ON SPACE-TIME ESTIMATES FOR THE SCHRÖDINGER OPERATOR

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ABSTRACT. We prove mixed-norm space-time estimates for solutions of the Schrödinger equation, with initial data in L^p-Sobolev or Besov spaces, and clarify the relation with adjoint restriction.

RÉSUMÉ. Nous obtenons des estimations en norme-mixte espace-temps pour l'équation de Schrödinger avec valeurs initiales dans des espaces de Sobolev ou de Besov. Nous éclaircissons également leurs relations avec celles de l'opérateur adjoint-restriction.

1. INTRODUCTION

We are concerned with regularity questions for the solution u of the initial value problem for the Schrödinger equation on $\mathbb{R}^d \times I$,

$$i\partial_t u + \Delta u = 0, \qquad u(\,\cdot\,, 0) = f,$$

where I is a compact time interval. When f is a Schwartz function, the solution can be written as $u = Uf$, with

$$Uf(x,t) \equiv e^{it\Delta} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi)\, e^{-it|\xi|^2 + i\langle x,\xi\rangle} d\xi\,; \tag{1.1}$$

here $\widehat{\ }$ denotes the Fourier transform defined by $\widehat{f}(\xi) = \int f(y)\, e^{-i\langle y,\xi\rangle} dy$.

Bounds for the solution in the spaces $L^r(I; L^q(\mathbb{R}^d))$ with initial data in L^2-Sobolev spaces have been extensively studied; these are known as 'Strichartz estimates' [32] and they play an important role in the study of the nonlinear equation (see for example [34]). In this paper we are instead concerned with bounds in the spaces $L^q(\mathbb{R}^d; L^r(I))$, equipped with the norm

$$\|u\|_{L^q(\mathbb{R}^d;L^r(I))} = \Big(\int_{\mathbb{R}^d} \Big(\int_I |u(x,t)|^r\, dt\Big)^{q/r} dx\Big)^{1/q}$$

when the initial datum is given in Sobolev spaces L^p_α, with norm $\|f\|_{L^p_\alpha} = \|(I-\Delta)^{\alpha/2} f\|_{L^p(\mathbb{R}^d)}$. We thus seek to prove the bound

$$\|Uf\|_{L^q(\mathbb{R}^d;L^r(I))} \leqslant C\|f\|_{L^p_\alpha(\mathbb{R}^d)}\,, \tag{1.2}$$

for suitable choices of p, q, r and α. Unlike the estimates in $L^r(I; L^q(\mathbb{R}^d))$, the inequality (1.2) is no longer invariant under Galilean transformations when $q \neq r$ which usually makes the problem more difficult.

Estimates with particular p, q and r are related to several well-known problems in harmonic analysis and various results have been obtained in specific cases. Notably, when $r = \infty$ and $p = 2$, (1.2) is the global version of the usual (local) maximal estimates which have been studied to prove pointwise convergence of Uf as $t \to 0$ (see for example [7],

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[37], [3], [35], [18]). Two of the authors [27] proved the sharp maximal estimates for some $p = q > 2$ which strengthen the fixed time estimates due to Fefferman–Stein [10] and Miyachi [21]. When $p = q > 2$ and $r = 2$, the problem is closely related to square function estimates for Bochner–Riesz operators, and also to $L^q(L^2)$ regularity of solutions for the wave equation (see [19] and §3.6). Finally, for $p = 2$, some $q, r \in (2,\infty)$ and $I = \mathbb{R}$, Planchon [24] considered a homogeneous version of the problem replacing L^p_α with the homogeneous space $\dot{H}^\alpha$, see also [16], [28], [35] for closely related results. In this article we obtain some new results on (1.2) for various choices of p, q and r and clarify the relations with the aforementioned problems.

Connection with adjoint restriction estimates. It is known that (1.2) is closely related to estimates for the adjoint restriction operator defined on a compact portion of the paraboloid in $\mathbb{R}^{d+1}$. Various maximal and smoothing estimates were obtained by relying on the adjoint restriction estimate, or its bilinear and multilinear variants (see [29], [11], [30], [37], [3], [14], [35], [18], [25], [4], [5]). Here we prove an actual equivalence of the space-time regularity estimates with estimates for the adjoint restriction operator, which allows us to extend the range of (1.2) by combining it with recent progress on the restriction problem [5]. A related result establishing the equivalence between adjoint restriction and Bochner–Riesz for paraboloids was found by Carbery [6].

Let $\mathcal{E}$ denote the adjoint restriction (or Fourier extension) operator given by

$$\mathcal{E}f(\xi,s) = \int_{|y|\leqslant 1} f(y)\, e^{is|y|^2 - i\langle \xi, y\rangle}\, dy, \quad (\xi,s) \in \mathbb{R}^d \times \mathbb{R}. \tag{1.3}$$

Definition. *We say that* $\mathrm{R}^*(p\to q)$ *holds if* $\mathcal{E} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^{d+1})$ *is bounded.*

The critical cases for adjoint restriction occur when $q = \frac{d+2}{d} p'$, and for a given q we denote the critical p by $p(q)$. In that case, it follows from the explicit formula

$$Uf(x,t) = \frac{1}{(4\pi i t)^{d/2}} \int \exp\Big(\frac{i|x-y|^2}{4t}\Big) f(y)\, dy \tag{1.4}$$

and scaling that $\mathrm{R}^*(p(q)\to q)$ implies the $L^{p(q)}(\mathbb{R}^d) \to L^q(\mathbb{R}^d \times I)$ boundedness of U. Moreover it was shown in [25] that it implies the $L^q_\alpha \to L^q(\mathbb{R}^d \times I)$ bound for $\alpha > d(1-\frac{2}{q}) - \frac{2}{q}$. We strengthen the connection between $\mathrm{R}^*(p\to q)$ and Schrödinger estimates by establishing an equivalence for general p, q. In order to formulate it we invoke Besov spaces $B^p_{\alpha,\nu}$. Recall that $\|f\|_{B^p_{\alpha,\nu}} = (\sum_{k\geqslant 0} 2^{k\alpha\nu} \|P_k f\|_p^\nu)^{1/\nu}$ where for $k \geqslant 1$, the operators P_k localize frequencies to annuli of width $\approx 2^k$ and $P_0 = I - \sum_{k\geqslant 1} P_k$.

Theorem 1.1. *Suppose* $2 \leqslant p \leqslant q < \infty$. *Then, for every* $\nu \in (0,2]$, *the following are equivalent:*

(i) $\mathrm{R}^*(p\to q)$ *holds.*
(ii) $U : B^p_{\alpha,\nu}(\mathbb{R}^d) \to L^q(\mathbb{R}^d \times I)$ *is bounded with* $\alpha = d\big(1 - \frac{1}{p} - \frac{1}{q}\big) - \frac{2}{q}$.

In §2 we shall also formulate more technical variants of Theorem 1.1 which are valid for mixed norm spaces.

We note that the restriction $\nu \leqslant 2$ in Theorem 1.1 is only needed for the implication $(i)\Rightarrow(ii)$. Moreover, the theorem implies that $\mathrm{R}^*(p\to q)$ holds if and only if for all $\lambda > 1$ the inequality

$$\|Uf\|_{L^q(\mathbb{R}^{d+1})} \lesssim \lambda^{d(1-\frac{1}{p}-\frac{1}{q})-\frac{2}{q}} \|f\|_p$$

holds for all $f \in L^p$ with frequency support in $\{\xi : \lambda/2 \leqslant |\xi| \leqslant 2\lambda\}$; of course for those f the parameter ν plays no role. For more general initial data recall that $B^p_{\alpha,\nu}$ is contained in the Sobolev space L^p_α for $\nu \leqslant \min\{2,p\}$, and vice versa, L^p_α is contained in $B^p_{\alpha,\nu}$ for $\nu \geqslant \max\{2,p\}$. It remains open whether the condition $\nu \leqslant 2$ is necessary and whether $B^p_{\alpha,2}$ can be replaced with L^p_α in Theorem 1.1. However it follows from a result in [19] (see §5.1 below) that if one is willing to give up an endpoint in the q-range then one can also obtain results on larger spaces including L^p_α, as well as mixed norm inequalities with $r \geqslant q$.

Corollary 1.2. *Let $2 < q_0 < \infty$, $1 \leqslant p_0 \leqslant q_0$, and suppose that $\mathrm{R}^*(p_0 \to q_0)$ holds. Let $q_0 < q < \infty$, $q \leqslant r \leqslant \infty$ and suppose that $0 \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{1}{p_0} - \frac{1}{q_0}$. Then*

$$\|Uf\|_{L^q(\mathbb{R}^d;L^r(I))} \leqslant C\|f\|_{B^p_{\alpha,q}(\mathbb{R}^d)}, \qquad \alpha = d\big(1 - \tfrac{1}{p} - \tfrac{1}{q}\big) - \tfrac{2}{r}.$$

By the trivial $\mathrm{R}^*(1 \to \infty)$ estimate and interpolation one can deduce the conclusion in the larger range $p_1(q) < p \leqslant q$, where $p_1(q) < p_0$ is defined by $\frac{1}{p_1(q)} = \frac{1}{p_0} + (1 - \frac{q_0}{q})(1 - \frac{1}{p_0})$. The recent progress on $\mathrm{R}^*(p \to p)$ by Bourgain and Guth [5], which employed the multilinear estimates of [2], can be used to prove new estimates of the form

$$\|Uf\|_{L^p(\mathbb{R}^d;L^r(I))} \leqslant C\|f\|_{B^p_{\alpha,p}(\mathbb{R}^d)}, \quad \alpha = d\big(1 - \tfrac{2}{p}\big) - \tfrac{2}{r}.$$

In two spatial dimensions their result implies that the displayed estimate holds in the case $r \geqslant p$ for $p \in (56/17, \infty)$ (see [5, pp. 1265]); moreover, in higher dimensions, it holds for the range $p \in (p_{BG}(d), \infty)$ with $p_{BG}(d) = 2 + \frac{12}{4d+1-k}$ if $d+1 \equiv k\ (mod\ 3)$, $k = -1, 0, 1$. This improves the result of [27], where the estimate was shown to hold in the range $p \in (\frac{2(d+3)}{d+1}, \infty)$ using the bilinear estimate of Tao [33]. We will also see that Bourgain and Guth's result can be combined with Tao's restriction bilinear estimate to obtain the critical restriction estimates $\mathrm{R}^*(p(q) \to q)$ for some range of q with $q < \frac{2(d+3)}{d+1}$ (see §5.2).

Necessary conditions. We now consider necessary conditions on p, q, r and α for (1.2) to hold. As previously mentioned, due to connections with other problems, conditions for specific choices of p, q and r are known, and examples in those special cases are also relevant when proving necessary conditions for general p, q and r. However we also establish additional conditions which seem to have not been noticed before. In particular the necessity of the strict inequalities in (v), (vi) in the following proposition are proved by constructions which involve the Besicovich set (see §3).

In what follows we set $\alpha_{cr}(p;q,r) := d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$.

Proposition 1.3. *Let $p, q, r \geqslant 2$ and suppose that there is a constant C such that*

$$\|Uf\|_{L^q(\mathbb{R}^d;L^r(I))} \leqslant C\|f\|_{L^p_\alpha(\mathbb{R}^d)},$$

holds whenever $f \in L^p_\alpha(\mathbb{R}^d)$. Then

- *(i)* $p \leqslant q$,
- *(ii)* $\alpha \geqslant \alpha_{cr}(p;q,r)$
- *(iii)* $\alpha \geqslant \frac{1}{q} - \frac{1}{r}$,
- *(iv)* $\alpha \geqslant \frac{1}{q} - \frac{1}{p}$,
- *(v)* $\alpha > \frac{1}{q} - \frac{1}{p}$ *if* $r > 2$,
- *(vi)* $\alpha > 0$ *if* $r = 2$, $p = q > 2$, $d \geqslant 2$.

The same conditions hold if we replace Sobolev norm L^p_α by the Besov norm of $B^p_{\alpha,\nu}$.

The condition (i) is a simple consequence of translation invariance. When $p = 2$, the condition (ii) coincides with (iii) if $\frac{d+1}{q} + \frac{1}{r} = \frac{d}{2}$. This is the condition in the endpoint version of Planchon's conjecture (*cf.* [24], [20]); that for these exponents $U : \dot{H}^\alpha(\mathbb{R}^d) \to L^q(\mathbb{R}^d; L^r(\mathbb{R}))$ with $\alpha = d\big(\frac{1}{2} - \frac{1}{q}\big) - \frac{2}{r}$ and $r \geqslant 2$. If $p = 2$ and $r = \infty$, then the conditions (iii) and (v) follow from the necessary conditions for Carleson's problem [7, 31], via an equivalence between local and global estimates [25].

The necessary conditions also naturally connect to those in the restriction and Bochner–Riesz problems. The necessity of the condition (vi) in dimensions $d \geqslant 2$ comes from the fact that a sharp square function estimate for the Schrödinger operator implies sharp bounds on Bochner–Riesz multipliers. When $p = q$ and $2 \leqslant r \leqslant q$, the condition $\alpha \geqslant \alpha_{cr}(p; p, r)$ is more restrictive than (vi) if $d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{r} > 0$. In particular, if $r = 2$ and $\alpha = \alpha_{cr}(p; p, 2)$, by (vi) the range $p > \frac{2d}{d-1}$ is necessary (as can be deduced from the connection to the Bochner–Riesz conjecture in $\mathbb{R}^d$), and for $r = p$, $\alpha = \alpha_{cr}(p; p, p)$ the range $p > \frac{2(d+1)}{d}$ is necessary (as can be deduced from the connection with the adjoint restriction problem for the paraboloid in $\mathbb{R}^{d+1}$, *cf.* Theorem 1.1). On the other hand, if $p < q$, $r = 2$, the condition $\alpha \geqslant \alpha_{cr}(p; q, 2)$ is more restrictive than (iv) if $\frac{d+1}{q} \leqslant \frac{d-1}{p'}$, the familiar range for the adjoint restriction theorem for the sphere in $\mathbb{R}^d$. Likewise if, $p < q = r$ then the condition $\alpha \geqslant \max\{0, \alpha_{cr}(p; q, q)\}$ implies $\frac{d+2}{q} \leqslant \frac{d}{p'}$, the range for the adjoint restriction theorem for the paraboloid in $\mathbb{R}^{d+1}$.

Remark (added March 2012). When $d \geqslant 5$, an additional necessary condition can be deduced from Bourgain's recent lower bounds for the Schrödinger maximal estimate. Precisely he showed that $\|UP_{2k}f\|_{L^q(B(0,1);L^\infty[0,2^{-2k}])} \leqslant C2^{2sk}\|P_{2k}f\|_2$ holds for $q \geqslant 2$ only if $s \geqslant 1/2 - 1/d$. By scaling this implies that $\|UP_kf\|_{L^q(\mathbb{R}^d;L^\infty[0,1])} \leqslant C2^{2sk}2^{kd(1/q-1/2)}\|P_kf\|_2$ can only hold if $s \geqslant 1/2 - 1/d$. By Sobolev imbedding this can be perturbed to give a necessary condition

$$\alpha \geqslant 1 - \tfrac{2}{d} - d\big(1 - \tfrac{1}{p} - \tfrac{1}{q}\big) - \tfrac{2}{r}$$

for $p, q, r \geqslant 2$, which is effective when p, q are close 2 and r is relatively large.

