SHARP LORENTZ SPACE ESTIMATES FOR ROUGH OPERATORS

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Abstract. We demonstrate the $(H^1, L^{1,2})$ or $(L^p, L^{p,2})$ mapping properties of several rough operators. In all cases these estimates are sharp in the sense that the Lorentz exponent 2 cannot be replaced by any lower number.

1. Introduction

In this paper we consider the endpoint behaviour on Hardy spaces of two classes of operators, namely singular integral operators with rough homogeneous kernels [4] and singular integral operators with convolution kernels supported on curves in the plane ([20], [27]). These operators fall outside the Calderón-Zygmund theory; however weak type $(L^1, L^{1,\infty})$ or $(H^1, L^{1,\infty})$ inequalities have been established in the previous literature ([7], [9], [16] [18], [25], [29]) We shall show that the target space $L^{1,\infty}$ can be improved to the Lorentz space $L^{1,2}$, possibly at the cost of moving to a stronger type of Hardy space (*e.g.* product H^1). Examples of Christ [8], [17] show that these types of results are optimal in the sense that one cannot replace $L^{1,2}$ by $L^{1,q}$ for any $q < 2$.

The space $L^{1,2}$ arises naturally as the interpolation space halfway between $L^{1,\infty}$ and L^1. As a gross caricature of how this space arises, suppose that we have a collection of functions f_i which are uniformly bounded in L^1, and whose maximal function $\sup_i |f_i|$ is in weak L^1, and we wish to estimate the quantity

$$\Big\| \sum_i \gamma_i f_i \Big\|_{L^{1,2}}$$

for some l^2 co-efficients γ_i. If the f_i are sufficiently orthogonal, we may hope to control this quantity by the square function

$$\Big\| \big(\sum_i |\gamma_i f_i|^2 \big)^{1/2} \Big\|_{L^{1,2}}. \tag{1.1}$$

However from our hypotheses we see that

$$\Big\| \big(\sum_i |\gamma_i f_i|^q \big)^{1/q} \Big\|_{L^{1,q}} \lesssim \Big(\sum_i |\gamma_i|^q \Big)^{1/q}$$

for $q = 1$ and $q = \infty$, and thus by interpolation for all $1 \le q \le \infty$ (*cf.* Lemma 2.2. below). Thus we expect to control (1.1) by the ℓ^2 norm of $\{\gamma_i\}$.

Our arguments will be based on more complicated versions of the above informal strategy. Generally, the L^1 estimates will be quite trivial, whereas the $L^{1,\infty}$ estimates will be variants of existing weak-type (1,1) estimates for rough operators in the literature (*e.g.* [7], [25]). We shall demonstrate this technique for

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two classes of operators. Firstly we show that the Hilbert transform on plane curves (t,t^m) maps product H^1 into $L^{1,2}$ or a related Hardy-Lorentz space; we also prove sharp $L^p \to L^{p,2}$ estimates for a related analytic family of hypersingular operators. Then we discuss homogeneous singular integrals with rough kernels in $\mathbb{R}^d$, satisfying an $L\log^2 L$ condition on the sphere, and show that these map the standard Hardy space H^1 to $L^{1,2}$.

We remark that a simple version of the above technique has been used by one of the authors in [23] to prove an endpoint version of the Hörmander multiplier theorem. Namely (stating only the one-dimensional version) if ϕ is a nonzero even smooth bump function then the condition $\sup_{t>0} \|\phi m(t\cdot)\|_{B^2_{1/2,1}}$ implies that the convolution operator with Fourier multiplier m maps H^1 to $L^{1,2}$ (and an example by Baernstein and Sawyer [1] shows that $L^{1,2}$ cannot be replaced by $L^{1,q}$ for $q<2$). The second author and Jim Wright [30] have recently improved this result by replacing the Besov space $B^2_{1/2,1}$ by the larger space $R^2_{1/2,2}$ defined in [24] improving on the known $(H^1, L^{1,\infty})$ result which is implicit in the latter paper.

The paper is structured as follows. After formulating our results in the current section we review some material about Hardy-Lorentz spaces and interpolation, in §2. In §3 we prove an abstract variant of a stopping time argument due to M. Christ which may be helpful elsewhere. §4 contains the main square-function estimate needed to prove our theorems on integrals along curves; in §5 we conclude the proof of these results. Rough homogeneous kernels are considered in §6 and §7.

Rough homogeneous convolution kernels.

Let K be a convolution kernel on the Euclidean space $\mathbb{R}^d$ and assume that K is homogeneous of degree $-d$ and that the restriction Ω to the unit sphere is integrable and has mean zero, $\int_{S^{d-1}} \Omega(\theta)d\sigma(\theta) = 0$. We may define the operator T_Ω of convolution with K on test functions at least by the usual method of principal values:

$$T_\Omega f(x) = p.v. \int \frac{\Omega(y/|y|)}{|y|^d} f(x-y)dy. \tag{1.2}$$

We consider the mapping properties of T_Ω, especially near the endpoint L^1. If Ω is somewhat regular (for example, if it is Hölder continuous or satisfies an appropriate L^1 Dini condition) then the standard Calderón-Zygmund theory shows that T is bounded on all L^p spaces, $1<p<\infty$, is of weak type $(1,1)$, and maps the Hardy space H^1 to L^1. If no regularity is assumed, but K is $L\log L$ on the sphere, then it was shown by Calderón-Zygmund [4] that T_Ω is bounded on L^p; in fact (see [25]) it is of weak type $(1,1)$. The behaviour at H^1 is more subtle, however, as an example of M. Christ shows (see also [17]). For the sake of illustration let us consider the case $d=2$. Let a be a smooth H^1 atom on the unit ball, which is smooth and radial, and let Ω_N be the lacunary function defined on the unit circle by

$$\Omega_N(\cos\alpha, \sin\alpha) \equiv G_N(\alpha) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi i C^j \alpha},$$

where C, N are large integers. Roughly speaking, the function $K*a(x)$ has magnitude $\sim N^{-1/2}|x|^{-d}$ whenever $|x| \sim C^j$ for some $j=1,\dots,N$. This shows that the L^1 norm (and indeed the $L^{1,q}$ quasi-norm for any $q<2$) of $K*a$ grows with N, even though Ω is in every L^p class, $p<\infty$, uniformly in N. Thus, the best result one can reasonably hope for is that T maps H^1 to the Lorentz space $L^{1,2}$, or the Hardy-Lorentz space $H^{1,2}$, the quasi-norm norm in the latter is the $L^{1,2}$ quasinorm of a suitable square-function or maximal operator used in the definition of H^1 (see §2 below).

The previous counterexample can be modified to include the case $\Omega \in L^\infty$. Take G_N as above, $\varepsilon>0$ and let $E_{\varepsilon,N} = \{\alpha : |G_N(\alpha)| > N^\varepsilon\}$. Define $G_{\varepsilon,N}(\alpha) = (G_N(\alpha)(1-\chi_{E_{\varepsilon,N}}(\alpha))$ and

$$\Omega_{\varepsilon,N}(\cos\alpha, \sin\alpha) = G_{\varepsilon,N}(\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} G_{\varepsilon,N}(s)ds.$$

Since G_N is in BMO with norm independent of N we have by the John-Nirenberg inequality that $|E_{\varepsilon,N}| = O(e^{-cN^\varepsilon})$, for some $c>0$. From this one checks that the L^1 norm of $T_{\Omega_N-\Omega_{N,\varepsilon}}a$ over the annulus $|x|\sim C^j$ is $O(N^{1/2}e^{-cN^\varepsilon}+2^{-j})$, hence negligible. Since on the other hand $\|\Omega_{N,\varepsilon}\|_\infty \lesssim N^\varepsilon$ this disproves a uniform $H^1\to L^{1,q}$ estimate for $q<2/(1+2\varepsilon)$.

Theorem 1.1. *Let $\Omega\in L\log^2 L(S^{d-1})$ and assume that $\int_{S^{d-1}}\Omega d\sigma(\theta)=0$. Then the operator T_Ω maps H^1 to $H^{1,2}$ and also to $L^{1,2}$.*

Remark 1.2. In fact we shall see that the $L\log^2 L$ condition can be strengthened to an $L\log L$ condition for a Littlewood-Paley square function (see Theorem 6.1 below)

Analogously we may also consider a maximal variant of T; here no cancellation is imposed. Let $\chi\in C_0^\infty(\mathbb{R}^d)$ and

$$\mathcal{M}_\Omega f(x)=\sup_{h>0}\Big|\int \frac{1}{h^d}\chi(\frac{y}{h})\Omega(\frac{y}{|y|})f(x-y)dy\Big|. \tag{1.3}$$

Theorem 1.3. *Let $\Omega\in L\log^2 L(S^{d-1})$. Then $\mathcal{M}_\Omega$ maps H^1 to $L^{1,2}$.*

Again, a modification of the above example shows that $\mathcal{M}_\Omega$ may fail to map H^1 into $L^{1,q}$ for $q<2$.

Integrals along curves in the plane.

In this subsection we shall always be working in the plane $\mathbb{R}^2$. Let $m>1$ be a real number; all constants may implicitly depend on m.

Define the Hilbert transform Hf and the maximal function Mf along the curve $(t,|t|^m)$ by

$$Hf(x)=p.v.\int f(x_1-t,x_2-|t|^m)\,\frac{dt}{t} \tag{1.4}$$

and

$$\mathcal{M}f(x)=\sup_{h>0}\Big|\int f(x_1-t,x_2-|t|^m)\frac{1}{h}\eta(\frac{t}{h})dt\Big|; \tag{1.5}$$

here η is a smooth function with compact support. These operators are invariant with respect to the scaling

$$(x_1,x_2)\mapsto(tx_1,t^m x_2),\quad t>0. \tag{1.6}$$

We shall work with the product type Hardy space on $\mathbb{R}^2$, considered by Chang and Fefferman [6] among others; we denote this space by H^1_{prod}. Moreover we denote by $H^{1,2}_{prod}$ the product-type Hardy-Lorentz space (see §2).

Theorem 1.4. *$\mathcal{M}$ maps H^1_{prod} to $L^{1,2}$, and H maps H^1_{prod} to $H^{1,2}_{prod}$ and to $L^{1,2}$.*

This should be compared with the results of Christ [7] who showed that M and H map the one-parameter Hardy space $H^1_{parabolic}$ (defined with respect to the dilations (1.6)) to $L^{1,\infty}$, see also Grafakos [16]. In fact, Christ [7] observes that $H^1_{parabolic}$ is not mapped to $L^{1,q}$ for $q<\infty$.

Now let $\gamma=(\gamma_1,\gamma_2)$ be a complex multi-index with $\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)\ge 0$, and define the (pseudo)-differentiation operator $\mathcal{D}^\gamma$ by

$$\widehat{\mathcal{D}^\gamma f}=|\xi^\gamma|\hat f=|\xi_1|^{\gamma_1}|\xi_2|^{\gamma_2}\hat f.$$

Consider the family of hypersingular operators H_γ defined by

$$H_\gamma f(x_1,x_2)=p.v.\int_{-\infty}^{\infty}\mathcal{D}^\gamma f(x_1-t,x_2-|t|^m)|t|^{\gamma_1+\gamma_2 m}\frac{dt}{t}. \tag{1.7}$$

The space L^p $(1 < p < 2)$ is not mapped to $L^{p,q}$ if $q < 2$ (see [8]); moreover this shows that H does not map H^1_{prod} to $L^{1,q}$ or any Hardy-Lorentz space $H^{1,q}$ for any $q < 2$. An angular Littlewood-Paley theory plays a role in this counterexample. Grafakos [16] proved using the methods in [7] that for $m = 2$, $\gamma_1 = 0$ and $\mathrm{Re}(\gamma_2) = 1 - 1/p$ the space L^p is mapped to $L^{p,p'}$ if $1 < p \le 2$. His method surely extends to the general case considered here.

An improved optimal result is

Theorem 1.5. *Suppose that* $\mathrm{Re}(\gamma_1) \ge 0$, $\mathrm{Re}(\gamma_2) \ge 0$ *and* $\mathrm{Re}(\gamma_1 + \gamma_2) = 1 - 1/p$.

- *If* $1 < p \le 2$ *then* H_γ *is bounded from* L^p *to* $L^{p,2}$.
- *If* $p = 1$ *then* H_γ *is bounded from* H^1_{prod} *to* $L^{1,2}$.

In both cases the bounds grow at most polynomially in $|\gamma|$.

