C_\varepsilon(\sqrt{\delta})^{1-d/q-\varepsilon}$ if $q < 2$ and operator norm $\le C_\varepsilon(\sqrt{\delta})^{-(d-2)/q-\varepsilon}$ if $q \ge 2$. The damping factor $\delta^{-1+d/q}$ is enough to prove L^q boundedness for $q < \frac{d+2}{2}$. Using for example the results in [43] this final estimate can be significantly improved but any such improvement seems currently to have no impact on our result, as the main contribution to the weight operator comes from the terms $\mathcal{W}^j_{q,\delta}$. □

Elementary convolution estimates. The following simple and standard convolution estimates will be used many times in the paper.

Lemma 2.2. *(i) Let $H(x) = (1 + |x|)^{-N}$ and let $N > d$. Then there is $C_{d,N} > 0$ so that for all $x \in \mathbb{R}^d$*

$$\sup_{t\ge 1} \sup_{0\le s\le 1} \int t^d H(ty) H(x - sy)\, dy \le C_{d,N} H(x). \tag{17}$$

(ii) Let $H^A(x) = |\det A| H(Ax)$. Then $H^A * H^A(x) \le C_{d,N} H^A(x)$ for all $x \in \mathbb{R}^d$.
(iii) Let $\ell \in \mathbb{N}$, let h_1, h_2 be kernels with

$$(1 + |x|)^\ell |h_1(x)| + \sum_{|\alpha|=\ell} |\partial^\alpha h_2(x)| \le H(x)$$

and assume that $\int P(x)h_1(x)\,dx = 0$ *for all polynomials* P *of degree* $\le \ell - 1$. *Let* $h_i^A(x) = |\det A| h_i(Ax)$. *Then for* $t \ge 1$

$$\left|t^d h_1^A(t\,\cdot) * h_2^A(x)\right| \le C_{d,N,\ell}\, t^{-\ell} H^A(x)\,.$$

Proof. Let $F_{t,s}(x) = \int t^d H(ty)H(x-sy)\,dy$ then clearly $F_{t,0}(x) \lesssim H(x)$ and for $|x| \le 1$ we have $F_{t,s}(x) \lesssim 1$ so that the assertion holds for $|x| \le 1$.

If $|x| \ge 1$ then $\sup_{|sy|\le |x|/2} H(x-sy) \lesssim H(x)$ and $\sup_{|sy|\ge 2|x|} H(x-sy) \lesssim H(x)$, so that

$$\int_{|sy|\notin[\frac{|x|}{2},2|x|]} t^d H(ty)H(x-sy)\,dy \lesssim \|t^d H(t\,\cdot)\|_1 H(x) \lesssim H(x),$$

for all $t > 0$.

Next, if $|sy| \approx |x|$ then $t^d H(ty) \approx t^d H(tx/s)$. Letting $B_{l,s}(x) = \{y : |sy - x| \le 2^l\}$ for $l \ge 0$, we have $|B_{l,s}(x)| \lesssim (2^l/s)^d$ and we may estimate (using $N > d$)

$$\int_{|x|/2\le|sy|\le 2|x|} H(x-sy)t^d H(ty)\,dy \lesssim t^{d-N}s^N|x|^{-N}\sum_{l\ge 0}\int_{B_{l,s}(x)} 2^{-lN}dy \lesssim t^{d-N}s^{N-d}|x|^{-N} \lesssim H(x)$$

since $t \ge 1$, $s \le 1$ and $|x| \ge 1$.

(ii) follows immediately by a change of variable. Similarly for *(iii)* we may reduce to the case where A is the identity. Then one can use Taylor's formula and the cancellation of h_1, and *(i)*, to estimate

$$\begin{aligned}|t^d h_1(t\,\cdot) * h_2(x)| &\lesssim \int_0^1 \frac{(1-s)^{\ell-1}}{(\ell-1)!}\int \left|t^d h_1(ty)\langle y, \nabla\rangle^\ell h_2(x-sy)\right| dy\,ds \\ &\lesssim t^{-\ell}\sup_{0\le s\le 1}\int t^d H(ty)H(x-sy)\,dy \lesssim t^{-\ell}H(x)\,.\end{aligned}$$

□

For the kernels in (11), Lemma 2.2 yields the bound

$$\mathcal{K}_t^{\theta,j} * \mathcal{K}_\tau^{\theta,j}(x) \le C_d \mathcal{K}_\tau^{\theta,j}(x)\,, \quad t > \tau\,. \tag{18}$$

Proof of Proposition 2.1. We shall use many times that $\|\mathcal{K}_t^{\theta,j}\|_1 \lesssim 1$, uniformly in θ, j, t. Moreover,

$$\mathcal{K}_t^{\theta,j}(x) \approx \mathcal{K}_{\tilde t}^{\tilde\theta,\tilde j}(x)\,, \quad |\theta - \tilde\theta| \le C2^{-j}, \quad |j - \tilde j| \le C, \quad C^{-1} \le t/\tilde t \le C\,, \tag{19}$$

where the implicit constants in the equivalence depend only on C and the dimension. Next, the class $\mathcal{S}_n^{\theta,j}$ is stable under small perturbation in the sense that given $A > 0$ there is a constant C, depending only on A, d and the parameter $\mathcal{N}$ in the definition (10), so that

$$\Psi \in \mathcal{S}_n^{\theta,j} \implies C^{-1}\Psi \in \mathcal{S}_{\tilde n}^{\tilde\theta,\tilde j} \text{ if } |\theta - \tilde\theta| \le A2^{-j},\ |j - \tilde j| \le A,\ |n - \tilde n| \le A. \tag{20}$$

In what follows we let Θ_j be a maximal 2^{-j-d}-separated subset of S^{d-1}. Clearly $\mathrm{card}(\Theta_j) \lesssim 2^{j(d-1)}$; moreover each $\tilde\theta \in S^{d-1}$ has distance $\le 2^{-j}$ to at least one $\theta \in \Theta_j$.

L^q-bounds for $\mathcal{W}^j_{q,\delta,k}$. We replace a sup in $\theta \in \Theta_j$ by an ℓ^q norm and interchange a summation and integration to estimate for any fixed j, k

$$\begin{aligned}
&\Big\| \Big(\sup_\theta \mathcal{K}^{\theta,j}_{2^{k+j}\delta} * \big|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g\big|^q \Big)^{1/q} \Big\|_q \\
&\lesssim \Big(\sum_{\theta\in\Theta_j} \iint \sup_{|\tilde\theta-\theta|\le 2^{-j}} \mathcal{K}^{\tilde\theta,j}_{2^{k+j}\delta}(y) \sup_{|\tilde\theta-\theta|\le 2^{-j}} \big|\mathcal{M}^{\tilde\theta,j}_{k-j} P_{k-j} g(x-y)\big|^q dy dx \Big)^{1/q} \\
&\lesssim \Big(\sum_{\theta\in\Theta_j} \big\|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g\big\|_q^q \Big)^{1/q}.
\end{aligned}$$

For the last inequality we have used (19) and (20).

We now need a further decomposition. Let η_0 be supported in $(-1,1)$ so that $\eta_0(s) = 1$ for $s \in (-1/2, 1/2)$ and let

$$\eta^{j,k}_\theta(\xi) = \eta_0\big(\sqrt{|2^{2j-k}\langle \xi,\theta\rangle|^2 + |2^{j-k}(\xi - \langle\xi,\theta\rangle\theta)|^2}\big)\,.$$

For $m \ge 0$ define operators $Q^{\theta,j,k}_m$, $P^{\theta,j,k}_m$ as follows. For $m = 0$ set

$$\widehat{Q^{\theta,j,k}_0 f}(\xi) = \widehat{P^{\theta,j,k}_0 f}(\xi) = \eta^{j,k}_\theta(\xi)\widehat f(\xi)$$

and, for $m \ge 1$, set

$$\begin{aligned}
\widehat{Q^{\theta,j,k}_m f}(\xi) &= \big(\eta^{j,k}_\theta(2^{-m}\xi) - \eta^{j,k}_\theta(2^{-m+1}\xi)\big)\widehat f(\xi)\,,\\
\widehat{P^{\theta,j,k}_m f}(\xi) &= \big(\eta^{j,k}_\theta(2^{-m}\xi) + \eta^{j,k}_\theta(2^{-m+1}\xi)\big)\widehat f(\xi)\,.
\end{aligned}$$

Then $\sum_{m=0}^\infty Q^{\theta,j,k}_m P^{\theta,j,k}_m$ is the identity; moreover $Q^{\theta,j,k}_m P_{k-j} = 0$ for $m > j+3$. Using the cancellation of $Q^{\theta,j,k}_m$ it is straightforward to derive the estimate

$$\sup_{\Psi\in\mathcal{S}^{\theta,j}_{k-j}} |Q^{\theta,j,k}_m \Psi(x)| \le C 2^{-2m} \frac{2^{(k-j)d} 2^{-j}}{(1 + 2^{k-2j}|\langle x,\theta\rangle| + 2^{k-j}|x - \langle x,\theta\rangle\theta|)^{d+1}}$$

from Lemma 2.2 and (12); in fact if $\mathcal{N} > d+3$ in (10) one can also get a gain of higher powers of 2^{-m}. Now, for fixed k, j, θ

$$\begin{aligned}
\Big(\sum_{\theta\in\Theta_j} \big\|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g\big\|_q^q \Big)^{1/q} &\lesssim \Big(\sum_{\theta\in\Theta_j} \Big\| \sup_{\Psi\in\mathcal{S}^{\theta,j}_{k-j}} \Big| \sum_{0\le m\le j+3} (Q^{\theta,j,k}_m \Psi) * (P^{\theta,j,k}_m P_{k-j} g)\Big|\Big\|_q^q \Big)^{1/q} \\
&\lesssim \sum_{0\le m\le j+3} 2^{-2m} \Big(\sum_{\theta\in\Theta_j} \big\|P^{\theta,j,k}_m P_{k-j} g\big\|_q^q \Big)^{1/q} \qquad (21)
\end{aligned}$$

and the desired bound follows when we establish the estimate

$$\Big(\sum_{\theta\in\Theta_j} \Big\|P^{\theta,j,k}_m P_{k-j} g\Big\|_q^q \Big)^{1/q} \lesssim \max\{2^{j\frac{d-2}{q}}, 2^{j(\frac{d}{q}-1)}\}\, 2^{m/q} \|g\|_q, \quad 1 \le q \le \infty, \qquad (22)$$

with the usual modification for $q = \infty$. To prove (22) we notice that by interpolation we only have to verify the cases $q = 1, 2, \infty$. The cases $q = \infty$ and $q = 1$ are immediate since the operators $P^{\theta,j,k}_m$ and P_{k-j} have convolution kernels with uniformly bounded L^1 norms and since $\mathrm{card}(\Theta_j) \lesssim 2^{j(d-1)}$. For $q = 2$ we use Plancherel's theorem. Write $\mathcal{F}[P^{\theta,j,k}_m P_{k-j} g] = a_{j,k,\theta,m}\widehat g$. Then the multiplier $a_{j,k,\theta,m}$ is supported in

$$\{\,\xi :\ C^{-1}2^{k-j-3} \le |\xi| \le C 2^{k-j+2}, \quad |\langle\xi,\theta\rangle| \le C 2^{k-2j+m}\,\}.$$

Since Θ_j is a 2^{-j-d}-separated set we see that for fixed j,k,m every $\xi\in\mathbb{R}^d$ is contained in no more than $C_d 2^{m+j(d-2)}$ of the sets $\operatorname{supp}(a_{j,k,\theta,m})$, which implies the bound (22).

L^q-bounds for $\mathcal{W}^j_{q,\delta}$. We argue as above and now replace the sup in $k\in\mathbb{Z}$, $\theta\in\Theta_j$ by an ℓ^q norm. This yields

$$\Big\|\Big(\sup_k\sup_\theta \mathcal{K}^{\theta,j}_{2^{k+j}\delta}*\big|\mathcal{M}^{\theta,j}_{k-j}P_{k-j}g(x)\big|^q\Big)^{1/q}\Big\|_q\lesssim\Big(\sum_k\sum_{\theta\in\Theta_j}\big\|\mathcal{M}^{\theta,j}_{k-j}P_{k-j}g\big\|_q^q\Big)^{1/q}. \tag{23}$$

A combination of (21) and (22) together with the use of the reproducing Littlewood–Paley cutoff $\widetilde{P}_{k-j}$ yields

$$\begin{aligned}&\Big(\sum_k\sum_{\theta\in\Theta_j}\Big\|\mathcal{M}^{\tilde\theta,j}_{k-j}P_{k-j}g\Big\|_q^q\Big)^{1/q}\\&\lesssim\sum_{0\le m\le j+3}2^{-2m}\Big(\sum_k\sum_{\theta\in\Theta_j}\big\|P^{\theta,j,k}_m P_{k-j}\widetilde{P}_{k-j}w\big\|_q^q\Big)^{1/q}\\&\lesssim\sum_{0\le m\le j+3}2^{-m(2-\frac1q)}\max\{2^{j\frac{d-2}{q}},2^{j(\frac dq-1)}\}\Big(\sum_k\big\|\widetilde{P}_{k-j}w\big\|_q^q\Big)^{1/q}\end{aligned}$$

and for $q\ge 2$ this is $\lesssim 2^{j(d-2)/q}\|w\|_q$.

Now consider the case $q<2$. By a standard linearization combined with an analytic family argument the claimed L^q estimates can be deduced from the Hardy-space estimate and the already proven L^2 estimate. We omit the details of the interpolation argument. For the Hardy-space estimate we need to prove an inequality for an L^2-atom, *i.e.* an L^2 function g_ρ supported on a ball $\{x:|x|\le\rho\}$ with $\|g_\rho\|_2\le\rho^{-d/2}$ and $\int g_\rho\,dx=0$. It suffices to verify

$$\|\mathcal{W}^j_{1,\delta}g_\rho\|_1\lesssim 2^{j(d-1)}[\log(\tfrac1\delta)]^{1/2}. \tag{24}$$

By the Schwarz inequality we have

$$\mathcal{W}^j_{1,\delta}g(x)\lesssim\mathcal{W}^j_{2,\delta}g(x)$$

and thus by the above L^2 estimates

$$\|\mathcal{W}^j_{1,\delta}g_\rho\|_{L^1(|x|\le 2\rho)}\lesssim\rho^{d/2}\|\mathcal{W}^j_{2,\delta}g_\rho\|_2\lesssim 2^{j(\frac d2-1)}\rho^{d/2}\|g_\rho\|_2\lesssim 2^{j(\frac d2-1)}.$$

On the complementary set we estimate

$$\|\mathcal{W}^j_{1,\delta}g_\rho\|_{L^1(|x|\ge 2\rho)}\lesssim\sum_{k\in\mathbb{Z}}\|\mathcal{W}^j_{1,\delta,k}g_\rho\|_{L^1(|x|\ge 2\rho)}.$$

From our previous bound for $\mathcal{W}^j_{q,\delta,k}$ and the cancellation of the atom,

$$\|\mathcal{W}^j_{1,\delta,k}g_\rho\|_1\lesssim 2^{j(d-1)}\|\widetilde{P}_{k-j}g_\rho\|_1\lesssim 2^{j(d-1)}\min\{1,2^{k-j}\rho\}.$$

Now if $\Psi\in\mathcal{S}^{\theta,j}_{k-j}$ then by (12) with $n=k-j$, (18), and $2^{2j}\delta\le 1$,

$$\mathcal{K}^{\theta,j}_{2^{k+j}\delta}*|\Psi|(x)\lesssim\mathcal{K}^{\theta,j}_{2^{k+j}\delta}*\mathcal{K}^{\theta,j}_{2^{k-j}}(x)\lesssim\frac{(2^k\delta)^d 2^{j(d-1)}}{(1+2^k\delta|\langle x,\theta\rangle|+2^j2^k\delta|x-\langle x,\theta\rangle\theta|)^{d+1}}.$$

Thus a favorable estimate holds for $2^k\rho\ge\delta^{-1}$, namely

$$\|\mathcal{W}^j_{1,\delta,k}g_\rho\|_{L^1(|x|\ge 2\rho)}\lesssim 2^{j(d-1)}(2^k\delta\rho)^{-1}.$$

These estimates can be summed for $2^k \le \rho^{-1}$ and $2^k \ge \delta^{-1}\rho^{-1}$. For the intermediate terms we use the L^1 bounds and the Schwarz inequality

$$\sum_{\rho^{-1}\le 2^k\le(\rho\delta)^{-1}} \|\mathcal{W}^j_{1,\delta,k} g_\rho\|_1 \lesssim 2^{j(d-1)} \sum_{\rho^{-1}\le 2^k\le(\rho\delta)^{-1}} \|\widetilde{P}_{k-j} g_\rho\|_1$$
$$\lesssim 2^{j(d-1)}[\log(\tfrac{1}{\delta})]^{1/2}\Big(\sum_k \|\widetilde{P}_{k-j} g_\rho\|_1^2\Big)^{1/2},$$

and by Minkowski's integral inequality and the Littlewood–Paley theorem on H^1 we have

$$\Big(\sum_k \|\widetilde{P}_{k-j} g_\rho\|_1^2\Big)^{1/2} \le \Big\|\Big(\sum_k |\widetilde{P}_{k-j} g_\rho|^2\Big)^{1/2}\Big\|_1 \lesssim \|g_\rho\|_{H^1} \lesssim 1. \tag{25}$$

Now collect the estimates and (24) is proved.