Results for $d = 1$ and $d = 2$. In one and two spatial dimensions, via more refined analysis based on bilinear technology, it is possible to obtain sharp estimates. First we state precise bounds for frequency localized functions in one spatial dimension.

Theorem 1.4. *For large λ, let*

$$\mathfrak{A}_\lambda(p; q, r) = \sup\Big\{\|Uf\|_{L^q(\mathbb{R};L^r(I))} \,:\, \|f\|_p \leqslant 1, \quad \operatorname{supp}\widehat{f} \subset \{\xi : \lambda/5 \leqslant |\xi| \leqslant 15\lambda\}\Big\}.$$

Then for $\lambda \gg 1$, the following norm equivalences hold:

(i) For $2 \leqslant r \leqslant p \leqslant q \leqslant \infty$,

$$\mathfrak{A}_\lambda(p; q, r) \;\approx\; \begin{cases} \lambda^{1/q-1/p}[\log \lambda]^{1/2-1/r} & \text{if} \quad \frac{1}{q} + \frac{1}{r} \geqslant \frac{1}{2}\,, \\ \lambda^{1-1/p-1/q-2/r} & \text{if} \quad \frac{1}{q} + \frac{1}{r} < \frac{1}{2}\,. \end{cases}$$

(ii) For $2 \leqslant p < r \leqslant q \leqslant \infty$,

$$\mathfrak{A}_\lambda(p; q, r) \;\approx\; \begin{cases} \lambda^{1/q-1/r} & \text{if} \quad \frac{2}{q} + \frac{1}{r} \geqslant 1 - \frac{1}{p}\,, \\ \lambda^{1-1/p-1/q-2/r} & \text{if} \quad \frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p}\,. \end{cases}$$

Again, using the result in §5.1 we obtain

Corollary 1.5. *Suppose that* $2 \leqslant r \leqslant p \leqslant q$, $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$, *or* $2 \leqslant p < r \leqslant q$, $\frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p}$. *Then* $U : B^p_{\alpha,q}(\mathbb{R}) \to L^q(\mathbb{R}; L^r(I))$ *is bounded with* $\alpha = 1 - \frac{1}{p} - \frac{1}{q} - \frac{2}{r}$.

To compare these results, recall that $B^p_{\alpha,q_1} \subset B^p_{\alpha,q_2}$ for $q_1 < q_2$, that $B^p_{\alpha,2} \subset L^p_\alpha \subset B^p_{\alpha,p}$ when $p \geqslant 2$, and that $B^p_{\alpha,p}$ is the same as the Sobolev–Slobodecki space $W^{\alpha,p}$ when $0 < \alpha < 1$. In higher dimensions, if one singles out the case $p = q$, one could hope to prove the following

Conjecture 1.6. *Let* $p \in [2, \infty)$, $r \in [2, \infty]$ *satisfy* $\frac{d}{p} + \frac{1}{r} < \frac{d}{2}$ *and* $\frac{2d+1}{p} + \frac{1}{r} < d$. *Then* $U : B^p_{\alpha,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d; L^r(I))$ *is bounded with* $\alpha = d(1 - \frac{2}{p}) - \frac{2}{r}$.

In [19], the conjecture was proven in the reduced range $p \in (\frac{2(d+2)}{d}, \infty)$, and for $d = 1$ it was proven in the range $p \in (4, \infty)$. In [27], the conjecture was proven for $p \in (\frac{2(d+3)}{d+1}, \infty)$ with $r \geqslant p$ (see [25] for a nonendpoint version).

Theorem 1.4 also provides the negative part of the following corollary. The positive part was proven in [19, Proposition 5.2].

Corollary 1.7. *Let* $2 \leqslant p < \infty$. *Then* $U : L^p(\mathbb{R}) \to L^p(\mathbb{R}; L^r(I))$ *is bounded if and only if* $r \leqslant 2$.

In two dimensions we can improve on the previously known range in p if r is large; this is closely related to results on maximal operators for L^2_α functions (*cf.* [24], [18], [26], [20]).

Theorem 1.8. *Let* $\frac{16}{5} < p < \infty$ *and* $4 \leqslant r \leqslant \infty$. *Then* $U : B^p_{\alpha,p}(\mathbb{R}^2) \to L^p(\mathbb{R}^2; L^r(I))$ *is bounded with* $\alpha = 2\big(1 - \frac{2}{p}\big) - \frac{2}{r}$.

The range in r can be further improved for $16/5 < p < 4$, by interpolating with the above mentioned $L^p(L^p(I))$ bounds for $p > 56/17$ (see [5]) and the $L^p(L^2(I))$ bounds of [19] for $p > 4$. Moreover one can obtain intermediate $L^p_\alpha \to L^q(L^r(I))$ bounds with the critical α by interpolating with the sharp $L^2 \to L^q(L^r)$ bounds in [20].

Organization of this paper. In the following section, we prove Theorem 1.1 and related mixed norm results. In §3 we discuss necessary conditions to show Proposition 1.3 and the lower bounds in Theorem 1.4. The upper bounds are proven in §4. In §5 we detail how to combine the frequency localized pieces to obtain estimates for Besov and Sobolev spaces, and in the final section we prove Theorem 1.8.

Notation. By $m(D)$ we denote the convolution operator with Fourier multiplier m; that is to say $m(D)f = (m\widehat{f}\,)^\vee$. For two nonnegative quantities A, B the notation $A \lesssim B$ is used for $A \leqslant CB$, with some unspecified constant C. We also use $A \approx B$ to indicate that $A \lesssim B$ and $B \lesssim A$.

2. $L^p \to L^q(L^r(I))$ bounds and the adjoint restriction operator

We formulate a more technical version of Theorem 1.1 that also applies to mixed-norm inequalities. In what follows let

$$\mathcal{A}(\rho) := \{\xi \in \mathbb{R}^d : 3\rho \leqslant |\xi| \leqslant 12\rho\}\,. \tag{2.1}$$

Theorem 2.1. *Let* $p, q, r \in [2, \infty]$, $p \leqslant q$, $\beta > -d(\frac{1}{2} - \frac{1}{p})$ *and* $0 < \nu \leqslant 1$. *Then the inequality*

$$\sup_{\lambda > 1} \lambda^{-\beta} \sup_{\|f\|_p \leqslant 1} \Big(\int_{\mathcal{A}(\lambda)} \Big(\int_\lambda^{2\lambda} |\mathcal{E}f(\tfrac{s}{\lambda}\xi, s)|^r ds \Big)^{q/r} d\xi \Big)^{1/q} < \infty \tag{2.2}$$

holds if and only if for $\gamma = d(1-\frac1p-\frac1q)-\frac2r+2\beta$,

$$\sup_{\|f\|_{B^p_{\gamma,\nu}}\leqslant 1}\Big\|\Big(\int_{-1}^{1}|e^{it\Delta}f|^r dt\Big)^{1/r}\Big\|_q<\infty\,. \tag{2.3}$$

If in addition $r<\infty$ *this equivalence remains valid for the range* $0<\nu\leqslant 2$.

Taking Theorem 2.1 for granted we can quickly give

Proof of Theorem 1.1. By Theorem 2.1 we just have to show that $\mathrm{R}^*(p\to q)$ is equivalent with (2.2) for large λ, in the case $q=r$ and $\beta=0$. Clearly the latter is implied by $\mathrm{R}^*(p\to q)$; this follows by a change of variables $(\eta,s)=(s\lambda^{-1}\xi,s)$ which has Jacobian bounded above and below in the region where $s\approx\lambda$.

Vice versa, supposing that (2.2) holds in the case $q=r$ and $\beta=0$, by the change of variables, we have that $\mathcal{E}: L^p(\mathbb{R}^d)\to L^q(W_\lambda)$, where

$$W_\lambda=\{\,(\xi,s)\,:\,s\in[\lambda,2\lambda],\quad x\in\mathcal{A}(s)\,\}.$$

For $\omega\in\mathbb{R}^{d+1}$ define $f^\omega(y)=e^{i\langle\omega,y\rangle-i\omega_{d+1}|y|^2}f(y)$ and observe that $\mathcal{E}f^\omega=\mathcal{E}f(\cdot-\omega)$. Thus using a finite number of translations we see that $\mathcal{E}: L^p(\mathbb{R}^d)\to L^q(B_\lambda)$, where B_λ is the ball in $\mathbb{R}^{d+1}$ of radius λ centred at the origin, and the operator norm is uniformly bounded in λ. Letting $\lambda\to\infty$ yields $\mathrm{R}^*(p\to q)$. □

We now proceed to prove Theorem 2.1.

Lemma 2.2. *Let* $p,q,r\in[2,\infty]$ *with* $p\leqslant q$ *and let* $\lambda\gg 1$. *Suppose that*

$$\Big(\int_{\mathcal{A}(\lambda^2)}\Big(\int_{\lambda^2}^{2\lambda^2}|\mathcal{E}f(\tfrac{s}{\lambda^2}\xi,s)|^r ds\Big)^{q/r}d\xi\Big)^{1/q}\leqslant A\|f\|_p \tag{2.4}$$

holds. Then, for $\psi\in C_c^\infty$ *with support in* $\{\xi: 4<|\xi|<5\}$,

$$\Big\|\Big(\int_{1/2}^{1}|e^{it\Delta}\psi(\tfrac{D}{\lambda})f|^r dt\Big)^{1/r}\Big\|_q\lesssim A\lambda^\alpha\|f\|_p\,,\quad \alpha=d\big(1-\tfrac1p-\tfrac1q\big)-\tfrac2r. \tag{2.5}$$

Proof. If f_λ is the characteristic function of a ball of radius $(100\lambda)^{-2}$ then $|\mathcal{E}(f_\lambda)(\frac{s}{\lambda^2}\xi,s)|\geqslant\lambda^{-2d}$ for $(\xi,s)\in\mathcal{A}(\lambda^2)\times[\lambda^2,2\lambda^2]$. The resulting lower bound $A\geqslant c\lambda^{2d(-1+1/p+1/q)+2/r}$ (which is far from being sharp) will be used repeatedly to dominate certain error terms which decay fast in λ.

The convolution kernel for $e^{it\Delta}\psi(\frac{D}{\lambda})$ can be written as

$$K_t^\lambda(x)\;=\;\Big(\frac{\lambda}{2\pi}\Big)^d\int\psi(\xi)\,e^{-it\lambda^2|\xi|^2+i\lambda\langle x,\xi\rangle}d\xi.$$

By integration by parts it follows that

$$|K_t^\lambda(x)|\leqslant C_N|x|^{-N},\quad\text{for } |x|\geqslant 11\lambda. \tag{2.6}$$

Hence, by a standard argument, (2.5) reduces to showing the inequality

$$\Big(\int_{|x|\leqslant 11\lambda}\Big(\int_{1/2}^{1}|K_t^\lambda*f|^r dt\Big)^{q/r}dx\Big)^{1/q}\lesssim A\lambda^\alpha\|f\|_p,\quad\alpha=d\big(1-\tfrac1p-\tfrac1q\big)-\tfrac2r \tag{2.7}$$

for f supported in the cube of sidelength $\lambda(2d)^{-1}$ centred at the origin. Indeed, suppose that (2.7) is verified, let $\mathfrak{Q}_\lambda=\{Q\}$ be a grid of cubes with sidelength $\lambda(2d)^{-1}$, and centres

x_Q, and let B_Q be the ball of radius 11λ centred at x_Q. Then we may estimate the $L^q(\mathbb{R}^d; L^r([1/2,1]))$ norm of $e^{it\Delta}\psi(\frac{D}{\lambda})$ by

$$\begin{aligned}(2.8)\qquad &\Big(\int \sum_Q \chi_{B_Q}(x)\Big(\int_{1/2}^1 |K_t^\lambda * [f\chi_Q](x)|^r dt\Big)^{q/r} dx\Big)^{1/q} \\ &+ \Big(\int \sum_Q \chi_Q(x)\Big(\int_{1/2}^1 |K_t^\lambda * [f\chi_{\mathbb{R}^d\setminus B_Q}](x)|^r dt\Big)^{q/r} dx\Big)^{1/q}\end{aligned}$$

by Minkowski's inequality in L^r. We use the finite overlap of the balls, the translation invariance of the operators and (2.7) to estimate the first term by

$$CA\lambda^\alpha\Big(\sum_Q \|f\chi_Q\|_p^q\Big)^{1/q} \lesssim CA\lambda^\alpha\|f\|_p$$

where for the last inequality we have used the assumption $p \leqslant q$. For the second term in (2.8) we use (2.6) with $N > 2d$ and then Young's inequality to bound it by

$$C\Big(\int\Big[\int_{|w|\geqslant 10\lambda} |w|^{-N}|f(x-w)|dw\Big]^q dx\Big)^{1/q} \lesssim \lambda^{-N+d(1-\frac{1}{p}+\frac{1}{q})}\|f\|_p \lesssim A\lambda^\alpha\|f\|_p.$$

We used the trivial lower bound for A in the last step.

Our task is now to prove (2.7). We use a stationary phase calculation to see that $K_t^\lambda = H_t^\lambda + E_t^\lambda$, where

$$H_t^\lambda(x) = \frac{e^{-i|x|^2/4t}}{(4\pi it)^{d/2}}\sum_{\nu=0}^M \psi_\nu\big(\frac{x}{2\lambda t}\big)\lambda^{-\nu}$$

and

$$|E_\lambda(x,t)| \leqslant C_L\lambda^{-L}$$

where we choose $L \gg d$. For the leading term $\psi_0 = \psi$, and the functions ψ_ν are obtained by letting certain differential operators act on ψ; thus $\psi_\nu(w) = 0$ for $|w| \leqslant 4$ and $|w| \geqslant 5$.