The following estimate for a localized averaging operator will follow from our proof. Let $\eta \in C_0^\infty(\mathbb{R})$ and define

$$Af(x_1, x_2) = \int \eta(t) f(x_1 - t, x_2 - |t|^m) dt. \tag{1.8}$$

Corollary 1.6. *Suppose* $m \ge 2$. *Then* A *maps* $L^{m,2}$ *boundedly to the Sobolev space* $L^m_{1/m}$.

Remarks 1.7.

(i) Suppose that $t \mapsto g(t)$ is a smooth curve passing through the origin and suppose that its curvature vanishes to at most order $m - 2$ at the origin. Then the statement of Corollary 1.8 remains true if $(t, |t|^m)$ is replaced by a $g(t)$ provided that η is supported in a sufficiently small neighborhood of the origin.

(ii) In the statements of Theorems 1.4 and 1.5 the curve $(t, |t|^m)$ can be replaced by $(t, |t|^m \mathrm{sign}\ (t))$.

(iii) A variant of this family H_γ was previously considered by Stein and Wainger [27] in their proof of L^p boundedness of the Hilbert transform. They worked with a distance function ρ, smooth and positive in $\mathbb{R}^2 \setminus \{0\}$ which is homogeneous of degree 1 with respect to the dilations (1.6) and considered the analytic family

$$\widetilde{H}_\alpha f(x_1, x_2) = \mathrm{p.v.} \int_{-\infty}^{\infty} \rho^\alpha(D) f(x_1 - t, x_2 - t^m) |t|^\alpha \frac{dt}{t}.$$

The result in [27] is that $\widetilde{H}_\alpha$ is bounded on L^p for $\alpha < 1 - 1/p$. Our proof of Theorem 1.3 shows that this result can be improved to $\widetilde{H}_\alpha : L^p \to L^{p,2}$ if $\alpha = 1 - 1/p$, $1 < p \le 2$.

(iv) The principal value singularity $p.v.\, t^{-1}|t|^{\gamma_1 + \gamma_2 m}$ in the definition of H_γ can be replaced by $\chi_+^{\gamma_1 + \gamma_2 m - 1} = \lim_{\varepsilon \to 0} e^{-\varepsilon t} (\Gamma(\gamma_1 + m\gamma_2))^{-1} t_+^{\gamma_1 + m\gamma_2 - 1}$. This requires only minor changes in the proof of Theorem 1.5.

2. Preliminaries

Notation. For two quantities a and b we write $a \lesssim b$ or $b \gtrsim a$ if there exists an absolute positive constant C so that $a \le Cb$. We shall consistently refer to the homogeneous quasi-norms on Lorentz and Hardy-Lorentz spaces as "norms", even when the triangle inequality with constant 1 fails. If I is a (dyadic) cube, then x_I will denote its center, and 2^{i_I} will denote its side-length. We somewhat abuse notation and use $2^s I$ to denote the cube with the same center as I and sidelength $2^{s + i_I}$. The Lebesgue measure of a set E will sometimes be denoted by $|E|$ and sometimes by $\mathrm{meas}(E)$.

2A. Hardy spaces. There are many equivalent characterizations of the isotropic Hardy-spaces ([13]), in terms of maximal functions, atomic decompositions and square-functions (see [26] for a rather complete treatment). We shall use several of them, but most relevant will be the characterization via Littlewood-Paley square-functions, which we choose as a definition.

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with the property that $\widehat{\Phi}$ is compactly supported and equal to 1 in a neighborhood of the origin. Let ϕ_k be defined by

$$\widehat{\phi}_k(\xi) = \widehat{\Phi}(2^{-k-1}\xi) - \widehat{\Phi}(2^{-k}\xi) \tag{2.1}$$

Consider the space $\mathcal{S}'_{restr}$ of tempered distributions which are *restricted at* ∞; it consists of all $f \in \mathcal{S}'$ with the property that $f*\phi \in L^r$ for $\phi \in \mathcal{S}$, for sufficiently large $r < \infty$ (we use the terminology of Stein [26, p.123]). This choice of the test function space allows one to derive versions of the Calderón reproducing formula (*e.g.* one excludes polynomials which have Fourier transforms supported at the origin). For $0 < p, q < \infty$ we define $H^{p,q}$ as the space consisting of tempered distributions restricted at ∞ which satisfy

$$\|f\|_{H^{p,q}} := \Big\| \Big(\sum_{k\in\mathbb{Z}} |\phi_k * f|^2 \Big)^{1/2} \Big\|_{L^{p,q}} < \infty \tag{2.2}$$

and write $H^p = H^{p,p}$. Using arguments in [13], [21] one can show that the definition does not depend on the particular choice of Φ. As shown in [21], [31] some aspects in the classical theory simplify by assuming (as we do here) that $\widehat{\Phi}$ has compact support. In particular for $b > 0$, $r > 0$ one has the inequality ([21])

$$\sup_{|y|\le 2^{-k}b} |\phi_k * f(x+y)| \le C_{b,r}(M[|\phi_k * f|^r](x))^{1/r} \tag{2.3}$$

and (2.3) allows us to take advantage of the Fefferman-Stein theorem concerning $L^p(\ell^r)$ estimates for the Hardy-Littlewood maximal function M ([12]). This carries over to Lorentz-spaces. Set

$$S_b f(x) = \Big(\sum_{k\in\mathbb{Z}} \sup_{|y|\le b2^{-k}} |\phi_k * f(x+y)|^2 \Big)^{1/2}$$

Since $\|g\|_{L^{p,q}} \approx \|g^a\|^{1/a}_{L^{p/a,q/a}}$ we obtain that for $f \in H^{p,q}$

$$\|f\|_{H^{p,q}} \approx \|S_b f\|_{L^{p,q}}. \tag{2.4}$$

The space $H^{p,q}$ is complete quasi-normed space. We note that the definition can be extended to Hilbert-space valued functions (in fact when proving estimates we may often reduce to finite-dimensional Hilbert spaces with possibly large dimension).

For the purpose of real interpolation consider the Peetre K-functional $K(t, f, H^{p_0}, H^{p_1})$, defined for $f \in H^{p_0} + H^{p_1}$ as the infimum of $\|f\|_{H^{p_0}} + t\|f\|_{H^{p_1}}$ over all decompositions $f = f_0 + f_1$ with $f_0 \in H^{p_0}$ and $f_1 \in H^{p_1}$. Then a straightforward modification of arguments by Jawerth and Torchinsky [19] yields the formula

$$K(t, f, H^{p_0}, H^{p_1}) \approx K(t, S_b f, L^{p_0}, L^{p_1}). \tag{2.5}$$

Consequently, by (2.4) and (2.5) one identifies $H^{p,q}$ with the real interpolation space $[H^{p_0}, H^{p_1}]_{\theta,q}$ if $0 < \theta < 1$ and $(1-\theta)/p_0 + \theta/p_1 = 1/p$ (see [2]), and the spaces $H^{p,q}$ can be identified with the spaces in [11], [15] defined by means of various maximal functions or square functions (see [32]).

Let $\{e_k\}$ be an orthonormal basis of ℓ^2. From standard Hardy space theory [26] we have

$$\Big\| \sum_k L_k f_k \Big\|_{H^{p,q}} \approx \Big\| \sum_k \widetilde{L}_k f_k e_k \Big\|_{L^{p,q}(\ell^2)} = \Big\| \Big(\sum_k |\widetilde{L}_k f_k|^2 \Big)^{1/2} \Big\|_{L^{p,q}}. \tag{2.6}$$

where L_k, $\widetilde{L}_k$ denote convolution with ϕ_k, $\widetilde{\phi}_k$; here $\widetilde{\phi}_k$ is as above and $\widetilde{\phi}_k = 2^{kd}\widetilde{\phi}_0(2^k\cdot)$ so that the Fourier transform of $\widetilde{\phi}$ equals one on the support of $\widehat{\phi}$.

Moreover if E is any finite subset of the integers we have

$$\Big\| \sum_{k\in E} L_k f_k \Big\|_{L^{p,q}} \le C \Big\| \sum_k L_k f_k \Big\|_{H^{p,q}} \tag{2.7}$$

where C does not depend on E. Note, however, that convergence in $L^{p,q}$ may not be compatible with convergence in the sense of tempered distributions, if $p < 1$ or $p = 1$, $q > 1$.

A Littlewood-Paley decomposition. It is shown in the classical theory that the above assumptions on Φ can be substantially weakened. A general result in this context is in [32]. To eliminate a number of technical error terms in the proof of Theorem 1.1 we shall work with Littlewood-Paley functions localized in space, and in order to have an analogue of the Calderón reproducing formula we will have to use a somewhat unusual version of the Littlewood-Paley decomposition:

Lemma 2.1. *Let r, N_0 be nonnegative integers and let $\varepsilon > 0$. Then for $s = 0, \dots, r$ there are radial functions $\Psi_{(s)}$, $\psi_{(s)}$ in $C_0^\infty(\mathbb{R}^d)$ with the following properties.*

(i) Ψ_s is supported on the ball of radius ε centered at the origin, and $\widehat{\Psi_s}(\xi) - 1 = O(|\xi|^{N_0})$ as $\xi \to 0$. Moreover $\psi_s = \Psi_s - 2^{-d}\Psi_s(2^{-1}\cdot)$ so that the moments of order $\le N_0$ of ψ_s vanish.

(ii) Define $\psi_s^k(x) = 2^{kd}\psi_s(2^k x)$ and let L_s^k be the operator of convolution with ψ_s^k. Then for every tempered distribution f restricted at ∞ we have

$$f = \sum_{k\in\mathbb{Z}} L_0^k \cdots L_r^k f; \tag{2.8}$$

moreover if S_r^0 denotes the operator of convolution with Ψ_r then

$$f = S_r^0 f + \sum_{k\ge 1} L_0^k \cdots L_r^k f. \tag{2.9}$$

The convergence in (2.8), (2.9) holds in the sense of tempered distributions.

Proof. Let Ψ be a radial bump function supported in $\{x : |x| \le 2^{-6r-6}\varepsilon\}$ so that $\widehat{\Psi} - 1 = O(|\xi|^{N+1})$, and let S_0^k be the operator of convolution with $2^{-dk}\Psi(2^{-k}\cdot)$. Let

$$L_0^k = S_0^k - S_0^{k-1}.$$

We recursively define for $s = 0, 1, \dots, r-1$

$$S_{s+1}^k = (2Id - (S_s^k)^2)(S_s^k)^2 \tag{2.10}$$

$$L_{s+1}^k = (2Id - (S_s^k)^2 - (S_s^{k-1})^2)(S_s^k + S_s^{k-1}) \tag{2.11}$$

and note the identity

$$S_{s+1}^k - S_{s+1}^{k-1} = (S_s^k - S_s^{k-1})L_{s+1}^k \tag{2.12}$$

so that $S_{s+1}^k - S_{s+1}^{k-1} = L_0^k \cdots L_{s+1}^k$. One can check inductively that each S_s^k is the operator of convolution with $2^{kd}\Psi^s(2^k\cdot)$ where the radial bump function Ψ^s is supported in $\{x : |x| \le 2^{-6(r-s+1)}\varepsilon\}$ and $\widehat{\Psi^s}(\xi) - 1 = O(|\xi|^{N_0+1})$ as $\xi \to 0$, and that the operators L_s^k, S_s^0 have all the desired properties. $\square$

Remark. We note that (2.6) holds if L_k, $\widetilde{L}_k$ are replaced by any of the operators L_s^k above, or perhaps by a composition of finitely many such operators. This remark holds under the condition that the number N_0 of vanishing moments is sufficiently large (in dependence of p; specifically we need $N_0 \ge n(1/p - 1)$).

Parabolic dilations. One may define Hardy spaces with respect to a nonisotropic dilation structure [3]. In this paper we need to consider such Hardy-spaces on $\mathbb{R}^2$ defined with respect to the scaling $(x_1, x_2) \mapsto (tx_1, t^m x_2)$, for a fixed real number $m > 1$.

If we redefine the function ϕ_k to be $\widehat{\phi}_k(\xi_1, \xi_2) = \widehat{\Phi}(2^{-(k+1)}\xi_1, 2^{-(k+1)m}\xi_2) - \widehat{\Phi}(2^{-k}\xi_1, 2^{-km}\xi_2)$ then the operator of convolution with ϕ_k is a Littlewood-Paley projection to the region $|\xi_1| + |\xi_2|^{1/m} \sim 2^k$. We may then define $H^p_{parabolic}$ as the space of distributions f restricted at ∞, for which $\|(\sum_k |\widetilde{\phi}_k * f|^2)^{1/2}\|_p$ is finite. Similarly one can define parabolic Hardy-Lorentz space and the obvious analogues of the statements in the previous subsections remain true.