The weak type estimate. We observe the pointwise estimate

$$\sup_k \mathcal{K}^{\theta',j}_{2^{k+j}\delta} * \big|\mathcal{M}^{\theta',j}_{k-j} P_{k-j} g\big|^q \lesssim M^{\theta,j}\big[\sup_k |\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g|^q\big], \quad |\theta-\theta'| \le 2^{-j},$$

where $M^{\theta,j}$ is the maximal operator associated to tubes of eccentricity 2^{-j}, with the long side pointing in the direction θ. For each (θ, j), $M^{\theta,j}$ is a rescaled version of the Hardy–Littlewood maximal function and therefore satisfies the standard weak-type $(1,1)$ inequality with a bound independent of θ and j.

We use the continuous embedding $\ell^q(L^{q,\infty}) \subset L^{q,\infty}(\ell^\infty)$ (see *e.g.* Lemma 2.1 in [18]) and dominate

$$\Big\|\sup_{0<\delta<1/2} |\mathcal{W}^j_{q,\delta} g|\Big\|_{L^{q,\infty}} \lesssim \Big(\sum_{\theta\in\Theta_j} \Big\|\big(M^{\theta,j}\big[\sup_k |\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g|^q\big]\big)^{1/q}\Big\|^q_{L^{q,\infty}}\Big)^{1/q}$$
$$\lesssim \Big(\sum_{\theta\in\Theta_j} \big\|\sup_k |\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g|\,\big\|^q_{L^q}\Big)^{1/q}, \tag{26}$$

and for the last inequality we have used the uniform weak-type $(1,1)$ bounds for the $M^{\theta,j}$ together with the identity $\|F^q\|_{L^{1,\infty}} = \|F\|^q_{L^{q,\infty}}$ for the usual quasinorms.

We may dominate (26) by

$$\Big(\sum_{\theta\in\Theta_j}\sum_k \big\|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g\big\|^q_q\Big)^{1/q}$$

which for $q \ge 2$ has already been estimated by $2^{j(d-2)/q}\|g\|_q$. Thus the asserted estimate follows in this range.

For $1 < q < 2$ we claim that

$$\Big(\sum_{\theta\in\Theta_j} \big\|\sup_k |\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g|\,\big\|^q_q\Big)^{1/q} \lesssim 2^{j(\frac{d}{q}-1)}(1+j)^{1/q-1/2}\|g\|_q.$$

This follows by complex interpolation from the estimate for $q = 2$ already proved above and an $H^1 \to \ell^1(L^1(\ell^\infty))$ bound. Again it suffices to consider an L^2-atom g_ρ supported on a ball of radius ρ centered at the origin and we need to check

$$\sum_{\theta\in\Theta_j} \big\|\sup_k |\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g_\rho|\,\big\|_1 \lesssim 2^{j(d-1)}(1+j)^{1/2}.$$

Now on the ball of radius 2ρ we use an L^2 estimate and the trivial estimation $|\mathcal{M}^{\theta,j}_{k-j}f| \lesssim M^{\theta,j}(f)$ to obtain

$$\sum_{\theta\in\Theta_j} \big\| \sup_k |\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g_\rho| \big\|_{L^1(|x|\le 2\rho)} \lesssim \rho^{d/2} \sum_{\theta\in\Theta_j} \big\| \sup_k |\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g_\rho| \big\|_2$$
$$\lesssim \rho^{d/2} \sum_{\theta\in\Theta_j} \Big\| \Big(\sum_k (M^{\theta,j}(P_{k-j}g_\rho))^2 \Big)^{1/2} \Big\|_2$$
$$\lesssim \rho^{d/2} \sum_{\theta\in\Theta_j} \Big(\sum_k \|P_{k-j} g_\rho\|_2^2 \Big)^{1/2} \lesssim 2^{j(d-1)} \rho^{d/2} \|g_\rho\|_2 \lesssim 2^{j(d-1)}\,.$$

For $|x| \ge 2\rho$ we replace the sup in k by the sum. By standard L^1 estimates and using the cancellation of the atom we have

$$\big\|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g_\rho\big\|_1 \lesssim \|P_{k-j} g_\rho\|_1 \lesssim \min\{2^{k-j}\rho, 1\}$$

and, using estimates of the kernels,

$$\big\|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g_\rho\big\|_{L^1(|x|\ge 2\rho)} \lesssim (2^{k-2j}\rho)^{-1} \quad \text{if } \ 2^k\rho > 2^j\,.$$

Thus $\sum_{\theta\in\Theta_j} \sum_{2^k\rho\notin[2^j,2^{2j}]} \big\|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g_\rho\big\|_{L^1(|x|\ge 2\rho)} \lesssim 2^{j(d-1)}$. Moreover, for the intermediate terms,

$$\sum_{\theta\in\Theta_j} \sum_{2^k\rho\in[2^j,2^{2j}]} \big\|\mathcal{M}^{\theta,j}_{k-j} P_{k-j} g_\rho\big\|_1 \lesssim 2^{j(d-1)} \sum_{2^k\rho\in[2^j,2^{2j}]} \|P_{k-j} g_\rho\|_1 \lesssim 2^{j(d-1)}(1+j)^{1/2},$$

by the argument in (25). We combine these estimates and the L^1 bound is proved. □

3. Multipliers and the bilinear adjoint restriction theorem

In this section we prove bilinear estimates for multiplier transformations, under suitable separation conditions. The proofs rely on

Tao's bilinear adjoint restriction theorem ([41], [23]). *Let $b > 1/2$ and $p > 2 + 4/d$. There exist $\varepsilon_\circ > 0$, $N_\circ \in \mathbb{N}$ and C, depending on b, p and d, that for all functions h defined on $I_b = [-b,b]^{d-1}$ and satisfying*

$$\sup_{\omega\in[-b,b]^{d-1}} \max_{\substack{|\alpha|\le N_\circ \\ i=1,2}} |\partial^\alpha_\omega h(\omega)| \le \varepsilon_\circ \tag{27}$$

the following holds: For all pairs of functions (F_1, F_2) with $\operatorname{dist}(\operatorname{supp}(F_1), \operatorname{supp}(F_2)) \ge 1/2$ and $F_i \in L^2([-b,b]^{d-1})$,

$$\Big(\int \Big| \prod_{i=1,2} \int_{I_b} F_i(\omega) \exp\big(\imath\langle x', \omega\rangle + \imath x_d(|\omega|^2/2 + h(\omega))\big) d\omega \Big|^{p/2} dx \Big)^{2/p} \lesssim C \prod_{i=1,2} \big\|F_i\big\|_{L^2}. \tag{28}$$

We will need to consider families of hypersurfaces which depend on a parameter s and which, for fixed s, are small perturbations of the paraboloids $\xi_d = |\xi'|^2/2$, where $\xi' = (\xi_1, \dots, \xi_{d-1})$. These lead to "elliptic" phase-functions as considered in [42], [41].

Definition. We denote by $\mathrm{Ell}(b, \varepsilon, N_\circ)$ the class of functions $(\xi', s) \mapsto \gamma(\xi', s)$ defined on $[-b,b] \times (-1,1)$ which are of the form

$$\gamma(\xi', s) = \frac{|\xi'|^2}{2} - s + h(\xi', s),$$

with

$$\sup_{\substack{\omega\in[-b,b]^{d-1}\\ s\in(-1,1)}} \max_{\substack{|\alpha|\le N_\circ\\ i=1,2}} |\partial^\alpha_{\omega,s} h(\omega,s)| \le \varepsilon. \tag{29}$$

We may and shall assume in what follows that $N_\circ$ is large, say $N_\circ > 10d$.

We now consider Fourier multipliers depending on a parameter $s \in (-1,1)$, supported in a tubular neighborhood of $(\xi', \gamma(\xi', s))$.

Lemma 3.1. *Let $p > 2 + 4/d$, $b > 1/2$. There are ε, $N_\circ$, depending on d, b, and p, so that the following holds for all $\delta_\circ < 1/2$.*

Let $\gamma \in \mathrm{Ell}(b, \varepsilon, N_\circ)$. For $|s| \le 1$ and $i = 1, 2$, let $(\xi, s) \mapsto a_i(\xi, s)$ be supported on $(-b, b)^{d-1} \times (-1, 1)$ and satisfy the conditions

$$|a_i(\xi, s)| \le 1\,, \tag{30a}$$

$$a_i(\xi, s) = 0 \quad \text{if } \ |\xi_d - \gamma(\xi', s)| \ge \delta_\circ, \tag{30b}$$

and

$$(\xi', \xi_d) \in \operatorname{supp} a_1(\,\cdot\,, s), \quad (\widetilde{\xi}', \widetilde{\xi}_d) \in \operatorname{supp} a_2(\,\cdot\,, s) \implies |\xi' - \widetilde{\xi}'| \ge 1\,. \tag{30c}$$

Then, for all pairs of L^2 functions (f_1, f_2),

$$\Big\| \int_{-1/2}^{1/2} \prod_{i=1,2} a_i(D, s) f_i \, ds \Big\|_{p/2} \lesssim \delta_\circ^2 \prod_{i=1,2} \|f_i\|_2\,. \tag{31}$$

Proof. For fixed s we introduce coordinates

$$\xi = \Gamma^s(\xi', \tau) := (\xi', \gamma(\xi', s) + \tau)$$

in the Fourier integral. We then need to estimate the $L^{p/2}$ norm of

$$\int_{-1/2}^{1/2} \prod_{i=1,2} \Big[\int_{-\delta_\circ}^{\delta_\circ} \int_{[-b,b]^{d-1}} [a_i \widehat{f_i}](\Gamma^s(\xi^{i\prime}, \tau^i)) e^{\imath(\langle x', \xi^{i\prime}\rangle + x_d(\gamma(\xi^{i\prime}, s) + \tau^i))} d\xi^{i\prime} d\tau^i \Big] ds;$$

here we denote by $(\xi^{i\prime}, \tau^i)$ the variables in the two different copies of $\mathbb{R}^d$. By Minkowski's integral inequality the $L^{p/2}$ norm is dominated by

$$\iiint_{[-\delta_\circ,\delta_\circ]^2 \times [-\frac12,\frac12]} \Big(\int \Big| \prod_{i=1,2} \Big[\int_{[-b,b]^{d-1}} [a_i \widehat{f_i}](\Gamma^s(\xi^{i\prime}, \tau^i) e^{\imath(\langle x', \xi^{i\prime}\rangle + x_d \gamma(\xi^i, s))} d\xi^i \Big] \Big|^{p/2} dx \Big)^{2/p} ds\, d\tau^1 d\tau^2.$$

By the bilinear adjoint restriction theorem and the boundedness of a_i this is estimated by

$$\iiint_{[-\delta_\circ,\delta_\circ]^2 \times [-\frac12,\frac12]} \prod_{i=1,2} \Big(\int_{[-b,b]^{d-1}} \big|\widehat{f_i}(\Gamma^s(\xi^{i\prime}, \tau^i))\big|^2 d\xi^{i\prime} \Big)^{1/2} ds\, d\tau^1 d\tau^2.$$

We apply the Schwarz inequality in the s variable. Then for fixed τ^1, τ^2, we change variables $(\xi', s) \mapsto \xi = (\xi', \gamma(\xi', s) + \tau^i)$, using that $\partial_s \gamma(\xi', s) = 1 + O(\varepsilon)$. Thus the last displayed

expression is estimated by

$$\iint\limits_{[-\delta_\circ,\delta_\circ]^2}\prod_{i=1,2}\Big(\int\limits_{[-b,b]^{d-1}\times[-1,1]}\big|\widehat{f}(\xi^{i\prime},\gamma(\xi^{i\prime},s)+\tau^i)\big|^2 ds\,d\xi^{i\prime}\Big)^{1/2}d\tau^1 d\tau^2$$
$$\lesssim \iint\limits_{[-\delta_\circ,\delta_\circ]^2}\prod_{i=1,2}\|\widehat{f_i}\|_2\,d\tau^1 d\tau^2 \lesssim \delta_\circ^2\prod_{i=1,2}\|f_i\|_2\,.$$

□

In what follows we will use the notation $\prod^*_{i=1,2}[\mathcal{E}_i] = \mathcal{E}_1\overline{\mathcal{E}_2}$ for products involving a complex conjugate.

Proposition 3.2. *Let $q\in[1,\frac{d+2}{2})$ and $b>1/2$. There are ε, $N_\circ$, depending on d, b and q, so that the following holds for all $\delta_\circ<1/2$.*

Let $\gamma\in\mathrm{Ell}(b,\varepsilon,N_\circ)$, $\delta_\circ<1/2$ and, for $|s|\le 1$, $i=1,2$, let $a_i(\,\cdot\,,s)$ be multipliers supported in $(-b,b)^{n-1}\times(-1,1)$ and satisfy the conditions

$$|\partial_\xi^\alpha a_i(\xi,s)|\le\delta_\circ^{-|\alpha|},\quad |\alpha|\le d+2, \tag{32a}$$

$$a_i(\xi,s)=0\quad \textit{if}\ \ |\xi_d-\gamma(\xi',s)|\ge\delta_\circ, \tag{32b}$$

and

$$(\xi',\xi_d)\in\operatorname{supp}a_1(\,\cdot\,,s),\quad (\widetilde{\xi}',\widetilde{\xi}_d)\in\operatorname{supp}a_2(\,\cdot\,,s)\implies|\xi'-\widetilde{\xi}'|\ge 1\,. \tag{32c}$$

Then

$$\Big|\int_{\mathbb{R}^d}\int_{-1/2}^{1/2}\prod_{i=1,2}^*[a_i(D,s)f_i(x)]w(x)\,dx\,ds\Big| \le C_N\delta_\circ^2\prod_{i=1,2}\Big(\int|f_i(x)|^2\Big(\int\frac{|w(x-y)|^q}{(1+\delta_\circ|y|)^{(d+1)q}}dy\Big)^{1/q}dx\Big)^{1/2}. \tag{33}$$

Proof. We dyadically decompose the kernel of the convolution operators. Let ρ_0 be a $C_c^\infty(\mathbb{R})$ function supported on $(-1,1)$ and equal to one on $[-1/2,1/2]$ and define, for $x\in\mathbb{R}^d$,

$$\Phi_0(x)=\rho_0(\delta_\circ|x|),\qquad \Phi_j(x)=\rho_0(2^{-j}\delta_\circ|x|)-\rho_0(2^{1-j}\delta_\circ|x|),\quad j\ge 1.$$

Then the $\{\Phi_j\}_{j=0}^\infty$ form a radial partition of unity. We thus need to bound the sum

$$\sum_{j_1,j_2\ge 0}\Big|\int_{\mathbb{R}^d}\int_{-1/2}^{1/2}\prod_{i=1,2}^*\Big[\int a_i(D-\eta^i,s)f_i(x)\widehat{\Phi}_{j_i}(\eta^i)\,d\eta^i\Big]w(x)\,dx\,ds\Big|.$$

By symmetry considerations it suffices to consider the terms with $0\le j_1\le j_2$. The desired estimate then follows if we can show that

$$\Big|\int_{\mathbb{R}^d}\int_{-1/2}^{1/2}\prod_{i=1,2}^*\Big[\int a_i(D-\eta^i,s)f_i(x)\widehat{\Phi}_{j_i}(\eta^i)\,d\eta^i\Big]ds\,w(x)\,dx\Big|\lesssim$$
$$C(L)\delta_\circ^2 2^{-j_2L}\prod_{i=1,2}\Big(\int|f_i(x)|^2\Big(\int_{|y|\le C2^{j_2}\delta_\circ^{-1}}|w(x+y)|^q dy\Big)^{1/q}dx\Big)^{1/2} \tag{34}$$

for $0\le j_1\le j_2$, $L\le d+2$. We shall first verify this inequality for $L=0$ and then provide the modification for $L=d+2$.