For the error term we use a trivial bound

$$\Big(\int_{|x|\leqslant 11\lambda}\Big(\int_{1/2}^1\Big[\int |E_\lambda(x-y,t)||f(y)|\,dy\Big]^r dt\Big)^{q/r} dx\Big)^{1/q} \lesssim \lambda^{d-L}\|f\|_p \lesssim A\lambda^\alpha\|f\|_p\,.$$

For the oscillatory terms we have to prove the inequality

$$(2.9)\qquad \Big(\int_{|x|\leqslant 11\lambda}\Big(\int_{1/2}^1\Big|\int \psi_\nu\big(\frac{x-y}{2\lambda t}\big)\exp\big(i\frac{|x-y|^2}{4t}\big)f(y)\,dy\Big|^r dt\Big)^{q/r} dx\Big)^{1/q} \lesssim A\lambda^\alpha\|f\|_p$$

whenever f is supported in $\{|y| \leqslant \lambda/2\}$. By a change of variable $t \mapsto u = 1/t$ (with $u \approx t \approx 1$) and the support properties for ψ_ν this follows from

$$\begin{aligned}(2.10)\qquad &\Big(\int_{\frac{7}{2}\lambda\leqslant|x|\leqslant\frac{21}{2}\lambda}\Big(\int_1^2\Big|\int_{|y|\leqslant\lambda/2}\psi_\nu\big(\frac{u(x-y)}{2\lambda}\big)\exp\big(i\frac{u}{4}(|y|^2-2\langle x,y\rangle)\big) \\ &\qquad\qquad \times f(y)dy\Big|^r du\Big)^{q/r} dx\Big)^{1/q} \lesssim A\lambda^\alpha\|f\|_p\end{aligned}$$

whenever f is supported in $\{|y| \leqslant \lambda/2\}$. We now use a parabolic scaling in the (x,u)-variables and set

$$x = \lambda^{-1}w, \quad u = \lambda^{-2}s; \qquad y = 2\lambda z.$$

The previous inequality becomes

$$\Big(\int_{\frac{7}{2}\lambda^2\leqslant|w|\leqslant\frac{21}{2}\lambda^2}\Big(\int_{\lambda^2}^{2\lambda^2}\Big|\int_{|z|\leqslant 1}\psi_\nu\big(\frac{sw-2\lambda^2 sz}{2\lambda^4}\big) \times \exp(i(s|z|^2-\langle\frac{sw}{\lambda^2},z\rangle))f(2\lambda z)(2\lambda)^d\,dz\Big|^r\frac{ds}{\lambda^2}\Big)^{q/r}\frac{dw}{\lambda^d}\Big)^{1/q} \lesssim A\lambda^\alpha\|f\|_p. \tag{2.11}$$

We have the Fourier series expansion $\psi_\nu(x)=\sum_{\ell\in\mathbb{Z}^d}c_{\ell,\nu}e^{i\langle\ell,x\rangle}$ for $x\in[-\frac{9}{10}\pi,\frac{9}{10}\pi]^d$ and for each ν the Fourier coefficients are rapidly decaying, $|c_{\ell,\nu}|\leqslant C_{N,\nu}(1+|\ell|)^{-N}$. Thus

$$\psi_\nu\big(\frac{sw-2\lambda^2 sz}{2\lambda^4}\big)=\sum_\ell c_{\ell,\nu}e^{i\lambda^{-4}\langle sw,\ell\rangle/2}e^{-i\lambda^{-2}s\langle z,\ell\rangle}.$$

Using Minkowski's inequality for the sum and the rapid decay of the Fourier coefficients the previous inequality (2.10) follows from

$$\Big(\int_{\frac{7}{2}\lambda^2\leqslant|w|\leqslant\frac{21}{2}\lambda^2}\Big(\int_{\lambda^2}^{2\lambda^2}\Big|\int_{|z|\leqslant 1}\exp(i(s|z|^2-\langle\frac{s(w+\ell)}{\lambda^2},z\rangle))f(2\lambda z)\,dz\Big|^r ds\Big)^{q/r}dw\Big)^{1/q} \lesssim (1+|\ell|)^M A\lambda^{\alpha-d+\frac{2}{r}+\frac{d}{q}}\|f\|_p. \tag{2.12}$$

The left hand side is trivially bounded by $C\lambda^{2/r+2d/q}$ and therefore the displayed inequality holds for $|\ell|\geqslant\lambda^2/4$. If $|\ell|\leqslant\lambda^2/4$, we change variables and see that for (2.12) we only need to show

$$\Big(\int_{3\lambda^2\leqslant|w|\leqslant 11\lambda^2}\Big(\int_{\lambda^2}^{2\lambda^2}\Big|\int_{|z|\leqslant 1}\exp(i(s|z|^2-\langle\frac{sw}{\lambda^2},z\rangle))g(z)\,dz\Big|^r ds\Big)^{q/r}dw\Big)^{1/q} \lesssim A\lambda^{\alpha-d+\frac{2}{r}+\frac{d}{q}}\lambda^{d/p}\|g\|_p.$$

The right hand side is just $A\|g\|_p$, so that this would follow from (2.4). □

Lemma 2.3. *Let $p,q,r\in[2,\infty]$ and $\lambda\gg 1$. Let $2<a_0<a_1$ and let η be a radial C_c^∞ function which satisfies $\eta(\xi)=1$ for $\frac{a_0-2}{4}\leqslant|\xi|\leqslant 2(a_1+2)$. Suppose*

$$\sup_{\|f\|_p\leqslant 1}\Big\|\Big(\int_{1/2}^1|e^{it\Delta}\eta(\tfrac{D}{\lambda})f|^r dt\Big)^{1/r}\Big\|_q\leqslant B\,. \tag{2.13}$$

Then

$$\Big(\int_{a_0\lambda^2\leqslant|\xi|\leqslant a_1\lambda^2}\Big(\int_{\lambda^2}^{2\lambda^2}\big|\mathcal{E}f\big(\tfrac{s}{\lambda^2}\xi,s\big)\big|^r ds\Big)^{q/r}d\xi\Big)^{1/q}\lesssim B\lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}\|f\|_p. \tag{2.14}$$

Proof. In what follows let $\alpha=d(1-\frac{1}{p}-\frac{1}{q})-\frac{2}{r}$. We begin by observing the lower bound $B\geqslant c\lambda^\alpha$ which follows from the example in §3.2.

By a change of variable $\xi=\lambda x$, $s=\lambda^2\rho$, $y=2\lambda z$ we see that (2.14) is equivalent with

$$\Big(\int_{a_0\lambda\leqslant|x|\leqslant a_1\lambda}\Big(\int_1^2\Big|\int_{|y|\leqslant 2\lambda}f(\tfrac{y}{2\lambda})e^{i(\rho|y|^2/4-\rho\langle x,y\rangle/2)}dy\Big|^2 d\rho\Big)^{q/r}dx\Big)^{1/q} \leqslant CB\lambda^{-\alpha}(2\lambda)^d\lambda^{-d/q-2/r}\|f\|_p.$$

By inverting $t = 1/\rho$ the previous inequality follows from

$$\Big(\int_{a_0\lambda\leqslant|x|\leqslant a_1\lambda}\Big(\int_{1/2}^1\Big|\frac{1}{(4\pi it)^{d/2}}\int_{|y|\leqslant 2\lambda}g(y)e^{\frac{i|x-y|^2}{4t}}dy\Big|^r dt\Big)^{q/r}dx\Big)^{1/q} \\ \lesssim B\lambda^{-\alpha}\lambda^{d-d/p-2/r}\lambda^{-d/p}\|g\|_p$$

which can be rewritten as

$$\Big(\int_{a_0\lambda\leqslant|x|\leqslant a_1\lambda}\Big(\int_{1/2}^1|e^{it\Delta}g(x)|^r dt\Big)^{q/r}dx\Big)^{1/q}\lesssim B\|g\|_p\,, \tag{2.15}$$

for g supported in $\{y : |y| \leqslant 2\lambda\}$. By assumption

$$\Big(\int_{a_0\lambda\leqslant|x|\leqslant a_1\lambda}\Big(\int_{1/2}^1\Big|e^{it\Delta}\eta(\tfrac{D}{\lambda})g(x)\Big|^r dt\Big)^{q/r}dx\Big)^{1/q}\leqslant B\|g\|_p$$

and thus (2.14) follows from the straightforward estimate

$$\Big(\int_{a_0\lambda\leqslant|x|\leqslant a_1\lambda}\Big(\int_{1/2}^1\Big|e^{it\Delta}(I-\eta(\tfrac{D}{\lambda}))g(x)\Big|^r dt\Big)^{q/r}dx\Big)^{1/q}\leqslant C_M\lambda^{-M}\|g\|_p\,, \tag{2.16}$$

whenever g is supported in $\{y : |y| \leqslant 2\lambda\}$.

To see (2.16) we decompose the multiplier. Let χ_0 be smooth and supported in $\{|\xi| < 2\}$ and $\chi_0(\xi) = 1$ for $|\xi| \leqslant 1$, and let $\chi_k(\xi) = \chi_0(2^{-k}\xi) - \chi_0(2^{1-k}\xi)$, for $k \geqslant 1$. Let

$$E_{\lambda,k}(x,t) = \frac{1}{(2\pi)^d}\int \chi_k(\tfrac{\xi}{\lambda})(1-\eta(\tfrac{\xi}{\lambda}))e^{-it|\xi|^2+i\langle x,\xi\rangle}d\xi$$

and we need to bound the expression

$$\big(I-\eta(\tfrac{D}{\lambda})\big)e^{it\Delta}g(x,t) = \sum_{k\geqslant 0}\int_{|y|\leqslant 2\lambda}E_{\lambda,k}(x-y)g(y)dy.$$

We now examine $\nabla_\xi(\langle x-y,\xi\rangle - t|\xi|^2) = x - y - 2t\xi$. Since $a_0 > 2$, for the relevant choices $a_0|\lambda| \leqslant |x| \leqslant a_1\lambda$, $1/2 \leqslant t \leqslant 1$, $|y| \leqslant 2\lambda$ we have

$$|x-y-2t\xi| \geqslant \begin{cases}\frac12(a_0-2)\lambda & \text{if } |\xi|\leqslant \frac{a_0-2}{4}\lambda,\\ \max\{\frac{|\xi|}{2},\ (a_1+2)\lambda\} & \text{if } |\xi|\geqslant (a_1+2)\lambda.\end{cases}$$

Since $1 - \eta(\lambda^{-1}\xi) = 0$ for $\frac{a_0-2}{4} \leqslant |\xi| \leqslant 2(a_1 + 2)$, after an N-fold integration by parts we find that $|E_{\lambda,k}(x - y,t)| \leqslant C_N(2^k\lambda)^{d-N}$ for this choice of x, y, t, and the estimate (2.16) follows. □

To complete the proof of Theorem 2.1 we also need the following scaling lemma.

Lemma 2.4. *Let $\gamma > d(\frac1p - \frac1q) - \frac2r$. Suppose that for $\lambda \gg 1$*

$$\Big\|\Big(\int_{1/2}^1|e^{it\Delta}\chi(\tfrac{D}{\lambda})f|^r dt\Big)^{1/q}\Big\|_q\lesssim\lambda^\gamma\|f\|_p, \tag{2.17}$$

where $\chi \in C_c^\infty$ is supported in $(1/2, 2)$ (with suitable bounds). Then, for $\lambda \gg 1$,

$$\Big\|\Big(\int_I|e^{it\Delta}\chi(\tfrac{D}{\lambda})f|^r dt\Big)^{1/r}\Big\|_q\lesssim\lambda^\gamma\|f\|_p. \tag{2.18}$$

Proof. It is easy to calculate that

$$\sup_{0\leqslant t\leqslant(8\lambda)^{-2}} |\mathcal{F}^{-1}[\chi(\tfrac{\cdot}{\lambda})\exp(-it|\cdot|^2)](x)| \leqslant C_N\lambda^d(1+\lambda|x|)^{-N}$$

and thus, by Young's inequality,

$$\begin{aligned}\Big\|\Big(\int_0^{(8\lambda)^{-2}} |e^{it\Delta}\chi(\tfrac{D}{\lambda})f|^r dt\Big)^{1/r}\Big\|_q &\lesssim \Big\|\lambda^{-2/r}\int \lambda^d(1+\lambda|y|)^{-N}|f(\cdot-y)|dy\Big\|_q \\ &\lesssim \lambda^{d(\frac1p-\frac1q)-\frac2r}\|f\|_p. \end{aligned}\tag{2.19}$$

Now letting $(8\lambda)^{-2}\leqslant b\leqslant 1$,

$$\Big(\int_{b/2}^b |e^{it\Delta}\chi(\tfrac{D}{\lambda})f(x)|^r dt\Big)^{1/r} = b^{1/r}\Big(\int_{1/2}^1 \Big|\chi(\tfrac{D}{b^{1/2}\lambda})e^{is\Delta}[f(b^{1/2}\cdot)](b^{-1/2}x)\Big|^r ds\Big)^{1/r}.$$

Thus by a change of variable (2.17) implies

$$\Big\|\Big(\int_{b/2}^b |e^{it\Delta}\chi(\tfrac{D}{\lambda})f|^r dt\Big)^{1/r}\Big\|_q \lesssim (\sqrt{b})^{-d(\frac1p-\frac1q)+\frac2r}(\lambda\sqrt{b})^\gamma\|f\|_p.$$

We choose $b=2^{-l}$, and since $\gamma>d(\frac1p-\frac1q)-\frac2r$ we may sum over l with $(8\lambda)^{-2}\leqslant 2^{-l}\leqslant 1$ and combine with (2.19). Hence we get

$$\Big\|\Big(\int_0^1 |e^{it\Delta}\chi(\tfrac{D}{\lambda})f|^r dt\Big)^{1/r}\Big\|_q \lesssim \lambda^\gamma\|f\|_p.$$

Now (2.18) with $I=[-1,1]$ follows using the formula $e^{-it\Delta}f=\overline{e^{it\Delta}\overline{f}}$, and the triangle inequality. Finally, by scaling, we can enlarge the time interval (so that the implicit constant is of course dependent on the interval), and we are done. □

Proof of Theorem 2.1. The implication (2.3) ⇒ (2.2), for all $\nu>0$, follows from Lemma 2.3.

For the implication (2.2) ⇒ (2.3) we decompose $f=\sum_{k=0}^\infty P_kf$, with the standard inhomogeneous decomposition, and assume for $k>1$ that supp $\widehat{P_kf}$ is contained in $\{\xi: 2^{k-1}\leqslant|\xi|\leqslant 2^{k+1}\}$ and supp $\widehat{P_0f}$ is contained in $\{\xi:|\xi|\leqslant 2\}$. We estimate $\chi(t)UP_kf(x,t)$ where $\chi\in C_c^\infty$ with $\chi(t)=1$ on $[-1,1]$. Let $\widetilde{P}_k$ have similar properties to P_k, with $\widetilde{P}_kP_k=P_k$. We prove the inequality

$$\Big\|\Big(\int|\chi(t)U\widetilde{P}_kf(\cdot,t)|^r dt\Big)^{1/r}\Big\|_q \lesssim 2^{k\gamma}\|f\|_p, \quad \gamma=d\big(1-\tfrac1p-\tfrac1q\big)-\tfrac2r+2\beta\,, \tag{2.20}$$

which we apply with P_kf in place of f. Now if $\beta>-d(1/2-1/p)$ then the restriction on γ in Lemma 2.4 is satisfied. Thus (2.20) follows by combining Lemmata 2.2 and 2.4 (together with a finite decomposition and mild rescaling). This immediately yields the implication (2.2) ⇒ (2.3) in the range $\nu\leqslant 1$.

If $r<\infty$ we can use Littlewood–Paley theory to extend this implication to the case $\nu=2$ (which implies the corresponding inequality for $\nu<2$). Let, for a function g on $\mathbb{R}^d\times\mathbb{R}$,

$$R_{2k}g(x,t)=\frac{1}{2\pi}\int\int\beta(2^{-2k}\tau)e^{i\tau(t-s)}d\tau\, g(x,s)\,ds,$$

where β is supported in $[1/10,10]$ and $\beta(\tau)=1$ for $\tau\in[1/8,8]$.