Product type Hardy spaces. Let $\{L_{k_1,k_2}\}_{k_1,k_2\in\mathbb{Z}}$ be a product Littlewood-Paley decomposition on $\mathbb{R}^2$, where L_{k_1,k_2} is a multiplier with symbol supported in the region $\{(\xi_1,\xi_2) : |\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}\}$; we may assume that L_{k_1,k_2} is the operator of convolution with $\phi_{k_1} \otimes \phi_{k_2}$ where ϕ_{k_1}, ϕ_{k_2} are as above (defined on the real line).

If $0 < p, q < \infty$, we define the product Hardy-Lorentz space $H^{p,q}_{prod}$ to be the quasi-Banach space which consists of all tempered distributions restricted at ∞ for which

$$\|f\|_{H^{p,q}_{prod}} = \Big\| \big(\sum_{k_1} \sum_{k_2} |L_{k_1,k_2} f|^2 \big)^{1/2} \Big\|_{L^{p,q}}$$

is finite. We define H^p_{prod} to be $H^{p,p}_{prod}$.

The formulas for interpolation of Hardy-Lorentz-spaces remain true; in fact (2.5) was proved in this context in [19]. Moreover analogues of (2.6), (2.7) remain true for the operators L_{k_1,k_2}. These can be proved by using the theory of product-type singular integral operators (see *e.g.* [6], [14]).

2B. Analytic interpolation in Lorentz spaces. We need a version of a theorem by Sagher [22] concerning analytic families of operators acting on Lorentz spaces. It has been observed in [23] and [16] that Sagher's arguments carry over to somewhat more general situations; we now recall the version which appeared in [16].

We denote by S the strip $S = \{z : 0 < \mathrm{Re}(z) < 1\}$ and by $\overline{S}$ its closure. A function g on $\overline{S}$ is said to be of *admissible* growth if there is $a < \pi$ so that $|g(z)| \lesssim \exp(e^{a|\mathrm{Im}(z)|})$ for $z \in \overline{S}$. Let X_0 and X_1 be two Banach spaces, compatible in the sense of interpolation theory, and assume that there is a subspace W of $X_0 \cap X_1$ which is dense in both X_0 and X_1. For $z \in \overline{S}$ let $\mathcal{T}_z$ be an operator which maps functions in W to measurable functions on $\mathbb{R}^n$; $\mathcal{T}_z$ is then called an analytic family if for any $f \in W$ and almost every $x \in \mathbb{R}^n$ the function $z \to \mathcal{T}_z f(x)$ is analytic in S and continuous and of admissible growth in $\overline{S}$. Now if

$$\|\mathcal{T}_z f\|_{L^{p_i,q_i}} \le C_i(z)\|f\|_{X_i}, \quad i = 0, 1, \tag{2.13}$$

and if $C_i(z)$ is of admissible growth then the result in [16] states that T_θ maps the complex interpolation space $[X_0, X_1]_\theta$ boundedly to L^{p_θ,q_θ}; here $(1/p_\theta, 1/q_\theta) = (1-\theta)(1/p_0, 1/q_0) + \theta(1/p_1, 1/q_1)$. We shall need the following consequence of this result.

Lemma 2.2. *For $k \in \mathbb{Z}$ and $z \in S$ let $T_{k,z}$ be an operator which maps functions in W to measurable functions on $\mathbb{R}^n$ and assume that $T_{k,z}$ is an analytic family, for each k. Suppose that for all $f \in W$*

$$\Big\| \sum_{k\in E} |T_{k,i\tau} f| \Big\|_{L^1} \le C(i\tau)\|f\|_{X_0} \tag{2.14}$$

$$\Big\| \sup_{k\in E} |T_{k,1+i\tau} f| \Big\|_{L^{1,\infty}} \le C(1+i\tau)\|f\|_{X_1} \tag{2.15}$$

for any finite subset $E \subset \mathbb{Z}$, with admissible constants $C(i\tau)$, $C(1+i\tau)$. Let $0 < \theta < 1$. Then

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |T_{k,\theta} f|^q \Big)^{1/q} \Big\|_{L^{1,q}} \lesssim \|f\|_{[X_0,X_1]_\theta} \tag{2.16}$$

if $1/q_\theta = 1 - \theta$.

Proof. Fix $\tilde f \in [X_0, X_1]_\theta$ and $E \subset \mathbb{Z}$ be finite. There are measurable functions g_k such that $\sum |g_k(x)|^{q'} \le 1$ and

$$\Big| \sum_{k\in E} T_{k,\theta}\tilde f(x) g_k(x) \Big| \ge \frac{1}{2} \Big(\sum_{k\in E} |T_{k,\theta}\tilde f(x)|^q \Big)^{1/q}$$

for almost every $x \in \mathbb{R}^n$. Define $g_{k,z}(x) = \frac{g_k(x)}{|g_k(x)|}|g_k(x)|^{q'z}$ if $g_k(x) \neq 0$, and $g_{k,z}(x) = 0$ if $g_k(x) = 0$.

Now define an analytic family by $\mathcal{T}_z f(x) = \sum_{k\in E} T_{k,z} f(x) g_{k,z}(x)$. Then the assumptions (2.14-15) imply the boundedness of $\mathcal{T}_{i\tau}$ from X_0 to L^1 and of $\mathcal{T}_{1+i\tau}$ from X_1 to $L^{1,\infty}$, with admissible constants. One deduces the boundedness of $\mathcal{T}_\theta$ from $[X_0, X_1]_\theta$ to $L^{1,q}$. The constants are independent of E and the choice of $\{g_k\}$. This implies

$$\Big\| \Big(\sum_{k\in E} |T_{k,\theta} \tilde{f}|^q \Big)^{1/q} \Big\|_{L^{1,q}} \leq C \|\tilde{f}\|_{[X_0,X_1]_\theta}$$

with C being independent of E and $\tilde{f}$. The finiteness assumption on E can be removed by applications of the monotone convergence theorem. □

2C. A vector-valued inequality. We shall use the following observation which can serve as an elementary substitute for the failing $L^p(\ell^1)$ inequality for the vector-valued Hardy-Littlewood maximal operator ([12]). It is just the dual version of a scalar maximal inequality.

Lemma 2.3. *Let $\Phi \in L^1(\mathbb{R}^d)$ so that for each $\theta \in S^{d-1}$ the function $r \mapsto |\Phi(r\theta)|$ is decreasing in $r > 0$. Let $\{t_k\}_{k\in\mathbb{Z}}$ be a collection of positive numbers and let P_k be the operator of convolution with $t_k^d \Phi(t_k \cdot)$. Then for $1 \leq p < \infty$*

$$\Big\| \sum_k |P_k f_k| \Big\|_p \leq C_p \|\Phi\|_1 \Big\| \sum_k |f_k| \Big\|_p. \tag{2.17}$$

Proof. We may assume that Φ is nonnegative. Then by duality the assertion follows immediately from the $L^{p'}$ boundedness of the maximal operator $w \mapsto \sup_k |P_k w|$; the latter is a consequence of the method of rotation and the bounds for the one-dimensional Hardy-Littlewood operator (see [26, p.72-73]). □

2D. Averaging functions in $L^{1,q}$. The triangle inequality fails in $L^{1,q}$ if $q > 1$, but the following Lemma, proved for $q = \infty$ by Stein and N. Weiss [28], can often serve as a substitute. For $1 < q < \infty$ the statement follows from the cases $q = 1$ and $q = \infty$ by interpolation.

Lemma 2.4. *Suppose that $\|f_i\|_{L^{1,q}} \leq 1$ and $\sum |c_i| \leq 1$. Then*

$$\Big\| \sum_i c_i f_i \Big\|_{L^{1,q}} \lesssim \sum_i |c_i| (1 + \log_+ |c_i|)^{1-\frac{1}{q}}.$$

3. A stopping time construction

We shall use an abstract form of the Calderón-Zygmund decomposition, in which no nesting or doubling properties are assumed. The argument is related to the stopping time construction in [7].

Lemma 3.1. *Let $\preceq$, $\subseteq$ be partial orders on a set Λ; we also use the notation $\prec$ synonymously with $\precneqq$. Let Γ be a finite subset of Λ, let ν be a non-negative measure on Γ, and let $A : \Lambda \to \mathbb{R}^+$ be a positive function.*

Assume that for each $\gamma \in \Gamma$ and $N > 0$ the set

$$\{\lambda \in \Lambda : A(\lambda) \leq N \text{ and } \gamma \subseteq \lambda\} \tag{3.1}$$

is finite.

Then one can find a subset $\mathcal{B}$ of Λ and a map $q : \Gamma \to \Lambda$ which have the following properties.

(1) *$\gamma \subseteq q(\gamma)$ for all $\gamma \in \Gamma$.*
(2) *If $q(\gamma) \notin \mathcal{B}$ then $q(\gamma) = \gamma$.*

(3)

$$\sum_{\lambda \in \mathcal{B}} A(\lambda) \leq \nu(\Gamma)$$

(4) *For all* $\lambda \in \Lambda$, *we have*

$$\nu(\{\gamma \in \Gamma : q(\gamma) \prec \lambda, \gamma \subseteq \lambda\}) < A(\lambda).$$

Proof. Define

$$\Lambda_* = \Gamma \cup \{\lambda \in \Lambda : A(\lambda) \leq \nu(\Gamma) \text{ and } \gamma \subseteq \lambda \text{ for some } \gamma \in \Gamma\} \tag{3.2}$$

By the finiteness of Γ and the finiteness assumption on the sets (3.1) the set Λ_* is finite. Suppose we have found q and $\mathcal{B}$ with properties (1)-(4) relatively to Λ_* then (1)-(4) are unchanged if Λ_* is enlarged to Λ. Hence it suffices to give a proof under the additional assumption that Λ is finite.

We now induct on the cardinality of Λ. The lemma is vacuously true when Λ is empty, with $\mathcal{B}$ being empty and q being the empty function.

Now suppose inductively that Λ is non-empty, and that the lemma is true for all sets Λ of lesser cardinality. Choose an element $\lambda_{max} \in \Lambda$ which is maximal with respect to the partial ordering $\preceq$, and let $\Lambda' = \Lambda - \{\lambda_{max}\}$. Define the set $\Gamma' \subset \Gamma$ by

$$\Gamma' = \Gamma \cap \Lambda'$$

if the estimate

$$\nu(\{\gamma \in \Gamma : \gamma \subseteq \lambda_{max}\}) < A(\lambda_{max}) \tag{3.3}$$

holds, and by

$$\Gamma' = \{\gamma \in \Gamma : \gamma \not\subseteq \lambda_{max}\}$$

otherwise.

Now apply the induction hypothesis with Λ replaced by Λ', Γ replaced by Γ', and A and ν replaced by their restrictions to Λ' and Γ' respectively. This gives us a set $\mathcal{B}' \subset \Lambda'$ and an assignment $q' : \Gamma' \to \Lambda'$ satisfying analogues (1')-(4') of the desired properties (1)-(4).

Define the subset $\mathcal{B}$ of Λ by $\mathcal{B} = \mathcal{B}'$ if (3.3) holds, and $\mathcal{B} = \mathcal{B}' \cup \{\lambda_{max}\}$ if (3.3) fails. Define $q : \Gamma \to \Lambda$ by setting $q(\gamma) = q'(\gamma)$ if $\gamma \in \Gamma'$, and $q(\gamma) = \lambda_{max}$ if $\gamma \in \Gamma \backslash \Gamma'$.

We now claim that (1)-(4) holds for these choices of $\mathcal{B}$ and q. The claims (1), (2) are easily verified from (1'), (2'), and the construction of $\mathcal{B}$ and q. If (3.3) holds then $\mathcal{B} = \mathcal{B}'$ and (3) follows from (3'); otherwise, $\mathcal{B} = \mathcal{B}' \cup \{\lambda_{max}\}$ and (3) follows from (3'), the construction of Γ', and the failure of (3.3).

It remains to verify (4). First suppose that $\lambda \neq \lambda_{max}$, so that $\lambda \in \Lambda'$. Then (4) follows from (4'), because the elements γ of $\Gamma \backslash \Gamma'$ satisfy $q(\gamma) = \lambda_{max}$ and thus cannot contribute to the left-hand side of (4) by the maximality of λ_{max}.

Now suppose that $\lambda = \lambda_{max}$. If (3.3) holds, then (4) is immediate. If (3.3) fails, then by construction the left-hand side of (4) is zero. Thus (4) holds in all cases, and the induction step is complete. □

We remark that the finiteness assumption (3.1) may be dropped if one is willing to replace the induction by transfinite induction (*i.e.* use Zorn's lemma). One can then prove this lemma for arbitrary Λ.