We now form a grid $\mathfrak{Q}(j_2)$ of dyadic cubes of sidelength $2^{j_2}\delta_\circ^{-1}$. For every $Q \in \mathfrak{Q}(j_2)$ let Q^* be the double cube with same center as Q. By the support properties of the kernels, and $j_1 \le j_2$, we have

$$\int a_i(D-\eta^i,s)[f_i\chi_{\mathbb{R}^d\setminus Q^*}](x)\widehat{\Phi}_{j_i}(\eta_i)\,d\eta_i \;=0\,,\qquad \text{if } x\in Q,\quad Q\in\mathfrak{Q}(j_2),\quad i=1,2.$$

Thus the left hand side of (34) is equal to

$$\Big|\sum_{Q\in\mathcal{Q}(j_2)}\int_Q\int_{-1/2}^{1/2}\prod_{i=1,2}^{*}\Big[\int a_i(D-\eta^i,s)[\chi_{Q^*}f_i](x)\widehat{\Phi}_{j_i}(\eta^i)\,d\eta^i\Big]ds\,w(x)\,dx\Big|\,. \tag{35}$$

We use the formula $m(D-\eta)f=\mathrm{Mod}_\eta m(D)[\mathrm{Mod}_{-\eta}f]$ where $\mathrm{Mod}_\eta g(x)=g(x)e^{\imath\langle x,\eta\rangle}$. In order to obtain (34) for $L=0$ (which is efficient for $j_2=0$) we use Hölder's inequality to estimate (35) by

$$\iint|\widehat{\Phi}_{j_1}(\eta^1)\widehat{\Phi}_{j_2}(\eta^2)|\sum_{Q\in\mathfrak{Q}(j_2)}\Big\|\int_{-1/2}^{1/2}\prod_{i=1,2}^{*}\Big[a_i(D,s)\mathrm{Mod}_{-\eta^i}[\chi_{Q^*}f_i]\Big]\Big\|_{q'}\|w\chi_Q\|_q d\eta^1 d\eta^2\,. \tag{36}$$

Now

$$\begin{aligned}&\Big\|\int_{-1/2}^{1/2}\prod_{i=1,2}^{*}\Big[a_i(D,s)\mathrm{Mod}_{-\eta^i}[\chi_{Q^*}f_i]\Big]\,ds\Big\|_{q'}\\ \le\;&\int_{-1/2}^{1/2}\Big\|\prod_{i=1,2}^{*}\Big[a_i(D,s)\mathrm{Mod}_{-\eta^i}[\chi_{Q^*}f_i]\Big]\Big\|_{q'}\,ds\\ =\;&\int_{-1/2}^{1/2}\Big\|\prod_{i=1,2}a_i(D,s)\mathrm{Mod}_{-\eta^i}[\chi_{Q^*}f_i]\Big\|_{q'}\,ds\,.\end{aligned}$$

By our assumption on q we have $q'=p/2$ for some $p>2+4/d$ and therefore we can use Lemma 3.1 to bound the last displayed expression by

$$\delta_\circ^2\prod_{i=1,2}\|f_i\chi_{Q^*}\|_2\,.$$

Since the L^1 norms of $\widehat{\Phi}_{j_1}$ are $O(1)$, uniformly in $\delta_\circ$, we have

$$\begin{aligned}(35)\;&\lesssim\delta_\circ^2\sum_{Q\in\mathfrak{Q}(j_2)}\|f_1\chi_{Q^*}\|_2\|f_2\chi_{Q^*}\|_2\|w\chi_Q\|_q\\ &\lesssim\delta_\circ^2\prod_{i=1,2}\Big(\sum_{Q\in\mathfrak{Q}(j_2)}\|f_i\chi_{Q^*}\|_2^2\|w\chi_Q\|_q\Big)^{1/2}\\ &\lesssim\delta_\circ^2\prod_{i=1,2}\Big(\int_{Q^*}|f_i(x)|^2\Big(\int_{|y|\le C2^{j_2}\delta_\circ^{-1}}|w(x+y)|^q dy\Big)^{1/q}dx\Big)^{1/2}\end{aligned}$$

which yields (34) for $L=0$.

We now turn to the case $j_2>0$ where we need to improve the above estimate by a factor of $C(L)2^{-j_2L}$ for $L\le d+2$. We expand the convolution $\int a_2(\xi-\eta,s)\widehat{\Phi}_{j_2}(\eta)d\eta$ by a Taylor expansion about $\eta=0$. Since Φ_{j_2} vanishes in a neighborhood of the origin the integrals

$\int \widehat{\Phi}_{j_2}(\eta)P(\eta)d\eta$ are zero for any polynomial P. Thus only the integral remainder term in the Taylor expansion survives and we obtain

$$a_2(\,\cdot\,,s) * \widehat{\Phi}_{j_2}(\xi) = \int_0^1 \frac{(1-\sigma)^{L-1}}{(L-1)!} \int \widehat{\Phi}_{j_2}(\eta) \langle -\eta, \nabla_\xi\rangle^L a_2(\xi-\sigma\eta,s) d\eta\, d\sigma\,.$$

We repeat the above argument in which we now have to bound

$$\int_{\eta^1}\int_{\eta^2}\int_0^1 \big|\widehat{\Phi}_{j_1}(\eta^1)\,d\eta^1\big|\,\big|\widehat{\Phi}_{j_2}(\eta^2)\,d\eta^2\big| \sum_{Q\in\mathcal{Q}(j_2)} \int_{x\in Q} |w(x)| \quad \times$$
$$\Big|\int_{-1/2}^{1/2} a_1(D-\eta^1,s)[\chi_{Q^*}f_1](x)\overline{\langle -\eta^2,\nabla\rangle^L a_2(D-\sigma\eta^2,s)[\chi_{Q^*}f_2](x)}\,ds\Big|\,dx\,d\sigma\,d\eta^1 d\eta^2$$

in place of (35). In the estimate we may replace $\langle -\eta^2,\nabla\rangle^L$ with $(\eta^2)^\alpha\partial_\xi^\alpha$, for any multiindex α with $|\alpha|=L$. As above we continue with Hölder's inequality, and this time Lemma 3.1 and the differentiability assumptions on a_2 yield for $|\alpha|=L$

$$\int_0^1 \Big\| \int_{-1/2}^{1/2} a_1(D,s)[\chi_{Q^*}\mathrm{Mod}_{-\eta^1}f_1]\overline{\partial_\xi^\alpha a_2(D,s)\mathrm{Mod}_{-\sigma\eta^2}[\chi_{Q^*}f_2]}\,ds\Big\|_{q'} d\sigma \lesssim \delta_\circ^{2-L}\prod_{i=1,2}\|f_i\chi_{Q^*}\|_2\,.$$

The loss of $\delta_\circ^{-L}$ in the previous formula is (more than) mitigated by

$$\iint |\widehat{\Phi}_{j_1}(\eta^1)(\eta^2)^\alpha\widehat{\Phi}_{j_2}(\eta^2)|\,d\eta^1\,d\eta^2 \lesssim 2^{-j_2L}\delta_\circ^L, \qquad |\alpha|=L.$$

Thus the above argument yields (34) for $j_2>0$. □

4. Proof of the weighted inequality

In this section we prove inequality (3) of Theorem 1.3. We mainly focus on a local inequality (with t-interval $[1,2]$) which for later application we formulate for slightly more general multipliers. Instead of S_t^δ we consider operators $\mathcal{S}_t^\delta$ defined by

$$\widehat{\mathcal{S}_t^\delta f}(\xi) = \phi\big(\delta^{-1}(1-\tfrac{|\xi|^2}{t^2})\beta(\xi,t)\big)\widehat{f}(\xi)$$

where ϕ is as in (1), β is a nonvanishing C^∞ function on the set of (ξ,t) with $1/2<t<5/2$ and $1/2\le|\xi|\le 4$. Of course $\beta(\xi,t)=1$ in Theorem 1.3.

Theorem 4.1. *Let $d\ge 2$ and $q\in[1,\frac{d+2}{2})$. Then, for $0<\delta<1/2$,*

$$\int_{\mathbb{R}^d}\int_1^2 |\mathcal{S}_t^\delta f(x)|^2\frac{dt}{t}\,w(x)\,dx \lesssim \delta^{2-d/q}\int_{\mathbb{R}^d}|f(x)|^2\,Ww(x)\,dx \tag{37}$$

with

$$Ww(x) = \sum_{1\le 2^{2j}<\delta^{-1}} 2^{-2j(\frac{d}{q}-1)}M\circ\mathcal{W}^j_{q,\delta,0}w(x) + \delta^{\frac{d}{q}-1}M\circ\mathfrak{M}_{\sqrt{\delta}}w(x).$$

We now show that Theorem 4.1 (with $\beta \equiv 1$) implies assertion (3) of Theorem 1.3. Let $W_k w(x) = W[w(2^{-k}\cdot)](2^k x)$. By (37) and rescaling we see that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^\infty |S_t^\delta f(x)|^2 \frac{dt}{t}\, w(x)\,dx \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d}\int_1^2 \big|S^\delta_{2^k s}\sum_{i=-2}^{2} P_{k+i} f(x)\big|^2 \frac{ds}{s}\, w(x)\,dx \\
&\lesssim \delta^{2-d/q} \sum_{k\in\mathbb{Z}}\sum_{i=-2}^{2} \int_{\mathbb{R}^d}\int_1^2 \big|P_{k+i}f(x)\big|^2 W_k w(x)\,dx\frac{ds}{s} \\
&\lesssim \delta^{2-d/q} \int_{\mathbb{R}^d}\int_1^2 |f(x)|^2 M(\sup_k |W_k w|^s)^{1/s}(x)\,dx
\end{aligned}$$

and the last inequality is a consequence of Coifman's improvement of the Córdoba–Fefferman weighted norm inequality for singular integrals (see for example [17, p. 417]). Now $M(\sup_k |W_k w|^s)^{1/s}(x) \lesssim \mathfrak{W}_{q,\delta} w$, by Minkowski's inequality (*cf.* (16)).

By a duality argument (using the boundedness results of Proposition 2.1 and the Marcinkiewicz interpolation theorem) Theorem 4.1 implies the following sharp L^p results.

Corollary 4.2. *Let $d \geq 2$ and $p \in (\frac{2(d+2)}{d}, \infty)$. Then, for $0 < \delta < 1/2$,*

$$\Big\| \Big(\int_1^2 |\mathcal{S}_t^\delta f(x)|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \lesssim \delta^{1-d(\frac12-\frac1p)} \|f\|_p.$$

Moreover if $\beta \equiv 1$ and S_t^δ is as in (2) then

$$\Big\| \Big(\int_0^\infty |S_t^\delta f(x)|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \lesssim \delta^{1-d(\frac12-\frac1p)} \|f\|_p.$$

The remainder of this section is devoted to the proof of Theorem 4.1.

Preliminary considerations. We begin with a rescaled variant of Proposition 3.2. Such rescaling arguments have been used in [42], [41], [22] and elsewhere. In what follows fix a function $\zeta \in C^\infty(\mathbb{R}^d)$ supported in $\{y : |y| \leq 1/8\}$ and define a convolution operator with homogeneous multiplier by

$$\widehat{Q^{\theta,j} f} = \zeta\big(2^j\big(\tfrac{\xi}{|\xi|} - \theta\big)\big) \tag{38}$$

In order to reduce estimates to Proposition 3.2 by rescaling we will need to localize all multipliers to a narrow sector $\{\xi : |\frac{\xi}{|\xi|} - u| \leq \varepsilon_1\}$ where u is a unit vector and ε_1 is a small constant.

Lemma 4.3. *Given $C > 1$ there are small $\varepsilon_1, \varepsilon_2 \in (0, 1/8)$ depending on q, d and C and the function β so that the following statement holds for $C2^{-j} < \varepsilon_1$ and $2^{2j}\delta < \varepsilon_2$.*

Let ψ be supported in a ball of radius ε_1 contained in $\{\xi : 1/4 \leq |\xi| \leq 4\}$ such that $\|\partial^\alpha \psi\|_\infty \leq 1$ for $|\alpha| \leq d+2$, let $\theta_1, \theta_2 \in S^{d-1}$ be such that $2^{-j-1} \leq |\theta_1 - \theta_2| \leq C2^{-j}$, let $t_0 \in [1,2]$ and J be an interval of length 2^{-2j-2} containing t_0. Then

$$\begin{aligned}
&\Big| \int\int_J \mathcal{S}_t^\delta Q^{\theta_1,j}\psi(D) f_1(x) \overline{\mathcal{S}_t^\delta Q^{\theta_2,j}\psi(D) f_2(x)}\, \frac{dt}{t}\, w(x)\,dx\Big| \\
&\qquad\qquad \lesssim \delta(2^{2j}\delta)^{1-d/q} \prod_{i=1,2} \Big(\int |f_i(x)|^2 \big(\mathcal{K}^{\theta_i,j}_{2^j\delta} * |w|^q(x)\big)^{1/q} dx\Big)^{1/2}.
\end{aligned}$$

Proof. Set $\vartheta = \frac{\theta_1+\theta_2}{|\theta_1+\theta_2|}$ and $e_d = (0,\dots,0,1)$. Take R_ϑ to be a rotation satisfying $R_\vartheta e_d = \vartheta$ and let it act on functions by $\mathcal{R}_\vartheta f(y) = f(R_\vartheta y)$. Let $\mathcal{A}_j f(y) = f(A_j y)$, where $A_j y = (2^j y_1, \dots, 2^j y_{d-1}, -2^{2j} y_d)$. Finally we set $\mathcal{D}_{t_0} f(y) = f(t_0 y)$ and, as before, the modulation Mod_a is defined by $\mathrm{Mod}_a g(x) = g(x) e^{i\langle x,a\rangle}$.

For fixed δ, j and $i = 1, 2$, the multiplier for $S_t^\delta Q^{\theta_i,j}\psi(D)$ is given by

$$m_{\theta_i}(\xi,t) = \psi(\xi)\phi\Big(\delta^{-1}(1-\tfrac{|\xi|^2}{t^2})\beta(\xi,t)\Big)\zeta\Big(2^j(\tfrac{\xi}{|\xi|}-\theta_i)\Big).$$

Let $\Xi_\vartheta(\eta) = t_0 R_\vartheta(e_d + A_{-j}\eta)$ and $t(s) = t_0(1+2^{-2j}s)$. A rescaled multiplier depending on the parameter $s = 2^{2j}(t_0^{-1}t - 1) \in [-1/2, 1/2]$ is defined by

$$\begin{aligned} M_{\theta_i}(\eta,s) &:= m_{\theta_i}(\Xi_\vartheta(\eta), t(s)) \\ &= \phi\Big(\delta^{-1}\big(1-\tfrac{|e_d+A_{-j}\eta|^2}{(1+2^{-2j}s)^2}\big)\beta\big(\Xi_\vartheta(\eta),t(s)\big)\Big)\,\zeta\big(2^j R_\vartheta\big[\tfrac{e_d+A_{-j}\eta}{|e_d+A_{-j}\eta|} - R_\vartheta^{-1}\theta_i\big]\big)\psi\big(\Xi_\vartheta(\eta)\big). \end{aligned}$$

Now compute

$$\begin{aligned} 1 - \frac{|e_d + A_{-j}\eta|^2}{(1+2^{-2j}s)^2} &= 2^{1-2j}\Big(\frac{s(1+2^{-2j-1}s) + \eta_d - |\eta'|^2/2 - \eta_d^2 2^{-2j-1}}{(1+2^{-2j}s)^2}\Big) \\ &= 2^{1-2j}\Big(s + \eta_d - \frac{|\eta'|^2}{2}\Big) + 2^{-4j} r_j(\eta,s), \end{aligned} \tag{39}$$

where r_j is a quadratic polynomial in η with coefficients uniformly bounded in s, j. Moreover the supports of the functions $\zeta\big(2^j R_\vartheta\big[\frac{e_d+A_{-j}\eta}{|e_d+A_{-j}\eta|} - R_\vartheta^{-1}\theta_i\big]\big)$, $i = 1, 2$, are uniformly separated and these functions have derivatives with bounds uniform in j.

Now let $b = 10dC$, and set $M = 1 + \sum_{|\alpha|\le 2} \max_{|\xi|\le b} \max_{|s|\le 1} |\partial_\eta^\alpha r_j(\eta,s)|$, for the error terms r_j in (39). Let ε be as in Proposition 3.2, and choose ε_1 small compared to $(2d)^{-1}(\varepsilon/M)^{1/2}$. By the assumed separation property we have

$$\mathcal{S}_t^\delta Q^{\theta_1,j}\psi(D) f_1 \, \overline{\mathcal{S}_t^\delta Q^{\theta_2,j}\psi(D) f_2} \equiv 0, \quad 2^{-j} \ge d\varepsilon_1\,.$$

For the relevant complementary range we have $2^{-2j}M < \varepsilon$ so that the functions $(\eta', s) \mapsto |\eta'|^2/2 - s + 2^{-2j} r_j(\eta', s)$ belong to $\mathrm{Ell}(b, \varepsilon, N_\circ)$. Since β is smooth and satisfies an inequality $2\varepsilon_2 \le |\beta(\xi,t)| \le (2\varepsilon_2)^{-1}$ for $1/4 \le |\xi| \le 4$, the formula (39) allows us to apply Proposition 3.2 with $a_i(D,s) = M_{\theta_i}(D,s)(1+2^{-2j}s)^{-1/2}$ and $\delta_\circ = (2\varepsilon_2)^{-1}\delta 2^{2j} < 1/2$.