The contribution $(I - R_{2k})[\chi U P_k f]$ is negligible. To see this one uses various standard integration by parts arguments, in particular the decay of $\int \chi(s) e^{is(|\xi|^2-\tau)} ds$ when $|\xi|^2 \gg \tau$ or $\tau \gg |\xi|^2$ to analyze the kernel. We omit the details which give

$$\Big\| \Big(\int_{\mathbb{R}} |(I - R_{2k})[\chi U P_k f]|^r dt \Big)^{1/r} \Big\|_{L^q(\mathbb{R}^d)} \lesssim C_N 2^{-kN} \|P_k f\|_p.$$

It thus remains to show

$$\Big\| \Big(\int_{\mathbb{R}} \Big| \sum_{k \geqslant 1} R_{2k}[\chi U P_k f] \Big|^r dt \Big)^{1/r} \Big\|_{L^q(\mathbb{R}^d)} \lesssim \Big(\sum_k \big[2^{k\gamma} \|P_k f\|_p \big]^2 \Big)^{1/2}. \tag{2.21}$$

Using Littlewood–Paley theory on $L^r(\mathbb{R})$ followed by applications of the triangle inequalities for $L^{r/2}$ and $L^{q/2}$ we see that the left hand side of (2.21) is controlled by a constant times

$$\begin{aligned} \Big\| \Big(\sum_k |\chi U P_k f|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^d; L^r(\mathbb{R}))} &= \Big\| \sum_k |\chi U P_k f|^2 \Big\|^{1/2}_{L^{q/2}(\mathbb{R}^d; L^{r/2}(\mathbb{R}))} \\ \leqslant \Big(\sum_k \big\| |\chi U P_k f|^2 \big\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}(\mathbb{R}))} \Big)^{1/2} &= \Big(\sum_k \|\chi U P_k f\|^2_{L^q(\mathbb{R}^d; L^r(\mathbb{R}))} \Big)^{1/2}. \end{aligned}$$

Now (2.21) follows from (2.20). □

3. Proof of Proposition 1.3

First we prove the easier necessary conditions (i)-(iv).

3.1. *The condition $p \leqslant q$.* This follows from the translation invariance (see an argument in [12]). More precisely, the $L^p_\alpha(\mathbb{R}^d) \to L^q(\mathbb{R}^d; L^r(I))$ boundedness is equivalent with the $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d; L^r(I))$ boundedness of the operator $U[(I-\Delta)^{\alpha/2} f]$ which commutes with translations on $\mathbb{R}^d$. Let $A = \sup_{\|f\|_p \leqslant 1} \|U[(I-\Delta)^{\alpha/2} f]\|_{L^q(L^r)}$. Then by the density argument, for $\epsilon > 0$ there is a $g \in C^\infty_c(\mathbb{R}^d)$ such that $A - \epsilon < \|U[(I-\Delta)^{\alpha/2} g]\|_{L^q(L^r)}$ and $\|g\|_p = 1$. One may test the inequality with $f = g + g(\cdot + ae_1)$. Letting $a \to \infty$, we see that $(A-\epsilon)2^{1/q} \leqslant A 2^{1/p}$, which gives $A2^{1/q} \leqslant A2^{1/p}$ by letting $\epsilon \to 0$, and thus $p \leqslant q$.

3.2. *The condition $\alpha \geqslant d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$.* This condition follows by a focusing example (see for example [25]). Let $\eta \in C^\infty_c$ be radial and supported in $\{\xi : 1 < |\xi| < 2\}$. Define for $\lambda \gg 1$, the function f_λ by $\widehat{f_\lambda}(\xi) = e^{i\frac{1}{2}|\xi|^2} \eta(\lambda^{-1}\xi)$. Then $\|f_\lambda\|_{L^p_\alpha} \lesssim \lambda^{\alpha + d/p}$. Moreover $|Uf(x,t)| \gtrsim \lambda^d$ if, for suitable $c > 0$, $|x| \leqslant c\lambda^{-1}$ and $|t - \frac{1}{2}| \leqslant c\lambda^{-2}$. For large λ this leads to the restriction $\alpha \geqslant d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$.

3.3. *The condition $\alpha \geqslant \frac{1}{q} - \frac{1}{r}$.* Let g_λ be defined by $\widehat{g_\lambda}(\xi) = \chi(|\xi - \lambda e_1|)$, χ supported in an ε-neighborhood of 0 (see [8], [27]), so that $\|g_\lambda\|_{L^p_\alpha} \lesssim \lambda^\alpha$. Also,

$$U g_\lambda(x,t) = \frac{1}{(2\pi)^d} \int \chi(|h|) e^{i\phi_\lambda(x,t,h)}\, dh$$

where $\phi_\lambda(x,t,h) = -t|h|^2 - t\lambda^2 + x_1\lambda + \langle x - 2t\lambda e_1, h\rangle$. Then $|Ug_\lambda(x,t)| \geqslant c_0 > 0$ if $|t - (2\lambda)^{-1}x_1| \leqslant c\lambda^{-1}$ for $0 \leqslant x_1 \leqslant \lambda$, $|x_i| \leqslant c$, $i = 2, \dots, d$. It follows that $\|Uf\|_{L^q(L^r(I))} \geqslant \lambda^{1/q - 1/r}$. Hence the condition $\alpha \geqslant 1/q - 1/r$ follows.

3.4. *The condition* $\alpha \geqslant \frac{1}{q}-\frac{1}{p}$. Let $\lambda \gg 1$ and set $\widehat{h_\lambda}(\eta) = \phi(|\eta'|)\lambda\phi(\lambda(\eta_1-\lambda))$ with $\phi \in C_c^\infty(\mathbb{R})$. Then $\|h_\lambda\|_{L^p_\alpha} \lesssim \lambda^\alpha\lambda^{1/p}$. Note that

$$Uh_\lambda(x,t) = \frac{1}{(2\pi)^d}\int e^{-it|\eta'|^2+i\langle x',\eta'\rangle)}\phi(|\eta'|)d\eta' e^{-i\lambda^2 t+i\lambda x_1}\int e^{i(-t\xi_1^2-2\lambda t\xi_1+x_1\xi_1)}\lambda\phi(\lambda\xi_1)d\xi_1,$$

so that $|Uh_\lambda(x,t)| \geqslant c > 0$ if $|t|, |x'| \leqslant c$ and $|x_1| \leqslant c\lambda$ for small enough $c > 0$. This shows the necessity of $\alpha \geqslant 1/q - 1/p$.

To show the conditions (v) and (vi), we use sharp bounds in the construction of Besicovich sets [15] and adapt Fefferman's argument for the disc multiplier [9] (see also [1]).

3.5. *The condition* $\alpha > \frac{1}{q}-\frac{1}{p}$ *if* $r > 2$. This follows from

Proposition 3.1. *Let* $p, q, r \in [2,\infty)$. *Let* η *be a radial* C_c^∞ *function satisfying* $\eta(\xi) = 1$ *for* $1/4 \leqslant |\xi| \leqslant 12$. *Define* $\mathfrak{a}_\lambda$ *by*

$$\mathfrak{a}_\lambda(p;q,r) = \sup_{\|f\|_p\leqslant 1}\Big\|\Big(\int_{1/2}^1 |e^{it\Delta}\eta(\tfrac{D}{\lambda})f|^r dt\Big)^{1/r}\Big\|_{L^q(\mathbb{R}^d)}. \tag{3.1}$$

Then for $\lambda \gg 1$,

$$\mathfrak{a}_\lambda(p;q,r) \geqslant c\lambda^{1/q-1/p}(\log\lambda)^{1/2-1/r}. \tag{3.2}$$

Proof. In what follows we set

$$\mathcal{A}_4(\lambda^2) = \{x : 3\lambda^2 \leqslant |\xi| \leqslant 4\lambda^2\}.$$

By Lemma 2.3, with parameters $a_0 = 3$, $a_1 = 4$, for $\lambda \gg 1$

$$\sup_{\|f\|_{L^p(\mathbb{R}^d)}\leqslant 1}\Big(\int_{\mathcal{A}_4(\lambda^2)}\Big(\int_{\lambda^2}^{2\lambda^2}\big|\mathcal{E}f\big(\tfrac{s}{\lambda^2}\xi,s\big)\big|^r ds\Big)^{\frac{q}{r}}d\xi\Big)^{\frac{1}{q}} \lesssim \mathfrak{a}_\lambda(p;q,r)\lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}.$$

Let

$$Tf(\xi,s) = \mathcal{E}f(\tfrac{s}{\lambda^2}\xi,s).$$

Using Khintchine's inequality we also get

$$\sup_{\|\{f_j\}\|_{L^p(\ell^2)}\leqslant 1}\Big(\int_{\mathcal{A}_4(\lambda^2)}\Big(\int_{\lambda^2}^{2\lambda^2}\Big(\sum_j |Tf_j|^2\Big)^{\frac{r}{2}}ds\Big)^{\frac{q}{r}}d\xi\Big)^{\frac{1}{q}} \lesssim \mathfrak{a}_\lambda(p;q,r)\lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}. \tag{3.3}$$

For integers $|j| \leqslant \lambda/10$, let $z^j = (\lambda^{-1}j, 0, \dots, 0)$ in $\mathbb{R}^d$. Let $I_j = \{y : |y-z^j| \leqslant (100d\lambda)^{-1}\}$. Let

$$R_j = \{(\xi,s) \in \mathbb{R}^{d+1} : |\xi_1 - 2j\lambda^{-1}s| \leqslant 10^{-1}\lambda,\, |\xi_i| \leqslant 10^{-1}\lambda, \quad i = 2,\dots,d,\, |s| \leqslant 100^{-1}\lambda^2\}.$$

For a pointwise lower bound we use the following lemma.

Lemma 3.2. *Let* $a \in \mathbb{R}^d$, $b \in \mathbb{R}$, *and* $g_j(y) = \chi_{I_j}(y)e^{i\langle a,y\rangle - ib|y|^2}$. *Then there is a constant* $c > 0$, *independent of* λ, j *so that*

$$\mathrm{Re}\,\big[e^{i\langle\xi-a,z^j\rangle - i(s-b)|z^j|^2}\mathcal{E}[g_j](\xi,s)\big] \geqslant c\lambda^{-d}, \text{ if } (\xi,s) \in R_j + (a,b).$$

Proof. Let $I_0 = \{y : |y| \leqslant (100d\lambda)^{-1}\}$. We have

$$\mathcal{E}g_j(\xi, s) = \int e^{is|y|^2 - i\langle \xi, y\rangle} g_j(y)\, dy = \int e^{-i\langle \xi - a, z^j + h\rangle + i(s-b)|z^j+h|^2} \chi_{I_j}(z^j + h) dh$$
$$= e^{-i\langle \xi - a, z^j\rangle} e^{i(s-b)|z^j|^2} \int e^{-i(\langle \xi - a - 2(s-b)z^j, h\rangle)} e^{i(s-b)|h|^2} \chi_{I_0}(h) dh$$

The pointwise lower bound follows quickly. □

Let N_λ to be the largest integer which is smaller than $\lambda/10$. By making use of the Besicovich set construction of Keich [15], there are vectors $v_j \in \mathbb{R}^{d+1}$ such that $v_j = a_j e_1 + b_j e_{d+1}$ for some $a_j, b_j \in \mathbb{R}$, $v_j + R_j \subset \{(\xi, s) : \lambda^2 \leqslant s \leqslant 2\lambda^2\}$, and

$$\mathrm{meas}\Big(\bigcup_{j=1}^{N_\lambda} (v_j + R_j) \Big) \lesssim \frac{\lambda^{d+3}}{\log \lambda}.$$

This is just an obvious extension of the two dimensional construction which gives a collection of rectangles $\{R_j^{[2]}\}$ and vectors (a_j, b_j) such that $\mathrm{meas}\big(\bigcup_{j=1}^{N_\lambda}(v_j + R_j^{[2]})\big) \lesssim \frac{\lambda^4}{\log \lambda}$ and $(a_j, b_j) + R_j^{[2]} \subset \{(\xi_1, s) : \lambda^2 \leqslant s \leqslant 2\lambda^2\}$.

Let $\Phi(\xi, s) = (\frac{s}{\lambda^2}\xi, s)$ which is 1–1 on $\mathcal{A}_4(\lambda^2) \times [\lambda^2, 2\lambda^2]$, and has Jacobian J_Φ with $|\det(J_\Phi(\xi, s))| \sim 1$. Let

$$V_j := \Phi^{-1}(v_j + R_j) \cap \big(\mathcal{A}_4(\lambda^2) \times [\lambda^2, 2\lambda^2]\big), \qquad E := \bigcup_{j=1,\ldots,N_\lambda} V_j.$$

Then it follows that

$$\lambda^{d+2} \lesssim \mathrm{meas}(V_j), \qquad \mathrm{meas}(E) \lesssim \frac{\lambda^{d+3}}{\log \lambda}\,. \tag{3.4}$$

Let $f_j(y) = \chi_{I_j}(y) e^{i\langle a_j, y\rangle - ib_j|y|^2}$. Then by Lemma 3.2,

$$|Tf_j(\xi)| \gtrsim \lambda^{-d}, \qquad \xi \in V_j, \tag{3.5}$$

and

$$\Big\| \Big(\sum |f_j|^2 \Big)^{1/2} \Big\|_p \lesssim \lambda^{(1-d)/p}. \tag{3.6}$$

We now modify arguments in [1]. By (3.4) and (3.5), we have

$$\lambda^{d+3} \lesssim N_\lambda \lambda^{d+2} \lesssim \sum_{j=1}^{N_\lambda} \mathrm{meas}(V_j) \tag{3.7}$$
$$= \int_E \sum_{j=1}^{N_\lambda} \chi_{V_j}(\xi, s)\, ds\, d\xi \lesssim \lambda^{2d} \int_E \sum_{j=1}^{N_\lambda} |Tf_j(\xi, s)|^2 ds\, d\xi,$$

and by applications of Hölder's inequality,

$$\lambda^{2d} \int_E \sum_{j=1}^{N_\lambda} |Tf_j(\xi, s)|^2 ds d\xi \lesssim \lambda^{2d} A \cdot B, \tag{3.8}$$

where

$$A = \Big(\int_{\mathcal{A}_4(\lambda^2)} \Big(\int_{\lambda^2}^{2\lambda^2} \Big(\sum_j |Tf_j(\xi,s)|^2\Big)^{\frac{r}{2}} ds\Big)^{\frac{q}{r}} d\xi\Big)^{\frac{2}{q}},$$

$$B = \Big(\int_{\mathcal{A}_4(\lambda^2)} \Big(\int_{\lambda^2}^{2\lambda^2} \chi_E(\xi,s)\,ds\Big)^{\frac{(q/2)'}{(r/2)'}} d\xi\Big)^{1-\frac{2}{q}}.$$