4. Integrals along plane curves

In this and the next section we shall always be working in the plane $\mathbb{R}^2$. We fix a real number $m > 1$, all constants may implicitly depend on m. We define $H^1_{parabolic}$ to be the one-parameter Hardy space with respect to the scaling (1.6).

The proofs of our results concerning plane curves are based on the following key estimate.

Proposition 4.1. *For each integer l let η_l be a C^∞ function with compact support in $[1/2, 2]$ or in $[-2, -1/2]$, with C^4 norms uniformly bounded in ℓ.*

Let $d\mu_l$ be the measure defined by

$$\int f d\mu_l = \int f(x_1 - t, x_2 - |t|^m) 2^l \eta_l(2^l t)\ dt.$$

Then for any vector-valued function $F = \{f_l\}_{l\in\mathbb{Z}}$,

$$\Big\| \Big(\sum_l |f_l * d\mu_l|^2 \Big)^{1/2} \Big\|_{L^{1,2}} \lesssim \|f\|_{H^1_{parabolic}(\ell^2)}. \tag{4.1}$$

We allow the f_l themselves to be Hilbert space valued functions, and $|\cdot|$ is then to be interpreted as the Hilbert space norm.

In the next section, we shall see how this proposition implies $L^{1,2}$ and $L^{p,2}$ mapping properties for the Hilbert transform on plane curves and similar objects; this will be done by exploiting the fact that the $d\mu_l$ have essentially disjoint frequency supports if some moment conditions are assumed on the η_l. The estimate (4.1) should be compared with the bound

$$\| \sup_l |f * d\mu_l| \|_{L^{1,\infty}} \lesssim \|f\|_{H^1_{parabolic}}$$

proven in Christ [7]. Our techniques shall be closely related to those in that paper.

Proof. We may decompose f atomically as $f = \sum_I c_I P_I(b_I)$, where the I are $2^k \times 2^{mk+\vartheta}$ rectangles with sides parallel to the axes, and k, $km + \vartheta$ are integers, $0 \le \vartheta < 1$. The c_I are non-negative numbers such that $\sum_I c_I \sim \|f\|_{H^1_{parabolic}(\ell^2)}$, the b_I satisfy $\|b_I\|_{L^2(\ell^2)} \lesssim |I|^{-1/2}$, and P_I is the projection operator defined by

$$P_I[b](x) = \Big(b(x) - \frac{1}{|I|} \int_I b(x) dx \Big) \chi_I(x).$$

Note that the definition of P_I makes sense as acting on scalar valued functions or on vector-valued functions, as above. By the translation trick in [7] (attributed to P. Jones) we may assume that the cubes I are dyadic. Henceforth we shall refer to the I as (parabolic) *cubes*. It thus suffices to show the estimate

$$\Big\| \Big(\sum_l \Big| \sum_I c_I P_I[b_{I,l}] * d\mu_l \Big|^2 \Big)^{1/2} \Big\|_{L^{1,2}} \lesssim (\sum_I c_I)(\sup_I |I|^{1/2}) \Big\| \Big(\sum_l |b_{I,l}|^2 \Big)^{1/2} \Big\|_2 \tag{4.2}$$

for arbitrary collections I of cubes, non-negative numbers c_I, and arbitrary measurable functions $b_{I,l}$. By limiting arguments it is sufficient to prove the analogue of (4.2), where the sums in l and the sums in I are extended over finite sets (with bounds independent of the cardinalities). Henceforth we make this finiteness assumption.

Fix the I and c_I. By complex interpolation (Lemma 2.2) it suffices to show that

$$\Big\| \Big(\sum_l \Big| \sum_I c_I P_I[b_{I,l}] * d\mu_l \Big|^q \Big)^{1/q} \Big\|_{L^{1,q}} \lesssim \big(\sum_I c_I \big) \big(\sup_I |I|^{1/q'} \big) \Big\| \Big(\sum_l |b_{I,l}|^q \Big)^{1/q} \Big\|_q \tag{4.3}$$

holds for $q=1$ and $q=\infty$ and all (complex) functions $b_{I,l}$.

When $q=1$, (4.3) simplifies to

$$\sum_l \sum_I c_I \big\| P_I[b_{I,l}] * d\mu_l \big\|_1 \lesssim \big(\sum_I c_I\big) \sup_I \sum_l \|b_{I,l}\|_1,$$

and the claim follows from Young's inequality, the finite mass of $d\mu_l$, and the fact that P_I is bounded on L^1. Thus it remains to prove the $q=\infty$ endpoint, namely

$$\Big\| \sup_l \big| \sum_I c_I P_I[b_{I,l}] * d\mu_l \big| \Big\|_{L^{1,\infty}} \lesssim \big(\sum_I c_I\big) \sup_I \sup_l |I| \, \|b_{I,l}\|_\infty.$$

We may assume that

$$\sup_I \sup_l |I| \, \|b_{I,l}\|_\infty \leq 1 \tag{4.4}$$

Writing $a_{I,l} = P_I[b_{I,l}]$, we thus see that $a_{I,l}$ is supported on I, has mean zero, and $\|a_{I,l}\|_\infty \lesssim |I|^{-1}$, and our task is now to show that

$$\operatorname{meas}\big(\{\sup_l \big| \sum_I c_I a_{I,l} * d\mu_l \big| \gtrsim \alpha\}\big) \lesssim \alpha^{-1} \sum_I c_I \tag{4.5}$$

for all $\alpha > 0$.

Fix $\alpha > 0$. We shall use a sort of Calderón-Zygmund decomposition and will first look at the "good" cubes contributing to a function which is $O(\alpha)$. Let $\mathcal{G}$ be the family of all I for which

$$M\big(\sum_{I'} c_{I'} \frac{\chi_{I'}}{|I'|}\big)(x) \leq \alpha \text{ for some } x \in I; \tag{4.6}$$

here M is the Hardy-Littlewood maximal operator with respect to the scaling (1.6).

We consider the contribution of the cubes in $\mathcal{G}$ to (4.5). The L^∞ norm of $\sum_{I\in\mathcal{G}} c_I \frac{\chi_I}{|I|}$ is $O(\alpha)$, to see this, consider for each for each x_0 the smallest cube in $\mathcal{G}$ containing x_0 and apply (4.6) for this cube. We now apply Chebyshev's inequality and the standard fact [28] that the maximal function associated to the curve $(t, |t|^m)$ is bounded on L^2. This yields

$$\begin{aligned}
&\operatorname{meas}\Big(\{x : \sup_l \big| \sum_{I\in\mathcal{G}} c_I a_{I,l} * d\mu_l \big| \geq \alpha\}\Big) \\
&\quad \leq \alpha^{-2} \Big\| \sup_l \big| \sum_{I\in\mathcal{G}} c_I a_{I,l} * d\mu_l \big| \Big\|_2^2 \lesssim \alpha^{-2} \Big\| \sup_l \sum_{I\in\mathcal{G}} c_I \frac{\chi_I}{|I|} * |d\mu_l| \Big\|_2^2 \\
&\quad \lesssim \alpha^{-2} \Big\| \sum_{I\in\mathcal{G}} c_I \frac{\chi_I}{|I|} \Big\|_2^2 \lesssim \alpha^{-1} \Big\| \sum_{I\in\mathcal{G}} c_I \frac{\chi_I}{|I|} \Big\|_1 \lesssim \alpha^{-1} \sum_I c_I.
\end{aligned} \tag{4.7}$$

Thus we may restrict our attention to the "bad" cubes. By the Hardy-Littlewood inequality, the $L^{1,\infty}$ norm of $M(\sum_I c_I \chi_I / |I|)$ is $O(\sum_I c_I)$, and so by the definition of $\mathcal{G}$

$$\operatorname{meas}\big(\cup_{I\notin\mathcal{G}} I\big) \lesssim \alpha^{-1} \sum_I c_I.$$

Let $C > 1$ and CI denote the cube expanded by C (with same center as I). By the Hardy-Littlewood inequality again we have

$$\text{meas}\big(\cup_{I \notin \mathcal{G}} CI \big) \lesssim \alpha^{-1} \sum_I c_I. \tag{4.8}$$

To complete the proof of (4.5) we shall prove the stronger square-function estimate

$$\text{meas}\Big(\big\{ x : \big(\sum_l \big| \sum_{I \notin \mathcal{G}} c_I a_{I,l} * d\mu_l(x) \big|^2 \big)^{1/2} \geq \alpha \big\} \Big) \lesssim \alpha^{-1} \sum_I c_I. \tag{4.9}$$

In order to prove (4.9) we use an abstract version of the Calderón-Zygmund decomposition based on Lemma 3.1. We first describe the sets Λ and Γ which occur in this lemma. If $m \geq 2$ we define Λ as the set of all dyadic rectangles Q of dimensions $2^\sigma \times 2^{\sigma+(m-1)\tau+\vartheta}$ for integers σ, τ and for $\vartheta \in [0,1)$, where $\sigma \leq \tau$, and $(m-1)\tau + \vartheta$ is the smallest integer $\geq (m-1)\tau$ (*i.e.* $\vartheta = 0$ if m is an integer). Note that σ, τ and ϑ are unique for each Q and we shall write $\sigma = \sigma(Q)$, $\tau = \tau(Q)$, $\vartheta = \vartheta(Q)$. If $1 < m < 2$ we define Λ similarly, with the additional requirement that we only admit those τ for which the fractional part of $(m-1)(\tau - \sigma)$ is $< m-1$; this is to ensure that $\tau(Q)$ is well defined. In both cases the subset Γ is the set of parabolic cubes I for which $c_I \neq 0$ and which do not belong to $\mathcal{G}$; by assumption Γ is finite. Note that one has $\tau(I) = \sigma(I)$ for parabolic cubes I.

We wish to partially order the set Λ by requiring $Q \prec Q'$ if $\tau(Q) < \tau(Q')$; note that then Q and Q' are incomparable under $\preceq$ if $\tau(Q) = \tau(Q')$ and $Q \neq Q'$. Finally we take set inclusion $\subseteq$ as the second partial order in Lemma 3.1.

We define the tendril $T(Q)$ to be the set

$$T(Q) = \{ x + (t, |t|^m) : x \in 2Q, |t| \leq 2^{\tau(Q)+2} \}. \tag{4.10}$$

Note that $|T(Q)| \sim 2^{\sigma(Q)+m\tau(Q)} + 2^{2\sigma(Q)+(m-1)\tau(Q)}$ for any rectangle Q parallel to the axes, and therefore

$$|T(Q)| \sim 2^{\sigma(Q)+m\tau(Q)} \qquad \text{for } Q \in \Lambda. \tag{4.11}$$

The function $A(Q)$ in Lemma 3.1 is then defined by

$$A(Q) = \alpha 2^{\sigma(Q)+m\tau(Q)},$$

and the measure ν is defined by

$$\nu(\{I\}) = c_I.$$

The finiteness condition in the proof of Lemma 3.1 is easily verified and we find a map $I \mapsto q(I)$ defined on Γ so that $I \subseteq q(I)$ and

$$\sum_{\substack{I \in \Gamma \\ q(I) \prec Q \\ I \subseteq Q}} c_I < \alpha |T(Q)| \tag{4.12}$$

for all $Q \in \Lambda$, and

$$\text{meas}\big(\cup_{I \in \Gamma} T(q(I)) \big) \lesssim \frac{1}{\alpha} \sum_I c_I + \text{meas}\big(\cup_{I \in \Gamma} T(I) \big);$$

the latter inequality follows from statements (2) (3), (4) of Lemma 3.1. Since for parabolic cubes I the tendril $T(I)$ is contained in a fixed dilate of I and since $\Gamma \cap \mathcal{G} = \emptyset$ one has actually

$$\text{meas}\big(\cup_{I \in \Gamma} T(q(I)) \big) \lesssim \frac{1}{\alpha} \sum_I c_I, \tag{4.13}$$

by (4.8).

For any I, l we see that $a_{I,l} * d\mu_l$ is supported in $T(q(I))$ if $l < \tau(q(I))$. In view of (4.13) the inequality (4.9) follows from

$$\operatorname{meas}\Big(\{x : (\sum_l |\sum_{I: l \geq \tau(q(I))} c_I a_{I,l} * d\mu_l|^2)^{1/2} \geq \alpha\}\Big) \lesssim \alpha^{-1} \sum_I c_I.$$

It suffices by Chebyshev's inequality to prove the L^2 estimate

$$\Big\| \Big(\sum_l \Big| \sum_{\substack{I \in \Gamma \\ l \geq \tau(q(I))}} c_I a_{I,l} * d\mu_l \Big|^2 \Big)^{1/2} \Big\|_2^2 \lesssim \alpha \sum_I c_I. \tag{4.14}$$

Let

$$\Gamma(m) = \{I \in \Gamma : \tau(q(I)) = m\}. \tag{4.15}$$

By the triangle inequality it suffices to show

$$\Big\| \Big(\sum_l |\sum_{I \in \Gamma(l-s)} c_I a_{I,l} * d\mu_l|^2 \Big)^{1/2} \Big\|_2^2 \lesssim 2^{-s} \alpha \sum_I c_I$$

for all $s \geq 0$.