We obtain

$$\begin{aligned} &\Big|\int_{\mathbb{R}^d}\int_{-1/2}^{1/2} \prod_{i=1,2}^{*} [M_{\theta_i}(D,s) f_i(x)] w(x)\,dx\,\frac{ds}{1+2^{-2j}s}\Big| \\ &\qquad\qquad \lesssim (2^{2j}\delta)^2 \prod_{i=1,2}\Big(\int |f_i(x)|^2 (2^{2j}\delta)^{-d/q} \mathcal{V}_{2^{2j}\delta} w(x)\,dx\Big)^{1/2}, \end{aligned}$$

where

$$\mathcal{V}_{\delta_\circ} w(x) = \Big(\int \frac{\delta_\circ^d |w(x-y)|^q}{(1+\delta_\circ|y|)^{(d+1)q}} dy\Big)^{1/q}.$$

Now, with $t = t_0(1+2^{-2j}s)$, we have $m_{\theta_i}(\xi,t) = M_{\theta_i}(A_j(R_\vartheta^{-1}t_0^{-1}\xi - e_d), s)$, so that

$$\mathcal{S}_t^\delta Q^{\theta_i,j}\psi(D) f_i(x) = \mathcal{D}_{t_0}\mathcal{R}_\vartheta^{-1}\mathrm{Mod}_{e_d}\mathcal{A}_{-j} M_{\theta_i}(D,s) g_i(x) \;\text{ with }\; g_i = \mathcal{A}_j \mathrm{Mod}_{-e_d}\mathcal{R}_\vartheta \mathcal{D}_{t_0}^{-1} f_i.$$

This leads to

$$\begin{aligned}
&\Big|\int\int_{I_{j,T}} \prod_{i=1,2}^{*} [\mathcal{S}_t^\delta Q^{\theta_i,j}\psi(D) f_i(x)] \frac{dt}{t}\, w(x)\,dx\Big| \\
&= 2^{-2j}\Big|\int\int_{-1/2}^{1/2} \prod_{i=1,2}^{*} [\mathcal{D}_{t_0}\mathcal{R}_\vartheta^{-1}\mathrm{Mod}_{e_d}\mathcal{A}_{-j} M_{\theta_i}(D,s) g_i(x)] \frac{ds}{1+2^{-2j}s}\, w(x)\,dx\Big| \\
&\lesssim 2^{-2j}(2^{2j}\delta)^2 \prod_{i=1,2}\Big(\int |f_i(x)|^2 (2^{2j}\delta)^{-d/q} \mathcal{D}_{t_0}\mathcal{R}_\vartheta^{-1}\mathcal{A}_j[\mathcal{V}_{2^{2j}\delta}(\mathcal{A}_j\mathcal{R}_\vartheta\mathcal{D}_{t_0}^{-1} w)](x)\,dx\Big)^{1/2} \\
&\lesssim 2^{2j(1-d/q)}\delta^{2-d/q} \prod_{i=1,2}\Big(\int |f_i(x)|^2 \big(\mathcal{K}^{\vartheta,j}_{2^j t_0\delta} * |w|^q(x)\big)^{1/q} dx\Big)^{1/2}.
\end{aligned}$$

The assertion now follows from $\mathcal{K}^{\vartheta,j}_{2^j t_0\delta}(x) \approx \mathcal{K}^{\theta_i,j}_{2^j\delta}(x)$ since $t_0 \in [1,2]$ and $|\vartheta - \theta_i| \lesssim 2^{-j}$ for $i = 1, 2$. □

A version of the following lemma is originally due to Carleson (unpublished); slightly different forms can be found in [10], [29] and [33]. For the sake of completeness we include the proof.

Lemma 4.4. *Let A be an invertible linear transformation and A^t its transpose. Suppose that $\{m_k\}_{k\in\mathbb{N}}$ have disjoint supports. Then for $s \geq 0$ and almost every $x \in \mathbb{R}^d$*

$$\Big(\sum_k \big|\mathcal{F}^{-1}[m_k(A^t\,\cdot)\widehat{f}](x)\big|^2\Big)^{1/2} \leq C \sup_k \|m_k\|_{L^2_s}\Big(\int \frac{\det(A^{-1})}{(1+|A^{-1}y|^2)^s}|f(x-y)|^2 dy\Big)^{1/2}.$$

Proof. We can assume that $m_k = 0$ for all but finitely many k. $\mathcal{F}^{-1}[m_k(A^t\,\cdot)\widehat{f}](x) = \mathcal{F}^{-1}[m_k\widehat{f(A\,\cdot)}](A^{-1}x)$ we may use a change of variable to reduce to the case where A is the identity transformation. Also, using an analytic interpolation argument, we may assume that $s \in \mathbb{N}\cup\{0\}$.

Let $g_k = \mathcal{F}^{-1}[m_k]$. Then $\sum_k \big|\mathcal{F}^{-1}[m_k\widehat{f}](x)\big|^2 = \sup_{\|a\|_{\ell^2(\mathbb{Z}^d)}\leq 1} |\sum_k a_k\, g_k * f(x)|^2$. Now, for each fixed $a \in \ell^2(\mathbb{Z}^d)$, we apply the Schwarz inequality in the convolution integral and then Plancherel's theorem to obtain

$$\Big|\sum_k a_k\, g_k * f(x)\Big| \lesssim \Big\|\sum_k a_k\, m_k\Big\|_{L^2_s}\Big(\int \frac{|f(x-y)|^2}{(1+|y|^2)^s} dy\Big)^{1/2}.$$

Thus, we are done as for $s \in \mathbb{N}\cup\{0\}$,

$$\Big\|\sum_k a_k\, m_k\Big\|^2_{L^2_s} \lesssim \sum_{|\alpha|\leq s}\Big\|\sum_k a_k D^\alpha m_k\Big\|^2_{L^2} \lesssim \|a\|^2_{\ell^2}\sup_k \|m_k\|^2_{L^2_s},$$

by the disjointness of the supports. □

Some reductions. We remark that it suffices to prove Theorem 4.1 only for very small values of δ, as by straightforward estimation

$$\int_{\mathbb{R}^d}\int_1^2 |\mathcal{S}_t^\delta f(x)|^2 \frac{dt}{t}\, w(x)\,dx \lesssim \delta^{-C}\int_{\mathbb{R}^d} |f(x)|^2\, Mw(x)\,dx$$

for a suitable power $C > 0$, and clearly $Mw \lesssim \mathfrak{M}_{\sqrt{\delta}} w$. In particular we may assume that δ is small compared to the constant ε_2 in Lemma 4.3.

We may replace $\mathcal{S}_t^\delta$ by

$$T_t^\delta = \psi(D)\mathcal{S}_t^\delta \tag{40}$$

where ψ is as in Lemma 4.3 (smooth and supported in a ball of radius ε_1). In view of the invariance properties of the weight operators one can use a partition of unity to deduce the weighted inequality (37) from the corresponding result for T_t^δ.

We now prepare for an application of Lemma 4.3 and decompose on the frequency side the product $F_1\overline{F_2}$ for suitable F_i (initially $F_i = T_t^\delta f_i$). We let $\chi_\circ$ to be a radial C_c^∞ function $\chi_\circ(\omega) = 1$ for $|\omega| \le 2^5$ and so that $\operatorname{supp}\chi_\circ$ is contained in $\{\omega : |\omega| \le 2^5 + 1\}$, moreover set,

$$\chi_1(\omega) = \chi_\circ(\omega) - \chi_\circ(2\omega).$$

We also let $j_\circ = j_\circ(\delta)$ denote the integer with

$$\sqrt{\delta/\varepsilon_2} \le 2^{-j_\circ} < 2\sqrt{\delta/\varepsilon_2}$$

where $\varepsilon_2 \in (0,1)$ is as in Lemma 4.3.

Define bilinear forms for pairs of Schwartz functions by

$$\begin{aligned}\mathfrak{B}_\circ[F_1,F_2](x) &= \frac{1}{(2\pi)^{2d}}\iint \chi_\circ(2^{j_\circ}(\xi-\eta))\widehat{F}_1(\xi)\widehat{F}_2(-\eta)e^{\imath\langle\xi-\eta,x\rangle}d\xi d\eta,\\ \mathcal{B}^j[F_1,F_2](x) &= \frac{1}{(2\pi)^{2d}}\iint \chi_1(2^{j}(\xi-\eta))\widehat{F}_1(\xi)\widehat{F}_2(-\eta)e^{\imath\langle\xi-\eta,x\rangle}d\xi d\eta.\end{aligned} \tag{41}$$

Then one easily verifies the decomposition

$$F_1\overline{F_2} = \mathfrak{B}_\circ[F_1,\overline{F_2}] + \sum_{j<j_\circ}\mathcal{B}^j[F_1,\overline{F_2}].$$

Later, in cases where the supports of the Fourier transforms of F_1 and F_2 are separated we wish to dispense with the frequency cutoff $\Phi_j(\xi-\eta)$ and replace $\mathfrak{B}_\circ$ or $\mathcal{B}_j$ by a product. This will be accomplished by using the identities

$$\begin{aligned}(2\pi)^d\mathfrak{B}_\circ[F_1,F_2](x) &= 2^{-j_0 d}\widehat{\chi}_\circ(2^{-j_0}\cdot) * [F_1F_2](x),\\ (2\pi)^d\mathcal{B}^j[F_1,F_2](x) &= 2^{-jd}\widehat{\chi}_1(2^{-j}\cdot) * [F_1F_2](x)\end{aligned} \tag{42}$$

which follow from the Fourier inversion formula (and the assumption that $\chi_\circ$ is radial).

The desired weighted norm inequality (37) follows from the following two propositions, applied for $f_1 = f_2 = f$. The proof of the first one is rather straightforward.

Proposition 4.5.

$$\Big|\int\!\!\int_1^2 \mathfrak{B}_\circ[T_t^\delta f_1, \overline{T_t^\delta f_2}](x)\frac{dt}{t}\,w(x)\,dx\Big| \lesssim \delta \prod_{i=1,2}\Big(\int |f_i(x)|^2 \int \frac{\delta^{d/2}\mathfrak{M}_{\sqrt{\delta}}w(x-y)}{(1+\delta^{1/2}|y|)^{d+1}}dy\,dx\Big)^{1/2}.$$

More substantial (and relying on §3) is

Proposition 4.6. *Let $q \in [1, \frac{d+2}{2})$. Then, for $j < j_\circ$,*

$$\begin{aligned}&\Big|\int\!\!\int_1^2 \mathcal{B}^j[T_t^\delta f_1, \overline{T_t^\delta f_2}](x)\frac{dt}{t}\,w(x)\,dx\Big|\\ &\qquad\lesssim \delta^{2-\frac{d}{q}}2^{-2j(\frac{d}{q}-1)}\prod_{i=1,2}\Big(\max_{|\nu|\le 8}\int |f_i(x)|^2 H_j * \mathcal{W}^{j-\nu}_{q,\delta,0}w(x)\,dx\Big)^{1/2},\end{aligned}$$

where $H_j(x) = 2^{-jd}(1+2^{-j}|x|)^{-d-1}$.

We now introduce some basic decompositions. As in the definition (38) let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be supported in $\{y : |y| \le 1/8\}$, now with the additional assumption that $\zeta(y) = 1$ for $|y| \le 1/9$. Let $\widetilde{\zeta} \in C_c^\infty(\mathbb{R}^d)$ be supported in $\{y : |y| \le 1/7\}$ so that $\widetilde{\zeta}(y) = 1$ for $|y| \le 1/8$; hence $\widetilde{\zeta}\zeta = \zeta$. Let φ be a smooth function supported in $[-9/8, 9/8]$ and equal to $1/2$ on $[-7/8, 7/8]$ such that

$$\sum_{n\in\mathbb{Z}} \varphi(\cdot - n) = 1.$$

We let Θ_j be a maximal 2^{-j-d}-separated set of S^{d-1} and define for $n \in \mathbb{Z}$, $l \in \mathbb{Z}$, operators via the Fourier transform by

$$\begin{aligned}\mathcal{F}[Q_n^{\theta,j} f](\xi) &= \frac{\zeta(2^j(\frac{\xi}{|\xi|} - \theta))}{\sum_{\theta'\in\Theta_j} \zeta^2(2^j(\frac{\xi}{|\xi|} - \theta'))}\varphi(2^j|\xi| - n)\widehat{f}(\xi),\\ \mathcal{F}[P_l^{\theta,j} f](\xi) &= \widetilde{\zeta}(2^j(\tfrac{\xi}{|\xi|} - \theta))\varphi(2^{2j}|\xi| - l)\widehat{f}(\xi);\end{aligned}$$

moreover, with $Q^{\theta,j}$ as in (38) set

$$\mathcal{Q}^{\theta,j} = \sum_n Q^{\theta,j} Q_n^{\theta,j}$$

so that

$$T_t^\delta f \;=\; \sum_{\theta\in\Theta_j} T_t^\delta \mathcal{Q}^{\theta,j} f \;=\; \sum_{\theta\in\Theta_j}\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} T_t^\delta Q^{\theta,j} Q_n^{\theta,j} P_l^{\theta,j} f,$$

for both cases $j < j_\circ$ and $j = j_\circ$.

Note that the multipliers for $Q^{\theta,j}$ and $\mathcal{Q}^{\theta,j}$ are contained in a sector of width $c2^{-j}$ around θ. The multiplier for $Q_n^{\theta,j}$ is contained in a $c2^{-j}$ ball centered around Ξ with $|\Xi| = 2^{-j}n + O(2^{-j})$. The multiplier for $P_l^{\theta,j}$ is contained in a plate with $(d-1)$ sides of length $O(2^{-j})$ and a short side of length $O(2^{-2j})$, the long sides being perpendicular to θ . Note that for $2^{-2j}m \in [1, 2]$

$$\begin{aligned} &T_t^\delta Q^{\theta,j} Q_n^{\theta,j} P_l^{\theta,j} \neq 0 \text{ for some } t \in [(m-1)2^{-2j}, (m+1)2^{-2j}],\\ &\qquad \Longrightarrow\ |m - l| \lesssim 1, \quad |2^{-2j}l - 2^{-j}n| \lesssim 2^{-j}\,. \end{aligned} \tag{43}$$

We also notice that, from the localization and separation properties of the cutoff functions $\chi_\circ(2^{j_\circ}(\xi - \eta))$ and $\chi_1(2^j(\xi - \eta))$ in (41),

$$\begin{aligned} &\mathfrak{B}_\circ(T_t^\delta Q^{\theta,j} g, \overline{T_t^\delta Q^{\theta',j} g}) \neq 0 \ \Longrightarrow\ \operatorname{dist}(\theta, \theta') \le C2^{-j_\circ}\,,\\ &\mathcal{B}_j(T_t^\delta Q^{\theta,j} g, \overline{T_t^\delta Q^{\theta',j} g}) \neq 0 \ \Longrightarrow\ 2^{-j+3} \le \operatorname{dist}(\theta, \theta') \le 2^{-j+6}\,. \end{aligned} \tag{44}$$

Proof of Proposition 4.5. We first observe by a straightforward integration by parts that for $t \in [1, 2]$ the convolution kernel associated to $T_t^\delta Q^{\theta,j_\circ}$ is dominated by

$$C_d \frac{\delta^{\frac{d+1}{2}}}{(1 + \delta|\langle x, \theta\rangle| + \delta^{1/2}|x - \langle x, \theta\rangle\theta|)^N} \lesssim \mathcal{K}^{\theta,j_\circ}_{2^{-j_\circ}}(x) \approx \mathcal{K}^{\theta,j_\circ}_{2^{j_\circ}\delta}(x).$$

On the right hand side we may also replace θ by any $\widetilde{\theta}$ with $|\theta - \widetilde{\theta}| \lesssim 2^{-j_\circ}$. For more compact notation we write

$$\mathcal{K}^{\theta,j_\circ} := \mathcal{K}^{\theta,j_\circ}_{2^{-j_\circ}}(x).$$

Now for $|\theta_1-\theta_2|\lesssim 2^{-j_\circ}$ and each $t\in[1,2]$, we have

$$\begin{aligned}(45)\qquad &\Big|\int \prod_{i=1,2}{}^{*}[T_t^\delta Q^{\theta_i,j_\circ}g_i(x)]w(x)\,dx\Big| \lesssim \int\Big[\prod_{i=1,2}(\mathcal{K}^{\theta_i,j_\circ}*|g_i|^2(x))^{1/2}\Big]|w(x)|dx\\ &\lesssim \prod_{i=1,2}\Big(\int \mathcal{K}^{\theta_i,j_\circ}*|g_i|^2(x)|w(x)|dx\Big)^{1/2}\lesssim \prod_{i=1,2}\Big(\int |g_i(y)|^2\,\mathcal{K}^{\theta_i,j_\circ}*|w|(y)\,dy\Big)^{1/2};\end{aligned}$$

here we used $\|\mathcal{K}^{\theta,j_\circ}\|_1=O(1)$ and applied the Schwarz inequality twice.