From (3.3) and (3.6) we obtain,

$$A \lesssim \Big(\lambda^{\frac{1-d}{p}} \mathfrak{a}_\lambda(p;q,r)\lambda^{-d+\frac{1}{p}+\frac{d}{q}+\frac{2}{r}}\Big)^2. \tag{3.9}$$

In order to estimate B we set

$$\mathfrak{v}(\xi) = \int_{\lambda^2}^{2\lambda^2} \chi_E(\xi,s)\,ds,$$

the measure of the vertical cross section of E at ξ. For $M>0$, we break

$$B \lesssim \Big(\int_{\{\xi\in\mathcal{A}_4(\lambda^2)\,:\,\mathfrak{v}(\xi)\leqslant M\}} \mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi\Big)^{1-\frac{2}{q}} + \Big(\int_{\{\xi\in\mathcal{A}_4(\lambda^2)\,:\,\mathfrak{v}(\xi)> M\}} \mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi\Big)^{1-\frac{2}{q}}.$$

From the construction of E it is obvious that $\mathfrak{v}$ is supported in a tube where $|\xi_1|\leqslant C\lambda^2$ and $|\xi_i|\leqslant C\lambda$, $2\leqslant i\leqslant d$, so that

$$\Big(\int_{\{\xi\in\mathcal{A}_4(\lambda^2)\,:\,\mathfrak{v}(\xi)\leqslant M\}} \mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi\Big)^{1-\frac{2}{q}} \lesssim M^{1-\frac{2}{r}}\lambda^{(d+1)(1-\frac{2}{q})}.$$

Moreover since $r\leqslant q$ and therefore $(1-\frac{(q/2)'}{(r/2)'})\geqslant 0$, by (3.4)

$$\begin{aligned}\Big(\int_{\{\xi\in\mathcal{A}_4(\lambda^2):\mathfrak{v}(\xi)> M\}} \mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi\Big)^{1-\frac{2}{q}} &\lesssim \Big(\int \mathfrak{v}(\xi) M^{\frac{(q/2)'}{(r/2)'}-1} d\xi\Big)^{1-\frac{2}{q}}\\ &\leqslant M^{\frac{2}{q}-\frac{2}{r}}\mathrm{meas}(E)^{1-\frac{2}{q}} \lesssim M^{\frac{2}{q}-\frac{2}{r}}\Big(\frac{\lambda^{d+3}}{\log\lambda}\Big)^{1-\frac{2}{q}}.\end{aligned}$$

Combining these two bounds, we have

$$B \lesssim M^{-2/r}\lambda^{(d+3)(1-\frac{2}{q})}\big[M\lambda^{-2(1-\frac{2}{q})} + M^{\frac{2}{q}}(\log\lambda)^{\frac{2}{q}-1}\big],$$

and choosing $M=\lambda^2(\log\lambda)^{-1}$, which optimizes the above, we obtain

$$B \lesssim \lambda^{(d+3)(1-\frac{2}{q})}\lambda^{\frac{4}{q}-\frac{4}{r}}(\log\lambda)^{\frac{2}{r}-1}. \tag{3.10}$$

Finally, we combine (3.10), (3.9), (3.8) and (3.7) to obtain

$$\lambda^{d+3} \lesssim \lambda^{2d}\lambda^{(d+3)(1-\frac{2}{q})}\lambda^{\frac{4}{q}-\frac{4}{r}}(\log\lambda)^{\frac{2}{r}-1}\big[\lambda^{\frac{1-d}{p}}\mathfrak{a}_\lambda(p;q,r)\lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}\big]^2,$$

which yields $\mathfrak{a}_\lambda(p;q,r)\geqslant c(\log\lambda)^{\frac{1}{2}-\frac{1}{r}}\lambda^{\frac{1}{q}-\frac{1}{p}}$. □

3.6. *Relation with Bochner–Riesz and the condition $\alpha > 0$ if $r = 2$, $p = q > 2$, $d \geqslant 2$.* The $L^p \to L^p(L^2(I))$ estimate implies sharp results for the Bochner–Riesz multiplier in the same way as the wave equation (*cf.* §7 in [22]).

For small $\delta > 0$, let us set $h_\delta(\xi) = \phi(\delta^{-1}(1-|\xi|^2))$ with $\phi \in C_c^\infty(-1,1)$. Let ψ be radial, supported in $\{1/2 < |\xi| < 2\}$ so that $\psi = 1$ on the support of h_δ. Then by the Fourier inversion formula and the support property of ψ it follows that

$$h_\delta(D)f = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta\widehat{\phi}(\delta s)\, e^{is} e^{is\Delta}\psi(D)f\, ds.$$

By the Schwarz inequality we get

$$|h_\delta(D)f| \leqslant \Big(\int |\delta\widehat{\phi}(\delta s)| ds\Big)^{1/2}\Big(\int |e^{is\Delta}\psi(D)f|^2 |\delta\widehat{\phi}(\delta s)| ds\Big)^{1/2}.$$

Thus we see that

$$\|h_\delta\|_{M_p} \lesssim \sup_{\|f\|_p\leqslant 1}\Big\|\Big(\int |e^{is\Delta}\psi(D)f|^2|\delta\widehat{\phi}(\delta s)|ds\Big)^{1/2}\Big\|_p,$$

which after rescaling becomes

$$\|h_\delta\|_{M_p} \lesssim \sup_{\|f\|_p\leqslant 1}\Big\|\Big(\int |e^{it\Delta}\psi(\sqrt{\delta}D)f|^2|\widehat{\phi}(t)|dt\Big)^{1/2}\Big\|_p.$$

Hence, using the rapid decay of $\widehat{\phi}$ and a further rescaling we see that the sharp bound $\|h_\delta\|_{M_p} \lesssim \delta^{1/2-d(1/2-1/p)}$, for $p > 2 + \frac{2}{d-1}$, would follow from $U : B^p_{\alpha,\nu} \to L^p(L^2(I))$, with $\alpha = d(1-\frac{2}{p}) - 1$, for any $\nu > 0$.

We see that the $L^p \to L^p(L^2(I))$ inequality for some $p > 2$ would imply that h_δ is a multiplier of $\mathcal{F}L^p$ with bounds independent of δ. However a variant of Fefferman's argument for the ball multiplier [9], based on a Kakeya set argument, shows that

$$\|h_\delta\|_{M_p} \gtrsim \log(1/\delta)^{1/2-1/p}. \tag{3.11}$$

This establishes the final necessary condition (vi) in Proposition 1.3. For completeness we include some details of the argument.

Proof of (3.11). By de Leeuw's theorem it suffices to prove the lower bound for $d = 2$. We may assume that $\delta < 10^{-10}$. By Khintchine's inequality, we have

$$\Big\|\big(\sum_\nu |h_\delta(D)f_\nu|^2\big)^{1/2}\Big\|_p \lesssim \|h_\delta\|_{M_p}\Big\|\big(\sum_\nu |f_\nu|^2\big)^{1/2}\Big\|_p. \tag{3.12}$$

For $\nu \in \mathbb{Z}\cap[-10^{-2}\delta^{-1/2},\, 10^{-2}\delta^{-1/2}]$, let us set

$$h_{\delta,\nu}(\xi) = h_\delta(\xi)\phi(\delta^{-1/2}\xi_1 - \nu)\chi_+(\xi), \quad \xi = (\xi_1,\xi_2) \in \mathbb{R}^2$$

where χ_+ is the characteristic function of the upper half plane. Define T_ν by $\widehat{T_\nu f} = h_{\delta,\nu}\widehat{f}$. Let η_ν be the inverse Fourier transform of a bump function which is supported on a ball of radius $C\delta^{-1/2}$ so that $\eta_\nu(\xi) = 1$ for ξ in the support of $h_{\delta,\nu}$. Define Φ_ν by $\widehat{\Phi}_\nu(\xi) = \eta_\nu(\xi)\phi(\delta^{-1/2}\xi_1 - \nu)\chi_+(\xi)$. Then $|\Phi_\nu(x)| \lesssim \delta^{-d/2}(1+\delta^{-1/2}|x|)^{-(d+1)}$ for the ν's under consideration, so that $\|\{\Phi_\nu * g_\nu\}\|_{L^p(\ell^2)} \lesssim \|\{g_\nu\}\|_{L^p(\ell^2)}$. Since $T_\nu g = h_\delta(D)[\Phi_\nu * g]$, inequality (3.12) applied to $f_\nu = \Phi_\nu * g_\nu$ implies that

$$\Big\|\big(\sum_\nu |T_\nu g_\nu|^2\big)^{1/2}\Big\|_p \lesssim \|h_\delta\|_{M_p}\Big\|\big(\sum_\nu |g_\nu|^2\big)^{1/2}\Big\|_p. \tag{3.13}$$

Let $\theta_\nu = (\delta^{1/2}\nu, \sqrt{1-\delta\nu^2})$, let $\theta_\nu^\perp$ be a unit vector perpendicular to θ_ν and

$$R_\nu = \big\{(x_1,x_2) : |\langle x, \theta_\nu\rangle| \leqslant 10^{-2}\delta^{-1},\ |\langle x, \theta_\nu^\perp\rangle| \leqslant 10^{-1}\delta^{-1/2}\big\}\,.$$

Letting $f_\nu(y) = \chi_{R_\nu}(y) e^{i\langle \theta_\nu, y\rangle}$, we have that

$$|e^{-i\langle x,\theta_\nu\rangle} T_\nu f_\nu(x)| \geqslant c > 0 \text{ for } x \in R_\nu\,. \tag{3.14}$$

Here we use again the sharp bounds in the construction of Besicovich sets [15]. There are vectors a_ν, $|\nu| \leqslant 10^{-2}\delta^{-1/2}$ so that with $E := \bigcup_\nu R_\nu$ the measure of E is $O(\delta^{-2}/\log\delta^{-1})$ but the corresponding translations $a_\nu + R^\nu$ have $O(1)$ overlap. Define $g_\nu(x) = f_\nu(x - a_\nu)$, which is supported in $a_\nu + R_\nu$. Then $|T_\nu g_\nu| \geqslant c$ on $a_\nu + R_\nu$. Thus we get

$$\delta^{-2} \lesssim \sum_\nu |R_\nu| \lesssim \sum_\nu \int \chi_{a_\nu + R_\nu}(x)\,dx \lesssim \int_E \sum_\nu |T_\nu g_\nu|^2 dx$$

and also by Hölder's inequality and (3.13) the last one in the above string of inequalities is bounded by

$$\mathrm{meas}(E)^{1-2/p} \Big\| \Big(\sum_\nu |T_\nu g_\nu|^2\Big)^{1/2}\Big\|_p^2 \lesssim \|h_\delta\|_{M_p}^2 \Big(\frac{\delta^{-2}}{\log\delta^{-1}}\Big)^{1-2/p} \Big\|\Big(\sum_\nu |g_\nu|^2\Big)^{1/2}\Big\|_p^2.$$

Now by the bounded overlap of the translated rectangles $a_\nu + R_\nu$, we see

$$\Big\|\Big(\sum_\nu |g_\nu|^2\Big)^{1/2}\Big\|_p^2 \lesssim \Big(\int \sum_\nu \chi_{a_\nu + R_\nu} dx\Big)^{2/p} \lesssim \Big(\sum_\nu |R_\nu|\Big)^{2/p} \lesssim \delta^{-4/p}.$$

Combining the three displayed inequalities we get $\delta^{-2} \lesssim \|h_\delta\|_{M_p}^2 (\delta^{-2}/\log\delta^{-1})^{1-2/p}\delta^{-4/p}$ and thus the desired (3.11). □

4. Proof of Theorem 1.4

The lower bounds for $\mathfrak{A}_\lambda(p;q,r)$ were established in the previous section, and here we prove the upper bounds, mainly by interpolation arguments. By Lemma 2.4, we can take $I = [1/2, 1]$.

4.1. *Proof of (i).* We consider the cases $\frac1q + \frac1r \geqslant \frac12$ and $\frac1q + \frac1r < \frac12$ separately.

The case $\frac1q + \frac1r \geqslant \frac12$. Note that the set

$$\big\{\,(\tfrac1p, \tfrac1q, \tfrac1r) \,:\, 2 \leqslant r \leqslant p \leqslant q \leqslant \infty, \quad \tfrac1q + \tfrac1r \geqslant \tfrac12\,\big\}$$

is the closed tetrahedron with vertices $(\frac14,\frac14,\frac14)$, $(\frac12,\frac12,\frac12)$, $(\frac12,0,\frac12)$, and $(0,0,\frac12)$. Hence by interpolation it is enough to show the estimate

$$\mathfrak{A}_\lambda(p;q,r) \lesssim \lambda^{\frac1q - \frac1p}[\log\lambda]^{\frac12 - \frac1r} \tag{4.1}$$

for $(p,q,r) = (4,4,4)$, $(2,2,2)$, $(2,\infty,2)$ and $(\infty,\infty,2)$. The estimate for $(p,q,r) = (2,2,2)$ is immediate from Plancherel's theorem. More generally we recall from [19] the estimate $\mathfrak{A}_\lambda(p;p,2) \lesssim 1$ with $2 \leqslant p \leqslant \infty$, which is related to a square-function estimate for equally spaced intervals. So we also get the estimates for $(p,q,r) = (\infty,\infty,2)$. For $(2,\infty,2)$ we choose a nonnegative $\chi_\circ \in C_c^\infty(\mathbb{R})$, so that $\chi_\circ(t) = 1$ on $[1/2,1]$. We need to estimate, for fixed x,

$$\int \chi_\circ(t) |U\eta(\tfrac{D}{\lambda}) f(x,t)|^2 dt = \frac{1}{(2\pi)^{2d}} \iint e^{ix(\xi - w)} \widehat{f}(\xi)\overline{\widehat{f}(w)}\eta(\tfrac{\xi}{\lambda})\overline{\eta(\tfrac{w}{\lambda})}\widehat{\chi_\circ}(|\xi|^2 - |w|^2)\,d\xi\,dw$$

and since $|\xi|+|w|\geqslant\lambda$, the above is bounded by

$$C_N\int_{\mathbb{R}}\int_{\mathbb{R}}\big(1+\lambda\big||\xi|-|w|\big|\big)^{-N}|\widehat{f}(\xi)||\widehat{f}(w)|\,d\xi\,dw\;\lesssim\;\lambda^{-1}\|\widehat{f}\|_2^2.$$

This gives the desired estimate for $(p,q,r)=(2,\infty,2)$. For $(p,q,r)=(4,4,4)$ we use the bound

$$\Big(\iint\Big|\psi(\xi,s)\int_{|y|\leqslant1}f(y)\,e^{i\lambda(s|y|^2-\xi y)}f(y)\,dy\Big|^4d\xi ds\Big)^{1/4}\lesssim\lambda^{-\frac12}(\log\lambda)^{\frac14}\|f\|_4,$$

where $\psi\in C_c^\infty$. This is implicit in [13] (see also [23] for more discussion and related issues). Then by rescaling, Lemma 2.2 and Lemma 2.4 we get (4.1) for $(p,q,r)=(4,4,4)$.