Fix s. It then suffices to show that for each l

$$\Big\| \sum_{I \in \Gamma(l-s)} c_I a_{I,l} * d\mu_l \Big\|_2^2 \lesssim 2^{-s} \alpha \sum_{I \in \Gamma(l-s)} c_I \tag{4.16}$$

for each l, since the claim follows by summing in l. By scaling (with respect to the parabolic dilations (1.6) and taking into account the definition of $\tau(Q)$ we see that it suffices to prove (4.16) for $l = 0$. Expanding the left-hand side of (4.16), we reduce to

$$\sum_{I, I' \in \Gamma(-s)} c_I c_{I'} |\langle a_{I,0} * d\mu_0, a_{I',0} * d\mu_0 \rangle| \lesssim 2^{-s} \alpha \sum_{I \in \Gamma(-s)} c_I.$$

By symmetry we may assume that $|I'| \leq |I|$. It then suffices to show that

$$\sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I|}} c_{I'} |\langle a_{I,0} * d\mu_0, a_{I',0} * d\mu_0 \rangle| \lesssim 2^{-s} \alpha, \tag{4.17}$$

for all $I \in \Gamma(-s)$.

Fix $I \in \Gamma(-s)$ with center x_I. I has dimension $2^{\tau(I)} \times 2^{m\tau(I) + \vartheta(I)}$; since $I \subseteq q(I)$ by Lemma 3.1, (1), and $\sigma(q(I)) \leq \tau(q(I))$ by definition of Λ we see that

$$\tau(I) \leq \tau(q(I)) = -s. \tag{4.18}$$

Rewrite the left-hand side of (4.12) as

$$\sum_{I' : |I'| \leq |I|, \tau(q(I')) = -s} c_{I'} |\langle a_{I,0} * F, a_{I',0} \rangle| \tag{4.19}$$

where $F = d\mu_0 * \widetilde{d\mu_0}$ (and $\tilde{}$ refers to reflection in the argument). Observe that F is supported on a sector

$$\{(x_1, x_2) : |x_2| \lesssim |x_1|\}$$

and obeys the estimates

$$|\nabla^\alpha F(x)| \lesssim |x|^{-1-|\alpha|}$$

for all multiindices α with $|\alpha| \leq 2$. From the size conditions on $a_{I,0}$, this implies

$$|a_{I,0} * F(x)| \lesssim 2^{-\tau(I)}$$

and by the moment conditions on $a_{I,0}$

$$|\nabla^\alpha (a_{I,0} * F)(x)| \lesssim 2^{\tau(I)} |x - x_I|^{-2-|\alpha|}, \quad \text{if } |x - x_I| \geq 2^{\tau(I)+1}, \ |\alpha| \leq 1.$$

This in turn implies from the size and moment conditions on $a_{I',0}$ and the assumption $|I'| \leq |I|$ that

$$|\langle a_{I,0} * F, a_{I',0} \rangle| \lesssim 2^{2\tau(I)} \mathrm{diam}(I \cup I')^{-3},$$

where the diameter is respect to the Euclidean metric.

Thus it suffices to show that

$$\sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I|}} c_{I'} \mathrm{diam}(I \cup I')^{-3} \lesssim 2^{-2\tau(I)} 2^{-s} \alpha. \tag{4.20}$$

For $\sigma \leq -s$, let $\mathcal{R}_{\sigma,s}$ be the set of dyadic rectangles of dimensions $(2^\sigma, 2^{\sigma-(m-1)(s-1)+\vartheta})$ so that $0 \leq \vartheta < 1$. Observe that $\mathcal{R}_{\sigma,s}$ is a subset of Λ consisting of rectangles R with $\tau(R) = -s+1$. Also let $\mathcal{W}_\sigma$ be the set of isotropic dyadic cubes of dimensions $(2^\sigma, 2^\sigma)$; then each $W \in \mathcal{W}_\sigma$ is a union of $\sim 2^{(m-1)(s-1)}$ rectangles in $\mathcal{R}_{\sigma,s}$, with disjoint interiors.

If $I' \in \Gamma(-s)$ with $|I'| \leq |I|$ then I' has dimensions $(2^{\sigma(I')}, 2^{\sigma(I')-(m-1)s})$ and $\sigma(I') \leq \sigma(I) = \tau(I)$, and therefore every such I' is contained in a unique rectangle $R \in \mathcal{R}_{\tau(I),s}$. Since $\tau(q(I')) = -s$ and $\tau(R) = -s+1$ we have from Lemma 3.1, (4),

$$\sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I| \\ I' \subseteq R}} c_{I'} \lesssim \alpha |T(R)| \lesssim \alpha 2^{\tau(I)-ms}$$

and therefore

$$\begin{aligned}
&\sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I|}} c_{I'} \mathrm{diam}(I \cup I')^{-3} \\
&\qquad = \sum_{W \in \mathcal{W}_{\tau(I)}} \sum_{\substack{R \in \mathcal{R}_{\tau(I),s} \\ R \subset W}} \sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I| \\ I' \subset R}} c_{I'} \mathrm{diam}(I \cup I')^{-3} \\
&\qquad \lesssim \alpha 2^{\tau(I)-ms} \sum_{W \in \mathcal{W}_{\tau(I)}} (2^{\tau(I)} + \mathrm{dist}(W, I))^{-3} \mathrm{card}(\{R \in \mathcal{R}_{\tau(I),s} : R \subset W\}) \\
&\qquad \lesssim \alpha 2^{\tau(I)-s} \sum_{W \in \mathcal{W}_{\tau(I)}} (2^{\tau(I)} + \mathrm{dist}(W, I))^{-3} \lesssim 2^{-2\tau(I)} \alpha 2^{-s}
\end{aligned}$$

which is (4.20). $\square$

5. Integrals along plane curves, cont.

We now prove Theorems 1.4 and 1.5. Following [5] we work with an angular Littlewood-Paley decomposition.

Let $\zeta \in C_0^\infty(\mathbb{R}_+)$ so that $\zeta(s) = 1$ if $s \in ((10^m m)^{-1}, 10^m m)$ and define Q_l by

$$\widehat{Q_l f}(\xi) = q_l(\xi)\widehat{f}(\xi) = \zeta(2^{l(m-1)}|\xi_1|/|\xi_2|)\widehat{f}(\xi). \tag{5.1}$$

The operators Q_l form a Littlewood-Paley family of multipliers supported in sectors. Note that $q_l(\xi) = 1$ whenever ξ is normal to the curves $(t, \pm|t|^m)$ if $2^{l-1} \le |t| \le 2^{l+1}$.

Let χ_0 be a smooth and even function on $\mathbb{R}$ so that $\chi_0(s) = 1$ if $|s| \le 1/2$ and $\chi_0(s) = 0$ of $|s| \ge 1$. Define $\mathcal{P}_l$ by $\widehat{\mathcal{P}_l f}(\xi) = \chi_0(|(2^{-l}\xi_1, 2^{-lm}\xi_2)|)\widehat{f}(\xi)$.

Observe that the multiplier q_l satisfies the estimates $\partial^\alpha q_l(\xi) = O(|\xi_1|^{-\alpha_1}|\xi_2|^{-\alpha_2})$ uniformly in l. Therefore by standard product theory we have the estimate

$$\big\|\{(Id - \mathcal{P}_l)Q_l f\}\big\|_{H^1_{prod}(\ell^2)} \lesssim \big\|\{Q_l f\}\big\|_{H^1_{prod}(\ell^2)} \lesssim \|f\|_{H^1_{prod}} \tag{5.2}$$

where f itself may be a Hilbert-space valued function.

We now consider the maximal function Mf. We show that

$$\|\sup_l |d\mu_l * f|\|_{L^{1,2}} \lesssim \|f\|_{H^1_{prod}}, \tag{5.3}$$

where $d\mu_l$ is a measure as in Proposition 4.1.

Given (5.3) we show the same bound for the nondyadic maximal function by a standard argument. After a straightforward application of Lemma 2.4 we may assume that η has support in $(-2^{-5}, 2^{-5})$ and vanishes in $(-2^{-6}, 2^{-6})$. Let $\tilde{\eta}$ be supported in $\cup \pm (2^{-8}, 2^{-3})$ and equal to 1 on $\cup \pm (-2^{-7}, 2^{-2})$. We use a Fourier expansion and write for $1/2 \le s \le 2$

$$\frac{1}{s}\eta(\frac{t}{s}) = \tilde{\eta}(t)\sum_{k\in\mathbb{Z}} c_k(s)e^{2\pi i k t}$$

where $c_k(s) = O((1+|k|)^{-N})$ uniformly in $s \in [1/2, 2]$. We set

$$d\mu_{k,l} = \int f(t, |t|^m)2^l\tilde{\eta}(2^l t)e^{2\pi i k 2^l t}dt.$$

and $M_k f(x) = \sup_l |f * d\mu_{k,l}|$. An application of (5.3) shows that M_k maps H^1 to $L^{1,2}$ with norm $O((1+|k|)^4)$ and since $Mf(x) \lesssim \sum_k (1+|k|)^{-N} M_k f(x)$ we obtain the inequality for the nondyadic maximal operator from another application of Lemma 2.4.

Now we turn to the proof of (5.3). As in [5] the idea is to approximate $d\mu_l$ by $Q_l(Id - \mathcal{P}_l)d\mu_l$ in order to apply Proposition 4.1 and (5.2).

Using straightforward integration by parts arguments we observe that the functions $\mathcal{P}_0 d\mu_0$ and $(Id - \mathcal{P}_0)(Id - Q_l)d\mu_0$ are Schwartz functions. By rescaling this, using (1.6), we see that the maximal functions $\sup_l |f * \mathcal{P}_l d\mu_l|$ and $\sup_l |f * (Id - \mathcal{P}_l)(Id - Q_l)d\mu_l|$ are dominated by nonisotropic version of the grand maximal function (with respect to (1.6)) which maps $H^1_{parabolic}$ and hence H^1_{prod} to L^1. It thus suffices to show that

$$\|\sup_l |f * (Id - \mathcal{P}_l)Q_l d\mu_l|\|_{L^{1,2}} \lesssim \|f\|_{H^1_{prod}}.$$

Writing $f_l = (Id - \mathcal{P}_l)Q_l f$, we can dominate the left-hand side by the $L^{1,2}$ norm of the square-function $(\sum_l |f_l * d\mu_l|^2)^{1/2}$. With this choice of f_l the inequality

$$\Big\| \Big(\sum_l |d\mu_l * f_l|^2 \Big)^{1/2} \Big\|_{L^{1,2}} \lesssim \|f\|_{H^1_{prod}} \tag{5.4}$$

follows from from Proposition 4.1, the embedding $H^1_{prod}(\ell^2) \subset H^1_{parabolic}(\ell^2)$ and (5.2).

Now consider the analytic family H_γ (and in particular the Hilbert transform $H = H_0$). We may decompose

$$H_\gamma f = \sum_l f * d\sigma_l^\gamma$$

where

$$\langle d\sigma_l^\gamma, f\rangle = \int f(t, |t|^m) 2^l \chi(2^l t) |t|^{\gamma_1 + \gamma_2 m} \frac{dt}{t}$$

and $\chi(t) = \chi_0(t) - \chi_0(t/2)$. Note that χ is an even function. The functions $\mathcal{P}_0 d\sigma_0^\gamma$ and $(Id - \mathcal{P}_0)(Id - Q_l) d\sigma_0^\gamma$ are Schwartz functions as before, but also have mean zero and so their Fourier transforms decay at 0 as well as infinity.