By Lemma 4.4

$$(46)\qquad \sum_l |P_l^{\theta,j_\circ}f(x)|^2\lesssim \mathcal{K}^{\theta_i,j_\circ}*|f|^2(x).$$

Now let $I_m^{j_\circ}=[2^{-2j_\circ}m,2^{-2j_\circ}(m+1)]$. It will be implicit in all m-summations that $I_m^{j_\circ}\subset[1,2]$. In what follows $\mathfrak{A}_{j_\circ}^m$ will be an index set consisting of (l_1,l_2,n_1,n_2) with $|l_i-m|\lesssim 1$, $|2^{-2j_\circ}l_i-2^{-j_\circ}n_i|\lesssim 2^{-j_\circ}$ for $i=1,2$. Then

$$\begin{aligned}&\Big|\sum_{\substack{\theta_1,\theta_2\in\Theta_j\\ |\theta_1-\theta_2|\le C2^{-j_\circ}}}\int\!\!\int_1^2\prod_{i=1,2}{}^{*}[T_t^\delta\mathcal{Q}^{\theta_i,j_\circ}f_i(x)]\frac{dt}{t}w(x)\,dx\Big|\\ &=\Big|\sum_{\substack{\theta_1,\theta_2\in\Theta_j\\ |\theta_1-\theta_2|\le C2^{-j_\circ}}}\sum_m\int_{I_m^j}\sum_{\substack{(l_1,l_2,n_1,n_2)\\ \in\mathfrak{A}_{j_\circ}^m}}\int\prod_{i=1,2}{}^{*}[T_t^\delta Q^{\theta_i,j_\circ},P_{l_i}^{\theta_i,j_\circ}Q_{n_i}^{\theta_i,j_\circ}f_i(x)]w(x)\,dx\frac{dt}{t}\Big|\\ &\lesssim 2^{-2j_\circ}\sum_{\substack{\theta_1,\theta_2\in\Theta_{j_\circ}\\ |\theta_1-\theta_2|\le C2^{-j_\circ}}}\sum_m\sum_{\substack{(l_1,l_2,n_1,n_2)\\ \in\mathfrak{A}_{j_\circ}^m}}\prod_{i=1,2}\Big(\int|P_{l_i}^{\theta_i,j_\circ}Q_{n_i}^{\theta_i,j_\circ}f_i(x)|^2\,\mathcal{K}^{\theta_i,j_\circ}*|w|(x)\,dy\Big)^{1/2};\end{aligned}$$

here we have applied (45) and carried out the t integration. We now notice that $|n_1-n_2|\lesssim 1$, $|l_1-l_2|\lesssim 1$ for $(l_1,l_2,n_1,n_2)\in\mathfrak{A}_{j_\circ}^m$ and that for fixed (l_1,l_2,n_1,n_2) there are only $O(1)$ integers m for which $(l_1,l_2,n_1,n_2)\in\mathfrak{A}_{j_\circ}^m$. Hence, by various applications of the Schwarz inequality and then by (46) the last displayed quantity is controlled by

$$\begin{aligned}&2^{-2j_\circ}\prod_{i=1,2}\Big(\sum_{\theta_i\in\Theta_{j_\circ}}\sum_{l_i}\sum_{n_i}\int\big|P_{l_i}^{\theta_i,j_\circ}Q_{n_i}^{\theta_i,j_\circ}f_i(x)\big|^2\,\mathcal{K}^{\theta_i,j_\circ}*|w|(x)\,dx\Big)^{1/2}\\ (47)\qquad &\lesssim 2^{-2j_\circ}\prod_{i=1,2}\Big(\sum_{\theta_i\in\Theta_{j_\circ}}\sum_{n_i}\int|Q_{n_i}^{\theta_i,j_\circ}f_i(x)|^2\,\mathcal{K}^{\theta_i,j_\circ}*\mathcal{K}^{\theta_i,j_\circ}*|w|(x)\,dy\Big)^{1/2}\end{aligned}$$

where, by Lemma 2.2, we may replace $\mathcal{K}^{\theta_i,j_\circ}*\mathcal{K}^{\theta_i,j_\circ}$ with $\mathcal{K}^{\theta_i,j_\circ}$.

By Lemma 4.4 we have, with $H_{j_\circ}(x)=2^{-j_\circ d}(1+2^{-j_\circ}|x|)^{-d-1}$,

$$\sum_{\theta\in\Theta_{j_\circ}}\sum_n|Q_n^{\theta,j_\circ}f|^2\lesssim H_{j_\circ}*|f|^2(x)$$

so that we may estimate the term (47) by

$$(48)\qquad 2^{-2j_\circ}\prod_{i=1,2}\Big(\int|f_i(x)|^2\,H_{j_\circ}*\sup_\theta[\mathcal{K}^{\theta,j_\circ}*|w|](x)\,dx\Big)^{1/2}.$$

Now by the definition of $\mathfrak{B}_\circ$, (42) and translation invariance

$$\begin{aligned}\mathfrak{B}_\circ[T_t^\delta f_1, \overline{T_t^\delta f_2}](x) &= \sum_{\substack{\theta_1,\theta_2\in\Theta_{j_\circ}\\ |\theta_1-\theta_2|\le C2^{-j_\circ}}} \mathfrak{B}_\circ[T_t^\delta \mathcal{Q}^{\theta_1,j_\circ} f_1, \overline{T_t^\delta \mathcal{Q}^{\theta_2,j_\circ} f_2}](x)\\ &= \sum_{\substack{\theta_1,\theta_2\in\Theta_{j_\circ}\\ |\theta_1-\theta_2|\le C2^{-j_\circ}}} \frac{1}{(2\pi)^d}\int 2^{-j_\circ d}\widehat{\chi}_\circ(2^{-j_\circ}h)\prod_{i=1,2}{}^*[T_t^\delta \mathcal{Q}^{\theta_i,j_\circ}\tau_h f_i](x)\,dh\end{aligned}$$

where $\tau_h f(x) = f(x-h)$. We combine this identity with the previous estimate and the obvious inequality $|2^{-j_\circ d}\widehat{\chi}_\circ(2^{-j_\circ}\cdot)| \lesssim H_{j_\circ}$ to obtain

$$\begin{aligned}&\Big|\int\int_1^2 \mathfrak{B}_\circ[T_t^\delta f_1, \overline{T_t^\delta f_2}](x)\frac{dt}{t}\,w(x)\,dx\Big|\\ &\lesssim \int H_{j_\circ}(h)\Big|\int\int_1^2 \sum_{\substack{\theta_1,\theta_2\in\Theta_{j_\circ}\\ |\theta_1-\theta_2|\le C2^{-j_\circ}}} \prod_{i=1,2}{}^*[T_t^\delta \mathcal{Q}^{\theta_i,j_\circ}\tau_h f_i](x)\frac{dt}{t}\,w(x)\,dx\Big|\,dh\\ &\lesssim \int H_{j_\circ}(h)2^{-2j_\circ}\prod_{i=1,2}\Big(\int |f_i(x-h)|^2\, H_{j_\circ}*\sup_\theta[\mathcal{K}^{\theta,j_\circ}*|w|](x)\,dx\Big)^{1/2}dh\,.\end{aligned}$$

Clearly $\sup_\theta[\mathcal{K}^{\theta,j_\circ}*|w|] \lesssim \mathfrak{M}_{\sqrt{\delta}}[w]$. By the Schwarz inequality and a subsequent change of variable the last displayed quantity is bounded by

$$C\delta\prod_{i=1,2}\Big(\sum_{\theta_i\in\Theta_{j_\circ}}\int |f_i(x)|^2\, H_{j_\circ}*H_{j_\circ}*\mathfrak{M}_{\sqrt{\delta}}[w](x)\,dx\Big)^{1/2}.$$

Since $H_{j_\circ}*H_{j_\circ}(x) \lesssim \delta^{d/2}(1+\delta^{1/2}|x|)^{-d-1}$, by Lemma 2.2, this concludes the proof of the proposition. □

Proof of Proposition 4.6. We fix $j < j_\circ$ so that $2^{2j}\delta < 1/2$. We define that $\theta\sim\theta'$ if $\theta\in\Theta_j$, $\theta'\in\Theta_j$ and $2^{-j+3}\le|\theta-\theta'|\le 2^{-j+6}$; this is the relevant range in (44).

Below we shall prove the estimate

$$\begin{aligned}(49)\quad &\Big|\int\int_1^2 \prod_{i=1,2}{}^*[T_t^\delta \mathcal{Q}^{\theta_i,j} f_i(x)]\frac{dt}{t}\,w(x)\,dx\Big|\\ &\qquad\lesssim \delta^{2-\frac{d}{q}}2^{-2j(\frac{d}{q}-1)}\prod_{i=1,2}\Big(\sum_n\int |Q_n^{\theta,j}f_i(x)|^2(\mathcal{K}_{2^j\delta}^{\theta_i,j}*|w|^q(x))^{1/q}dx\Big)^{1/2}\end{aligned}$$

for any θ_1,θ_2 with $\theta_1\sim\theta_2$. We first show why (49) implies the asserted estimate.

It is crucial to observe that for $\theta_1\sim\theta_2$ and any g_1,g_2 the Fourier transform of the product $T_t^\delta Q^{\theta_1,j}g_1\,\overline{T_t^\delta Q^{\theta_2,j}g_2}$ is supported in

$$\{\xi : 2^{-j+2}\le|\xi|\le 2^{-j+6},\ |\langle\xi,\vartheta\rangle|\le 2^{-2j+12}\}\,;$$

here $\vartheta = \frac{\theta_1+\theta_2}{|\theta_1+\theta_2|}$. Let $\eta_0\in C_c^\infty(\mathbb{R})$ be even and supported in $[-2^{13},2^{13}]$ so that $\eta_0(s)=1$ for $|s|\le 2^{12}$ and let $\eta_1\in C_c^\infty(\mathbb{R})$ be supported in $[2,128]$ so that $\eta_1(s)=1$ for $s\in[4,64]$. Consider the even Schwartz function $\Psi\equiv\Psi^{j,\theta_1,\theta_2}$ defined by

$$\widehat{\Psi^{j,\theta_1,\theta_2}}(\xi) = \eta_0(2^{2j}\langle\xi,\vartheta\rangle)\eta_1(2^j|\xi|).$$

Note that there is a constant C (independent of j, θ_1, θ_2) so that

$$C^{-1}\Psi^{j,\theta_1,\theta_2} \in \mathcal{S}^{\theta_i,j}_{j-\nu}$$

for $\nu = 0, \dots, 8$, $i = 1, 2$.

From these considerations it follows that

$$\int \prod_{i=1,2}{}^{*} [T^\delta_t \mathcal{Q}^{\theta_i,j} f_i](x) w(x)\, dx = \sum_{|\nu|\le 8} \int \prod_{i=1,2}{}^{*} [T^\delta_t \mathcal{Q}^{\theta_i,j} f_i](x) \Psi^{j,\theta_1,\theta_2} * P_{\nu-j} w(x)\, dx$$

and therefore, for any θ_1, θ_2 with $\theta_1 \sim \theta_2$, (49) can be changed to

$$\Big| \int \int_1^2 \prod_{i=1,2}{}^{*} [T^\delta_t \mathcal{Q}^{\theta_i,j}](x) \frac{dt}{t}\, w(x)\, dx \Big| \lesssim \prod_{i=1,2} \Big(\sum_n \int |Q^{\theta_i,j}_n f_i(x)|^2 U^{\theta_i,j}_{q,\delta} w(x)\, dx \Big)^{1/2}, \tag{50}$$

where

$$U^{\theta,j}_{q,\delta} w(x) = \delta^{2-\frac{d}{q}} 2^{-2j(\frac{d}{q}-1)} \sum_{|\nu|\le 8} \big(\mathcal{K}^{\theta,j}_{2^j\delta} * \sup_{\Psi \in \mathcal{S}^{\theta,j}_{\nu-j}} |\Psi * P_{\nu-j} w|^q(x) \big)^{1/q}.$$

By the definition of $\mathcal{B}_j$ we have

$$\begin{aligned} \mathcal{B}^j[T^\delta_t f_1, \overline{T^\delta_t f_2}](x) &= \sum_{\theta_1\sim\theta_2} \mathcal{B}^j[T^\delta_t \mathcal{Q}^{\theta_1,j} f_1, \overline{T^\delta_t \mathcal{Q}^{\theta_2,j} f_2}](x) \\ &= \sum_{\theta_1\sim\theta_2} (2\pi)^{-d} \int_h 2^{-jd} \widehat{\chi}_1(2^{-j}h) \prod_{i=1,2}{}^{*} [T^\delta_t \mathcal{Q}^{\theta_i,j} \tau_h f_i](x) dh\,. \end{aligned}$$

Therefore (50) yields

$$\begin{aligned} &\Big| \int \int_1^2 \mathcal{B}^j[T^\delta_t f_1, \overline{T^\delta_t f_2}](x) \frac{dt}{t}\, w(x)\, dx \Big| \\ &\lesssim \int H_j(h) \prod_{i=1,2} \Big(\sum_{\theta_i\in\Theta_j} \sum_n \int |Q^{\theta,j}_n f_i(x-h)|^2 U^{\theta_i,j}_{q,\delta} w(x)\, dx \Big)^{1/2} \\ &\lesssim \prod_{i=1,2} \Big(\sum_{\theta_i\in\Theta_j} \sum_n \int |Q^{\theta,j}_n f_i(x)|^2 H_j * [\sup_\theta U^{\theta,j}_{q,\delta} w](x)\, dx \Big)^{1/2}, \end{aligned}$$

by the Schwarz inequality. Now, for $|\nu| \le 8$, observe $\mathcal{K}^{\theta,j}_{2^j\delta} \approx \mathcal{K}^{\theta,j-\nu}_{2^{j-\nu}\delta}$ and therefore $\sup_\theta U^{\theta,j}_{q,\delta} w(x) \lesssim \delta^{2-\frac{d}{q}} 2^{-2j(\frac{d}{q}-1)} \mathcal{W}^{j-\nu}_{q,\delta,0} w(x)$. Moreover, by Lemma 4.4,

$$\sum_{\theta_i\in\Theta_j} \sum_n |Q^{\theta,j}_n f(x)|^2 \lesssim H_j * |f|^2(x)\,.$$

Hence, using the Schwarz inequality again, we get

$$\begin{aligned} &\Big| \int \int_1^2 \mathcal{B}^j[T^\delta_t f_1, \overline{T^\delta_t f_2}](x) \frac{dt}{t}\, w(x)\, dx \Big| \\ &\qquad \lesssim \delta^{2-\frac{d}{q}} 2^{-2j(\frac{d}{q}-1)} \max_{|\nu|\le 8} \prod_{i=1,2} \Big(\int |f_i(x)|^2 H_j * H_j * \mathcal{W}^{j-\nu}_{q,\delta,0} w(x)\, dx \Big)^{1/2} \end{aligned}$$

and since $H_j * H_j \lesssim H_j$ we obtain the asserted estimate.

Proof of (49). We argue as in the proof of Proposition 4.5 and rely on Lemma 4.3. We let $I^j_m = [2^{-2j}m, 2^{-2j}(m+1)$ if this interval is a subset of $[1,2]$ (otherwise $I^j_m = \emptyset$). Define the index sets $\mathfrak{A}^m_j$ as in the proof of Proposition 4.5 (with j instead of $j_\circ$).

Then the right hand side of (49) is equal to

$$\Big|\sum_m \sum_{(l_1,l_2,n_1,n_2)\in\mathfrak{A}^m_j} \int\int_{I^j_m} \prod_{i=1,2}{}^{*} [T^\delta_t Q^{\theta,j} Q^{\theta_i,j}_{n_i} P^{\theta,j}_{l_i} f_i(x)] \frac{dt}{t}\, w(x)\, dx\Big|$$

and, by Lemma 4.3, this is estimated by

$$\sum_m \sum_{\substack{|l_1-m|\lesssim 1\\ |l_2-m|\lesssim 1}} \sum_{|n_1-n_2|\lesssim 1} \delta^{2-\frac{d}{q}} 2^{-2j(\frac{d}{q}-1)} \prod_{i=1,2} \Big(\int \big|P^{\theta_i,j}_{l_i} Q^{\theta_i,j}_{n_i} f_i(x)\big|^2 \big(\mathcal{K}^{\theta_i,j}_{2^j\delta} * |w|^q(x)\big)^{1/q}\, dx\Big)^{1/2}$$

$$\lesssim \delta^{2-\frac{d}{q}} 2^{-2j(\frac{d}{q}-1)} \prod_{i=1,2} \Big(\sum_{n_i}\sum_{l_i} \int \big|P^{\theta_i,j}_{l_i} Q^{\theta_i,j}_{n_i} f_i(x)\big|^2 \big(\mathcal{K}^{\theta_i,j}_{2^j\delta} * |w|^q(x)\big)^{1/q}\, dx\Big)^{1/2}$$

$$\lesssim \delta^{2-\frac{d}{q}} 2^{-2j(\frac{d}{q}-1)} \prod_{i=1,2} \Big(\sum_{n_i} \int \big|Q^{\theta_i,j}_{n} f_i(x)\big|^2 \mathcal{K}^{\theta_i,j}_{2^{-j}} * \big(\mathcal{K}^{\theta_i,j}_{2^j\delta} * |w|^q(x)\big)^{1/q}\, dx\Big)^{1/2};$$

here we have used the Schwarz inequality and

$$\sum_l |P^{\theta,j}_l g(x)|^2 \lesssim \mathcal{K}^{\theta,j}_{2^{-j}} * |g|^2(x)$$

which is a consequence of Lemma 4.4. We also have

$$\mathcal{K}^{\theta,j}_{2^{-j}} * \big(\mathcal{K}^{\theta,j}_{2^j\delta} * |w|^q(x)\big)^{1/q} \lesssim \big(\mathcal{K}^{\theta,j}_{2^{-j}} * \mathcal{K}^{\theta,j}_{2^j\delta} * |w|^q(x)\big)^{1/q} \lesssim \big(\mathcal{K}^{\theta,j}_{2^j\delta} * |w|^q(x)\big)^{1/q};$$

where the first estimate follows from Hölder's inequality and the second from (18) and the assumption $2^j\delta < 2^{-j}$. This completes the proof of (49) and thus the proposition is established. □

5. $L^p(L^2)$ estimates for solutions of Schrödinger and wave equations

Proposition 5.1. *Let $d \geq 2$ and $p \in (\frac{2(d+2)}{d}, \infty]$, or $d = 1$ and $p = \infty$, and let $a \in (0,\infty)$. Let I denote a compact interval of time. Then for $k \geq 1$,*

$$\Big\| \Big(\int_I |U^a_t P_k f|^2\, dt\Big)^{1/2}\Big\|_p \lesssim 2^{ka\lambda(p)} \|f\|_{L^p}, \quad \lambda(p) = d\Big(\frac12 - \frac1p\Big) - \frac12. \tag{51}$$