The case $\frac1q+\frac1r<\frac12$. We begin as before by observing that the set

$$\Delta_1=\big\{\,(\tfrac1p,\tfrac1q,\tfrac1r)\;:\;2\leqslant r\leqslant p\leqslant q\leqslant\infty,\quad \tfrac1q+\tfrac1r<\tfrac12\,\big\}$$

is the closed tetrahedron with vertices $(0,0,0)$, $(\frac14,\frac14,\frac14)$ $(\frac12,0,\frac12)$ and $(0,0,\frac12)$, from which the triangle with vertices $(\frac14,\frac14,\frac14)$ $(\frac12,0,\frac12)$, and $(0,0,\frac12)$ is removed. We use a bilinear analogue of our adjoint restriction operator, and rely on rather elementary estimates from [13]. Define χ_ℓ so that $\sum_{\ell\in\mathbb{Z}}\chi_\ell\equiv1$, $\chi_\ell=\chi_1(2^\ell\cdot)$ and χ_1 is supported in $(2,8)$. Let

$$\mathfrak{B}_{\lambda,\ell}[f,g]\;=\;\iint_{[-1,1]^2}e^{is(|y|^2+|z|^2)-i\frac{s}{\lambda^2}\xi(y+z)}\chi_\ell(|y-z|)f(y)g(z)\,dydz,$$

so that

$$(\mathcal{E}f\mathcal{E}f)(\frac{s}{\lambda^2}\xi,s)=\sum_{\ell\geqslant0}\mathfrak{B}_{\lambda,\ell}(f,f)(\xi,s).$$

We shall verify that for $\ell\geqslant0$

$$\|\mathfrak{B}_{\lambda,\ell}(f,g)\|_{L^{q/2}(\mathcal{A}(\lambda^2);L^{r/2}[\lambda^2,2\lambda^2])}\lesssim2^{-2\ell(\frac12-\frac1q-\frac1r)}\|f\|_p\|g\|_p \tag{4.2}$$

when $(\frac1p,\frac1q,\frac1r)$ is contained in the closed tetrahedron with vertices $(0,0,0)$, $(\frac14,\frac14,\frac14)$ $(\frac12,0,\frac12)$ and $(0,0,\frac12)$. By summing a geometric series, this yields (2.4) for $(\frac1p,\frac1q,\frac1r)\in\Delta_1$, which by Lemmata 2.2 and 2.4, yields the desired

$$\mathfrak{A}_\lambda(p;q,r)\lesssim\lambda^{1-\frac1p-\frac1q-\frac2r}. \tag{4.3}$$

We remark that conversely, if (4.3) holds, then we can use Lemma 2.3 and a Fourier expansion of $\chi_\ell(y-z)$ to bound the left hand side of (4.2) by $C\|f\|_p\|g\|_p$, with C independent of ℓ.

It remains to show (4.2). By interpolation it is enough to do this with $(p,q,r)=(\infty,\infty,\infty)$, $(4,4,4)$ $(2,\infty,2)$, and $(\infty,\infty,2)$. The last two estimates were already obtained; note that the bounds (4.1) and (4.3) coincide for the cases $(p,q,r)=(2,\infty,2)$ and $(\infty,\infty,2)$ and the bounds for (4.2) are independent of ℓ. Hence from the bounds (4.1) previously obtained and the discussion above we have the required bounds for $(p,q,r)=(2,\infty,2)$, and $(\infty,\infty,2)$. We note that the argument for the proof of the endpoint adjoint restriction theorem in [13] gives

$$\|B_{\lambda,\ell}(f,g)\|_{L^2_{\xi,s}}\lesssim\|f\|_4\|g\|_4, \tag{4.4}$$

uniformly in $\ell\geqslant0$, where $B_{\lambda,\ell}(f,g)(\xi,s)=\mathfrak{B}(f,g)(\frac{\lambda^2}{s}\xi,s)$, and by a change of variables we obtain (4.2) holds with $(p,q,r)=(4,4,4)$. To get the inequality (4.2) for $(p,q,r)=(\infty,\infty,\infty)$ we need to integrate $\chi_\ell(|y-z|)$ over $[-1,1]^2$ which yields the gain of $2^{-\ell}$.

4.2. *Proof of (ii).* We also consider the cases $1-\frac{1}{p}>\frac{2}{q}+\frac{1}{r}$ and $1-\frac{1}{p}\leqslant\frac{2}{q}+\frac{1}{r}$ separately.

The case $1-\frac{1}{p}\leqslant\frac{2}{q}+\frac{1}{r}$. We note that the set

$$\Delta_2=\big\{\,(\tfrac{1}{p},\tfrac{1}{q},\tfrac{1}{r})\,:\,2\leqslant p<r\leqslant q\leqslant\infty,\quad \tfrac{2}{q}+\tfrac{1}{r}\geqslant 1-\tfrac{1}{p}\,\big\}$$

is the closed tetrahedron with vertices $(\frac12,\frac12,\frac12)$, $(\frac14,\frac14,\frac14)$ $(\frac12,\frac16,\frac16)$ and $(\frac12,0,\frac12)$, from which the face with vertices $(\frac12,\frac12,\frac12)$, $(\frac14,\frac14,\frac14)$ and $(\frac12,0,\frac12)$ is removed. Note that from the previous bounds (4.1) and (4.3) we already have the required bounds

$$\mathfrak{A}_\lambda(p;q,r)\lesssim\lambda^{\frac{1}{q}-\frac{1}{r}} \tag{4.5}$$

for $(p,q,r)=(2,2,2)$ and $(2,\infty,2)$. Obviously Δ_2 is contained in the convex hull of $(\frac12,0,\frac12)$, $(\frac12,\frac12,\frac12)$, and the half open line segment $[(\frac12,\frac16,\frac16),(\frac14,\frac14,\frac14))$. Hence by it is enough to show (4.5) for $(\frac1p,\frac1q,\frac1r)$ contained in the half closed line segment $[(\frac12,\frac16,\frac16),(\frac14,\frac14,\frac14))$. But these follow from Lemmata 2.2 and 2.4, combined with the restriction estimate for the parabola which gives (2.4) for $(\frac1p,\frac1q,\frac1r)\in[(\frac12,\frac16,\frac16),(\frac14,\frac14,\frac14))$.

The case $1-\frac{1}{p}>\frac{2}{q}+\frac{1}{r}$. We note that the set

$$\big\{\,(\tfrac{1}{p},\tfrac{1}{q},\tfrac{1}{r})\,:\,2\leqslant p<r\leqslant q\leqslant\infty,\quad \tfrac{2}{q}+\tfrac{1}{r}<1-\tfrac{1}{p}\,\big\}$$

is contained in the quadrangular pyramid $\mathcal{Q}$ with vertices $(0,0,0)$, $(\frac12,0,0)$, $(\frac14,\frac14,\frac14)$ $(\frac12,\frac16,\frac16)$, and $(\frac12,0,\frac12)$. We need to show (4.3) for $(\frac1p,\frac1q,\frac1r)$ contained in the above set. Repeating the above argument, the asserted estimates follow if we establish, for $\ell\geqslant0$ and $(\frac1p,\frac1q,\frac1r)\in\mathcal{Q}$,

$$\|\mathfrak{B}_{\lambda,\ell}(f,g)\|_{L^{q/2}(\mathcal{A}(\lambda^2);L^{r/2}[\lambda^2,2\lambda^2])}\lesssim 2^{-\ell(1-\frac1p-\frac2q-\frac1r)}\|f\|_p\|g\|_p. \tag{4.6}$$

We only need to verify it for $(p,q,r)=(\infty,\infty,\infty)$, $(4,4,4)$, $(2,\infty,2)$, $(2,6,6)$, and $(2,\infty,\infty)$. The first three cases were already obtained when we showed (4.2), and the case $(p,q,r)=(2,6,6)$ follows from the linear adjoint restriction estimate for the parabola as before. Finally the case $(p,q,r)=(2,\infty,\infty)$ with a gain of $2^{-\ell/2}$ follows from the Schwarz inequality, and so we are done.

5. Sharper regularity results

5.1. *Combining frequency localized pieces.* One can use the uniform regularity results for the frequency localized pieces to prove sharper bounds such as Sobolev estimates by using arguments based on the Fefferman–Stein #-function. Let φ be a radial smooth function supported in $\{\xi:1/4<|\xi|<4\}$, not identically 0. Let $I=[-1,1]$ and

$$\Gamma(p;q,r)=\sup_{\lambda>1}\lambda^{-d(-\frac1p-\frac1q)+\frac2r}\big\|U\varphi\big(\tfrac{D}{\lambda}\big)\big\|_{L^p\to L^q(\mathbb{R}^d;L^r(I))} \tag{5.1}$$

It is not hard to verify that the finiteness of $\Gamma(p;q,r)$ is independent of the particular choice of φ. The following statement is a special case of the result in the appendix of [19].

Proposition 5.1. *Let $p_0,q_0,r_0\in[1,\infty]$, $q\in(q_0,\infty)$, $r_0\leqslant r<\infty$, $p_0\leqslant q_0$ and assume $1/p_0-1/q_0=1/p-1/q$. Suppose that $\Gamma(p_0;q_0,r_0)<\infty$. Then*

$$\Big\|\Big(\int_I|Uf(\cdot,t)|^r dt\Big)^{1/r}\Big\|_{L^q(\mathbb{R}^d)}\lesssim\|f\|_{B^p_{\alpha,q}(\mathbb{R}^d)},\quad \alpha=d\big(1-\tfrac1p-\tfrac1q\big)-\tfrac2r\,. \tag{5.2}$$

If $f \in B^p_{s,q}(\mathbb{R}^d)$ with $s = d(1-\frac{1}{p}-\frac{1}{q})$, then for almost every $x \in \mathbb{R}^d$ the function $t \mapsto U^a f(x,t)$ is locally in $B^q_{1/q,\nu}(\mathbb{R})$, and (thus) continuous, and

$$\big\| \sup_{t\in I} |Uf(\cdot,t)| \,\big\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{B^p_{s,q}(\mathbb{R}^d)}, \quad s = d\big(1-\tfrac{1}{p}-\tfrac{1}{q}\big)\,.$$

The Sobolev estimates follow from this since for $q \geqslant p \geqslant 2$ one has $L^p_\alpha \subset B^p_{\alpha,p} \subset B^p_{\alpha,q}$. We note that the result in [19] is slightly sharper. Namely the left hand side of (5.2) can be replaced by the $L^q(\mathbb{R}^d)$ norm of $(\sum_{k>0}(\int_I |P_k Uf(\,\cdot\,,t)|^r dt)^{\nu/r})^{1/\nu}$, where $\nu > 0$.

Proof of Corollaries 1.5 and 1.2. Proposition 5.1 implies the validity of the corollaries given their analogues for frequency localized functions (namely Theorems 1.4 and 1.1). For Corollary 1.2 we use that $\mathrm{R}^*(p_0 \to q_0)$ implies $\mathrm{R}^*(p \to q_0)$ for all $p \geqslant p_0$. $\square$

5.2. *A remark on recent results by Bourgain and Guth.* As mentioned in the introduction, the recent results in [5] on $\mathrm{R}^*(q \to q)$ give results on the sharp $L^q_\alpha \to L^q(\mathbb{R}^d \times I)$ boundedness of U. In a restricted range they also imply new results on $R^*(p \to q)$ with the best possible $p = p(q)$ which Tao [33] had proved for $q > \frac{2(d+3)}{d+1}$, and likewise one then obtains corresponding results for the Schrödinger operator. The following statement is proved by a simple interpolation argument for bilinear operators.

Proposition 5.2. *Suppose that $\mathrm{R}^*(q_0 \to q_0)$ holds for some $q_0 \in (2, \frac{2(d+3)}{d+1})$. Then*
(i) $\mathrm{R}^*(p \to q)$ holds with $q = \frac{d+2}{d}p'$ provided that

$$q > q_* := \frac{2(d+3)}{d+1}\Big(1-\gamma(d,q_0)\Big), \qquad \textit{where} \quad \gamma(d,q_0) = \frac{\frac{1}{q_0}-\frac{d+1}{2(d+3)}}{\frac{d+1}{2d}-\frac{d+2}{dq_0}}.$$

(ii) Let $q_* < q < \infty$, $q \leqslant r \leqslant \infty$ and suppose that $0 \leqslant \frac{1}{p}-\frac{1}{q} < 1-\frac{2(d+1)}{dq_*}$. Then $U : L^p_\alpha(\mathbb{R}^d) \to L^q(\mathbb{R}^d; L^r(I))$ is bounded with $\alpha = d\big(1-\frac{1}{p}-\frac{1}{q}\big)-\frac{2}{r}$.

In two dimensions $\mathrm{R}^*(q \to q)$ was proven in [5] for $q > 56/17$ and the sharp inequality $\mathrm{R}^*(p \to q)$ for $q = 2p'$ follows for $q > 13/4$.

Proof of Proposition 5.2. By Theorem 1.1 and Proposition 5.1 it suffices to prove the first part.

Let E_1 and E_2 be 1/2-separated sets in the unit ball of $\mathbb{R}^d$ and define $\mathcal{E}_i f = \mathcal{E}[f\chi_{E_i}]$. By Theorem 2.2 in [36], it suffices to prove the estimate

$$\big\|\mathcal{E}_1 f_1 \mathcal{E}_2 f_2\big\|_{q/2} \lesssim \|f_1\|_p \|f_2\|_p \tag{5.3}$$

for $q > q_*$ and p in a neighbourhood of $\frac{dq}{dq-d-2}$ (i.e. the p which satisfies $q = \frac{d+2}{d}p'$).

By hypothesis and Hölder's inequality, (5.3) holds with $p \geqslant q = q_0$. By Tao's theorem (5.3) holds with $p \geqslant 2$ and $q/2 > \frac{d+3}{d+1}$. The theorem then follows by interpolation of bilinear operators. Indeed, we determine $\theta \in (0,1)$ and $q_* \in (q_0, \frac{2(d+3)}{d+1})$ by

$$\frac{1-\theta}{2} + \frac{\theta}{q_0} = 1 - \frac{d+2}{dq_*}, \qquad (1-\theta)\frac{d+1}{d+3} + \theta\frac{2}{q_0} = \frac{2}{q_*}.$$

We compute $\theta = \big(\frac{d+2}{dq_*}-\frac{1}{2}\big)/\big(\frac{1}{2}-\frac{1}{q_0}\big)$ and $\theta = \big(\frac{1}{q_*}-\frac{d+1}{2(d+3)}\big)/\big(\frac{1}{q_*}-\frac{d+1}{2(d+3)}\big)$, from which we obtain $1/q_* = \big(\frac{d+1}{2(d+3)}-\frac{b}{2}\big)/\big(1-\frac{d+2}{d}b\big)$ with $b = \big(\frac{1}{q_0}-\frac{d+1}{2(d+3)}\big)/\big(\frac{1}{2}-\frac{1}{q_0}\big)$. A further computation shows that q_* is equal to $\frac{2(d+3)}{d+1}\big(1-\gamma(d,q_0)\big)$ as in the statement of the lemma. $\square$

6. Proof of Theorem 1.8

Definition. Fix $d \geqslant 1$, and let $p, q, r \in [2,\infty]$. For $N > 1$, let

$$\Lambda_{p,q,r}(N,\rho) \equiv \Lambda_{p,q,r}(N,\rho;d) = \sup \big\| Uf_1\, Uf_2 \big\|_{L^{q/2}(\mathbb{R}^d, L^{r/2}[0,\rho])}$$

where the supremum is taken over all pairs of function (f_1, f_2) whose Fourier transforms are supported in 1-separated subsets of $\{\xi : |\xi - Ne_1| \leqslant 2d\}$, and which satisfy $\|f_1\|_p, \|f_2\|_p \leqslant 1$.