Summing this, we see that $\mathcal{D}^\gamma \sum_l (Id - \mathcal{P}_l)(Id - Q_l) d\sigma_l$ and $\mathcal{D}^\gamma \sum_l \mathcal{P}_l d\sigma_l$ are standard product type Calderón-Zygmund kernels and so convolution with these kernels will preserve L_p, $1 < p \leq 2$ and H^1_{prod}. It thus suffices to show that

$$\Big\| \sum_l (Id - \mathcal{P}_l) Q_l \mathcal{D}^\gamma d\sigma_l^\gamma * f \Big\|_{H^{1,2}_{prod}} \lesssim \|f\|_{H^1_{prod}} \quad \text{if } \mathrm{Re}(\gamma_1 + \gamma_2 m) = 0 \tag{5.5}$$

and

$$\Big\| \sum_l (Id - \mathcal{P}_l) Q_l \mathcal{D}^\gamma d\sigma_l^\gamma * f \Big\|_2 \lesssim \|f\|_2 \quad \text{if } \mathrm{Re}(\gamma_1 + \gamma_2 m) = 1/2 \tag{5.6}$$

with constants depending polynomially on γ.

To see (5.6) we note that a standard stationary phase calculation yields that $|\widehat{d\sigma_0^\gamma}(\xi)| \lesssim (1 + |\xi|)^{-1/2}$. By scale invariance we obtain the uniform L^2 boundedness of the operators with convolution kernels $(Id - \mathcal{P}_l)\mathcal{D}^\gamma d\sigma_l^\gamma$ if $\mathrm{Re}(\gamma_1 + m\gamma_2) = 1/2$. The inequality (5.6) follows now from the almost orthogonality of the operators Q_l.

In order to prove (5.5) it suffices to show that

$$\Big\| \Big(\sum_{k_1,k_2} \big| \sum_l (Id - \mathcal{P}_l) Q_l L_{k_1,k_2} f * d\sigma_l^\gamma \big|^2 \Big)^{1/2} \Big\|_{L^{1,2}} \lesssim \Big\| \Big(\sum_{k_1,k_2} |L_{k_1,k_2} f|^2 \Big)^{1/2} \Big\|_1, \tag{5.7}$$

by the square function characterization of $H^{1,2}_{prod}$; here L_{k_1,k_2} are as in §2. For each k_1, k_2 there are at most $O(1)$ indices l for which $(Id - \mathcal{P}_l) Q_l L_{k_1,k_2}$ does not vanish, so we may majorize the left-hand side of (5.7) by

$$\Big\| \Big(\sum_{k_1,k_2} \sum_l \big| (Id - \mathcal{P}_l) Q_l L_{k_1,k_2} f * d\sigma_l^\gamma \big|^2 \Big)^{1/2} \Big\|_{L^{1,2}}.$$

By Proposition 4.1 we may majorize this in turn by

$$\big\| \{ (Id - \mathcal{P}_l) Q_l L_{k_1,k_2} f \}_{l,k_1,k_2 \in \mathbb{Z}} \big\|_{H^1_{parabolic}(\ell^2)}.$$

But this is bounded by $\|f\|_{H^1_{prod}}$, by standard arguments similar to the proof of (5.2) above. This concludes the proof of Theorem 1.5. To see that the Hilbert transform H maps H^1_{prod} to $L^{1,2}$ we use in addition the product version of inequality (2.7).

Finally we prove Corollary 1.6. Define the measures $d\nu_l^\alpha$ by

$$\int f d\nu_l^\alpha = \int f(t, |t|^m) 2^l (\chi(2^l t)) \eta(t) |t|^{m\alpha} \frac{dt}{t}$$

and set $d\nu_l = d\nu_l^{1/m}$. We use duality and prove that convolution with $(Id - \Delta)^{1/2m} \sum_l d\nu_l$ maps $L^{m'}$ to $L^{m',2}$.

It is easy to see that for $\theta_1 + \theta_2 < 1$, $\theta_1 \geq 0$, $\theta_2 \geq 0$ the functions $(Id - \Delta)^{\theta/2} \sum_l (Id - \mathcal{P}_l)(Id - Q_l) d\nu_l * f$ and $(Id - \Delta)^{\theta/2} \sum_l \mathcal{P}_l d\nu_l * f$ are dominated by a constant times the nonisotropic Hardy-Littlewood maximal function of f.

Let $\widetilde{Q}_l = \widetilde{q}_l(D)$ is defined similarly as Q_l but with $q_l \widetilde{q}_l = q_l$. Observe that in view of the compact support of η we have $d\nu_l^\alpha = 0$ if $l > C_1$ for suitable C_1. Moreover, if $l \leq C_1$, we see, using the definition of Q_l and the Marcinkiewicz multiplier theorem that for $\alpha \geq 0$, that

$$\left\| (Id - \Delta)^{\alpha/2} (Id - \mathcal{P}_l) \widetilde{Q}_l g \right\|_{L^{m',2}} \lesssim \left\| \mathcal{D}_2^\alpha Q_l g \right\|_{L^{m',2}}.$$

Thus it remains to show that

$$\left\| \{ \mathcal{D}_2^\alpha Q_l d\nu_l^\alpha * f \} \right\|_{H^{p,2}_{prod}(\ell^2)} \lesssim \|f\|_{H^p_{prod}}, \quad \mathrm{Re}(\alpha) = 1 - 1/p,$$

for $1 \leq p \leq 2$. This is done by a reprise of the arguments above.

6. Rough homogeneous kernels: Preliminary reductions

Let χ_0 be a radial bump function which is 1 on $\{x : |x| \leq 1/2\}$ and zero on $\{x : |x| > 1\}$, and $\chi(x) = \chi_0(x) - \chi_0(x/2)$ is then a function on the unit annulus. We also denote by $\tilde{\chi}(t)$ the restriction of χ to the positive real line $\mathbb{R}^+$.

In what follows we shall work with the Littlewood-Paley operators introduced in Lemma 2.1 (with $r = 3$) and decompose the identity as $Id = \sum_k L_0^k L_1^k L_2^k L_3^k$; we assume that the numbers N_0, ε in Lemma 2.1 are chosen so that $N_0 \geq 100d$ and $\varepsilon \leq 10^{-10d}$.

Let δ_j be the dilation operator defined by

$$\delta_j g(x) = 2^{-jd} g(2^{-j} x),$$

and let $\mathcal{A}$ be the averaging operator defined by

$$\mathcal{A} g(x) = C^{-1} \int \tilde{\chi}(t) t^{-d} g(t^{-1} x) \frac{dt}{t},$$

where $C = \int \tilde{\chi}(t) \frac{dt}{t}$ is a normalization constant.

Since K is homogeneous of degree $-d$ we have the decomposition

$$K = \sum_j \delta_j \mathcal{A}[K\chi]. \tag{6.1}$$

If the restriction Ω of K to the unit sphere belongs to $L \log^2 L(S^{d-1})$ then $K\chi \in L \log^2 L(\mathbb{R}^d)$ and, since standard Calderón-Zygmund operators map $L \log^2 L$ to $L \log L$ the $L \log^2 L$ assumption for $K\chi$ is implied by

$$\Big(\sum_k |L_0^k (K\chi)|^2 \Big)^{1/2} \in L \log L. \tag{6.2}$$

In the present and subsequent section we prove the following stronger version of Theorem 1.1.

Theorem 6.1. *Let K be homogeneous of degree $-d$ and assume that the restriction Ω to S^{d-1} is an integrable function satisfying $\int \Omega d\sigma = 0$. Suppose that (6.2) holds. Then the operator T_Ω maps H^1 boundedly to $L^{1,2}$ and also to the Hardy-Lorentz space $H^{1,2}$.*

We also have

Theorem 6.2. *Let $K_0(r\theta) = \tilde{\chi}(r)\Omega(\theta)$ and assume $\Omega \in L^1(S^{d-1})$ and $(\sum_k |L_0^k(K_0)|^2)^{1/2} \in L\log L$. Then M_Ω maps H^1 boundedly to $L^{1,2}$.*

We shall prove Theorem 6.1. To prove Theorem 6.2 we use the argument in §5 to reduce to a version which involves only dyadic dilations. The proof of the relevant estimate for this dyadic maximal operator is similar to the proof of Theorem 6.1 and therefore omitted.

Let $\mathcal{T}$ be the operator defined by

$$\mathcal{T}f = \sum_j \delta_j \mathcal{A}[K\chi] * f \tag{6.3}$$

We now have to show that $\mathcal{T}$ is bounded from H^1 to $H^{1,2}$. The $H^1 \to L^{1,2}$ boundedness follows then from (2.7) and limiting arguments. In our proof of (6.3) we shall assume that the sum in j is actually finite, but prove a bound which is independent of this finiteness assumption. Again a limiting argument proves the general case.

We now decompose f in the standard manner as $f = \sum c_I a_I$, where c_I are nonnegative constants such that $\sum_I c_I \lesssim \|f\|_{H^1}$, and a_I is an atom supported on I with mean zero and such that $\|a_I\|_\infty \lesssim |I|^{-1}$ ([26]). The center of the atom will be denoted by x_I and we may assume that each atom has sidelength 2^{i_I} where i_I is an integer.

For technical reasons we wish to suppress low frequencies in our atoms. Let

$$\widetilde{a}_I = \sum_{l \geq -C_0} L_0^{l-i_I} L_1^{l-i_I} L_2^{l-i_I} L_3^{l-i_I} a_I,$$

We assume

$$\Big\| \Big(\sum_k |L_0^k(K\chi)|^2 \Big)^{1/2} \Big\|_{L\log L} \leq 1$$

(working with the norm $\|g\|_{L\log^\gamma L} = \inf\{\lambda > 0 : \int \frac{|g(t)|}{\lambda} \log^\gamma(e + \frac{|g(x)|}{\lambda}) dx \leq 1\}$) and we shall prove that

$$\Big\| \sum_I c_I \sum_j \delta_j \mathcal{A}[(K\chi)] * \widetilde{a}_I \Big\|_{H^{1,2}} \leq B \sum_I c_I \tag{6.4}$$

where B is a constant depending only on d. Now the cancellation of the atoms shows that $\|a_I - \tilde{a}_I\|_{H^1} \lesssim 2^{-\varepsilon C_0}$, and so

$$\|f - \sum_I c_I \tilde{a}_I\| \lesssim 2^{-\varepsilon C_0} \|f\|_{H^1}. \tag{6.5}$$

Let $\|\mathcal{T}\|$ denote the $H^1 \to H^{1,2}$ operator-norm, which because of our finiteness assumptions is a priori finite. (6.5) implies

$$\|\mathcal{T}f\|_{H^{1,2}} \lesssim 2^{-\epsilon C_0} \|\mathcal{T}\| \|f\|_{H^1} + B \sum c_I.$$

Therefore, if C_0 in the definition of the $\widetilde{a}_I$ is chosen large enough, this implies that $\|\mathcal{T}\| \lesssim B$.

In what follows we may assume

$$\sum c_I \leq 1. \tag{6.6}$$

We now dispose of the contributions when $j \leq i_I + 2C_0$. We claim this portion is not only in $H^{1,2}$ but is actually in H^1. Since H^1 is a Banach space we may restrict ourselves to a single cube I, so that it suffices to show that

$$\Big\| \sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * \tilde{a}_I \Big\|_{H^1} \lesssim 1.$$

This we rewrite as

$$\Big\| \sum_{l \geq -C_0} L_0^{l-i_I} L_1^{l-i_I} L_2^{l-i_I} [\sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I] \Big\|_{H^1} \lesssim 1.$$

By the analogue of (2.6) for the Littlewood-Paley operators $L_0^k L_1^k L_2^k$ it thus suffices to show

$$\Big\| \Big(\sum_{l \geq -C_0} | \sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I |^2 \Big)^{1/2} \Big\|_1 \lesssim 1.$$

Since the expression inside the norm is supported in a fixed dilate of I, it suffices by the Cauchy-Schwarz inequality to bound

$$\Big\| \Big(\sum_{l \geq -C_0} | \sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I |^2 \Big)^{1/2} \Big\|_2 \lesssim |I|^{-1/2}.$$

By modifying the method of rotations argument in [4] we see that the operator with convolution kernel $\sum_{j < i_I + 2C_0} \delta_{j \leq i_I + s}[K\chi]$ is bounded on L^2; hence the above reduces to

$$\Big(\sum_{l \geq -C_0} \| L_3^{l-i_I} a_I \|_2^2 \Big)^{1/2} \lesssim |I|^{-1/2}. \tag{6.7}$$

But this follows from the L^2 estimates on a_I and the almost orthogonality of the $L_3^{l-i_I}$.