Proof of Theorem 1.5. The result is an immediate consequence of Proposition 5.1 and the case $q = 2$, $r = 1$ of Proposition A.3. □

Proof of Proposition 5.1. We may assume that 2^k is large. Let ϕ be an even real-valued function so that $\widehat{\phi}(t) > c > 0$ for $t \in I$ and $\operatorname{supp}\phi \subset [-1/4, 1/4]$. It suffices to estimate the $L^p(\mathbb{R}^d, L^2(\mathbb{R}))$ norm of $\widehat{\phi}(t) U^a_t P_k f(x)$. For fixed x the $L^2(\mathbb{R})$ norm of this expression is equal to

$$\frac{1}{(2\pi)^{1/2}} \Big(\int_{\mathbb{R}} \Big| \int \widehat{\phi}(t) \exp(-\imath t\tau) U^a_t P_k f(x)\, dt\Big|^2 d\tau\Big)^{1/2}$$

$$= \frac{1}{(2\pi)^{1/2}} \Big(\int_{2^{(k-3)a}}^{2^{(k+3)a}} \Big| \mathcal{F}^{-1}\big[\phi(|\cdot|^a - \tau)\chi(2^{-k}|\cdot|)\widehat{f}\big](x)\Big|^2 d\tau\Big)^{1/2}.$$

By a finite splitting we may replace the integral over $[2^{(k-3)a}, 2^{(k+3)a}]$ by an integral over $((2^kT)^a, (2^{k+1}T)^a)$ with $T \approx 1$. After changing variables $\tau = (2^kTr)^a$ it suffices to show that

$$\Big\| \Big(\int_1^2 \Big| \mathcal{F}^{-1}\big[\phi(|\cdot|^a - (2^kTr)^a)\chi(2^{-k}|\cdot|)\widehat{f}\big]\Big|^2 dr\Big)^{1/2}\Big\|_p \lesssim 2^{a(\lambda(p)-\frac12)k}\|f\|_p,$$

or, after scaling and setting $\delta = (2^kT)^{-a}$,

$$\Big\| \Big(\int_1^2 \Big| \mathcal{F}^{-1}\big[\phi(\delta^{-1}(|\cdot|^a - r^a))\chi(T|\cdot|)\widehat{f}\big]\Big|^2 dr\Big)^{1/2}\Big\|_p \lesssim \delta^{\frac12-\lambda(p)}\|f\|_p\,. \tag{52}$$

But as ϕ is even, $\phi(\delta^{-1}(|\xi|^a - r^a)) = \phi\big(\delta^{-1}\beta(\xi,r)(1-|\xi|^2/r^2)\big)$ where $\beta(\xi,r) = r^2\frac{r^a-|\xi|^a}{r^2-|\xi|^2}$ is smooth for ξ away from the origin, and nonvanishing. Thus Corollary 4.2 may be applied and we get the L^p inequality (52) for $d \geq 2$ and $p > 2 + 4/d$.

The case $d = 1$, $p = \infty$ is more straightforward; the estimate

$$\Big(\int_1^2 \big| \mathcal{F}^{-1}\big[\phi(\delta^{-1}(|\cdot|^a - r^a))\chi_1(T|\cdot|)\widehat{f}\big](x)\big|^2 dr\Big)^{1/2} \lesssim \delta^{1/2}\|f\|_\infty \tag{53}$$

for $T \approx 1$ follows from

$$\sum_{0\leq n<\delta^{-1}} \big| \mathcal{F}^{-1}\big[\phi(\delta^{-1}(|\cdot|^a - (1+n\delta+\sigma)^a)\chi_1(T|\cdot|)\widehat{f}\big](x)\big|^2 \lesssim \|f\|_\infty^2, \qquad 0 < \sigma \leq \delta,$$

and integration in σ. The last displayed inequality however is a consequence of Lemma 4.4. □

We finish by stating a global variant of the one-dimensional square function estimate which does not use Sobolev spaces and which we will not use elsewhere in the paper.

Proposition 5.2. *Let $d = 1$, $p \in [2,\infty)$, $a \in (0,\infty)$, and let I be a compact interval. Then*

$$\Big\| \Big(\int_I |U_t^a f|^2\, dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

Moreover

$$\Big\| \Big(\int_I |U_t^a f|^2\, dt\Big)^{1/2}\Big\|_{BMO(\mathbb{R})} \lesssim \|f\|_{L^\infty(\mathbb{R})}.$$

Given the reduction in the localized case, in the proof of Proposition 5.1, the $L^p(L^2)$ estimates can be deduced from a regularized version of Rubio de Francia's square function estimate [30] associated to arbitrary disjoint collection of intervals. The L^∞-BMO estimate can be obtained from Sjölin's proof [34] of that estimate. We omit the details.

6. An $L^p(L^q)$ estimate

We state the $L^p(L^q)$ estimates alluded to in the introduction. We work with the norm

$$\|u\|_{L^p(L^q(I))} = \Big\| \Big(\int_I |u(\cdot,t)|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$

in $L^p(\mathbb{R}^d; L^q(I))$, with the usual modification $\|u\|_{L^p(L^\infty(I))} = \|\sup_{t\in I}|u(\cdot,t)|\|_p$ if $q = \infty$.

Theorem 6.1. *Let $a \in (1,\infty)$ and let I be a compact interval of time. Then*

$$\big\|U^a f\big\|_{L^p(L^q(I))} \lesssim \|f\|_{L^p_s}, \quad \frac{s}{a} = d\Big(\frac12 - \frac1p\Big) - \frac1q \tag{54}$$

holds true in each of the following three cases:

(i) $d = 1$, $4 < p < \infty$, $\frac{2p}{p-2} < q \le \infty$,
(ii) $d \ge 2$, $\frac{2(d+3)}{d+1} < p \le \frac{2(d+2)}{d}$, $\frac{2p}{(d+1)p-2(d+2)} < q \le \infty$.
(iii) $d \ge 2$, $\frac{2(d+2)}{d} < p < \infty$, $2 \le q \le \infty$.

Remark. The statements (i) and (iii) also hold for $0 < a < 1$.

Proof. The stated results for $p \le q \le \infty$ are in [28]. Consider the inequality

$$\sup_{k\in\mathbb{N}} 2^{-ka(\frac{d}{2}-\frac{d}{p}-\frac{1}{q})} \big\| U^a[P_k f] \big\|_{L^p(L^q(I))} \lesssim \|f\|_{L^p} \tag{55}$$

which holds for $2(d+3)/(d+1) < p \le q \le \infty$, $d \ge 2$ and $4 < p \le q \le \infty$, $d = 1$, by [28]. It holds for $q = 2$ if $p = \infty$, $d = 1$ and $2+4/d < p \le \infty$ if $d \ge 2$, by Theorem 1.5. By complex interpolation (55) also holds for $d \ge 2$, $\frac{2(d+3)}{d+1} < p \le \frac{2(d+2)}{d}$ and $\frac{1}{q} < \frac{d+1}{2} - \frac{d+2}{p}$ which is equivalent with $\frac{2p}{(d+1)p-2(d+2)} < q \le \infty$. Moreover for $d = 1$ complex interpolation shows that (55) holds for $\frac{2}{q} < 1 - \frac{2}{p}$ (*i.e.* $\frac{2p}{p-2} < q \le \infty$). Finally we may combine the dyadic pieces by using Proposition A.3 in the appendix. □

Remark. In two space dimensions, for $a = 2$, this result is further improved in [24] to include an $L^p(L^\infty)$ bound for $p > 16/5$.

Appendix A. Combining frequency localized pieces

We state a variant of results by Fefferman and Stein [16] and Miyachi [25] which is motivated by its application to prove Theorems 1.5 and 6.1. The approach extends and somewhat simplifies the one in [28] (see also [32], [27] for related results).

Let $\mathcal{B}$ be a Banach space with norm $|\cdot|_{\mathcal{B}}$; in our application $\mathcal{B} = L^q(I)$ for a compact interval I. We consider convolution operators T_k, with $k \in \mathbb{N}$, mapping $L^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d, \mathcal{B})$, a space of $\mathcal{B}$-valued functions. We define T_k by

$$T_k f(x) = h_k * f(x) = \int h_k(x-y) f(y)\, dy,$$

where for each k we make, for simplicity, the *a-priori* assumption that $h_k \in L^1(\mathbb{R}^d, \mathcal{B})$ but we do not assume a bound on these L^1 norms. We shall be interested in situations where, for some $a > 0$, the part of the kernel h_k supported in $|x| \ge C_1 2^{k(a-1)}$ can be neglected. In particular, this is true of U^a_t as defined in (7).

In what follows let $\rho_k \in C^1(\mathbb{R}^d)$ be such that

$$\begin{aligned} |\rho_k(x)| + 2^{-k}|\nabla \rho_k(x)| &\le 2^{kd}, \\ \operatorname{supp} \rho_k &\subset \{x : |x| \le 2^{-k}\}. \end{aligned} \tag{56}$$

We define R_k on $\mathcal{B}$-valued functions g by

$$R_k g(x) = \rho_k * g(x) = \int \rho_k(y) g(x-y)\, dy. \tag{57}$$

In applications the operators R_k often arise from dyadic frequency decompositions, however no cancelation condition on ρ_k is needed in the following result.

Theorem A.1. *Let $p_0 \in (1,\infty)$ and $a \in (0,\infty)$. With T_k and R_k defined as above, let*

$$A(p_0) := \sup_{k>0} 2^{kad/p_0} \|T_k\|_{L^{p_0} \to L^{p_0}(\mathcal{B})} \tag{58}$$

and

$$B(p_0) := \sup_{k>0} 2^{kad/p_0} \int_{|x|\geq C_1 2^{k(a-1)}} |h_k(x)|_{\mathcal{B}} dx \tag{59}$$

for some fixed constant $C_1 \geq 1$. Then for all $p \in (p_0, \infty)$ and $r > 0$, there exists $C = C(p_0, p, r, C_0, C_1, d, \varepsilon)$ so that

$$\Big\| \Big(\sum_{k>0} 2^{kadr/p} |R_k T_k f_k|_{\mathcal{B}}^r \Big)^{1/r} \Big\|_p \leq C A(p_0) \Big(1 + \frac{B(p_0)}{A(p_0)} \Big)^{1-p_0/p} \Big(\sum_{k>0} \|f_k\|_p^p \Big)^{1/p} . \tag{60}$$

Moreover,

$$\Big\| \Big(\sum_{k>0} |R_k T_k f_k|_{\mathcal{B}}^r \Big)^{1/r} \Big\|_{BMO} \leq C\big(A(p_0) + B(p_0) \big) \sup_{k>0} \|f_k\|_\infty . \tag{61}$$

Before we begin with the proof we state a preliminary Lemma.

Lemma A.2. *Define*

$$\mathcal{T}_k f(x) = 2^{kd} \int_{|x-y|\leq 2^{-k}} |T_k f(y)|_{\mathcal{B}} dy$$

and with the notation as in (58) *and* (59) *let*

$$\mathcal{A}(p, p_0) = C_1^{d(1/p_0 - 1/p)} A(p_0) + A(p_0)^{p_0/p} B(p_0)^{1-p_0/p} . \tag{62}$$

Then, for $p_0 \leq p \leq \infty$,

$$\|\mathcal{T}_k f\|_{L^p(\mathcal{B})} \lesssim 2^{-kad/p} \mathcal{A}(p, p_0) \|f\|_p .$$

Proof. We interpolate between $p = p_0$ and $p = \infty$. Since $\mathcal{A}(p_0, p_0) = A(p_0)$ the inequality is immediate for $p = p_0$ from assumption (58).

To prove the inequality for $p = \infty$ we choose a grid $\mathfrak{Q}_a^k$ of cubes $\mathcal{Q}$ of sidelength $2^{k(a-1)}$, so that the cubes in $\mathfrak{Q}_a^k$ have disjoint interior and $\sum_{\mathcal{Q}\in\mathfrak{Q}_a^k} \chi_{\mathcal{Q}} = 1$ almost everywhere. For each $\mathcal{Q} \in \mathfrak{Q}_a^k$ let $\mathcal{Q}^*$ be the cube with same center as $\mathcal{Q}$ and sidelength $10dC_1 2^{k(a-1)}$. It then suffices to show that for each cube $\mathcal{Q} \in \mathfrak{Q}_a^k$

$$\chi_{\mathcal{Q}}(x) |\mathcal{T}_k f(x)|_{\mathcal{B}} \lesssim \big(C_1^{d/p_0} A(p_0) + B(p_0) \big) \|f\|_\infty \tag{63}$$

for every x. Given $\mathcal{Q}$ we split $f = b_{\mathcal{Q}} + g_{\mathcal{Q}}$ where $b_{\mathcal{Q}} = f\chi_{\mathcal{Q}^*}$ and $g_{\mathcal{Q}} = f\chi_{\mathbb{R}^d\setminus\mathcal{Q}^*}$. For $b_{\mathcal{Q}}$ we apply Hölder's inequality and use assumption (58), so that

$$\begin{aligned} |\mathcal{T}_k b_{\mathcal{Q}}(x)|_{\mathcal{B}} &\lesssim 2^{kd/p_0} \big\| |T_k b_{\mathcal{Q}}|_{\mathcal{B}} \big\|_{p_0} \lesssim 2^{kd/p_0} A(p_0) 2^{-kad/p_0} \|b_{\mathcal{Q}}\|_{p_0} \\ &\lesssim A(p_0) 2^{kd/p_0} 2^{-kad/p_0} |\mathcal{Q}^*|^{1/p_0} \|f\|_\infty \lesssim C(d) C_1^{d/p_0} A(p_0) \|f\|_\infty \end{aligned}$$

since $|\mathcal{Q}^*| \approx C_1^d 2^{k(a-1)d}$.

For $g_{\mathcal{Q}}$ we note that when $x \in \mathcal{Q}$, $w \notin \mathcal{Q}^*$ and $|x - z| \leq 2^{-k}$, then $|z - w| \geq C_1 2^{k(a-1)}$, so we can use (59) to estimate

$$\begin{aligned} |\mathcal{T}_k g_{\mathcal{Q}}(x)|_{\mathcal{B}} &\leq 2^{kd} \int_{|x-z|\leq 2^{-k}} \int_{|z-w|\geq C_1 2^{k(a-1)}} |h_k(z-w)|_{\mathcal{B}} |f(w)| \, dw \, dz \\ &\lesssim B(p_0) \|f\|_\infty . \end{aligned}$$

Combining the two estimates, we get (63). □

Proof of Theorem A.1. We may assume $r \le 1$ and that the summation in k is extended over a finite set. We proceed as in [27] and, by the Fefferman-Stein theorem [16] on the #-maximal operator and the inequality $\big|\,|u|_{\mathcal B}^r - |v|_{\mathcal B}^r\big| \le |u-v|_{\mathcal B}^r$ we get

$$\begin{aligned}&\Big\|\Big(\sum_k |2^{kad/p} R_k T_k f_k|_{\mathcal B}^r\Big)^{1/r}\Big\|_p\\ &\lesssim \Big\| \sup_{Q: x\in Q} \sum_k 2^{kadr/p} \fint_Q \fint_Q |R_k T_k f_k(y) - R_k T_k f_k(z)|_{\mathcal B}^r \, dz\, dy\Big\|_{L^{p/r}(dx)}^{1/r}\end{aligned}$$

Let $x \mapsto Q(x)$ depend measurably on x, so that for each x the cube $Q(x)$ is centered at x and has sidelength in $[2^{L(x)}, 2^{L(x)+1})$. It suffices to estimate the L^p norm of

$$\Big(\sum_k 2^{kadr/p} \fint_{Q(x)} \fint_{Q(x)} |R_k T_k f_k(y) - R_k T_k f_k(z)|_{\mathcal B}^r \, dz\, dy\Big)^{1/r}.$$

We let $F = \{f_k\}_{k>0}$ and estimate the displayed expression by $\sum_{i=1}^3 \mathfrak{S}_i F(x)$ where

$$\begin{aligned}\mathfrak{S}_1 F(x) &= \Big(\sum_{k+L(x)\le 0} 2^{kadr/p} \fint_{Q(x)} \fint_{Q(x)} |R_k T_k f_k(y) - R_k T_k f_k(z)|_{\mathcal B}^r \, dz\, dy\Big)^{1/r},\\ \mathfrak{S}_2 F(x) &= \Big(\sum_{\substack{k+L(x)>0\\ k(a-1)\le L(x)}} 2^{kadr/p} \fint_{Q(x)} |R_k T_k f_k(y)|_{\mathcal B}^r \, dy\Big)^{1/r},\\ \mathfrak{S}_3 F(x) &= \Big(\sum_{\substack{k+L(x)>0\\ k(a-1)> L(x)}} 2^{kadr/p} \fint_{Q(x)} |R_k T_k f_k(y)|_{\mathcal B}^r \, dy\Big)^{1/r}.\end{aligned}$$