We remark that the unit vector e_1 does not play a special role here. It could be replaced by any unit vector, by rotational invariance.

By considering two bump functions, it is easy to calculate that

$$\Lambda_{p,q,r}(N,\rho) \gtrsim N^{\frac{2}{q}-\frac{2}{r}}, \quad 1 \leqslant p,q,r \leqslant \infty, \tag{6.1}$$

whenever $\rho > 1$, and significant for Theorem 1.8 is the following two dimensional estimate,

$$\sup_{\rho>1} \Lambda_{2,q,r}(N,\rho;2) \lesssim N^{\frac{2}{q}-\frac{2}{r}}, \quad q > 16/5, \quad r \geqslant 4\,, \tag{6.2}$$

which was proven in [20] (see also [18] and [26] for related previous results). We will combine this with the following two lemmata.

Lemma 6.1. *Let $p_0 \leqslant p \leqslant q \leqslant r$ and $\varepsilon_{\rm o} > 0$. Then, for $N, \rho > 1$,*

$$\Lambda_{p,q,r}(N,\rho) \lesssim N^{\varepsilon_{\rm o}} \rho^{2d(\frac{1}{p_0}-\frac{1}{p})} \Lambda_{p_0,q,r}(N,\rho)\,. \tag{6.3}$$

Lemma 6.2. *Let $2 \leqslant p \leqslant q \leqslant r \leqslant \frac{2q}{q-2}$ and $\varepsilon > 0$. Let $\psi \in C_c^\infty$ be supported in the annulus $\{\xi \in \mathbb{R}^d : 1/2 \leqslant |\xi| \leqslant 2\}$. Then, for $\lambda > 1$,*

$$\begin{aligned} &\big\| U\psi(\tfrac{D}{\lambda}) f \big\|_{L^q(\mathbb{R}^d; L^r[0,1])} \\ &\quad \lesssim \Big(\lambda^{\frac{4}{q}-2d(\frac{1}{p}-\frac{1}{q})} + \sup_{1<N<\lambda} N^{\frac{4}{r}-2d(\frac{1}{p}-\frac{1}{q})+\varepsilon} \Lambda_{p,q,r}(N, C\lambda^2/N^2) \Big)^{1/2} \lambda^{-\frac{2}{r}+d(\frac{1}{p}-\frac{1}{q})} \|f\|_p\,. \end{aligned} \tag{6.4}$$

Lemma 6.1 relies on a localization argument such as in [17] and Lemma 6.2 relies on a by now standard scaling argument in [36] which reduces estimates for bilinear operators with separation assumptions to estimates for linear operators.

We may combine (6.3), with $p_0 = 2$, and (6.4) to obtain

Corollary 6.3. *Let $2 \leqslant p \leqslant q \leqslant r \leqslant \frac{2q}{q-2}$. Suppose that*

$$\sup_{\rho>1} \Lambda_{2,q,r}(N,\rho;d) \lesssim N^\gamma, \quad \text{for some} \quad \gamma < 2d\big(1-\tfrac{1}{p}-\tfrac{1}{q}\big) - \tfrac{4}{r}. \tag{6.5}$$

Then if $d(1-\frac{1}{p}-\frac{1}{q}) - \frac{2}{q} \geqslant 0$, then for all $\lambda > 1$,

$$\big\| U\psi(\tfrac{D}{\lambda}) f \big\|_{L^q(\mathbb{R}^d; L^r[0,1])} \lesssim \lambda^{d(1-\frac{1}{p}-\frac{1}{q})-\frac{2}{r}} \|f\|_p\,. \tag{6.6}$$

Supposing this for the moment we give the

Proof of Theorem 1.8. By Proposition 5.1 it suffices to prove, in two spatial dimensions, the estimate (6.6) for $p = q > 16/5$ and $r \geqslant 4$. Using (6.2), we put $\gamma = 2/q - 2/r$ and verify that the condition (6.5) with $d = 2$ in the range $p = q > 16/5$ and $r \geqslant 4$. Thus (6.6) holds in this range, and we are done. □

Proof of Lemma 6.1. Let η_1, η_2 be smooth, supported in balls of diameter $1/2$ which are contained in $\{\xi : |\xi - Ne_1| \leqslant 2d\}$, and which are separated by $1/2$. Define the operators S_1, S_2 by $\widehat{S_i f}(\xi,t) = \eta_i(\xi)\widehat{Uf}(\xi)$, $i = 1, 2$. It suffices to prove that $\|S_1 f_1\, S_2 f_2\|_{L^{q/2}(\mathbb{R}^d,\, L^{r/2}[0,\rho])}$ is dominated by $\|f_1\|_p \|f_2\|_p$ times a constant multiple of the expression on the right hand side of (6.3).

We partition $\mathbb{R}^d$ into cubes $\mathcal{Q}_\nu$ of side ρ with centre $\rho\nu \in \rho\mathbb{Z}^d$, and define

$$\mathcal{P}_\nu = \{(x,t) \in \mathbb{R}^d \times [0,\rho] \,:\, x - 2tNe_1 \in \mathcal{Q}_\nu\}. \tag{6.7}$$

The parallelipipeds form a partition of $\mathbb{R}^d \times [0,\rho]$. For fixed x the intervals $I_\nu^x = \{t : (x,t) \in \mathcal{P}_\nu\}$ are disjoint. Thus

$$\|F\|^{q/2}_{L^{q/2}(\mathbb{R}^d;\, L^{r/2}[0,\rho])} \leqslant \int_{\mathbb{R}^d} \Big(\sum_\nu \int_{I_\nu^x} |F(x,t)|^{r/2} dt \Big)^{q/r} dx \leqslant \sum_\nu \big\| \chi_{\mathcal{P}_\nu} F \big\|^{q/2}_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\rho])};$$

here we used the triangle inequality for $\|\cdot\|^{q/r}_{\ell^{q/r}}$ as $q/r \leqslant 1$.

Taking $F = S_1 f_1 S_2 f_2$, and denoting by $\mathcal{Q}_\nu^*$, the enlarged cube with side $50 d\rho N^\varepsilon$, where $0 < \varepsilon < 4d\varepsilon_{\rm o}$, we obtain

$$\begin{aligned}\big\|S_1 f_1\, S_2 f_2\big\|^{q/2}_{L^{q/2}(\mathbb{R}^d;\, L^{r/2}[0,\rho])} &\leqslant \sum_\nu \|\chi_{\mathcal{P}_\nu} S_1 f_1\, S_2 f_2\|^{q/2}_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\rho])} \\ &\lesssim \sum_\nu (I_\nu^{q/2} + II_\nu^{q/2} + III_\nu^{q/2} + IV_\nu^{q/2}),\end{aligned}$$

where

$$\begin{aligned} I_\nu &= \big\|\chi_{\mathcal{P}_\nu} S_1[f_1 \chi_{\mathcal{Q}_\nu^*}]\, S_2[f_2 \chi_{\mathcal{Q}_\nu^*}]\big\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\rho])}, \\ II_\nu &= \big\|\chi_{\mathcal{P}_\nu} S_1[f_1 \chi_{\mathbb{R}^d\setminus\mathcal{Q}_\nu^*}]\, S_2[f_2 \chi_{\mathcal{Q}_\nu^*}]\big\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\rho])}, \\ III_\nu &= \big\|\chi_{\mathcal{P}_\nu} S_1[f_1 \chi_{\mathcal{Q}_\nu^*}]\, S_2[f_2 \chi_{\mathbb{R}^d\setminus\mathcal{Q}_\nu^*}]\big\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\rho])}, \\ IV_\nu &= \big\|\chi_{\mathcal{P}_\nu} S_1[f_1 \chi_{\mathbb{R}^d\setminus\mathcal{Q}_\nu^*}]\, S_2[f_2 \chi_{\mathbb{R}^d\setminus\mathcal{Q}_\nu^*}]\big\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\rho])}. \end{aligned} \tag{6.8}$$

First we consider the main terms I_ν. By Hölder's inequality,

$$I_\nu \leqslant \Lambda_{p_0,q,r}(N,\rho) \prod_{i=1}^2 \|f_i \chi_{\mathcal{Q}_\nu^*}\|_{p_0} \lesssim \Lambda_{p_0,q,r}(N,\rho) (\rho N^\varepsilon)^{2d(\frac{1}{p_0} - \frac{1}{p})} \prod_{i=1}^2 \|f_i \chi_{\mathcal{Q}_\nu^*}\|_p$$

We use the Schwarz inequality, the embedding $\ell^p \subset \ell^q$, $p \leqslant q$, and the fact that every x is contained in only $O(N^{\varepsilon d})$ of the cubes $\mathcal{Q}_\nu^*$ to get

$$\sum_\nu \prod_{i=1}^2 \|f_i \chi_{\mathcal{Q}_\nu^*}\|_p^{q/2} \leqslant \prod_{i=1}^2 \Big(\sum_\nu \|f_i \chi_{\mathcal{Q}_\nu^*}\|_p^q \Big)^{1/2} \lesssim N^{\varepsilon d} \prod_{i=1}^2 \|f_i\|_p^{q/2}.$$

Combining the previous two estimates we bound

$$\big(\sum_\nu I_\nu^{q/2}\big)^{2/q} \lesssim N^{2d\varepsilon(\frac{1}{p_0} - \frac{1}{p} + \frac{1}{q})} \rho^{2d(\frac{1}{p_0} - \frac{1}{p})} \Lambda_{p_0,q,r}(N,\rho) \prod_{i=1}^2 \|f_i\|_p. \tag{6.9}$$

We use very crude estimates to handle the remaining three terms which can to be dominated by $C_{M,\varepsilon}(N^\varepsilon\rho)^{-M}\|f_1\|_p\|f_2\|_p$, which finishes the proof since $\Lambda_{p_0,q,r}(N,\rho) \gtrsim N^{\frac{2}{q} - \frac{2}{r}}$ by (6.1).

We only give the argument to bound $\sum_\nu II_\nu^{q/2}$ as the other terms are handled similarly. By the Schwarz inequality we estimate $\sum_\nu II_\nu^{q/2}$ by

$$\Big(\sum_\nu \big\|\chi_{\mathcal{P}_\nu} S_1[f_1\chi_{\mathbb{R}^d\setminus\mathcal{Q}_\nu^*}]\big\|^q_{L^q(\mathbb{R}^d;L^r[0,\rho])}\Big)^{1/2}\Big(\sum_\nu \big\|S_2[f_2\chi_{\mathcal{Q}_\nu^*}]\big\|^q_{L^q(\mathbb{R}^d;L^r[0,\rho])}\Big)^{1/2}. \tag{6.10}$$

For the second factor we use a wasteful bound, namely that the $L^p \to L^q(\mathbb{R}^d;L^r[0,\rho])$ operator norm of S_2 is $O(\rho^{1/r}N^d)$. Consequently, the second factor in (6.10) can be bounded by $C\rho^{q/(2r)}N^{d(\varepsilon+q/2)}\|f_2\|_p^{q/2}$.

We consider the first factor in (6.10) and write $S_1f(x,t)=\mathcal{K}_t * f(x)$ where

$$\mathcal{K}_t(y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\chi(\xi-Ne_1)e^{-it|\xi|^2+i\langle y,\xi\rangle}\,d\xi$$

with $\chi\in C_c^\infty$ equal to 1 on the ball of radius $2d$ centred at the origin. Integration by parts yields that for every $t\in[0,\rho]$

$$|\mathcal{K}_t(y)|\leqslant C_M|y-2tNe_1|^{-M} \quad \text{ if } \ |y-2tNe_1|\geqslant 4d\rho.$$

Let $\mathfrak{c}_\nu$ be the centre of $\mathcal{Q}_\nu^*$. If $x-y\in\mathbb{R}^d\setminus\mathcal{Q}_\nu^*$ and $(x,t)\in\mathcal{P}_\nu$, then $|x-y-\mathfrak{c}_\nu|\geqslant 10d\rho N^\varepsilon$, $|x-2tNe_1-\mathfrak{c}_\nu|\leqslant 2d\rho N^\varepsilon$, and therefore also $|y-2tNe_1|\geqslant 8d\rho N^\varepsilon$. Thus for this choice of (x,t) and y we have

$$\big|S_1[f_1\chi_{\mathbb{R}^d\setminus\mathcal{Q}_\nu^*}]\big|\lesssim(\rho N^\varepsilon)^{-M+d+1}\int_{|y-2tNe_1|\geqslant 8d\rho N^\varepsilon}\frac{|f_1(x-y)|}{|y-2tNe_1|^{d+1}}dy$$

and the integral is bounded by $(\rho N)^{d+1}\int(1+|y|)^{-d-1}|f_1(x-y)|dy$. Here we use $\rho>1$. Now let $\mathcal{Q}_\nu^{**}$ be the cube of sidelength $\rho(2+N)$ centred at $\mathfrak{c}_\nu$; then $\mathcal{Q}_\nu^{**}\times[0,\rho]$ contains $\mathcal{P}_\nu$. Letting $\mathcal{C}_{\rho,N}:=\rho^{1/r}(\rho N^\varepsilon)^{-M_1+d+1}(\rho N)^{d+1}$, we have

$$\sum_\nu\big\|\chi_{\mathcal{P}_\nu}S_1[f_1\chi_{\mathbb{R}^d\setminus\mathcal{Q}_\nu^*}]\big\|^q_{L^q(\mathbb{R}^d;L^r[0,\rho])}\lesssim\mathcal{C}^q_{\rho,N}\sum_\nu\int_{\mathcal{Q}_\nu^{**}}\Big|\int\frac{|f_1(x-y)|}{(1+|y|)^{d+1}}dy\Big|^q dx$$

which is $\lesssim\mathcal{C}^q_{\rho,N}(\rho N)^{(d+1)}\|f_1\|_p^q$; here one uses Young's inequality and the fact that each $x\in\mathbb{R}^d$ is contained in at most $O((\rho N)^{d+1})$ of the cubes $\mathcal{Q}_\nu^{**}$. Collecting the estimates yields the crude bound

$$\sum_\nu II_\nu^{q/2}\leqslant C_M(\rho N^\varepsilon)^{-M}(\rho N)^{10dq}\|f_1\|_p^{q/2}\|f_2\|_p^{q/2},$$

and we conclude by choosing M sufficiently large. □

Proof of Lemma 6.2. For $j\geqslant 0$, we write

$$A(j,\lambda):=2^{2j(\frac{2}{r}-d(\frac1p-\frac1q))}\sup_{2^{j-1}\leqslant N\leqslant 2^{j+1}}\Lambda_{p,q,r}(N,C\lambda^2 2^{-2j+1}).$$