We now turn to the contributions $j > i_I + 2C_0$ and we wish to establish

$$\Big\| \sum_I c_I \sum_{l \geq -C_0} L_0^{l-i_I} L_1^{l-i_I} L_2^{l-i_I} [\sum_{j > i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I] \Big\|_{H^{1,2}} \lesssim 1.$$

We set $a_{I,l} = L_3^{l-i_I} a_I$ and let $\{e_j\}$ be the standard orthonormal basis of unit vectors in ℓ^2. By the remark following Lemma 2.1 we reduce to show that

$$\Big\| \sum_I c_I \sum_{l \geq -C_0} L_1^{l-i_I} [\sum_{j > i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_2^{l-i_I} a_{I,l}] e_{l-i_I} \Big\|_{L^{1,2}(\ell^2)} \lesssim 1.$$

By Lemma 2.1 we may decompose

$$K\chi = S_1^0(K\chi) + \sum_{k=1}^{\infty} L_1^k L_0^k (K\chi).$$

One easily checks that the convolution operator with kernel $K = \sum_j \delta_j \mathcal{A}[S_1^0 K\chi]$ is a standard Calderón-Zygmund operator. Indeed using the cancellation of the functions $L_2^{l-i_I} a_{I,l}$ it is easy to see that for a fixed cube I

$$\Big\| \Big(\sum_{l \geq -C_0} \Big| \sum_{j > i_I + 2C_0} \delta_j [\mathcal{A} S_1^0 (K\chi)] * L_2^{l-i_I} a_{I,l} \Big|^2 \Big)^{1/2} \Big\|_1 \lesssim 1,$$

and the resulting $H^1 \to L^1(\ell^2)$ inequality follows for this part.

Therefore it suffices to prove that

$$\Big\|\sum_I c_I \sum_{j>i_I+2C_0}\sum_{l\ge -C_0} L_1^{l-i_I}\delta_j\mathcal{A}(\sum_{k>0} L_1^k K^k) * L_2^{l-i_I} a_{I,l} e_{l-i_I}\Big\|_{L^{1,2}(\ell^2)} \lesssim 1, \tag{6.8}$$

where still $a_{I,l}=L_3^{l-i_I}a_I$, and $K^k=L_0^k(K\chi)$.

We can rewrite the desired estimate for this portion using the identity

$$L_1^m\delta_j=\delta_j L_1^{j+m}.$$

Consequently we have to prove for $q=2$ the inequality

$$\begin{aligned}\Big\|\sum_I\sum_{j>2C_0+i_I}\sum_{l\ge -C_0} c_I\delta_j(L_1^{l-i_I+j}\mathcal{A}[\sum_{k>0}L_1^kK^k]) * L_2^{l-i_I}a_{I,l}e_{l-i_I}\Big\|_{L^{1,q}(\ell^q)}\\ \lesssim \sup_I |I|^{1-1/q}\Big(\sum_l \|a_{I,l}\|_q^q\Big)^{1/q}\Big\|\big(\sum_k |K^k|^q\big)^{1/q}\Big\|_{L\log^{2-\frac{2}{q}}L}\end{aligned} \tag{6.9}$$

for *arbitrary* measurable functions K^k on $\{x: 1/4\le|x|\le 4\}$ and $a_{I,l}$ on CI. (6.8) follows then by using also (6.7).

We shall deduce the inequality for $q=2$ from the inequality (6.9) for $q=1$ and the obvious modification of (6.9) for $q=\infty$.

Notice that

$$\begin{aligned}\big\|L_1^{l-i_I+j}\mathcal{A}[L_1^kK^k]\big\|_{L^1\to L^1} &\le \int|\tilde\chi(t)|t^{-d}\big\|\psi_1^{l-i_I+j} * t^{-d}\psi_1^k(t^{-1}\cdot)\big\|_1\|K^k\|_1 dt\\ &\lesssim 2^{-|l-i_I+j-k|}\|K_k\|_1\end{aligned} \tag{6.10}$$

where we have used the cancellation of the Littlewood-Paley kernels. The last estimate immediately implies (6.9) for $q=1$. The nontrivial part concerns the estimate for $q=\infty$ which is proved in the next section. From these two estimates we deduce (6.9) for $q=2$ by complex interpolation, using Lemma 2.2. Assuming

$$\Big\|\big(\sum_k|K^k|^2\big)^{1/2}\Big\|_{L\log L}\le 1,$$

we consider the analytic family $K_z=\{K_z^k\}_{k\in\mathbb{Z}}$ defined by

$$K_z^k(x)=K^k(x)|K^k(x)|^{1-2z}\big|K(x)\big|_{\ell^2}^{2z-1}\big[\log(e+|K(x)|_{\ell^2})\big]^{1-2z}$$

if $K^k(x)\ne 0$ and by $K_z^k(x)=0$ otherwise. Then $\|K_{i\tau}\|_{L^1(\ell^1)}\lesssim 1$ and $\|K_{1+i\tau}\|_{L\log^2L(\ell^\infty)}\lesssim 1$. The rest is straightforward.

7. Rough homogeneous kernels: The weak type estimate

We are now proving the analogue of (6.9) for $q=\infty$. In addition to (6.6) we may also suppose that

$$\sup_I\sup_l\|a_{I,l}\|_\infty\le 1,\qquad \big\|\sup_k|K_k|\big\|_{L\log^2L}\le 1 \tag{7.1}$$

and show that for $\alpha>0$

$$\text{(7.2)}\qquad \operatorname{meas}\Big(\{x: |\sum_I \sum_{j>2C_0+i_I} \sum_{l\ge -C_0} c_I\delta_j(L_1^{l-i_I+j}\mathcal{A}[\sum_{k>0} L_1^k K^k]) * L_2^{l-i_I} a_{I,l} e_{l-i_I}\big|_{\ell^\infty} > \alpha\}\Big) \lesssim \alpha^{-1}.$$

Let $F = \sum_I c_I \frac{\chi_I}{|I|}$. Since $\|F\|_1 \lesssim 1$, we may apply the standard dyadic Calderón-Zygmund decomposition to F at level α, and obtain a collection of disjoint dyadic cubes $\mathcal{J} = \{J\}$ such that $\sum_J |J| \lesssim \alpha$, $\int_J F(x)\,dx \lesssim \alpha|J|$, and such that F is $O(\alpha)$ outside of $\bigcup_J J$.

To every dyadic cube I we assign a nonnegative integer t_I as follows. If I is not contained in any of the J, then $t_I = 0$. If I is a subset of a cube $J \in \mathcal{J}$, then t_I is chosen so that the sidelength of J is 2^{t_I} times the sidelength of I. One can view t_I as a stopping time; roughly speaking, $2^{t_I} I$ is the largest dilate of I on which the mean of F is greater than α, or I if no such dilate exists.

The contribution of the terms in (7.2) for which $j < i_I + t_I + 2C_0$ is contained inside the exceptional set $\bigcup_J CJ$, which has measure $O(\alpha)$. We can therefore restrict ourselves to the case $j \ge i_I + t_I + 2C_0$. We change the summation variable to $s = j - i_I - t_I \ge 2C_0$. Thus for the expression

$$\text{(7.3)}\qquad \mathcal{E}(x) = \sum_I \sum_{s\ge 2C_0} \sum_l c_I \sum_{k>0} \delta_{i_I+t_I+s}(L_1^{l+s+t_I}\mathcal{A}[L_1^k K^k]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

we have to show that the measure of the set $\{x : |\mathcal{E}(x)|_{\ell^\infty} > \alpha\}$ is $O(\alpha^{-1})$. This will be estimated by further splitting the expression $\mathcal{E}(x)$ into four pieces and then by applying of Chebyshev's inequality and L^1 or L^2 estimates for the individual pieces.

We now describe this splitting. Let

$$\text{(7.4)}\qquad M(x) = \sup_{k>0} |K^k(x)|.$$

We break up the functions K^k into a bounded part and an integrable part (this truncation has first been used in [9]). Let $\varepsilon_0 > 0$ be a constant to be chosen later ($\varepsilon_0 = 10^{-2}$, say, works). For all k write $K^k = 2^{\varepsilon_0(s+l)} K^k_{l,s,I} + R^k_{l,s,I}$, where $|K^k_{l,s,I}(x)| \le 1$ and the remainder $R^k_{l,s,I}$ is the restriction of K^k to the set $\{x : M(x) \ge 2^{\varepsilon_0(s+l)}\}$. We split

$$\mathcal{E}(x) = \mathcal{E}_1(x) + \mathcal{E}_2(x) + \mathcal{E}_3(x) + \mathcal{E}_4(x)$$

where

(7.5.1)
$$\mathcal{E}_1(x) = \sum_I \sum_{s\ge 2C_0} \sum_{l\ge -C_0} c_I \sum_{\substack{k>0\\ |k-l-s-t_I|\ge s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I}\mathcal{A}[L_1^k K^k]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

(7.5.2)
$$\mathcal{E}_2(x) = \sum_I \sum_{s\ge 2C_0} \sum_{l\ge -C_0} c_I \sum_{\substack{k>0\\ |k-l-s-t_I|< s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I}\mathcal{A}[L_1^k R^k_{l,s,I}]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

(7.5.3)
$$\mathcal{E}_3(x) = \sum_I \sum_{l\ge 2C_0} \sum_{2C_0\le s\le l} c_I 2^{\varepsilon_0(s+l)} \sum_{\substack{k>0\\ |k-l-s-t_I|< s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I}\mathcal{A}[L_1^k K^k_{l,s,I}]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

(7.5.4)
$$\mathcal{E}_4(x) = \sum_I \sum_{s\ge 2C_0} \sum_{-C_0\le l< s} c_I 2^{\varepsilon_0(s+l)} \sum_{\substack{k>0\\ |k-l-s-t_I|< s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I}\mathcal{A}[L_1^k K^k_{l,s,I}]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

It suffices to show that for $i = 1,2,3,4$ the measure of the set $\{x : |\mathcal{E}_i(x)|_{\ell^\infty} > \alpha/4\}$ is $O(\alpha^{-1})$. By Chebyshev's inequality and the continuous imbedding $\ell^1 \subset \ell^2 \subset \ell^\infty$ it suffices to show that

$$\|\mathcal{E}_1\|_{L^1(\ell^1)} + \|\mathcal{E}_2\|_{L^1(\ell^1)} + \|\mathcal{E}_3\|_{L^1(\ell^1)} \lesssim 1 \tag{7.6}$$

and

$$\|\mathcal{E}_4\|_{L^2(\ell^2)} \lesssim \alpha. \tag{7.7}$$

The estimation of $\mathcal{E}_1$ and $\mathcal{E}_2$ is straightforward. Since $\|(L_1^{l+s+t_I}\mathcal{A}[L_1^k K^k])\|_{L^1\to L^1} \lesssim 2^{-|k-l-s-t_I|}$ we get

$$\begin{aligned}\|\mathcal{E}_1\|_{L^1(\ell^1)} &\lesssim \sum_I \sum_{s\geq 2C_0}\sum_{l\geq -C_0} c_I \sum_{|k-l-s-t_I|\geq s+l} 2^{-|k-l-s-t_I|}\|L_2^{l-i_I}a_{I,l}\|_1\\ &\lesssim \sum_I c_I \sum_{s\geq 2C_0}\sum_{l\geq -C_0} 2^{-s-l}\lesssim 1.\end{aligned} \tag{7.8}$$

Next, by the definition of $R^k_{l,s,I}$

$$\|L_1^{l+s+t_I}\mathcal{A}[L_1^k R^k_{l,s,I}]\|_1 \lesssim 2^{-|k-l-s-t_I|}\int_{x:M(x)\geq 2^{\varepsilon_0(s+l)}} M(x)dx$$

and therefore

$$\begin{aligned}\|\mathcal{E}_2\|_{L^1(\ell^1)} &\lesssim \sum_I \sum_{s\geq 2C_0}\sum_{l\geq -C_0} c_I \sum_{|k-l-s-t_I|\leq s+l} 2^{-|k-l-s-t_I|}\int_{x:M(x)\geq 2^{\varepsilon_0(s+l)}} M(x)dx\\ &\lesssim \sum_I c_I \int |M(x)|\log^2(e+|M(x)|)dx \lesssim 1.\end{aligned} \tag{7.9}$$

The following Lemma is crucial for the estimation of $\mathcal{E}_3$.

Lemma 7.1. *Suppose that g is a bounded function supported in $\{x : 1/4 \leq |x| \leq 4\}$ and a is supported in a cube I with sidelength 2^{i_I}; moreover $\|a\|_\infty \leq |I|^{-1}$. Then for $m \geq 0$*

$$\big\|\delta_{i_I+m}[L^{l+m}\mathcal{A}g] * a\big\|_1 \lesssim 2^{-l/2}\|g\|_\infty$$

Proof. We may assume $\|g\|_\infty \leq 1$. Let $\mathcal{V}_m = \{\nu\}$ be a maximal 2^{-m}-separated subset of unit vectors in $\mathbb{R}^d$; its cardinality is $O(2^{m(d-1)})$. We may split $g = \sum_\nu g_{m,\nu}$ where $g_{m,\nu}$ is supported in the sector $\{x : |\frac{x}{|x|} - \nu| \lesssim 2^{-m+10}\}$ (and in the annulus where $1/4 \leq |x| \leq 4$).