Set $\|F\|_p \equiv \|F\|_{L^p(\mathbb{R}^d\times\mathbb{N})} = (\sum_k \|f_k\|_p^p)^{1/p}$ for $p<\infty$ and $\|F\|_\infty = \sup_k \|f_k\|_\infty$. For $p > p_0$ we will bound the L^p norms of $\mathfrak{S}_i F$ by $C\mathcal{A}(p,p_0)\|F\|_p$. In the proofs we shall use the notation

$$\omega_k(x) = 2^{kd} \chi_{\{|x|\le \sqrt{d} 2^{-k+3}\}}(x).$$

L^p bound for $\mathfrak{S}_1(F)$. Using the estimate (56) for $\nabla\rho_k$ we see that for $y, z \in Q(x)$

$$|R_k T_k f_k(y) - R_k T_k f_k(z)|_{\mathcal B} \lesssim 2^{L(x)+k} \int \omega_k(x-u) |T_k f_k(u)|_{\mathcal B} du\,.$$

Using the embedding of $\ell^{p/r} \in \ell^\infty$ we estimate

$$\begin{aligned}\mathfrak{S}_1 F(x)^r &\lesssim \sup_L \Big[\sum_{k+L\le 0} 2^{k+L} \int \omega_k(x-u) |2^{kad/p} T_k f_k(u)|_{\mathcal B} du\Big]^r\\ &\lesssim \Big(\sum_L \Big[\sum_{0<k\le -L} 2^{k+L} \int \omega_k(x-u) |2^{kad/p} T_k f_k(u)|_{\mathcal B} du\Big]^p\Big)^{r/p}\end{aligned}$$

and therefore, with the change of summation variable $n = -L-k$ and Minkowski's inequality

$$\begin{aligned}&\|\mathfrak{S}_1 F\|_p = \big\|[\mathfrak{S}_1 F]^r\big\|_{p/r}^{1/r}\\ &\lesssim \sum_{n>0} 2^{-n} \Big(\sum_{L>-n} \Big\| 2^{-(L+n)d} \int_{|y|\le C 2^{L+n}} 2^{-(L+n)ad/p} |T_{-L-n} f_{-L-n}(\cdot - y)|_{\mathcal B}\, dy\Big\|_p^p\Big)^{1/p}.\end{aligned}$$

By Lemma A.2 this can be estimated by

$$\mathcal{A}(p,p_0)\sum_{n>0}2^{-n}\Big(\sum_{L>-n}\|f_{-L-n}\|_p^p\Big)^{1/p}\lesssim\mathcal{A}(p,p_0)\|F\|_p.$$

L^p bound for $\mathfrak{S}_2(F)$. The L^p bound for $\mathfrak{S}_2F$ follows by interpolating the inequalities

$$\begin{aligned}\|\mathfrak{S}_2F\|_{p_0}&\lesssim A(p_0)\|F\|_{p_0}\,,\\ \|\mathfrak{S}_2F\|_\infty&\lesssim (A(p_0)+B(p_0))\|F\|_\infty.\end{aligned}\tag{64}$$

For the L^{p_0} bound, we sum a geometric series, using $p>p_0$, to estimate

$$\Big(\fint_{Q(x)}\sum_{\substack{k+L(x)>0\\ k(a-1)\le L(x)}}2^{kadr/p}|R_kT_kf_k(y)|_{\mathcal{B}}^r dy\Big)^{1/r}\lesssim\big(M\big[\sup_{k>0}2^{kadr/p_0}|R_kT_kf_k|_{\mathcal{B}}^r\big](x)\big)^{1/r}$$

and by the $L^{p_0/r}$ boundedness of the Hardy–Littlewood operator we get

$$\begin{aligned}\|\mathfrak{S}_2F\|_{p_0}&\lesssim\big\|\sup_{k>0}2^{kadr/p_0}|R_kT_kf_k|_{\mathcal{B}}^r\big\|_{p_0/r}^{1/r}=\big\|\sup_{k>0}2^{kad/p_0}|R_kT_kf_k|_{\mathcal{B}}\big\|_{p_0}\\ &\lesssim\Big(\sum_{k>0}\big\|2^{kad/p_0}|R_kT_kf_k|_{\mathcal{B}}\big\|_{p_0}^{p_0}\Big)^{1/p_0}\lesssim\Big(\sum_{k>0}\big\|2^{kad/p_0}|T_kf_k|_{\mathcal{B}}\big\|_{p_0}^{p_0}\Big)^{1/p_0}\lesssim A(p_0)\|F\|_{p_0}\,.\end{aligned}$$

For the L^∞ bound, we fix x, $Q=Q(x)$, $L=L(x)$ and let $y_Q=y_{Q(x)}$ be the center of Q. By Hölder's inequality and (56),

$$\mathfrak{S}_2F(x)\lesssim\Big(\sum_{\substack{k+L>0\\ k(a-1)\le L}}2^{kadr/p}\Big[\fint_Q\int\omega_k(y-z)\big|T_kf_k](z)\big|_{\mathcal{B}}\,dz\,dy\Big]^r\Big)^{1/r}.$$

Let Q^* be the $C_12^{10}d$ dilate of Q with respect to y_Q. We may estimate the last displayed expression by $\mathcal{E}^{\rm near}+\mathcal{E}^{\rm far}$ where

$$\begin{aligned}\mathcal{E}^{\rm near}&=\Big(\sum_{\substack{k+L>0\\ k(a-1)\le L}}2^{kadr/p}\Big[\fint_Q\int\omega_k(y-z)\big|T_k[f_k\chi_{Q^*}](z)\big|_{\mathcal{B}}\,dz\,dy\Big]^r\Big)^{1/r},\\ \mathcal{E}^{\rm far}&=\Big(\sum_{\substack{k+L>0\\ k(a-1)\le L}}2^{kadr/p}\Big[\fint_Q\int\omega_k(y-z)\big|T_k[f_k\chi_{\mathbb{R}^d\setminus Q^*}](z)\big|_{\mathcal{B}}\,dz\,dy\Big]^r\Big)^{1/r},\end{aligned}$$

and it suffices to check that

$$\mathcal{E}^{\rm near}\lesssim C_1^{d/p_0}A(p_0)\|F\|_\infty\,,\tag{65}$$

$$\mathcal{E}^{\rm far}\lesssim B(p_0)\|F\|_\infty\,.\tag{66}$$

To prove (65) we apply Hölder's inequality, use $p_0<p$ and assumption (58):

$$\begin{aligned}\mathcal{E}^{\rm near}&\lesssim\Big(\sum_k2^{kadr/p}\Big(\frac{C_1^d}{|Q^*|}\int\big|T_k[f_k\chi_{Q^*}](z)\big|_{\mathcal{B}}^{p_0}dz\Big)^{r/p_0}\Big)^{1/r}\\ &\lesssim C_1^{d/p_0}\sup_k2^{kad/p_0}|Q^*|^{-1/p_0}\big\|T_k[f_k\chi_{Q^*}]\big\|_{L^{p_0}(\mathcal{B})}\\ &\lesssim A(p_0)C_1^{d/p_0}\sup_k|Q^*|^{-1/p_0}\big\|f_k\chi_{Q^*}\big\|_{L^{p_0}(\mathcal{B})}\lesssim A(p_0)C_1^{d/p_0}\|F\|_\infty\,.\end{aligned}$$

To prove (66) we use assumption (59). Note that if $y \in Q$, $|y-z| \le \sqrt{d}2^{-k+3}$ (and $k(a-1) \le L$), $w \in \mathbb{R}^d \setminus Q^*$, then $|z-w| \ge C_1 2^{k(a-1)}$. Thus

$$\begin{aligned}
&⨍_Q \int \omega_k(y-z)\big|T_k[f_k\chi_{\mathbb{R}^d\setminus Q^*}](z)\big|_{\mathcal{B}}\,dz\,dy\\
&\lesssim ⨍_Q \int \omega_k(y-z)\int_{|z-w|\ge C_1 2^{k(a-1)}} |h_k(z-w)|_{\mathcal{B}}\|f_k\|_\infty dw\,dz\,dy\\
&\lesssim 2^{-kad/p_0}B(p_0)\|F\|_\infty
\end{aligned}$$

and (66) follows.

L^p bound for $\mathfrak{S}_3(F)$. Let B_L be the ball of radius $10d2^L$ centered at the origin. We may estimate

$$\begin{aligned}
\big\|\mathfrak{S}_3(F)\big\|_p &\lesssim \Big\|\sup_L \frac{\chi_{B_L}}{|B_L|} * \Big[\sum_{\substack{k+L>0\\ k(a-1)>L}} 2^{kadr/p}|R_kT_kf_k|_{\mathcal{B}}^r\Big]\Big\|_{p/r}^{1/r}\\
&\lesssim \Big(\sum_{n>0}\Big\|\sup_{L<(1-a^{-1})n}\frac{\chi_{B_L}}{|B_L|} * \big[2^{(n-L)adr/p}|R_{n-L}T_{n-L}f_{n-L}|_{\mathcal{B}}^r\big]\Big\|_{p/r}\Big)^{1/r}
\end{aligned}$$

by Minkowski's inequality. By Hölder's inequality on each ball B_L we see that the last expression is dominated by

$$\Big(\sum_{n>0}\Big\|\sup_{L<(1-a^{-1})n}\frac{\chi_{B_L}}{|B_L|} * 2^{(n-L)ad/p}|R_{n-L}T_{n-L}f_{n-L}|_{\mathcal{B}}\Big\|_p^r\Big)^{1/r}.$$

Now for $n>0$ we have $\chi_{B_L} * \omega_{n-L}(x) \lesssim \chi_{B_{L+1}}$. Thus we get

$$\big\|\mathfrak{S}_3(F)\big\|_p \lesssim \Big(\sum_{n>0}\|\mathfrak{S}_{3,n}F\|_p^r\Big)^{1/r}$$

where

$$\mathfrak{S}_{3,n}F(x) = \sup_{L<(1-a^{-1})n} 2^{-Ld}\chi_{B_{L+1}} * 2^{(n-L)ad/p}|T_{n-L}f_{n-L}|_{\mathcal{B}}.$$

It thus suffices to prove

$$\|\mathfrak{S}_{3,n}F\|_p \lesssim 2^{-nd(\frac{1}{p_0}-\frac{1}{p})}\mathcal{A}(p,p_0)\|F\|_p\,. \tag{67}$$

We shall use an analytic interpolation argument and for this it is necessary to linearize the operator. For any bounded linear functional $\lambda \in \mathcal{B}^*$ we denote by $\langle v,\lambda\rangle$ the action of λ on $v \in \mathcal{B}$. Let $(x,y) \to u_L(x,y)$ be any measurable function with values in $\mathcal{B}^*$, so that $\|u_L\|_\infty \le 1$. After replacing a sup in L by an ℓ^p norm and interchanging an integral and a summation it then suffices to bound

$$\Big(\sum_{L<(1-a^{-1})n}\Big\|2^{(n-L)ad/p}2^{-Ld}\int \chi_{B_{L+1}}(y)\big\langle T_{n-L}f_{n-L}(\,\cdot\,-y), u_L(\,\cdot\,,y)\big\rangle dy\Big\|_p^p\Big)^{1/p}$$

by the right hand side of (67), with a constant uniform in the choices of the u_L. In what follows we fix such a choice.

Define an analytic family

$$\mathcal{G}_{L,n}^z F(x) = 2^{(n-L)ad(1-z)/p_0}2^{-Ld}\int \chi_{B_{L+1}}(y)\big\langle T_{n-L}f_{n-L}(x-y), u_L(x,y)\big\rangle dy.$$

We then show that for $p_0 \le \widetilde{p} \le \infty$

$$\Big(\sum_{L<(1-a^{-1})n} \big\| \mathcal{G}^z_{L,n} F \big\|_{\widetilde{p}}^{\widetilde{p}} \Big)^{1/\widetilde{p}} \lesssim 2^{-nd(\frac{1}{p_0}-\frac{1}{\widetilde{p}})} \mathcal{A}(\widetilde{p}, p_0) \|F\|_{\widetilde{p}}\,, \quad \frac{1-\operatorname{Re}(z)}{p_0} = \frac{1}{\widetilde{p}} \tag{68}$$

and the required L^p estimate follows if we let $z = (1-p_0/p)$. By Stein's theorem on analytic families of operators it suffices to show (68) for $\operatorname{Re}(z) = 0$, $\widetilde{p} = p_0$ and $\operatorname{Re}(z) = 1$, $\widetilde{p} = \infty$.

First, for $\widetilde{p} = p_0$, $z = \imath\gamma$ we bound

$$\Big(\sum_{L<(1-a^{-1})n} \big\| \mathcal{G}^{\imath\gamma}_{L,n} F \big\|_{p_0}^{p_0} \Big)^{1/p_0} \lesssim \Big(\sum_{L<(1-a^{-1})n} 2^{(n-L)ad} \big\| T_{n-L} f_{n-L} \big\|^{p_0}_{L^{p_0}(\mathcal{B})} \Big)^{1/p_0}$$
$$\lesssim A(p_0) \Big(\sum_{L<n} \big\| f_{n-L} \big\|_{p_0}^{p_0} \Big)^{1/p_0} \lesssim A(p_0) \|F\|_{p_0}$$

which is (68) for $\operatorname{Re}(z) = 0$.

Now let $\widetilde{p} = \infty$, $\operatorname{Re}(z) = 1$. The required bound for $\mathcal{G}^{1+\imath\gamma}_{L,n}$ follows if we can show that for any fixed x_0 and fixed $L < (1-a^{-1})n$

$$2^{-Ld} \chi_{B_{L+1}} * \big| T_{n-L} f_{n-L} \big|_{\mathcal{B}}(x_0) \lesssim 2^{-nd/p_0} (C_1^{d/p_0} A(p_0) + B(p_0)) \|F\|_\infty\,. \tag{69}$$

Let $\mathcal{Q}^*$ be a cube of sidelength $20dC_1 2^{(n-L)(a-1)}$ centered at x_0; recall the inequality $(n-L)(a-1) > L$. We dominate the left hand side of (69) by $C(\mathcal{E}^{\text{near}}_n + \mathcal{E}^{\text{far}}_n)$, where

$$\mathcal{E}^{\text{near}}_n = \int 2^{-Ld} \chi_{B_{L+1}}(x_0 - z) \big| T_{n-L}[f_{n-L} \chi_{\mathcal{Q}^*}](z) \big|_{\mathcal{B}}\, dz\,,$$
$$\mathcal{E}^{\text{far}}_n = \int 2^{-Ld} \chi_{B_{L+1}}(x_0 - z) \big| T_{n-L}[f_{n-L} \chi_{\mathbb{R}^d \setminus \mathcal{Q}^*}](z) \big|_{\mathcal{B}}\, dz\,.$$

By Hölder's inequality

$$\mathcal{E}^{\text{near}}_n \lesssim \Big(2^{-Ld} \int \big| T_{n-L}[f_{n-L}\chi_{\mathcal{Q}^*}](z) \big|_{\mathcal{B}}^{p_0}\, dz \Big)^{1/p_0}$$
$$\lesssim A(p_0) 2^{-(n-L)ad/p_0} 2^{-Ld/p_0} \big\| f_{n-L} \chi_{\mathcal{Q}^*} \big\|_{p_0}$$

and since $\|f_{n-L}\chi_{\mathcal{Q}^*}\|_{p_0} \lesssim C_1^{d/p_0} 2^{(n-L)(a-1)d/p_0} \|f_{n-L}\|_\infty$ this yields

$$\mathcal{E}^{\text{near}}_n \lesssim C_1^{d/p_0} A(p_0) 2^{-nd/p_0} \|F\|_\infty.$$

Next observe that if $x_0 - z \in B_{L+1}$ and $y \in \mathbb{R}^d \setminus \mathcal{Q}^*$ then $|z-y| \ge C_1 2^{(n-L)(a-1)}$ and thus

$$\mathcal{E}^{\text{far}}_n \lesssim \int 2^{-Ld} \chi_{B_{L+1}}(x_0 - z) \int_{|z-y| \ge C_1 2^{(n-L)(a-1)}} |h_{n-L}(z-y)|_{\mathcal{B}} |f_{n-L}(y)|\, dy\, dz$$
$$\lesssim 2^{-(n-L)ad/p_0} B(p_0) \|f_{n-L}\|_\infty\,.$$

Since by assumption $aL < (a-1)n$ we get

$$\mathcal{E}^{\text{far}}_n \lesssim B(p_0) 2^{-nd/p_0} \|F\|_\infty\,.$$

The estimates for $\mathcal{E}^{\text{near}}_n$ and $\mathcal{E}^{\text{far}}_n$ yield (69). This finishes the estimation of $\mathfrak{S}_{3,n}(F)$ and thus concludes the proof of the theorem.

□

Applications. We show how for the semigroups U_t^a one can use Theorem A.1 to prove global estimates from frequency localized versions.

Let $L^p(L^q) \equiv L^p(\mathbb{R}^d, L^q[-1/2,1/2])$, and let $\varphi \in C^\infty(\mathbb{R})$ be supported in $(1/4,4)$ so that φ is not identically zero. Define

$$\Gamma_a(p,q) = \sup_{R>1} R^{-a(\frac{d}{2}-\frac{d}{p}-\frac{1}{q})} \big\| U^a \varphi\big(R^{-1}|D|\big) \big\|_{L^p \to L^p(L^q)} \tag{70}$$

Clearly this definition depends on φ, however the finiteness of $\Gamma_a(p,q)$ is independent of the particular φ used.