Define $T=U\psi(D)$, and thus $U\psi(\frac{D}{\lambda})f(x,t)=T[f(\lambda^{-1}\cdot)](\lambda x,\lambda^2t)$. By scaling,

$$\|U\psi(\tfrac{D}{\lambda})\|_{L^p\to L^q(\mathbb{R}^d;L^r[0,1])}=\lambda^{-\frac2r+d(\frac1p-\frac1q)}\|T\|_{L^p\to L^q(\mathbb{R}^d;L^r[0,\lambda^2])}\,, \tag{6.11}$$

so that the statement of the lemma is an immediate consequence of

$$\|T\|_{L^p\to L^q(\mathbb{R}^d;L^r[0,\lambda^2])}\lesssim\Big(\lambda^{\frac4q-2d(\frac1p-\frac1q)}+\sum_{1\leqslant 2^j\leqslant\lambda}A(j,\lambda)\Big)^{1/2}. \tag{6.12}$$

Now by scaling we have that

$$\big\|Tf_1\,Tf_2\big\|_{L^{q/2}(\mathbb{R}^d;L^{r/2}[0,\lambda^2])}\lesssim A(j,\lambda)\prod_{i=1}^2\|f_i\|_p, \tag{6.13}$$

whenever $\widehat{f_1}$ and $\widehat{f_2}$ are supported in a 2^{-j+1} ball, contained in $\{\xi:1/2<|\xi|\leqslant 2\}$, and their supports are 2^{-j}-separated. We will also require the following simpler estimates

$$\big\|Tf_1\,Tf_2\big\|_{L^{q/2}(\mathbb{R}^d;L^{r/2}[0,\lambda^2])}\lesssim \lambda^{\frac{4}{q}-2d(\frac1p-\frac1q)}\prod_{i=1}^2\|f_i\|_p, \tag{6.14}$$

whenever $\widehat{f_1}$ and $\widehat{f_2}$ are supported in an ball of radius λ^{-1}, contained in $\{\xi:1/2<|\xi|\leqslant 2\}$. By the Schwarz inequality, this follows from $\big\|Tf_1\big\|_{L^q(\mathbb{R}^d;L^r[0,\lambda^2])}\lesssim \lambda^{\frac{2}{q}-d(\frac1p-\frac1q)}\|f_1\|_p$. Let $t\mapsto\varpi(t)$ be a Schwartz function which is positive on $[0,4d]$ and whose Fourier transform is supported in $[-1,1]$. By scaling and rotation this would follow from

$$\big\|\varpi Tf\big\|_{L^q(\mathbb{R}^d;L^r(\mathbb{R}))}\lesssim \lambda^{\frac2q-\frac2r}\|f\|_p \tag{6.15}$$

whenever $\widehat{f}$ is supported in $\{\xi:|\xi-\lambda e_1|\leqslant 2d\}$. By a change of variables and trivial estimates it is easy to see (6.15) for $1\leqslant p\leqslant q=r\leqslant\infty$. The estimate for $r>q$ follows by applying Bernstein's inequality in t since the temporal Fourier transform of ϖTf is contained in $\{s:s\sim\lambda^2\}$.

We now argue similarly as in [36]. Write $\|Tf\|^2_{L^q(\mathbb{R}^d;L^r[0,\lambda^2])}=\|Tf\,Tf\|_{L^{q/2}(\mathbb{R}^d;L^{r/2}[0,\lambda^2])}$. For each j, $1\leqslant 2^j\leqslant 2\lambda$, we tile $\mathbb{R}^d$ with dyadic cubes $s^j_\ell=\prod_{i=1}^d[2^{-j}\ell_i,2^{-j}\ell_{i+1})$ of sidelength 2^{-j}, indexed by $\ell\in\mathbb{Z}^d$. For j, $1\leqslant 2^j\leqslant\lambda$, we write $\ell\sim_j\tilde\ell$ if s^j_ℓ and $s^j_{\tilde\ell}$ have adjacent parents, but are not adjacent. When $\lambda<2^j\leqslant 2\lambda$, we mean by $\ell\sim_j\tilde\ell$ that the distance between s^j_ℓ and $s^j_{\tilde\ell}$ is $\lesssim\lambda^{-1}$. Then, we then can write for every $(\xi,\eta)\in\mathbb{R}^d$, with $\xi\neq\eta$,

$$\sum_{1\leqslant 2^j\leqslant 2\lambda}\ \sum_{\substack{(\ell,\tilde\ell)\\ \ell\sim_j\tilde\ell}}\chi_{s^j_\ell}(\xi)\chi_{s^j_{\tilde\ell}}(\eta)=1 \tag{6.16}$$

Define P^j_ℓ by $\widehat{P^j_\ell f}=\chi_{s^j_\ell}\widehat{f}$; then the operators P^j_ℓ are bounded on L^p, $1<p<\infty$, with operator norms independent of ℓ and j. For any Schwartz function f we have by (6.16)

$$[Tf(x,t)]^2=\sum_{1\leqslant 2^j\leqslant 2\lambda}\ \sum_{(\ell,\tilde\ell):\ell\sim_j\tilde\ell}TP^j_\ell f(x,t)\,TP^j_{\tilde\ell}f(x,t)$$

Let $\varphi\in C^\infty_c$ be supported in $[-1,1]^d$, satisfying $\sum_{\mathfrak{z}\in\mathbb{Z}^d}\varphi(\xi-\mathfrak{z})=1$ for all $\xi\in\mathbb{R}^d$. Define $\widetilde{P}^j_{\mathfrak{z}}$ as acting on $L^a(L^b)$ functions by $\widehat{\widetilde{P}^j_{\mathfrak{z}}G}(\xi,t)=\varphi(2^j\xi-\mathfrak{z})\widehat{G}(\xi,t)$. We use the inequality

$$\Big\|\sum_{\mathfrak{z}}\widetilde{P}^j_{\mathfrak{z}}G_{\mathfrak{z}}\Big\|_{L^a(L^b)}\leqslant C\big\|\{G_{\mathfrak{z}}\}\big\|_{\ell^a(L^a(L^b))},\quad 1\leqslant a\leqslant 2,\quad a\leqslant b\leqslant a', \tag{6.17}$$

The constant C in (6.17) is independent of j. The inequality follows from Plancherel's theorem in the case $a=b=2$, and from an application of Minkowski's inequality in the case $a=1$, $1\leqslant b\leqslant\infty$. The intermediate cases follow by interpolation. Note that for any j and any $\mathfrak{z}\in\mathfrak{Z}^d$ the number of pairs $(\ell,\tilde\ell)$ with $\ell\sim_j\tilde\ell$ for which $\widetilde{P}^j_{\mathfrak{z}}[TP^j_\ell f\,TP^j_{\tilde\ell}f]\neq 0$ is uniformly bounded (independent of j, $\mathfrak{z}$, f). Thus inequality (6.17) applied with $a = q/2$, $b = r/2$, implies

$$\|Tf\|^2_{L^q(L^r[0,\lambda^2])} \lesssim \sum_{1\leqslant 2^j\leqslant 2\lambda} \Big(\sum_{\ell\sim_j\tilde\ell} \|TP^j_\ell f\; TP^j_{\tilde\ell} f\|^{q/2}_{L^{q/2}(L^{r/2}[0,\lambda^2])}\Big)^{2/q}; \tag{6.18}$$

here we use that $1 \leqslant q/2 \leqslant r/2 \leqslant (q/2)'$ (i.e. $q \leqslant r \leqslant \frac{2q}{q-2}$ which implies that $q/2 \leqslant 2$.)

Now by (6.13) and (6.14) the right hand side of (6.18) is dominated by a constant times

$$\sum_{1\leqslant 2^j\leqslant\lambda} A(j,\lambda)\Big(\sum_{\ell\sim_j\tilde\ell}\|P^j_\ell f\|_p^{q/2}\|P^j_{\tilde\ell}f\|_p^{q/2}\Big)^{2/q} + \lambda^{\frac4q-2d(\frac1p-\frac1q)}\Big(\sum_{\ell\sim_{j_0}\tilde\ell}\|P^{j_0}_\ell f\|_p^{q/2}\|P^{j_0}_{\tilde\ell}f\|_p^{q/2}\Big)^{2/q}$$
$$\lesssim \lambda^{\frac4q-2d(\frac1p-\frac1q)}\Big(\sum_\ell\|P^{j_0}_\ell f\|_p^q\Big)^{2/q} + \sum_{1\leqslant 2^j\leqslant\lambda} A(j,\lambda)\Big(\sum_\ell \|P^j_\ell f\|_p^q\Big)^{2/q}.$$

Here j_0 is the integer such that $\lambda < 2^{j_0} \leqslant 2\lambda$, and we have used the Schwarz inequality and the fact that for each (j,ℓ) the number of $\tilde\ell$ with $\ell\sim_j\tilde\ell$ is uniformly bounded. Since $2 \leqslant p \leqslant q$, we also have

$$\Big(\sum_\ell\|P^j_\ell f\|_p^q\Big)^{1/q} \lesssim \|f\|_p,$$

and thus we have shown (6.12). □

References

[1] W. Beckner, A. Carbery, S. Semmes, and F. Soria, *A note on the restriction of the Fourier Transform to spheres,* Bull. London. Math. Soc. **21** (1989), 394–398.

[2] J. Bennett, A. Carbery and T. Tao, On the multilinear restriction and Kakeya conjectures, Acta Math. **196** (2006), no. 2, 261–302.

[3] J. Bourgain, *Some new estimates on oscillatory integrals,* Essays on Fourier analysis in honor of Elias M. Stein, 83?112, Princeton Math. Ser., 42, Princeton Univ. Press, Princeton, NJ, 1995.

[4] ———, *On the Schrödinger maximal function in higher dimension*, arXiv:1201.3342.

[5] J. Bourgain and L. Guth, *Bounds on oscillatory integral operators based on multilinear estimates,* Geom. Funct. Anal. **21** (2011), 1239–1295.

[6] A. Carbery *Restriction implies Bochner–Riesz for paraboloids,* Math. Proc. Cambridge Philos. Soc. **111** (1992), no. 3, 525–529.

[7] L. Carleson, *Some analytic problems related to statistical mechanics,* Euclidean harmonic analysis, 5–45, Lecture Notes in Math., 779, Springer, Berlin, 1980.

[8] B.E.J. Dahlberg and C.E. Kenig, *A note on the almost everywhere behavior of solutions to the Schrödinger equation,* Harmonic analysis, 205–209, Lecture Notes in Math., 908, Springer, Berlin.

[9] C. Fefferman, *The multiplier problem for the ball,* Ann. of Math. (2) **94** (1971), 330–336.

[10] C. Fefferman and E. Stein, *H^p spaces of several variables,* Acta Math. **129** (1972), no. 3-4, 137–193.

[11] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schrodinger equation revisited* , Ann. Inst. H. Poincare Anal. Non Lineare **2** (1985), 309-327.

[12] L. Hörmander, *Estimates for translation invariant operators in L^p spaces,* Acta Math. **104** (1960), 93–140.

[13] ———, *Oscillatory integrals and multipliers on FL^p,* Ark. Mat. **11** (1973), 1–11.

[14] M. Keel and T. Tao, *Endpoint Strichartz estimates,* Amer. J. Math. **120** (1998), 955–980.

[15] U. Keich, *On L^p bounds for Kakeya maximal functions and the Minkowski dimension in $\mathbb{R}^2$,* Bull. London Math. Soc. **31** (1999), no. 2, 213?-221.

[16] C.E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations* , Indiana Univ. Math. J. **40** (1991), no. 1, 33–69.

[17] S. Lee, *Improved bounds for Bochner–Riesz and maximal Bochner–Riesz operators,* Duke Math. J. **122** (2004), 205–232.

[18] ______, *On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$,* Int. Math. Res. Not., (2006), Art. ID 32597, 21.
[19] S. Lee, K.M. Rogers and A. Seeger, *Improved bounds for Stein's square functions,* `arXiv:1012.2159`, Proc. London. Math. Soc., doi:10.1112/plms/pdr067.
[20] S. Lee, K.M. Rogers and A. Vargas, *An endpoint space–time estimate for the Schrödinger equation,* Adv. Math. **226** (2011), 4266–4285.
[21] A. Miyachi, *On some singular Fourier multipliers,* J. Fac. Sci. Univ. Tokyo Sect. IA Math., **28** (1981), 267–315.
[22] G. Mockenhaupt, A. Seeger and C. D. Sogge. *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates,* J. Amer. Math. Soc. **6** (1993), no. 1, 65?-130.
[23] D. Müller and A. Seeger, *Regularity properties of wave propagation on conic manifolds and applications to spectral multipliers,* Adv. Math. **161** (2001), no. 1, 41?130.
[24] F. Planchon, *Dispersive estimates and the 2D cubic NLS equation,* J. Anal. Math. **86** (2002), 319?-334.
[25] K.M. Rogers, *A local smoothing estimate for the Schrödinger equation,* Adv. Math. **219** (2008), no. 6, 2105–2122.
[26] ______, *Strichartz estimates via the Schrödinger maximal operator,* Math. Ann. **343** (2009), no.3, 604–622.
[27] K.M. Rogers and A. Seeger, *Endpoint maximal and smoothing estimates for Schrödinger equations,* J. Reine Angew. Math., **640** (2010), 47-66.
[28] A. Ruiz and L. Vega, On local regularity of Schrödinger equations, Internat. Math. Res. Notices **1993**, no. 1, 13–27.
[29] P. Sjölin, *Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in R^2* , Studia Math. **51** (1974), 169–182.
[30] ______, *Regularity of solutions to the Schrödinger equation,* Duke Math. J., **55** (1987), 699–715.
[31] ______, *A counter-example concerning maximal estimates for solutions to equations of Schrödinger type* , Indiana Univ. Math. J. **47** (1998), no. 2, 593–599.
[32] R. Strichartz, *Restrictions of Fourier transform to quadratic surfaces and decay of solutions of wave Equations,* Duke Math. J. **44** (1977), 705-714.
[33] T. Tao, *A sharp bilinear restriction estimate for paraboloids,* Geom. Funct. Anal. **13** (2003), 1359–1384.
[34] T. Tao, *Nonlinear dispersive equations,* Local and global analysis. CBMS 106, eds: AMS, 2006.
[35] T. Tao and A. Vargas, *A bilinear approach to cone multipliers. II. Applications,* Geom. Funct. Anal. **10** (2000), no. 1, 216–258.
[36] T. Tao, A. Vargas and L. Vega. *A bilinear approach to the restriction and Kakeya conjectures,* J. Amer. Math. Soc. **11** (1998), 967–1000.
[37] L. Vega, *Schrödinger equations: pointwise convergence to the initial data,* Proc. Amer. Math. Soc. **102** (1988), no. 4, 874–878.

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