Now $\delta_{i_I+m}[L^{l+m}\mathcal{A}g] * a$ is supported in a rectangle of dimensions $C_1 2^{i_I} \times \cdots \times C_1 2^{i_I} \times C_1 2^{i_I+m}$. Therefore by the Cauchy-Schwarz inequality

$$\begin{aligned}\big\|\delta_{i_I+m}[L_1^{l+m}\mathcal{A}g] * a\big\|_1 &\lesssim \sum_{\nu\in\mathcal{V}_m} 2^{(i_I d+m)/2}\|\delta_{i_I+m}[L_1^{l+m}\mathcal{A}g_{m,\nu}] * a\big\|_1\\ &\lesssim |I|^{1/2}2^{md/2}\Big(\sum_{\nu\in\mathcal{V}_m}\big\|\delta_{i_I+m}[L_1^{l+m}\mathcal{A}g_{m,\nu}] * a\big\|_2^2\Big)^{1/2}.\end{aligned} \tag{7.10}$$

We estimate this sum using Plancherel's theorem. For $\xi \in (\mathbb{R}^d)^*$

$$
\begin{aligned}
|\widehat{\mathcal{A}g}_{m,\nu}(-\xi)| &= \Big| \int_{r=1/4}^{4} \int_{\theta} g_{m,\nu}(r\theta) r^{d-1} \int \chi(\tau) e^{i\tau\langle r\theta,\xi\rangle} d\tau \, d\theta dr \Big| \\
&\lesssim \|g\|_\infty \int_{1/4}^{4} \int_{|\theta-\nu|\le 2^{-m+10}} (1+|\langle\theta,\xi\rangle|)^{-N} \, d\theta dr. \\
&\lesssim 2^{-m(d-1)/2} \Big(\int_{|\theta-\nu|\le 2^{-m+10}} (1+|\langle\theta,\xi\rangle|)^{-2N} \, d\theta \Big)^{1/2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_{\nu\in\mathcal{V}_m} \big\| \delta_{i_I+m}[L_1^{l+m} \mathcal{A} g_{m,\nu}] * a \big\|_2^2 \\
&\quad\lesssim 2^{-m(d-1)} \sum_{\nu\in\mathcal{V}_m} \int \big|\widehat{\psi_1^{l+m}}(2^{i_I+m}\xi)\big|^2 \int_{|\theta-\nu|\le 2^{-m+10}} (1+|\langle\theta, 2^{i_I+m}\xi\rangle|)^{-2N} d\theta \, |\widehat{a}(\xi)|^2 d\xi \\
&\quad\lesssim 2^{-m(d-1)} \int \big|\widehat{\psi_1^{l+m}}(2^{i_I+m}\xi)\big|^2 \int_{S^{d-1}} \big|(1+|\langle\theta, 2^{i_I+m}\xi\rangle|)^{-2N} d\theta \, |\widehat{a}(\xi)|^2 d\xi \\
&\quad\lesssim 2^{-m(d-1)} \int \big|\widehat{\psi_1}(\frac{\xi}{2^{l-i_I}})\big|^2 \, \min\{1, 2^{-i_I-m}|\xi^{-1}|\} \, |\widehat{a}(\xi)|^2 d\xi \\
&\quad\lesssim 2^{-m(d-1)} 2^{-(m+l)} \|\widehat{a}\|_2^2 \lesssim 2^{-md-l} |I|^{-1}, \qquad (7.11)
\end{aligned}
$$

by Plancherel's theorem and the estimate $|\widehat{\psi_1}(\xi)| \lesssim \min\{|\xi|^2, |\xi|^{-2}\}$.

The asserted estimate follows from (7.10) and (7.11). □

We now estimate the $L^1(\ell^1)$ norm of $\mathcal{E}_3$. To apply Lemma 7.1 we note that $L_2^{l-i_I} a_{I,l}$ is supported in a fixed dilate of I and $\|L_2^{l-i_I} a_{I,l}\|_\infty \lesssim |I|^{-1}$. Moreover $\|L_1^k K_{l,s,I}^k\|_\infty \lesssim 1$, uniformly in k, l, s, I. Hence

$$
\|\mathcal{E}_3\|_{L^1(\ell_1)} \lesssim \sum_I c_I \sum_{l\ge 2C_0} \sum_{2C_0\ge s\le l} 2^{\varepsilon_0(s+l)} \sum_{\substack{k>0 \\ |k-l-s-t_I|<s+l}} 2^{-l/2} \|L_1^k K_{l,s,I}^k\|_\infty \lesssim 1. \qquad (7.12)
$$

Finally we turn to the estimation of $\|\mathcal{E}_4\|_{L^2(\ell^2)}$. We first observe the basic estimate

Lemma 7.2.

$$
\Big\| \sum_I c_I \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|} \Big\|_2 \lesssim \alpha^{1/2}.
$$

Proof. Consider first those cubes I for which $t_I = 0$. It is easy to see that this contribution is bounded pointwise by $\min(F, C\alpha)$ for some constant C, and so the claim follows since $\|F\|_1 \lesssim 1$.

Now consider the cubes I for which $t_I > 0$. This part is majorized pointwise by

$$
\Big\| \sum_J \chi_{CJ} \Big\|_2 \lesssim \Big\| \sum_J \chi_J \Big\|_2 = \Big(\sum_J |J| \Big)^{1/2} \lesssim \alpha^{1/2},
$$

where for the first inequality we have used Lemma 2.3. □

The claimed estimate for $\mathcal{E}_4$ will follow from

Lemma 7.3. *Let g_I be bounded and supported on $\{x : 1/4 \le |x| \le 4\}$ and set $b_{I,l} = L_2^{l-i_I} a_{I,l}$. Assume $l \ge -C_0$, $s \ge 0$. Then for suitable $\varepsilon > 0$*

$$\Big\| \sum_I c_I \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A} g_I) * b_{I,l} \Big\|_2 \lesssim \sup_I \|g_I\|_\infty 2^{-s\varepsilon} \alpha^{1/2}.$$

Proof. This inequality is closely related to one in [25] and we shall adapt the proof here. Let $\mathcal{V}_s = \{\nu\}$ be a maximal 2^{-s}-separated subset of the unit sphere S^{d-1}; the cardinality of this set is $O(2^{(d-1)s})$. We decompose $g_I = \sum_\nu g_{I,\nu}$, where each $g_{I,\nu}$ is a bounded function on the sector

$$\mathfrak{S}_\nu^s = \{x : 1/4 \le |x| \le 4, \angle(x, \nu) \le 2^{-s}\}; \tag{7.13}$$

here we used $\angle(x, \nu)$ to denote the angle x and ν make at the origin.

We introduce a localization in Fourier space to a conic neigborhood of the hyperplane perpendicular to ν, namely

$$\Sigma_\nu^s = \{\xi : |\langle \xi, \nu \rangle| \le 2^{-s/2} |\xi|\}$$

(The exact choice of aperture $2^{-s/2}$ is unimportant as long as it is well between 2^{-s} and 1). We define the multiplier Q_ν^s whose symbol m_ν is homogeneous of degree 0, and equals 1 on Σ_ν^s and vanishes outside a slight widening of Σ_ν^s.

We then reduce to showing that

$$\Big\| \sum_I c_I \sum_{\nu \in \mathcal{V}_s} Q_\nu^s \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A} g_{I,\nu}) * b_{I,l} \Big\|_2 \lesssim \sup_I \|g_I\|_\infty 2^{-s\varepsilon} \alpha^{1/2} \tag{7.14}$$

and, for fixed ν,

$$\Big\| \sum_I c_I (Id - Q_\nu^s) \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A} g_{I,\nu}) * b_{I,l} \Big\|_2 \lesssim \sup_{I,\nu} \|g_{I,\nu}\|_\infty 2^{-sN} \alpha^{1/2} \tag{7.15}$$

where $N \le N_0/10$ (recall that $N_0 \ge 100d$). The estimate (7.15) is favorable if $N > d-1$.

To prove (7.15) we show the estimate

$$\big| (Id - Q_\nu^s) \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A} g_{I,\nu})(x) \big| \lesssim \|g_{I,\nu}\|_\infty 2^{-sN} \frac{2^{-(i_I+t_I)d}}{(1 + 2^{-(i_I+t_I)}|x|)^N} \tag{7.16}$$

for all $\nu \in \mathcal{V}_s$. From (7.16) we may estimate

$$\big| (Id - Q_\nu^s) \delta_{i_I+t_I+s}(L^{l+t_I+s} \mathcal{A} g_{I,\nu}) * b_{I,l} \big| \lesssim 2^{-Ns} H_I * \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|}$$

where H_I is the L^1 dilate of a radially decreasing L^1 function. By Lemma 2.3 and Lemma 7.2 the left hand side of (7.15) is dominated by

$$2^{-sN} \Big\| \sum_I c_I \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|} \Big\|_2 \lesssim 2^{-sN} \alpha^{1/2}.$$

We now show (7.16). Fix ν. Rescaling so that $i_I + t_I + s = 0$, it suffices to show that

$$|L_1^j (Id - Q_\nu^s) \mathcal{A} h(x)| \lesssim 2^{-(N+d)s} \|h\|_{L^\infty(\mathfrak{S}_\nu^s)} (1 + |x|)^{-N}$$

for all $j \geq l + t_I + s \geq s$ and all bounded h supported on $\mathfrak{S}^s_\nu$.

Fix j, x. We expand the left-hand side as

$$\Big|(2\pi)^{-d} \int_{\mathfrak{S}^\nu_s} h(z) \iiint (1 - m_\nu(\xi)) e^{i\langle \xi, x - 2^{-j} y - tz\rangle} \psi_1(y)\tilde{\chi}(t)\, d\xi dy \frac{dt}{t}\, dz\Big|$$

where the moments of ψ_1 vanish up to order N_0 and $\tilde{\chi}$ is supported where $1/4 \leq t \leq 4$. The decay in x follows from the fact that the phase is non-stationary in the ξ variable when $|x| \gg 1$.

Now we demonstrate the 2^{-Ns} bound; we may assume that $|x| \ll 2^{s/5}$. Since h is supported in $\mathfrak{S}^s_\nu$ and m_ν equals 1 on Σ_ν we see that for each $|\xi| \gtrsim 2^j$, the phase is non-stationary in the t variable (with a gradient of at least $2^{\varepsilon s}$). For $|\xi| \lesssim 2^j$ one picks up a loss of $(2^j/|\xi|)^C$, but this can be compensated for by the moment conditions on ψ_1, since $j \geq s$.

To show (7.14) we use the fact that the Q^s_ν have some weak orthogonality. More precisely, we have for any functions f_ν that

$$\Big\| \sum_\nu Q^s_\nu f_\nu \Big\|_2^2 \lesssim 2^{-\varepsilon s} 2^{(d-1)s} \sum_\nu \|f_\nu\|_2^2; \tag{7.17}$$

as in [25] this estimate is easily proven from Plancherel's theorem, the Cauchy-Schwarz inequality, and geometrical considerations. Because of this orthogonality, and Lemma 7.2, it now suffices to show that

$$\Big\| \sum_I c_I \delta_{i_I + t_I + s} \mathcal{A} g_{I,\nu} * a_I \Big\|_2 \lesssim 2^{-(d-1)s} \Big\| \sum_I c_I \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|} \Big\|_2, \tag{7.18}$$

uniformly in $\nu \in \mathcal{V}_s$.

Fix ν. Let R^s_ν be the rectangle centered at the origin, with dimensions $C_1 2^{-s} \times \cdots \times C_1 2^{-s} \times C_1$ so that the long side is parallel to ν. Then, if C_1 is chosen large enough there is the uniform pointwise estimate

$$\big|\delta_{i + t_I + s}[\mathcal{A} g_{I,\nu}] * a_I\big| \lesssim 2^{-s(d-1)} \|g_{I,\nu}\|_\infty\, \delta_{i_I + t_I + s}\big(\frac{\chi_{R^s_\nu}}{|R^s_\nu|}\big) * \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|}.$$

Thus (7.18) follows from Lemma 2.3. This completes the proof of (7.14) and the Lemma. □

The estimate (7.7) is an immediate consequence of Lemma 7.3. The estimate (7.6) holds by (7.8), (7.9) and (7.12) and thus we have proved the asserted weak type inequality.

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