Proposition A.3. *Let $p_0, q_0 \in [1,\infty]$, $p \in (p_0,\infty)$, $q \ge q_0$, $r \in (0,\infty)$, and let I be a compact interval. Suppose that $\Gamma_a(p_0,q_0)$ is finite. Then for $s/a = d(1/2-1/p) - 1/q$,*

$$\Big\| \Big(\sum_{k>0} \Big(\int_I |P_k U^a f(\,\cdot\,,t)|^q dt \Big)^{r/q} \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\sum_{k>0} 2^{ksp} \|P_k f\|_p^p \Big)^{1/p}, \tag{71}$$

and if $s/a = d/2 - 1/q$ then

$$\Big\| \Big(\sum_{k>0} \Big(\int_I |P_k U^a f(\,\cdot\,,t)|^q dt \Big)^{r/q} \Big)^{1/r} \Big\|_{BMO(\mathbb{R}^d)} \lesssim \sup_{k>0} 2^{ks} \|P_k f\|_\infty \,. \tag{72}$$

Moreover, if $t \mapsto \varpi(t)$ is smooth and compactly supported then

$$\big\| \varpi\, U^a f \big\|_{L^p(\mathbb{R}^d, B^q_{\gamma,r}(\mathbb{R}))} \lesssim \|f\|_{B^p_{s+a\gamma,p}(\mathbb{R}^d)}. \tag{73}$$

If $f \in B^p_{s,p}(\mathbb{R}^d)$ with $s = ad(1/2-1/p)$, then for almost every $x \in \mathbb{R}^d$ the function $t \mapsto U^a f(x,t)$ is locally in $B^q_{1/q,r}(\mathbb{R})$, and (thus) continuous, and we have the maximal inequality

$$\big\| \sup_{t\in I} |U^a f(\,\cdot\,,t)| \big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{B^p_{ad(1/2-1/p),p}(\mathbb{R}^d)}. \tag{74}$$

We note that the constants implicit in (71) and (73) depend on p, q, q_0, I, a, d, v, however if we take $I = [-1/2, 1/2]$ they are independent of q and q_0.

For the proof of Proposition A.3 we need a standard imbedding result.

Lemma A.4. *Let $p \in [1,\infty]$ and $1 \le q_0 \le q \le \infty$. Then*

$$\Gamma_a(p,q) \lesssim \Gamma_a(p,q_0).$$

Proof. Let h be in $C^1(-1/2,1/2)$. By the fundamental theorem of calculus,

$$|h(t)|^{q_0} \le |h(\tau)|^{q_0} + q_0 \int_{-1/2}^{1/2} |h(y)|^{q_0-1} |h'(y)|\, dy.$$

for all $t, \tau \in (-1/2,1/2)$. Integrating in $\tau \in (-1/2,1/2)$ and applying Hölder's inequality yields

$$\sup_{-1/2<t<1/2} |h(t)|^{q_0} \le \|h\|_{q_0}^{q_0} + q_0 \|h\|_{q_0}^{q_0-1} \|h'\|_{q_0}$$

where the L^q norms are on $(-1/2,1/2)$. Now, as $\|h\|_q^q \le \|h\|_\infty^{q-q_0} \|h\|_{q_0}^{q_0}$, we have

$$\|h\|_q \le 2^{1/q} \Big(\|h\|_{q_0} + q_0^{\frac{1}{q_0}-\frac{1}{q}} \|h\|_{q_0}^{1-\frac{1}{q_0}+\frac{1}{q}} \|h'\|_{q_0}^{\frac{1}{q_0}-\frac{1}{q}} \Big).$$

Setting $\mathcal{U}_R f(x,t) := U^a[\varphi(R^{-1}|D|)f](x,t)$, for fixed x we apply the displayed inequality with $h(t) = \mathcal{U}_R f(x,t)$, then integrate and apply Hölder's inequality in x to get

$$\|\mathcal{U}_R f\|_{L^p(L^q)} \le 2^{\frac{1}{p}+\frac{1}{q}} \Big(\|\mathcal{U}_R f\|_{L^p(L^{q_0})} + q_0^{\frac{1}{q_0}-\frac{1}{q}} \|\mathcal{U}_R f\|_{L^p(L^{q_0})}^{\frac{1}{q_0'}+\frac{1}{q}} \|\partial_t \mathcal{U}_R f\|_{L^p(L^{q_0})}^{\frac{1}{q_0}-\frac{1}{q}} \Big).$$

Now by definition $\partial_t U^a = \imath(-\Delta)^{a/2}U^a$, so that

$$\|\mathcal{U}_R f\|_{L^p(L^{q_0})} + R^{-a}\|\partial_t \mathcal{U}_R\|_{L^p(L^{q_0})} \lesssim \Gamma_a(p,q_0) R^{a(\frac{d}{2}-\frac{d}{p}-\frac{1}{q_0})}\|f\|_p$$

and substituting these bounds into the displayed inequality implies the assertion. □

Proof of Proposition A.3. We can reduce to the situation where $I = [-1/2, 1/2]$ or $\varpi \in C_c^\infty((-1/2,1/2))$, by a change of variables argument. By Lemma A.4 we may assume $q_0 = q$.

To prove (71) we apply Theorem A.1. Let ρ be a radial $C^\infty(\mathbb{R}^d)$-function which is compactly supported in the ball of radius $1/2$ centered at 0, with the property that $\widehat{\rho}$ is positive on $\operatorname{supp}\chi(|\cdot|)$. Denote by R_k the operator of convolution with $2^{kd}\rho(2^k\,\cdot)$. Let $L_k = \varphi(2^{-k}|D|)$ where φ is chosen so that $\varphi(|\cdot|)\widehat{\rho} = 1$ on $\operatorname{supp}\chi(|\cdot|)$. Thus $R_k L_k P_k = P_k$.

Now let $\mathcal{B} = L^q[-1/2,1/2]$ and let $T_k f(x,t) = 2^{-ka(\frac{d}{2}-\frac{1}{q})} L_k U^a f(x,t)$. Then the hypothesis that $\Gamma_a(p_0,q_0)$ is finite enables us to conclude that

$$A(p_0) := \sup_{k>0} 2^{kad/p_0}\|T_k\|_{L^{p_0}\to L^{p_0}(\mathcal{B})} < \infty.$$

For fixed t let h_k^t be the convolution kernel for $2^{-ka(\frac{d}{2}-\frac{1}{q})} L_k U_t^a$;

$$(2\pi)^{-d} h_k^t(x) = 2^{-ka(\frac{d}{2}-\frac{1}{q})} 2^{kd}\int_{\mathbb{R}^d} \varphi(|\xi|) e^{\imath(2^k\langle x,\xi\rangle + 2^{ak}t|\xi|^a)} d\xi.$$

An N-fold integration by parts yields

$$2^{ka(\frac{d}{2}-\frac{1}{q})}|h_k^t(x)| \le C_N 2^{k(d-N)}|x|^{-N}, \quad |x| \ge 2^{(a-1)k+4}, \quad t\in[0,1],$$

and thus condition (59) is satisfied with $C_1 = 2^5$. Thus, by Theorem A.1 we obtain the inequality

$$\Big\| \Big(\sum_{k>0} 2^{kadr/p}\|R_k T_k f_k\|_{L^q([-1/2,1/2])}^r \Big)^{1/r} \Big\|_p \lesssim \Big(\sum_{k>0} \|f_k\|_p^p \Big)^{1/p} \tag{75}$$

and if we apply it with $f_k = 2^{ks}P_k f$, $s = a(\frac{d}{2} - \frac{d}{p} - \frac{1}{q})$, then (71) follows. The assertion (72) is obtained in the same way.

Now consider a standard inhomogeneous Littlewood–Paley decomposition $\{\mathcal{P}_k\}_{k=0}^\infty$ on $L^p(\mathbb{R}^d)$ so that $\mathcal{P}_k = P_k$ for $k>0$ and where $\mathcal{P}_0$ localizes to frequencies with $|\xi|\le 2$. For the estimation of $\mathcal{P}_0 U^a$ standard multiplier arguments apply. We also need to consider a similar inhomogeneous Littlewood–Paley decomposition in the t variable, which we denote by $\{\mathcal{L}_j\}_{j=0}^\infty$. Then inequality (73) can be rewritten as

$$\Big\| \Big(\sum_{j=0}^\infty 2^{j\gamma r} \big\|\mathcal{L}_j[\varpi U^a f]\big\|_{L^q(\mathbb{R})}^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\sum_k 2^{k(s+a\gamma)p} \big\|\mathcal{P}_k f\big\|_{L^p(\mathbb{R}^d)}^p \Big)^{1/p}. \tag{76}$$

We claim that there is a constant M for which

$$\big\|\mathcal{L}_j[\varpi U^a \mathcal{P}_k g]\big\|_{L^p(L^q)} \le C_N \min\{2^{-jN}, 2^{-kaN}\}\|g\|_p \quad \text{whenever } |ka-j|\ge M, \tag{77}$$

so that for the essential terms k and j are coupled via $|ka-j|\le M$. This would mean that a t derivative of order α could be traded with an x derivative of order $a\alpha$, so that (76) would follow from (75). Thus it remains to prove (77). Note that for $k>0$, $j>0$, the convolution kernel of $g\to \mathcal{L}_j[\varpi U^a \mathcal{P}_k g](\,\cdot\,,t)$ can be written as

$$\frac{1}{(2\pi)^{d+1}} \iint \Big\{ \int \varpi(s) e^{\imath s(|\xi|^a - \tau)} \chi_1(2^{-j}|\tau|)\chi_1(2^{-k}|\xi|)\, ds \Big\} e^{\imath(\langle x,\xi\rangle + t\tau)} d\tau\, d\xi$$

and similar formulas hold if either $k = 0$ or $j = 0$. One checks that if $|ka - j| \gg 1$ then for ξ and τ in the support of the indicated cutoff functions the inequalities $||\xi|^a - \tau| \geq c \max\{|\xi|^a, |\tau|\}$ hold. We perform $N + d + 1$ integration by parts in s. For t large, we follow this by integrations by parts in τ. This easily yields (77).

The final assertions of the proposition are a consequence of the fact that for $r \leq 1$ the space $B^q_{1/q,r}(\mathbb{R})$ is imbedded in the space of bounded continuous functions. □

References

[1] J. Bourgain, *Besicovitch type maximal operators and applications to Fourier analysis,* Geom. Funct. Anal., **1** (1991), no. 2, 147–187.

[2] A. Carbery, *The boundedness of the maximal Bochner–Riesz operator on $L^4(\mathbb{R}^2)$,* Duke Math. J. **50** (1983), no. 2, 409–416.

[3] ———, *A weighted inequality for the maximal Bochner–Riesz operator on $\mathbb{R}^2$,* Trans. Amer. Math. Soc., **287** (1985), no. 2, 673–680.

[4] ———, *Radial Fourier multipliers and associated maximal functions,* Recent progress in Fourier analysis (El Escorial, 1983), 49–56, North-Holland Math. Stud., 111, North-Holland, Amsterdam, 1985.

[5] A. Carbery, G. Gasper, W. Trebels, *Radial Fourier multipliers of $L^p(R^2)$,* Proc. Nat. Acad. Sci. U.S.A. **81** (1984), no. 10, Phys. Sci., 3254–3255.

[6] A. Carbery, A. Seeger, *Weighted inequalities for Bochner–Riesz means in the plane,* Q. J. Math. **51** (2000), no. 2, 155–167.

[7] M. Christ, *On almost everywhere convergence of Bochner–Riesz means in higher dimensions,* Proc. Amer. Math. Soc. **95** (1985), 16–20.

[8] A. Cordoba, *The Kakeya maximal function and the spherical summation multipliers,* Amer. J. Math. 99 (1977), 1–22.

[9] ———, *A note on Bochner–Riesz operators,* Duke Math. J. **46** (1979), no. 3, 505–511.

[10] ———, *Geometric Fourier analysis,* Ann. Inst. Fourier (Grenoble) 32 (1982), no. 3, vii, 215–226.

[11] A. Córdoba and C. Fefferman, *A weighted norm inequality for singular integrals,* Studia Math. **57** (1976), no. 1, 97–101.

[12] H. Dappa and W. Trebels, *On maximal functions generated by Fourier multipliers,* Ark. Mat. **23** (1985), no. 2, 241–259.

[13] C. Fefferman, *The multiplier problem for the ball,* Ann. of Math. (2) **94** (1971), 330–336.

[14] ———, *A note on spherical summation multipliers,* Israel J. Math., **15**, (1973), 44-52.

[15] C. Fefferman and E.M. Stein, *Some maximal inequalities,* Amer. J. Math. **93** (1971), 107?115.

[16] ———, *H^p spaces of several variables,* Acta Math. **129** (1972), no. 3-4, 137–193.

[17] J. García-Cuerva and J.L. Rubio de Francia, Weighted norm inequalities and related topics, North-Holland, 1985.

[18] G. Garrigós and A. Seeger, *Characterizations of Hankel multipliers,* Math. Ann. **342**, no. 1, (2008), 31-68.

[19] Y. Heo, F. Nazarov and A. Seeger, *Radial Fourier multipliers in high dimensions,* Acta Math., to appear.

[20] S. Igari and S. Kuratsubo, *A sufficient condition for L^p-multipliers,* Pacific J. Math. **38** (1971), 85–88.

[21] M. Kaneko, G. Sunouchi, *On the Littlewood–Paley and Marcinkiewicz functions in higher dimensions,* Tôhoku Math. J. (2) **37** (1985), no. 3, 343–365.

[22] S. Lee, *Improved bounds for Bochner–Riesz and maximal Bochner–Riesz operators,* Duke Math. J. **122** (2004), 205–232.

[23] ———, *Bilinear restriction estimates for surfaces with curvatures of different signs,* Trans. Amer. Math. Soc. **358** (2006), no. 8, 3511–3533.

[24] S. Lee, K.M. Rogers and A. Seeger, *Maximal and mixed norm estimates for solutions of the Schrödinger equation,* in preparation.

[25] A. Miyachi, *On some singular Fourier multipliers,* J. Fac. Sci. Univ. Tokyo Sect. IA Math., **28** (1981), 267–315.

[26] G. Mockenhaupt, A. Seeger, C.D. Sogge, Christopher, *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates.* J. Amer. Math. Soc. **6** (1993), no. 1, 65–130.

[27] M. Pramanik, K.M. Rogers, A. Seeger, *A Calderón-Zygmund estimate with applications to generalized Radon transforms and Fourier integral operators,* Studia Mathematica, to appear.

[28] K.M. Rogers and A. Seeger, *Endpoint maximal and smoothing estimates for Schrödinger equations,* J. Reine Angew. Math., **640** (2010), 47-66.
[29] J.L. Rubio de Francia, *Estimates for some square functions of Littlewood–Paley type,* Publ. Sec. Mat. Univ. Autónoma Barcelona **27** (1983), no. 2, 81–108.
[30] ———, *A Littlewood–Paley inequality for arbitrary intervals,* Rev. Mat. Iberoamericana 1 (1985), no. 2, 1–14.
[31] A. Seeger, *On quasiradial Fourier multipliers and their maximal functions,* J. Reine Angew. Math. 370 (1986), 61–73.
[32] ———, *Remarks on singular convolution operators,* Studia Math. **97** (1990), 91–114.
[33] ———, *Endpoint inequalities for Bochner–Riesz multipliers in the plane,* Pacific J. Math. **174** (1996), 543–553.
[34] P. Sjölin, *A note on Littlewood–Paley decompositions with arbitrary intervals,* J. Approx. Theory **48** (1986), 328–334.
[35] E.M. Stein, *Localization and summability of multiple Fourier series,* Acta Math. **100** (1958), 93–147.
[36] ———,*Maximal functions. I. Spherical means.* Proc. Nat. Acad. Sci. U.S.A. **73** (1976), no. 7, 2174–2175.
[37] ———, *Some problems in harmonic analysis,* in *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, 3–20, Amer. Math. Soc., Providence, R.I.
[38] E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, Princeton, NJ, 1971.
[39] G. Sunouchi, *On the Littlewood–Paley function g^* of multiple Fourier integrals and Hankel multiplier transformations,* Tôhoku Math. J. (2) **19** (1967), 496–511.
[40] ———, *On the functions of Littlewood–Paley and Marcinkiewicz,* Tôhoku Math. J. (2) **36** (1984), no. 4, 505–519.
[41] T. Tao, *A sharp bilinear restriction estimate for paraboloids,* Geom. Funct. Anal. **13** (2003), 1359–1384.
[42] T. Tao, A. Vargas and L. Vega. *A bilinear approach to the restriction and Kakeya conjectures,* J. Amer. Math. Soc. **11** (1998), 967–1000.
[43] T. Wolff, *An improved bound for Kakeya type maximal functions,* Rev. Mat. Iberoamericana **11** (1995), no. 3, 651–674.

Sanghyuk Lee, School of Mathematical Sciences, Seoul National University, Seoul 151-742, Korea
E-mail address: shlee@math.snu.ac.kr

Keith Rogers, Instituto de Ciencias Matematicas CSIC-UAM-UC3M-UCM, Madrid 28049, Spain
E-mail address: keith.rogers@icmat.es

Andreas Seeger, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI, 53706, USA
E-mail address: seeger@math.wisc.edu